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Duality Respecting Representations and Compatible Complexity Measures for Gammoids

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Duality Respecting Representations and Compatible Complexity Measures for Gammoids

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Abstract

We show that every gammoid has special digraph representations, such that a representation of the dual of the gammoid may be easily obtained by reversing all arcs. In an informal sense, the duality notion of a poset applied to the digraph of a special representation of a gammoid commutes with the operation of forming the dual of that gammoid. We use these special representations in order to define a complexity measure for gammoids, such that the classes of gammoids with bounded complexity are closed under duality, minors, and direct sums.

Keywords. gammoids, digraphs, duality, complexity measure

A well-known result due to J.H. Mason is that the class of gammoids is closed under duality, minors, and direct sums [5]. Furthermore, it has been shown by D. Mayhew that every gammoid is also a minor of an excluded minor for the class of gammoids [6], which indicates that handling the class of all gammoids may get very involved. In this work, we introduce a notion of complexity for gammoids which may be used to define subclasses of gammoids with bounded complexity,

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that still have the desirable property of being closed under duality, minors, and direct sums; yet their representations have a more limited number of arcs than the general class of gammoids.

1 Preliminaries

In this work, we consider *matroids* to be pairs $M = (E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a system of independent subsets of E subject to the usual axioms ([7], Sec. 1.1). If $M = (E, \mathcal{I})$ is a matroid and $X \subseteq E$, then the restriction of M to X shall be denoted by $M|X$ ([7], Sec. 1.3), and the contraction of M to X shall be denoted by $M.X$ ([7], Sec. 3.1). Furthermore, the notion of a *digraph* shall be synonymous with what is described more precisely as *finite simple directed graph* that may have some loops, i.e. a digraph is a pair $D = (V, A)$ where V is a finite set and $A \subseteq V \times V$. Every digraph $D = (V, A)$ has a unique *opposite digraph* $D^{\mathrm{opp}} = (V, A^{\mathrm{opp}})$ where $(u, v) \in A^{\mathrm{opp}}$ if and only if $(v, u) \in A$. All standard notions related to digraphs in this work are in accordance with the definitions found in [2]. A *path* in $D = (V, A)$ is a non-empty and non-repeating sequence $p = p_1 p_2 \ldots p_n$ of vertices $p_i \in V$ such that for each $1 \leq i < n$, $(p_i, p_{i+1}) \in A$. By convention, we shall denote p_n by p_{-1}. Furthermore, the set of vertices traversed by a path p shall be denoted by $|p| = \{p_1, p_2, \ldots, p_n\}$ and the set of all paths in D shall be denoted by $\mathbf{P}(D)$.

Definition 1.1. Let $D = (V, A)$ be a digraph, and $X, Y \subseteq V$. A *routing* from X to Y in D is a family of paths $R \subseteq \mathbf{P}(D)$ such that

(i) for each $x \in X$ there is some $p \in R$ with $p_1 = x$,

(ii) for all $p \in R$ the end vertex $p_{-1} \in Y$, and

(iii) for all $p, q \in R$, either $p = q$ or $|p| \cap |q| = \emptyset$.

We shall write $R\colon X \rightrightarrows Y$ in D as a shorthand for "R is a routing from X to Y in D", and if no confusion is possible, we just write $X \rightrightarrows Y$ instead of R and $R\colon X \rightrightarrows Y$. A routing R is called *linking* from X to Y, if it is a routing onto Y, i.e. whenever $Y = \{p_{-1} \mid p \in R\}$.

Definition 1.2. Let $D = (V, A)$ be a digraph, $E \subseteq V$, and $T \subseteq V$. The *gammoid represented by* (D, T, E) is defined to be the matroid $\Gamma(D, T, E) = (E, \mathcal{I})$ where

$$\mathcal{I} = \{X \subseteq E \mid \text{there is a routing } X \rightrightarrows T \text{ in D}\}.$$

The elements of T are usually called *sinks* in this context, although they are not required to be actual sinks of the digraph D. To avoid confusion, we shall call the elements of T *targets* in this work. A matroid $M' = (E', \mathcal{I}')$ is called *gammoid*, if there is a digraph $D' = (V', A')$ and a set $T' \subseteq V'$ such that $M' = \Gamma(D', T', E')$.

Theorem 1.3 ([5], Corollary 4.1.2)**.** *Let $M = (E, \mathcal{I})$ be a gammoid and $B \subseteq E$ a base of M. Then there is a digraph $D = (V, A)$ such that $M = \Gamma(D, B, E)$.*

For a proof, see J.H. Mason's seminal paper *On a Class of Matroids Arising From Paths in Graphs* [5].

2 Special Representations

Definition 2.1. Let (D, T, E) be a representation of a gammoid. We say that (D, T, E) is a *duality respecting representation*, if

$$\Gamma(D^{\mathrm{opp}}, E\backslash T, E) = (\Gamma(D, T, E))^*$$

where $(\Gamma(D, T, E))^*$ denotes the dual matroid of $\Gamma(D, T, E)$.

Lemma 2.2. *Let (D, T, E) be a representation of a gammoid with $T \subseteq E$, such that every $e \in E\backslash T$ is a source of D, and every $t \in T$ is a sink of D. Then (D, T, E) is a duality respecting representation.*

Proof. We have to show that the bases of $N = \Gamma(D^{\mathrm{opp}}, E\backslash T, E)$ are precisely the complements of the bases of $M = \Gamma(D, T, E)$ ([7], Thm. 2.1.1). Let $B \subseteq E$ be a base of M, then there is a linking $L\colon B \rightrightarrows T$ in D, and since T consists of sinks, we have that the single vertex paths $\{x \in \mathbf{P}(D) \mid x \in T \cap B\} \subseteq L$. Further, let $L^{\mathrm{opp}} = \{p_n p_{n-1} \ldots p_1 \mid p_1 p_2 \ldots p_n \in L\}$. Then L^{opp} is a linking from T to B in D^{opp} which routes $T\backslash B$ to $B\backslash T$. The special property of D, that $E\backslash T$ consists of sources and that T consists of sinks, implies, that for all $p \in L$, we have $|p| \cap E = \{p_1, p_{-1}\}$. Observe that thus

$$R = \{p \in L^{\mathrm{opp}} \mid p_1 \in T\backslash B\} \cup \{x \in \mathbf{P}(D^{\mathrm{opp}}) \mid x \in E\backslash (T \cup B)\}$$

is a linking from $E\backslash B = (T \,\dot{\cup}\, (E\backslash T))\,\backslash B$ onto $E\backslash T$ in D^{opp}, thus $E\backslash B$ is a base of N. An analog argument yields that for every base B' of N, $E\backslash B'$ is a base of M. Therefore $\Gamma(D^{\mathrm{opp}}, E\backslash T, E) = (\Gamma(D, T, E))^*$. $\square$

Definition 2.3. Let M be a gammoid and (D,T,E) with $D=(V,A)$ be a representation of M. Then (D,T,E) is a *standard representation of* M, if (D,T,E) is a duality respecting representation, $T\subseteq E$, every $t\in T$ is a sink in D, and every $e\in E\backslash T$ is a source in D.

The name *standard representation* is justified, since the real matrix $A\in\mathbb{R}^{T\times E}$ obtained from D through the Lindström Lemma [4, 1] is a *standard matrix representation* of $\Gamma(D,T,E)$ up to possibly rearranging the columns ([8], p.137).

Theorem 2.4. *Let $M=(E,\mathcal{I})$ be a gammoid, and $B\subseteq E$ a base of M. There is a digraph $D=(V,A)$ such that (D,B,E) is a standard representation of M.*

Proof. Let $D_0=(V_0,A_0)$ be a digraph such that $\Gamma(D_0,B,E)=M$ (Theorem 1.3). Furthermore, let V be a set with $E\subseteq V$ such that there is an injective map $'\colon V_0\longrightarrow V\backslash E,\ v\mapsto v'$. Without loss of generality we may assume that $V=E\,\dot\cup\,V_0'$. We define the digraph $D=(V,A)$ such that

$$A=\{(u',v')\mid(u,v)\in A_0\}\cup\{(b',b)\mid b\in B\}\cup\{(e,e')\mid e\in E\backslash B\}.$$

For every $X\subseteq E$, we obtain that by construction, there is a routing $X\rightrightarrows B$ in D_0 if and only if there is a routing $X\rightrightarrows B$ in D. Therefore (D,B,E) is a representation of M with the additional property that every $e\in E\backslash B$ is a source in D, and every $b\in B$ is a sink in D. Thus (D,B,E) is a duality respecting representation of M (Lemma 2.2). $\square$

3 Gammoids with Low Arc-Complexity

Definition 3.1. Let M be a gammoid. The *arc-complexity of* M is defined to be

$$\mathrm{C}_A(M)=\min\left\{|A|\;\middle|\;((V,A),T,E)\text{ is a standard representation of }M\right\}.$$

Lemma 3.2. *Let $M=(E,\mathcal{I})$ be a gammoid, $X\subseteq E$. Then the inequalities $\mathrm{C}_A(M|X)\leq\mathrm{C}_A(M)$, $\mathrm{C}_A(M.X)\leq\mathrm{C}_A(M)$, and $\mathrm{C}_A(M)=\mathrm{C}_A(M^*)$ hold.*

Proof. Let M be a gammoid and let (D,T,E) be a standard representation of M with $D=(V,A)$ for which $|A|$ is minimal among all standard representations of M. Then $(D^{\mathrm{opp}},E\backslash T,E)$ is a standard representation of M^* that uses the same number of arcs. Thus $\mathrm{C}_A(M)=\mathrm{C}_A(M^*)$ holds for all gammoids M. Let $X\subseteq E$. If $T\subseteq X$, then (D,T,X) is a standard representation of $M|X$

and therefore $\mathrm{C}_A(M|X) \leq \mathrm{C}_A(M)$. Otherwise let $Y = T\backslash X$, and let $B_0 \subseteq X$ be a set of maximal cardinality such that there is a routing $R_0 \colon B_0 \rightrightarrows Y$ in D. Let $D' = (V, A')$ be the digraph that arises from D by a sequence of operations as described in Theorem 4.1.1 [5] and Corollary 4.1.2 [5] with respect to the routing R_0. Observe that every $b \in B_0$ is a sink in D' and that $|A'| = |A|$. We argue that $(D', (T \cap X) \cup B_0, X)$ is a standard representation of $M|X$: Let $Y_0 = \{p_{-1} \mid p \in R_0\}$ be the set of targets that are entered by the routing R_0. It follows from Corollary 4.1.2 [5] that the triple $(D', (T \cap X) \cup B_0 \cup (Y\backslash Y_0), E)$ is a representation of M. The sequence of operations we carried out on D preserves all those sources and sinks of D, which are not visited by a path $p \in R_0$. So we obtain that every $e \in E\backslash (T \cup B_0)$ is a source in D', and that every $t \in T \cap X$ is a sink in D'. Thus the set $T' = (T \cap X) \cup B_0$ consists of sinks in D', and the set $X\backslash T' \subseteq E\backslash (T \cup B_0)$ consists of sources in D'. Therefore $(D', (T \cap X) \cup B_0, X)$ is a standard representation, and we give an indirect argument that $(D', (T \cap X) \cup B_0, X)$ represents $M|X$. Clearly, $(D', (T \cap X) \cup B_0 \cup (Y\backslash Y_0), X)$ is a representation of $M|X$. Since we assume that $(D', (T \cap X) \cup B_0, X)$ does not represent $M|X$, there must be a set $X_0 \subseteq X$ such that there is a routing $Q_0 \colon X_0 \rightrightarrows (T \cap X) \cup B_0 \cup (Y\backslash Y_0)$ and such that there is no routing $X_0 \rightrightarrows (T \cap X) \cup B_0$, both in D'. Thus there is a path $q \in Q_0$ with $q_{-1} \in Y\backslash Y_0$ and $q_1 \in X$. Consequently we have a routing $Q_1' = \{q\} \cup \{b \in \mathbf{P}(D') \mid b \in B_0\}$ in D'. This implies that there is a routing $B_0 \cup \{q_1\} \rightrightarrows Y$ in D, a contradiction to the maximal cardinality of the choice of B_0 above. Thus our assumption must be wrong, and $(D', (T \cap X) \cup B_0, X)$ is a standard representation of $M|X$. Consequently $\mathrm{C}_A(M|X) \leq \mathrm{C}_A(M)$ holds again. Finally, we have $\mathrm{C}_A(M.X) = \mathrm{C}_A\left((M^*|X)^*\right) = \mathrm{C}_A\left(M^*|X\right) \leq \mathrm{C}_A(M^*) = \mathrm{C}_A(M)$. $\square$

Definition 3.3. Let $f \colon \mathbb{N} \longrightarrow \mathbb{N}\backslash \{0\}$ be a function. We say that f is *super-additive*, if for all $n, m \in \mathbb{N}\backslash \{0\}$

$$f(n+m) \geq f(n) + f(m)$$

holds.

Definition 3.4. Let $f \colon \mathbb{N} \longrightarrow \mathbb{N}\backslash \{0\}$ be a super-additive function, and let $M = (E, \mathcal{I})$ be a gammoid. The *f-width of M* shall be

$$\mathrm{W}_f(M) = \max\left\{ \frac{\mathrm{C}_A\left((M.Y)\,|X)\right)}{f\left(|X|\right)} \;\middle|\; X \subseteq Y \subseteq E \right\}.$$

Theorem 3.5. *Let $f \colon \mathbb{N} \longrightarrow \mathbb{N}\backslash \{0\}$ be a super-additive function, and let $0 < q \in \mathbb{Q}$. Let $\mathcal{W}_{f,q}$ denote the class of gammoids M with $\mathrm{W}_f(M) \leq q$. The class $\mathcal{W}_{f,q}$ is closed under duality, minors, and direct sums.*

Proof. Let $M = (E, \mathcal{I})$ be a gammoid and $X \subseteq Y \subseteq E$. It is obvious from Definition 3.4 that $\mathrm{W}_f\left((M.Y)\,|X\right) \leq \mathrm{W}_f(M)$, and consequently $\mathcal{W}_{f,q}$ is closed under minors. Since $\mathrm{C}_A(M) = \mathrm{C}_A(M^*)$ and since every minor of M^* is the dual of a minor of M ([7], Prop. 3.1.26), we obtain that $\mathrm{W}_f(M) = \mathrm{W}_f(M^*)$. Thus $\mathcal{W}_{f,q}$ is closed under duality.

Now, let $M = (E, \mathcal{I})$ and $N = (E', \mathcal{I}')$ with $E \cap E' = \emptyset$ and $M, N \in \mathcal{W}_{f,q}$. The cases where either $E = \emptyset$ or $E' = \emptyset$ are trivial, now let $E \neq \emptyset \neq E'$. Furthermore, let $X \subseteq Y \subseteq E \cup E'$. The direct sum commutes with the forming of minors in the sense that

$$((M \oplus N).Y)\,|X \quad = \quad \big((M.Y \cap E)\,|X \cap E\big) \oplus ((N.Y \cap E')|X \cap E')\,.$$

Let (D, T, E) and (D', T', E') be representations of M and N where $D = (V, A)$ and $D' = (V', A')$ such that $V \cap V' = \emptyset$. Then $\big((V \cup V', A \cup A'), T \cup T', E \cup E'\big)$ is a representation of $M \oplus N$, and consequently $\mathrm{C}_A(M \oplus N) \leq \mathrm{C}_A(M) + \mathrm{C}_A(N)$ holds for all gammoids M and N, thus we have

$$\mathrm{C}_A\left(((M \oplus N).Y)\,|X\right) \quad \leq \quad \mathrm{C}_A\left(M_{X,Y}\right) + \mathrm{C}_A\left(N_{X,Y}\right)$$

where $M_{X,Y} = (M.Y \cap E)\,|X \cap E$ and $N_{X,Y} = (N.Y \cap E')|X \cap E'$. The cases where $\mathrm{C}_A(M_{X,Y}) = 0$ or $\mathrm{C}_A(N_{X,Y}) = 0$ are trivial, so we may assume that $X \cap E \neq \emptyset \neq X \cap E'$. We use the super-additivity of f at $(*)$ in order to derive

$$\begin{aligned} \frac{\mathrm{C}_A\left(((M \oplus N).Y)\,|X\right)}{f\left(|X|\right)} \quad &\leq \quad \frac{\mathrm{C}_A\left(M_{X,Y}\right) + \mathrm{C}_A\left(N_{X,Y}\right)}{f\left(|X|\right)} \\ &\leq \quad \frac{q \cdot f\left(|X \cap E|\right) + q \cdot f\left(|X \cap E'|\right)}{f\left(|X|\right)} \quad \overset{(*)}{\leq} \quad q. \end{aligned}$$

As a consequence we obtain $\mathrm{W}_f(M \oplus N) \leq q$, and therefore $\mathcal{W}_{f,q}$ is closed under direct sums. $\square$

4 Further Remarks and Open Problems

Let $r, n \in \mathbb{N}$ with $n \geq r$, the uniform matroid of rank r on n elements is the matroid $U_{r,n} = (\{1, 2, \ldots, n\}, \mathcal{I}_{r,n})$ where $\mathcal{I}_{r,n} = \left\{X \subseteq \{1, 2, \ldots, n\} \;\middle|\; |X| \leq r\right\}$. Let $T = \{1, 2, \ldots, r\}$, $X = \{r+1, r+2, \ldots, n\}$, and $D = (X \cup T, X \times T)$. Then $U_{r,n} = \Gamma(D, T, T \cup X)$. Thus $\mathrm{C}_A\left(U_{r,n}\right) \leq r \cdot (n - r)$. Unfortunately, we were not able to find a known result in graph or digraph theory that implies:

Conjecture 4.1.

$$\mathrm{C}_A\left(U_{r,n}\right) = r \cdot (n - r).$$

A slightly weaker version is the following:

Conjecture 4.2. *For every $q \in \mathbb{Q}$ there is a gammoid $M = (E, \mathcal{I})$ with*

$$\mathrm{C}_A(M) \geq q \cdot |E|\,.$$

For the rest of this work, we set $f\colon \mathbb{N} \longrightarrow \mathbb{N}\backslash \{0\},\ x \mapsto \max\{1, x\}$, we denote W_f by W, and we fix an arbitrary choice of $q \in \mathbb{Q}$ with $q > 0$. Clearly, if Conjecture 4.2 holds, then $(\mathcal{W}_{f,i})_{i\in\mathbb{N}\backslash\{0\}}$ is strictly monotonous sequence of subclasses of the class of gammoids, such that every subclass is closed under duality, minors, and direct sums. For which super-additive f and $q \in \mathbb{Q}\backslash \{0\}$ may $\mathcal{W}_{f,q}$ be characterized by finitely many excluded minors? For which such classes can we list a sufficient (possibly infinite) set of excluded minors that decide class membership of $\mathcal{W}_{f,q}$?

A consequence of a result of S. Kratsch and M. Wahlström ([3], Thm. 3) is, that if a matroid $M = (E, \mathcal{I})$ is a gammoid, then there is a representation (D, T, E) of M with $D = (V, A)$ and $|V| \leq \mathrm{rk}_M(E)^2 \cdot |E| + \mathrm{rk}_M(E) + |E|$. It is easy to see that if $M \in \mathcal{W}_{f,q}$, then there is a representation (D, T, E) of M with $D = (V, A)$ and $|V| \leq \lfloor 2q \cdot f\left(|E|\right)\rfloor$, since every arc is only incident with at most two vertices. Therefore, deciding $\mathcal{W}_{f,q}$-membership with an exhaustive digraph search appears to be easier than deciding gammoid-membership with an exhaustive digraph search.

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