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A strong perfect digraph theorem

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Abstract

The *clique number* $\omega(D)$ of a digraph D is the size of the largest bidirectionally complete subdigraph of D. D is *perfect* if, for any induced subdigraph H of D, the dichromatic number $\chi(H)$ equals the clique number $\omega(H)$. Using the Strong Perfect Graph Theorem [7] we give a characterization of perfect digraphs by a set of forbidden induced subdigraphs. Modifying a recent proof of Bang-Jensen et al. [2] we show that the recognition of perfect digraphs is co-$\mathcal{NP}$-complete.

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1 Introduction

A conjecture that kept mathematicians busy with for a long time was Berge's Conjecture (cf. [3, 4]) which says that a graph is perfect if and only if it is a Berge graph, i.e., it does neither contain odd holes nor odd antiholes as induced subgraphs. After many partial results, the most famous being Lovasz' proof of the (Weak) Perfect Graph Theorem [10] stating that a graph is perfect if and only if its complement is perfect, a proof of Berge's Conjecture was published in 2006 by Chudnovsky et al. [7], so that Berge's Conjecture is now known as the Strong Perfect Graph Theorem (SPGT). In this note we show that the SPGT can be generalized in an easy and natural way to digraphs if the underlying coloring parameter, the chromatic number, is replaced by the dichromatic number introduced by Neumann-Lara [11].

The importance of perfect graphs lies in computer science. Many problems that are $\mathcal{NP}$-complete for graphs in general are polynomially solvable for perfect graphs, e.g. the maximum clique problem, the maximum stable set problem, the graph coloring problem and the minimum clique covering problem (see [9] resp. [8]). This is applicable, since the members of many important classes of graphs are known to be perfect, e.g. bipartite graphs and their line graphs, split graphs, chordal graphs, and comparability graphs. Bokal et al. [5] proved that 2-coloring a digraph feasibly is an $\mathcal{NP}$-complete problem. By our results, k-coloring of perfect digraphs is in $\mathcal{P}$ for any k. It remains an open question whether there are other classical problems which are hard on general digraphs but efficiently solvable on perfect digraphs.

By a result of Chudnovsky et al. [6] the recognition of Berge graphs is in $\mathcal{P}$, and so, by the SPGT [7], the same holds for the recognition of perfect graphs. In contrast to this, the recognition of induced directed cycles of length at least 3, which are a main obstruction for perfect digraphs, is $\mathcal{NP}$-complete by a result of Bang-Jensen et al. [2]. We will show in a very similar way that the recognition of perfect digraphs is co-$\mathcal{NP}$-complete.

2 Main result

We start with some definitions. For basic terminology we refer to Bang-Jensen and Gutin [1]. For the rest of the paper, we only consider digraphs without loops. Let $D = (V, A)$ be a digraph. The *dichromatic number* $\chi(D)$ of D is the smallest cardinality $|C|$ of a color set C, so that it is possible to assign a color from C to each vertex of D such that for every color $c \in C$ the subdigraph induced by the vertices colored with c is acyclic, i.e. it does not contain a directed cycle. The *clique number* $\omega(D)$ of D is the size of the largest induced subdigraph in which for any two distinct vertices v and w both arcs (v, w) and (w, v) exist. The clique number is an obvious lower bound for the dichromatic number. D is called *perfect* if, for any induced subdigraph H of D, $\chi(H) = \omega(H)$.

An (undirected) graph $G = (V, E)$ can be considered as the symmetric digraph $D_G = (V, A)$ with $A = \{(v, w), (w, v) \mid vw \in E\}$. In the following, we will not distinguish between G and D_G. In this way, the dichromatic number of a graph G is its chromatic number $\chi(G)$, the clique number of G is its usual clique number $\omega(G)$, and G is perfect as a digraph if and only if G is perfect as a graph. For us, an *edge* vw in a digraph $D = (V, A)$ is the set $\{(v, w), (w, v)\} \subseteq A$ of two antiparallel arcs, and a *single arc* in D is an arc $(v, w) \in A$ with $(w, v) \notin A$. The *oriented part* $O(D)$ of a digraph $D = (V, A)$ is the digraph (V, A_1) where A_1 is the set of all single arcs of D, and the *symmetric part* $S(D)$ of D is the digraph (V, A_2) where A_2 is the union of all edges of D. Obviously, $S(D)$ is a graph, and by definition we have

Observation 1. *For any digraph D, $\omega(D) = \omega(S(D))$.*

In the formulation of the SPGT and in our generalization some special types of graphs resp. digraphs are needed. An *odd hole* is an undirected cycle C_n with an odd number $n \geq 5$ of vertices. An *odd antihole* is the complement of an odd hole (without loops). A *filled odd hole/antihole* is a digraph H, so that $S(H)$ is an odd hole/antihole. For $n \geq 3$, the directed cycle on n vertices is denoted by $\vec{C}_n$. Furthermore, for a digraph $D = (V, A)$ and $V' \subseteq V$, by $D[V']$

we denote the subdigraph of D induced by the vertices of V'. Now we are ready to formulate our main result.

Theorem 2. *A digraph $D = (V, A)$ is perfect if and only if $S(D)$ is perfect and D does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph.*

Proof. Assume $S(D)$ is not perfect. Then there is an induced subgraph $H = (V', E')$ of $S(D)$ with $\omega(H) < \chi(H)$. Since $S(D[V']) = H$, we conclude by Observation 1,

$$\omega(D[V']) = \omega(S(D[V'])) = \omega(H) < \chi(H) = \chi(S(D[V'])) \leq \chi(D[V']),$$

therefore D is not perfect. If D contains a directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph, then D is obviously not perfect, since $\omega(\vec{C}_n) = 1 < 2 = \chi(\vec{C}_n)$.

Now assume that $S(D)$ is perfect but D is not perfect. It suffices to show that D contains an induced directed cycle of length at least 3. Let $H = (V', A')$ be an induced subdigraph of D such that $\omega(H) < \chi(H)$. Then there is a proper coloring of $S(H) = S(D)[V']$ with $\omega(S(H))$ colors, i.e., by Observation 1, with $\omega(H)$ colors. This cannot be a feasible coloring for the digraph H. Hence there is a (not necessarily induced) monochromatic directed cycle $\vec{C}_n$ with $n \geq 3$ in $O(H)$. Let C be such a cycle of minimal length. C cannot have a chord that is an edge vw, since both terminal vertices v and w of vw are colored in distinct colors. By minimality, C does not have a chord that is a single arc. Therefore, C is an induced directed cycle (of length at least 3) in H, and thus in D. □

We actually have proven:

Remark 3. *If D is a perfect digraph, then any feasible coloring of $S(D)$ is also a feasible coloring for D.*

Corollary 4. *A digraph $D = (V, A)$ is perfect if and only if it does neither contain a filled odd hole, nor a filled odd antihole, nor a directed cycle $\vec{C}_n$ with $n \geq 3$ as induced subdigraph.*

Proof. If D contains any configuration of the three forbidden types, D is obviously not perfect, since each of these configurations is not perfect.

Assume, D does not contain any of these configurations. Then $S(D)$ does neither contain odd holes nor odd antiholes, therefore, by the Strong Perfect Graph Theorem [7], $S(D)$ is perfect. Using Theorem 2, we conclude that D is perfect. □

Corollary 5. *k-coloring of perfect digraphs is in $\mathcal{P}$ for any $k \geq 1$.*

Proof. By Remark 3 it follows that a coloring of a perfect digraph D with $\omega(D)$ colors can be obtained by coloring the perfect graph $S(D)$, which is possible in polynomial time (see [8]). □

The preceding result does not depend on an efficient recognition of perfect digraphs. Moreover, the recognition problem for perfect digraphs is a hard problem. In order to test, whether a digraph D is perfect, by Theorem 2 we have to test

1. whether $S(D)$ is perfect, and
2. whether D does not contain an induced directed cycle $\vec{C}_n$, $n \geq 3$.

The first can indeed be tested efficiently by the results of Chudnovsky et al. [6] and the SPGT [7], but the second is a co-$\mathcal{NP}$-complete problem by a recent result of Bang-Jensen et al. ([2], Theorem 11). The proof of Bang-Jensen et al. can be easily modified to prove the following.

Theorem 6. *The recognition of perfect digraphs is co-$\mathcal{NP}$-complete.*

Proof. We reduce 3-SAT to non-perfect digraph recognition. We consider an instance of 3-SAT

$$F = \bigwedge_{i=1}^{m} C_i = \bigwedge_{i=1}^{m} (l_{i1} \vee l_{i2} \vee l_{i3}) \qquad \text{with } l_{ij} \in \{x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n\}.$$

For each variable x_k we construct a variable gadget $VG(k)$ and for each clause C_i a clause gadget $CG(i)$, as shown in Fig. 1. These gadgets are very similar to those used in Theorem 11 of the paper of Bang-Jensen et al. [2], only the edges (which are redundant for correctness of the reduction) are missing here. The rest of the construction is the same as in [2]: We form a chain of variable gadgets by introducing vertices b_0 and a_{n+1} and the arcs (b_k, a_{k+1}) for $k \in \{0, 1, \ldots, n\}$, and a chain of clause gadgets by introducing the vertices d_0 and c_{m+1} and the arcs (d_i, c_{i+1}) for $i \in \{0, 1, \ldots, m\}$. We close the two chains to form a ring by introducing the arcs (a_{n+1}, d_0) and (c_{m+1}, b_0). Finally, for each literal l_{ij} (which is x_k or $\overline{x}_k$) we connect the vertex l_{ij} in the clause gadget $CG(i)$ with the vertex $\overline{l_{ij}}$ in the variable gadget $VG(k)$ by an edge. This completes the construction of the digraph $D(F)$.

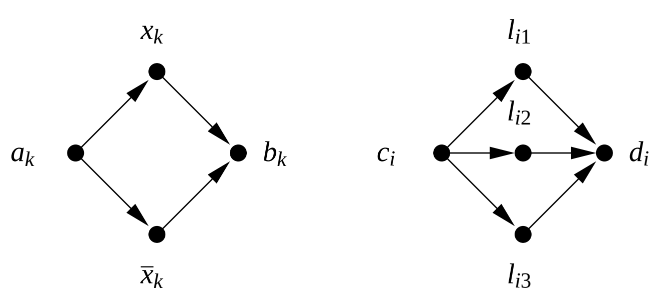

Figure 1: Variable gadget $VG(k)$ (left) and clause gadget $CG(i)$ (right)

We remark that $S(D(F))$ is a forest of stars, thus bipartite and hence perfect, so by Theorem 2 testing whether $D(F)$ is not perfect and testing whether $D(F)$ has an induced directed cycle of length at least 3 is the same. We have to show that $D(F)$ has an induced directed cycle if and only if F is satisfiable. Let $(z_1, \dots, z_n) \in \{0,1\}^n$ be an assignment satisfying F. Then the directed path through all y_k with

$$y_k = \begin{cases} x_k & \text{if } z_k = 1 \\ \overline{x}_k & \text{if } z_k = 0 \end{cases}$$

can be extended to an induced directed cycle through the clause gadgets by the construction, since in every clause there is a literal y_k the adjacent edge of which is only connected to the vertex $\overline{y}_k$ in $VG(k)$. On the other hand, if there is an induced directed cycle through the ring using the literals $(y_1, \dots, y_n)$ in the variable gadgets, then $(z_1, \dots, z_n)$ with $z_k = 1$ if $y_k = x_k$ and $z_k = 0$ if $y_k = \overline{x}_k$ is an assignment satisfying F, since for every clause some literal y_k that lies on the cycle is satisfied. $\square$

Note that the perfectness of digraphs does not behave as well as the perfectness of graphs in a second aspect: there is no analogon to Lovasz' Weak Perfect Graph Theorem [10]. A digraph may be perfect but its complement may be not perfect. An easy instance of this type is the directed 4-cycle $\vec{C}_4$, which is not perfect, and its complement H, which is perfect.

Open question 7. *Are there any interesting special classes of perfect digraphs with efficient algorithms for problems different from coloring?*

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