Diskrete Mathematik und Optimierung

Winfried Hochstättler:

The toric ideal of a cographic matroid is generated by quadrics

Technical Report feu-dmo023.10
Contact: `winfried.hochstaettler@fernuni-hagen.de`

FernUniversität in Hagen
Fakultät für Mathematik und Informatik
Lehrgebiet für Diskrete Mathematik und Optimierung
D – 58084 Hagen

2000 Mathematics Subject Classification: 05B35, 05C99
Keywords: matroid, basis exchange, toric ideal, cographic matroid

The toric ideal of a cographic matroid is generated by quadrics

Winfried Hochstättler

January 16, 2012

Abstract

The purpose of this note is to extend the results obtained for graphic matroids by Jonah Blasiak in [1] to cographic matroids.

1 Introduction

Neil White [4] conjectured that, given a matroid M on a finite set E, any two n-tuples of bases of M with the same ground set (taking multiplicities into account) are connected by a series of symmetric set swaps, as well as by a series of symmetric single element swaps. In algebraic terms this means that the toric ideal I_M of a matroid is generated by the quadratic binomials $y_{B_1}y_{B_2} - y_{B_1'}y_{B_2'}$, where $(B_1, B_2), (B_1', B_2')$ are pairs of bases connected by single element swaps, i.e. $(B_1', B_2') = (B_1 - e \cup f, B_2 - f \cup e)$ for suitable $e, f \in E$. Jonah Blasiak [1] proved that, if a class of matroids is closed under multiplication of elements, and in that class the so called n-base graph and the single exchange graph, to be defined below, are always connected, then White's conjecture holds for that class. Moreover, he proved that both graphs are always connected for matroids in the class of graphic matroids. Kashiwabara [3] verified White's conjecture for matroids of rank 3 and Schweig [2] for lattice path matroids.

It is obvious that the single exchange graph of a matroid is isomorphic to the single exchange graph of its dual. Moreover we show that, given a matroid M, which has a partition of its ground set into n bases, the n-base graph of the dual of M, where $(n-2)$ additional parallel copies have been added to each element, denoted by $\mathfrak{S}_n((n-1)M^*)$ maps homomorphically onto $\mathfrak{S}_n(M)$ (augmented by loops). Hence, if $\mathfrak{S}_n((n-1)M^*)$ is connected, $\mathfrak{S}_n(M)$ must be connected as well. Using Blasiak's results, this implies that White's Conjecture holds for cographic matroids.

We assume familiarity with matroid theory and with [1], in particular with the definition of the toric ideal I_M of a matroid, although we will not really use it.

2 Blasiak's Results

We call a matroid M on a finite set E an *n-base-tuple* if $|E| = nr(M)$, a 2-base tuple is also called a *base pair*. On an *n-base-tuple* M we define a graph $\mathfrak{S}_n(M)$ on the set of partitions $(B_1, \ldots, B_n)$ of E into bases, where $(B_1, \ldots, B_n)$ and $(B'_1, \ldots, B'_n)$ are adjacent if $B_i = B'_j$ for some $i, j \in \{1, \ldots, n\}$. We say that a class $\mathfrak{C}$ of matroids is *closed under multiplication of elements*, if given a matroid M from $\mathfrak{C}$, deleting an element from M or adding an element parallel to a given element in M yields another matroid from $\mathfrak{C}$.

Proposition 2.1 ([1]). *Let $\mathfrak{C}$ be a class of matroids closed under multiplication of elements. Suppose that for each $n \geq 3$ and each n-base-tuple $M \in \mathfrak{C}$ the n-base graph $\mathfrak{S}_n(M)$ is connected. Then for every matroid $M \in \mathfrak{C}$ the toric ideal I_M is generated by quadratic binomials.* □

Given a base pair, the *single exchange graph* $\mathfrak{S}(M)$ is defined on the partitions of the ground set into bases, and two vertices (B_1, B_2) and (B'_1, B'_2) are adjacent if one base partition arises from the other by single element swaps, i.e. $|B_1 \cap B'_1| = |B_2 \cap B'_2| = r(M) - 1$.

Proposition 2.2 ([1]). *If for a base pair $M \in \mathfrak{C}$ the single exchange graph $\mathfrak{S}(M)$ is connected, then for every matroid in $\mathfrak{C}$ the quadratic binomials of I_M are in the ideal generated by the binomials $y_{B_1}y_{B_2} - y_{B'_1}y_{B'_2}$ such that the pair of bases $(B'_1,\ B'_2)$ can be obtained from the pair $(B_1,\ B_2)$ by single element swaps.* □

Theorem 2.1 ([1]). *If M is a graphic matroid and an n-base tuple for $n \geq 3$, then the n-base graph $\mathfrak{S}_n(M)$ is connected.* □

Theorem 2.2 ([1]). *If M is a graphic matroid and a base pair, then the single exchange graph $\mathfrak{S}(M)$ is connected.* □

Putting these pieces together yields:

Theorem 2.3 ([1]). *If M is a graphic matroid, then the toric ideal I_M is generated by the quadratic binomials $y_{B_1}y_{B_2} - y_{B'_1}y_{B'_2}$ such that the pair of bases $B'_1,\ B'_2$ can be obtained from the pair $B_1,\ B_2$ by single element swaps.* □

3 A Dual Construction

Given a matroid N on a finite set E, and for $n \in \mathbb{N}$ we define the matroid $(n-1)N$ by adding $(n-2)$ parallel copies to each element $e \in E$.

In the following let M be an n-base-tuple with ground set E. Let M^* denote the matroid dual to M. Let F denote the ground set of $(n-1)M^*$.

Proposition 3.1. *$(n-1)M^*$ is an n-base tuple.*

Proof. We have $r((n-1)M^*) = r(M^*) = |E| - r(M) = (n-1)r(M)$ and $|F| = (n-1)|E| = (n-1)nr(M) = nr((n-1)M^*)$. $\square$

Let $\pi_F : F \to E$ denote the projection which maps each copy $e_i \in F$ of an element from E to its original $e \in E$. Let $\mathfrak{S}_n(M)'$ denote the graph which arises from $\mathfrak{S}_n(M)$ by attaching a loop to each of its vertices. We define the following projection from $\mathfrak{S}_n((n-1)M^*)$ to $\mathfrak{S}_n(M)'$. For a partition $(D_1, \dots, D_n)$ of F into bases let

$$\pi((D_1, \dots, D_n)) = (E \setminus \pi_F(D_1), \dots, E \setminus \pi_F(D_n)).$$

Proposition 3.2. *We claim that π defines a surjective graph homomorphism from $\mathfrak{S}_n((n-1)M^*)$ to $\mathfrak{S}_n(M)'$.*

Proof. Since D_i is a basis of $(n-1)M^*$ only if $\pi_F(D_i)$ is a basis of M^* if and only if $E \setminus \pi_F(D_i)$ is a basis of M, we conclude that $(E \setminus \pi_F(D_1), \dots, E \setminus \pi_F(D_n))$ is an n-tuple of bases of M. We have to show that they are disjoint. Assume $e \in (E \setminus \pi_F(D_i)) \cap (E \setminus \pi_F(D_j))$ for some $i \neq j$. Since each basis of $(n-1)M^*$ may contain at most one copy of e we conclude that F contains at most $n-2$ copies of e, a contradiction. Hence, π is well defined.

Assume that $(D_1, \dots, D_n)$ and $(D'_1, \dots, D'_n)$ are two adjacent vertices of $\mathfrak{S}_n((n-1)M^*)$. Hence we have $D_i = D'_j$ for some $i, j \in \{1, \dots, n\}$ implying $E \setminus \pi_F(D_i) = E \setminus \pi_F(D'_j)$. So $\pi(D_1, \dots, D_n)$ and $\pi(D'_1, \dots, D'_n)$ are also adjacent, and π is a graph homomorphism.

To verify surjectivity choose a partition $(B_1, \dots, B_n)$ of the ground set of M into bases. Assume that the copies of each element e are numbered as $e^1, \dots, e^{n-1}$ and for $A \subseteq E$ denote by A^i the copies of elements of A with index i. Hence $F = E^1 \dot\cup \dots \dot\cup E^{n-1}$. We claim that $(E^1 \setminus B_1^1, \dots, E^{n-1} \setminus B_{n-1}^{n-1}, \bigcup_{i=1}^{n-1} B_i^i)$ is a partition of the elements of $(n-1)M^*$ into bases. Since B_i is a basis of M if and only if $E \setminus B_i$ is basis of M^*,we conclude that $E^i \setminus B_i^i$ is a basis of $(n-1)M^*$. Since $\pi_F(\bigcup_{i=1}^{n-1} B_i^i) = E \setminus B_n$, and the sets are obviously pairwise disjoint, we conclude that $(E^1 \setminus B_1^1, \dots, E^{n-1} \setminus B_{n-1}^{n-1}, \bigcup_{i=1}^{n-1} B_i^i)$ is a vertex of $\mathfrak{S}_n((n-1)M^*)$ which maps to $(B_1, \dots, B_n)$. Hence π is surjective. $\square$

4 Cographic matroids

Theorem 4.1. *If M is a cographic matroid, then the toric ideal I_M is generated by the quadratic binomials $y_{B_1}y_{B_2} - y_{B'_1}y_{B'_2}$ such that the pair of bases (B'_1, B'_2) can be obtained from the pair (B_1, B_2) by single element swaps.*

Proof. Clearly, the class of cographic matroids is closed under multiplication. If M is a cographic base pair and $G = (V, E)$ a graph with cographic matroid M, then each partition of E into bases of M is a partition into spanning trees of G. Hence, $\mathfrak{S}(M) \cong \mathfrak{S}(M(G))$ where $M(G)$ denotes the graphic matroid of G. Thus, the cographic analogue of Theorem 2.2 is seen to hold. Now assume $n \geq 3$, and M is an n-base-tuple. If $(B_1, \dots, B_n)$ and $(B'_1, \dots, B'_n)$ are two vertices of $\mathfrak{S}_n(M)$, we choose vertices $(D_1, \dots, D_n)$ and $(D'_1, \dots, D'_n)$ of $\mathfrak{S}_n((n-1)M^*)$

such that $\pi((D_1, \dots, D_n)) = (B_1, \dots, B_n)$ and $\pi((D'_1, \dots, D'_n)) = (B'_1, \dots, B'_n)$. Since $(n-1)M^*$ is graphic, by Theorem 2.1 we have a path from $(D_1, \dots, D_n)$ to $(D'_1, \dots, D'_n)$ in $\mathfrak{S}_n((n-1)M^*)$. By Proposition 3.2 the path projects via π to a trail (a not necessary simple path) from $(B_1, \dots, B_n)$ to $(B'_1, \dots, B'_n)$. Hence $\mathfrak{S}_n(M)$ is connected and the claim follows from Proposition 2.1 and Proposition 2.2. □

References

[1] Jonah Blasiak. The toric ideal of a graphic matroid is generated by quadrics. *Combinatorica*, 28(3):283–297, 2008.

[2] Jay and Schweig. Toric ideals of lattice path matroids and polymatroids. *Journal of Pure and Applied Algebra*, 215(11):2660 – 2665, 2011.

[3] Kenji Kashiwabara. The toric ideal of a matroid of rank 3 is generated by quadrics. *Electron. J. Combin.*, 17(1):Research Paper 28, 12, 2010.

[4] Neil L. White. A unique exchange property for bases. *Linear Algebra and its Applications*, 31:81–91, June 1980.

see [1] for further references.