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Analysis of an Inconsistency Measure based on Inferences in Subsets

Bachelor's Thesis

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Zusammenfassung

Ein Inkonsistenzmaß bewertet den Grad der Inkonsistenz einer Wissensbasis. In den letzten Jahren wurden viele Maße zur Inkonsistenzmessung entwickelt. Zur Bewertung ihrer Qualität und Eigenschaften wurden unterschiedliche Ansätze vorgeschlagen, die von Rationalitätspostulaten über Expressivitätsmerkmale bis hin zur Komplexitätsanalyse reichen. Diese Bachelorarbeit untersucht ein neues Inkonsistenzmaß, das Entailment Inkonsistenzmaß, das auf Schlussfolgerungen in Teilmengen einer Wissensbasis definiert wird. Die Arbeit gibt einen Überblick über Bewertungsmethoden für Inkonsistenzmaße, insbesondere Rationalitätspostulate, Expressivität und Komplexität. Das vorgeschlagene Inkonsistenzmaß wird dann im Hinblick auf diese Methoden bewertet.

Die Ergebnisse zeigen, dass das Entailment Inkonsistenzmaß fünf von insgesamt 20 analysierten Rationalitätspostulaten erfüllt. Dieses Ergebnis ist schlechter im Vergleich zu den Ergebnissen bereits vorgeschlagener Inkonsistenzmaße. Die Expressivität ist für alle vier Expressivitätscharakteristiken maximal und damit höher als die Expressivität vieler bereits existierender Inkonsistenzmaße. In Bezug auf die Komplexität ist das resultierende Entailment Inkonsistenzmaß aufwändig zu berechnen, da es sich wahrscheinlich jenseits der dritten Ebene der polynomiellen Hierarchie befindet. Mit diesem Ergebnis gehört das neue Maß zu den Inkonsistenzenmaßen mit der höchsten Komplexitätsbewertung.

Abstract

An inconsistency measure evaluates the degree of inconsistency of a knowledge base. Many inconsistency measures have been developed in the last years. For the evaluation of their quality and properties, different approaches have been proposed, which range from rationality postulates, expressivity characteristics and computational complexity analysis. This bachelor thesis defines and investigates a new inconsistency measure, the entailment inconsistency measure, that is based on inferences in subsets of a given knowledge base. Besides giving an overview of evaluation methods for inconsistency measures, in particular with regard to rationality postulates, expressivity and complexity, the newly proposed inconsistency measure is evaluated with regard to these methods.

Results show that the entailment inconsistency measure satisfies five of in total 20 analysed rationality postulates. This result is inferior with regard to the results of already proposed inconsistency measures. Expressivity is maximal for all four expressivity characteristics and thus higher than many already existing inconsistency measures. Regarding computational complexity, the entailment inconsistency measure is computationally demanding with likely being beyond the third level of the polynomial hierarchy. With this result, it belongs to the computationally most demanding inconsistency measures.

Contents

1 Introduction

Inconsistency typically arises from conflicting pieces of information that cannot simultaneously hold true. In the realm of knowledge management, inconsistency is commonly perceived as undesirable. The traditional view is that knowledge bases and information systems should ideally maintain perfect consistency. However, reality shows a different picture: Inconsistency does not occur occasionally, or as an exception, but is commonly prevalent in real-world data [GH91, GH93, Hun03].

On the one hand, information about underlying inconsistencies is needed to resolve these inconsistencies in order to restore consistency. On the other hand, reasoning with inconsistent information, so dealing with the inconsistency instead of removing it, is as well needed for certain systems. That is, because contrary to conventional beliefs, inconsistencies can offer additional information. By recognizing this, there arises a need to assess whether an inconsistency exists and which level of severity the inconsistency has. With that information, one can decide what to do with the inconsistency [GH91].

Inconsistency measurement is a discipline which aims to quantify and assess the extent of inconsistencies. It answers the questions about the amount of inconsistency in information and the severity of this inconsistency. Furthermore, it comprises research about methodologies for evaluating the characteristics of these inconsistency measures characteristics and their quality [HK06, Thi09, Thi17b].

The contribution of this bachelor thesis is to give a definition of a new inconsistency measure that is based on inferences in subsets. The new inconsistency measure is then evaluated by current evaluation methods for inconsistency measures, in particular rationality postulates, expressivity and computational complexity. The results are discussed with regard to already existing inconsistency measures.

The thesis is structured as follows. Section 2 give some preliminaries on propositional logic and computational complexity that are necessary for the following parts. In Section 3 an overview about the current state of research on inconsistency measurement is given by presenting existing approaches with already proposed inconsistency measures and discussing the evaluation methods of rationality postulates, expressivity and complexity. Section 4 introduces the new inconsistency measure and evaluates the measure by the mentioned evaluation methods. A summary and outlook is given in Section 5.

2 Preliminaries

For the proofs, definitions and results in the following sections some preliminaries are needed, which are given in this section.

2.1 Propositional Logic

Let $\mathcal{L}(\mathsf{At})$ be a propositional language with At being a finite set of propositional atoms. $\mathcal{L}(\mathsf{At})$ uses the classical logical connectives $\{\wedge, \vee, \rightarrow, \neg\}$. Let ϕ be an arbitrary formula, then $\mathsf{At}(\phi)$ is the set of atoms appearing in ϕ. A literal is an atom a or its negation $\neg a$. A clause is a disjunction of literals. A formula ϕ in *conjunctive normal form* (CNF) is a conjunction of clauses. A knowledge base $\mathcal{K}$ is a finite set of formulae $\mathcal{K} \subseteq \mathcal{L}(\mathsf{At})$. Let $\mathbb{K}$ be the set of all knowledge bases.

Semantics for a propositional language is given by interpretations. An interpretation w is a function from At to $\{\mathsf{true}, \mathsf{false}\}$. Let $\Omega(\mathsf{At})$ be the set of all possible interpretations for At. An interpretation w is a model of an atom $a \in \mathsf{At}$, denoted by $w \models a$, if and only if $w(a) = \mathsf{true}$. The satisfaction relation $\models$ is extended to formulae: An interpretation w is a model of a formula ϕ, denoted by $w \models \phi$, if and only if $w(\phi) = \mathsf{true}$. For $\Phi \subseteq L(\mathsf{At})$ we define $w \models \Phi$ if and only if $w \models \phi$ for every $\phi \in \Phi$. The set of models $\mathsf{Mod}(\phi) = \{w \in \Omega(\mathsf{At}) \mid w \models \phi\}$ for every formula or set of formulae ϕ. If $\mathsf{Mod}(\phi) = \emptyset$, denoted by $\phi \models \perp$, ϕ is inconsistent.

The following paragraph deals with basic notions for inconsistency measurement. Minimal inconsistent subsets represent one approach for determining inconsistency in a knowledge base. A minimal inconsistent subset is, by definition, inconsistent and a subset of the knowledge base. It is minimal, thus has no strict subset which is inconsistent [HK04, GH08].

Definition 1. *A set $M \subseteq \mathcal{K}$ is called a minimal inconsistent subset* (MI) *of $\mathcal{K}$ if $M \models \perp$ and there is no $M' \subset M$ with $M' \models \perp$. Let $\mathsf{MI}(\mathcal{K})$ be the set of all* MIs *of $\mathcal{K}$.*

A formula of a knowledge base is called a free formula if it does not belong to any minimal inconsistent subset of the knowledge base and is therefore not participating in any conflict [HK04, GH08].

Definition 2. *A formula $\phi \in \mathcal{K}$ is called a free formula if $\phi \notin \bigcup \mathsf{MI}(\mathcal{K})$. Let $\mathsf{Free}(\mathcal{K})$ be the set of all free formulae of $\mathcal{K}$.*

A formula of a knowledge base is called a safe formula if its signature is disjoint from the signature of the rest of the knowledge base [Thi09, Thi18]. A safe formula is always a free formula [Thi18].

Definition 3. *A formula $\phi \in \mathcal{K}$ is called a safe formula if it is consistent and $\mathsf{At}(\phi) \cap \mathsf{At}(\mathcal{K} \setminus \{\phi\}) = \emptyset$. Let $\mathsf{Safe}(\mathcal{K})$ be the set of all safe formulae of $\mathcal{K}$.*

Last, the definitions of the length of a formula and the length of a knowledge base are given.

Definition 4. *Let ϕ be a formula. The length of ϕ, $\mathsf{len}(\phi)$, is defined as*

$$\mathsf{len}(\phi) = \begin{cases} 1 & \text{if } \phi \in \mathsf{At} \\ 1 + \mathsf{len}(\phi') & \text{if } \phi = \neg\phi' \\ 1 + \mathsf{len}(\phi_1) + \mathsf{len}(\phi_2) & \text{if } \phi = \phi_1 \wedge \phi_2 \\ 1 + \mathsf{len}(\phi_1) + \mathsf{len}(\phi_2) & \text{if } \phi = \phi_1 \vee \phi_2 \end{cases}$$

The length of a knowledge base $\mathsf{len}(\mathcal{K})$ is then defined as $\mathsf{len}(\mathcal{K}) = \sum_{\phi \in \mathcal{K}} \mathsf{len}(\phi)$.

2.2 Computational Complexity

P is the class of decision problems for which the positive and negative instances can be accepted by a deterministic Turing machine in polynomial time, whereas NP is the class of decision problems for which the instances can be accepted in polynomial time by a nondeterministic Turing machine. coNP is the class of the decision problems whose complement is in NP [Pap94].

The canonical NP-complete problem is the satisfiability problem SAT which answers, whether a satisfying interpretation exists for a propositional formula. The complement of this problem is UNSAT, which is the canonical coNP-complete problem. The problem asks whether a given propositional formula has no satisfying model [Pap94].

SAT	**Input**:	a formula ϕ in CNF
	Output:	TRUE iff $\mathsf{Mod}(\phi) \neq \emptyset$

UNSAT	**Input**:	a formula ϕ in CNF
	Output:	TRUE iff $\mathsf{Mod}(\phi) = \emptyset$

For any two complexity classes $\mathcal{C}$ and $\mathcal{D}$, the class $\mathcal{C}^{\mathcal{D}}$ is the class of decision problems solvable in class $\mathcal{C}$ with access to an oracle Turing machine for some problem complete in $\mathcal{D}$. The oracle Turing machine is able to solve any of these problems and answers instantaneously. With that, the polynomial hierarchy is defined in the following [Pap94].

Definition 5. *Let* $\Sigma_0^p = \Delta_0^p = \Pi_0^p = \mathsf{P}$. *Then for* $i \geq 0$

$$\Sigma_{i+1}^p = \mathsf{NP}^{\Sigma_i^p}$$
$$\Pi_{i+1}^p = \mathsf{coNP}^{\Sigma_i^p}$$
$$\Delta_{i+1}^p = \mathsf{P}^{\Sigma_i^p}$$

Specifically, we have $\Sigma_1^p = \mathsf{NP}$, $\Pi_1^p = \mathsf{coNP}$ *and* $\Delta_1^p = \mathsf{P}$.

The complexity class D_i^p contains all decision problems which are the conjunction of a decision problem in Σ_i^p and a decision problem in Π_i^p. coD_i^p is the class of the decision problems whose complement is in D_i^p.

Functional problems are problems which answer is not simply TRUE or FALSE, but the answer is more complex. E.g., the problem is not about answering whether there is a satisfying interpretation for a formula, but to find this interpretation. FP is the class of functional problems that can be solved by a deterministic Turing machine in polynomial time, whereas FNP is the class of function problems that can be solved by a non-deterministic Turing machine in polynomial time [Pap94]. A typical problem of the latter class is FSAT which asks for a model of a given propositional formula.

FSAT	**Input**:	a formula ϕ in CNF
	Output:	$\mathsf{Mod}(\phi)$

The class $\mathsf{FP}^{\mathcal{C}[f(n)]}$ is the class of all functional problems solvable by a deterministic polynomial-time Turing machine with $\mathcal{O}(f(n))$ calls to an oracle Turing machine complete for some problem in class $\mathcal{C}$.

As we will see with the definition of the new inconsistency measure in Section 4.1, it is necessary to analyse problems associated with counting. For this, counting complexity classes are introduced. Counting decision problems are problems which ask, whether there are at least or exactly a given number of solutions. These make them intuitively harder than decision problems that ask "only", if there is any solution. First, the predicate-based counting quantifier C is defined in the following which is needed for the definition of the complexity classes [Wag86].

Definition 6. C *is defined for a predicate $H(x, y)$ with free variables x and y as* $\mathsf{C}^k_y H(x, y) \leftrightarrow |\{y \mid H(x, y) \textit{ is true}\}| \geq k$.

This means, the counting quantifier is true for the predicate $H(x, y)$ and bound k iff the number of values of y, for which the predicate holds, is at least k. The following definition is about the polynomially bounded version of the counting quantifier [Wag86].

Definition 7. *For a class of problems K a problem A is in $\mathsf{C}K$ iff there is a problem $B \in K$, a polynomial-time computable function f, and a polynomial p with* $x \in A \leftrightarrow \mathsf{C}^{f(x)}_{|y| \leq p(|x|)}(x, y) \in B$.

This means, that an instance x is in problem A iff the number of values of y, which is polynomially bounded by x, for which the predicate holds and the predicate being in B, is at least $f(x)$. The canonical problem for the class CP is, whether there are at least k models for a Boolean formula (this problem seems to have no explicit name).

Input:	a formula ϕ in CNF
Output:	TRUE iff $|\mathsf{Mod}(\phi)| \geq k$

For the counting polynomial time hierarchy, we have $\mathsf{C}\Sigma^p_i = \mathsf{C}\Pi^p_i = \mathsf{coC}\Sigma^p_i$ for all $i \geq 0$. The class which will be mainly used in the complexity analysis section is CNP. A variation of this class is $\mathsf{C}_{=}\mathsf{NP}$, for which there is no lower bound k for the counting quantifier, but the number should be exactly the value k [Wag86].

Lastly, preliminaries for functional counting problems are introduced. Functional counting problems ask for the number of solutions. A solution is defined by a witness function w. The cardinality of witnesses $|w|$ is then the output of the associated counting problem. The class #P is the class of functional problems that ask for the number of accepting paths of nondeterministic Turing machines in polynomial time [Val79]. This also means, the class encompasses underlying decision problems that are in NP. The canonical #P-complete problem is the counting satisfiability problem

#SAT which asks for the number of models for a propositional formula [Pap94]. The model is in this case the witness w.

#SAT	**Input**:	a formula ϕ in CNF
	Output:	$\lvert\mathsf{Mod}(\phi)\rvert$

Hemaspaandra and Vollmer expanded the work by Valiant with a predicate based framework [HV95]. If $\mathcal{C}$ is a complexity class of decision problems, then $\#{\cdot}\mathcal{C}$ is the class of all counting problems with witness function w, which meets the following two conditions: for every input string x, every $y \in w(x)$ is polynomially bounded by x and the decision problem of deciding $y \in w(x)$ for given strings x and y is in $\mathcal{C}$ [HV95]. With this notion, we have $\#\mathsf{P} = \#{\cdot}\mathsf{P}$ and for higher complexity classes $\#\Sigma_i^p = \#\Pi_i^p = \#{\cdot}\Pi_i^p$ for $i \geq 1$, so particularly $\#\mathsf{NP} = \#{\cdot}\mathsf{coNP}$.

An example for this class is #CIRCUMSCRIPTION [DHK05]. The underlying decision problem asks for the minimal models of a given formula. Durand showed that this problem is $\#{\cdot}\mathsf{coNP}$-complete.

#CIRCUMSCRIPTION	**Input**:	a formula ϕ in CNF
	Output:	$\lvert\mathsf{MinMod}(\phi)\rvert$

For showing hardness of the previous example and in general for counting problems, Durand et. al introduced *substractive reductions* [DHK05]. The basic idea behind these reductions is to overcount the number of solutions for a problem and then subtract the surplus.

Definition 8. *Let $\#{\cdot}A$ and $\#{\cdot}B$ be two counting problems with the witness functions $w_A(x)$ and $w_B(x)$. A counting problem $\#{\cdot}A$ reduces to the counting problem $\#{\cdot}B$ via strong substractive reduction, written as $\#{\cdot}A \leq_{ssr} \#{\cdot}B$, if there exist two polynomial-time computable functions f and g such that for each x: $w_B(f(x)) \subseteq w_B(g(x))$ and $|w_A(x)| = |w_B(g(x))| - |w_B(f(x))|$.*

With these substractive reductions completeness for counting problems in $\#{\cdot}\mathsf{coNP}$ and higher counting complexity classes can be proved.

3 State of Research on Inconsistency Measurement

Having introduced the preliminaries in the previous chapter, this chapter gives an overview of the research about inconsistency measures and their evaluation methods.

3.1 Measuring Inconsistency

Inconsistency measures are basically functions that assign a non-negative real value to a given knowledge base [Thi09, GH11, Bes14], see Definition 9. By that, they quantify the severity of inconsistency of the knowledge base. The informal convention is, that the higher the value of the inconsistency measure is, the higher is the inconsistency in the knowledge base. An inconsistency value of zero means that there is no inconsistency in the knowledge base.

Definition 9. *An inconsistency measure $\mathcal{I}$ is any function $\mathcal{I} : \mathbb{K} \to \mathbb{R}^{\infty}_{\geq 0}$.*

An example of a very simple inconsistency measure is the drastic inconsistency measure $\mathcal{I}_d$ which assigns the value 0 to consistent knowledge bases and the value 1 to inconsistent knowledge bases [HK08].

Definition 10. $\mathcal{I}_d(\mathcal{K}) = \begin{cases} 1 & \textit{if } \mathcal{K} \models \perp \\ 0 & \textit{otherwise} \end{cases}$

Example. *Let $\mathcal{K}_1 = \{a, b\}$ and $\mathcal{K}_2 = \{a, \neg a\}$. Then, $\mathcal{I}_d(\mathcal{K}_1) = 0$ and $\mathcal{I}_d(\mathcal{K}_2) = 1$.*

Inconsistency measures can be divided into approaches that are formula-based, respectively syntactic, and measures that are language-based, respectively semantic [HK04, HK06, HDBGK18, Thi19]. However, this is not a unanimously agreed on division, shown by the fact that there are several measures that fall in neither of both categories [Thi19].

Syntactic approaches, on the one hand, use syntactic objects to quantify inconsistency. They mainly focus on conflicts between or within formulae in the knowledge base. Common approaches of the syntactic approach are measurements using minimum inconsistent subsets [Hun04, HK08, MLJ11, GH11, JMR14, JMR$^+$16, DGHK19] and closely related to that, measures which focus on maximal consistent subsets [GH11, ARSO15, ASOR17, DGHK19]. Other inconsistency measurement approaches use so called minimal proofs [JR13] or are based on forgetting, meaning restoring a consistent knowledge base [Bes16, Sal19].

Three exemplary inconsistency measures are the MI-inconsistency measure, the MI^{C}-inconsistency measure, and the D_f-inconsistency measure. The MI-inconsistency measure is based on the number of minimal inconsistent subsets [HK08] .

Definition 11. *The* MI*-inconsistency measure is defined as $\mathcal{I}_{\mathsf{MI}} = |\mathsf{MI}(\mathcal{K})|$ for $K \in \mathbb{K}$.*

The MI^{C}-inconsistency measure is, as well, based on minimal inconsistent subsets, but takes the size of each minimal inconsistent subset into account [HK08].

Definition 12. *The MI^{C}-inconsistency measure is defined as $\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}} = \sum_{M \in \mathsf{MI}(\mathcal{K})} \frac{1}{|M|}$ for $K \in \mathbb{K}$.*

The D_f-inconsistency measure is, as well, based on minimal inconsistent subsets, and further on consistent subsets [MLJB11]. Both their sizes are taken into account. For the definition of the inconsistency measure, the following definitions are needed first.

For $\mathcal{K} \in \mathbb{K}$, $\mathsf{MI}^i(\mathcal{K}) = \{M \in \mathsf{MI}(\mathcal{K}) \mid |M| = i\}$, and for consistent subsets $\mathsf{CN}^i(\mathcal{K}) = \{C \in \mathcal{K} \mid |C| = i \wedge C \models \perp\}$. With that, $R_i(\mathcal{K}) = 0$ if $|\mathsf{MI}^i(\mathcal{K})| + |\mathsf{CN}^i(\mathcal{K})| = 0$ and otherwise $R_i(\mathcal{K}) = |\mathsf{MI}^i(\mathcal{K})|/(|\mathsf{MI}^i(\mathcal{K})| + |\mathsf{CN}^i(\mathcal{K})|)$.

Definition 13. *The D_f-inconsistency measure is defined as* $\mathcal{I}_{D_f} = 1 - \Pi_{i=1}^{|\mathcal{K}|}(1 - R_i(\mathcal{K})/i)$ *for* $K \in \mathbb{K}$.

Semantic approaches, on the other hand, focus on conflicts between atoms of the language. This means that the approaches typically use non-classical logics [Thi17a, GH23]. The inconsistency measures are typically based on multi-valued logics, including quasi-classical logic [Hun00, Hun02, XLMQ10], 3-valued logic [XLMQ10, GH11, GH23], 4-valued logic [Hun06, GH08, MQX$^+$09, XLMQ10], probabilistic logic [Kni02, Thi09, Thi13], and fuzzy logic [Thi17a].

An example is the contension inconsistency measure $\mathcal{I}_c$ which is based on three-valued logic with the interpretation v that is a function from At to $\{T, F, B\}$ [GH11]. T and F are the classical truth values and B stands for *both*, which means that there is a conflict for the respective atom. An interpretation satisfies a formula ϕ, if $v(\phi) = T$ or $v(\phi) = B$, denoted by $v \models^3 \phi$. The contension inconsistency measure is defined on interpretations of a knowledge base that assign the truth value B to a minimum number of atoms.

Definition 14. *The contension inconsistency measure is defined as* $\mathcal{I}_c = min\{v^{-1}(B) \mid v \models^3 \mathcal{K}\}$ *for* $K \in \mathbb{K}$.

There are as well inconsistency measures that do not fall in one of the two categories described above. These measures include the mv-inconsistency measure [XM12], the hitting-set inconsistency measure [Thi16b], distance-based measures [GH13, GH17], and measures that are based on extended propositional logic [Gra20, Gra23].

The mv-inconsistency measure $\mathcal{I}_{mv}$, as an example from this group, is based on the atoms occurring in minimal inconsistent subsets in relation to the whole knowledge base [XM12].

Definition 15. *The mv-inconsistency measure is defined as* $\mathcal{I}_{mv} = \frac{|\bigcup_{M \in \mathsf{MI}(\mathcal{K})} \mathsf{At}(M)|}{|\mathsf{At}(\mathcal{K}|}$ *for* $K \in \mathbb{K}$.

Having introduced the current state of research about measuring inconsistency including example measures, the following section is about methods to evaluate inconsistency measures. The introduced inconsistency measures serve again as examples.

3.2 Evaluation of Inconsistency measures

Besides defining inconsistency measures, researchers have focused on describing characteristics of these measures for their evaluation and comparison. Evaluation methods for inconsistency measures comprise rationality postulates, expressivity, and complexity which are described in detail in the following sections.

3.2.1 Rationality Postulates

Rationality postulates have been proposed by various researchers to define favorable characteristics of inconsistency measures. Hunter and Konieczny proposed the first five of the following rationality postulates [HK06]. These have been extended by various other researchers over time. In the following, 20 rationality postulates will be introduced and described.

The first postulate is *Consistency* [HK06, HK08, HK10, Thi09, GH11]. This postulate can be seen as the most basic requirement for inconsistency measurement. It states that only consistent knowledge bases get the value zero, and inconsistent knowledge bases a value greater than zero. The postulate guarantees that inconsistency, instead of information, is measured [Thi09]. This means, those inconsistency measures that fulfill the postulate are able to quantify differences in inconsistencies but do not differentiate between consistent knowledge bases, as all consistent knowledge bases are assigned inconsistency value zero. This postulate is inherently the basis for inconsistency measures.

Postulate (Consistency). $\mathcal{I}(\mathcal{K}) = 0$ *iff* $\mathcal{K}$ *is consistent.*

The postulate of *Normalization* states that an inconsistency value shall be between zero and one [HK06, HK10]. This postulate is usually not regarded mandatory for inconsistency measures [HK06, HK10]. However, it is favorable as it allows comparability between different inconsistency measures due to normalization [Thi09]. An example of a normalized inconsistency measure is the mv-inconsistency measure [XM12, Thi17b]. An example for a measure that does not satisfy *Normalization* is $\mathcal{I}_{\mathsf{MI}}$ [HK08].

Postulate (Normalization). $0 \leq \mathcal{I}(\mathcal{K}) \leq 1$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a, b, c \wedge \neg b, d\}$. *Then* $\mathcal{I}_{mv}(\mathcal{K}) = 1/2$, *but* $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = 2$.

Monotony states that with the growth of a knowledge base, i.e., new formulae are added, the inconsistency value cannot decrease [HK06, HK08, HK10, Thi09, GH11]. An example for an inconsistency measure that satisfies *Monotony* is the measure I_{MI} [HK08, Thi13]. Contrary, the exemplary measure I_{mv} does not satisfy this postulate [Thi17b].

Postulate (Monotony). *If* $\mathcal{K} \subseteq \mathcal{K}'$ *then* $\mathcal{I}(\mathcal{K}) \leq \mathcal{I}(\mathcal{K}')$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a, b\}$ *and* $\mathcal{K}' = \{a \wedge \neg a, b, c \wedge \neg b, d\}$. *Then* $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = 1$ *and* $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}') = 2$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a\}$ *and* $\mathcal{K}' = \{a \wedge \neg a, b\}$. *Then* $\mathcal{I}_{mv}(\mathcal{K}) = 1$ *and* $\mathcal{I}_{mv}(\mathcal{K}') = 1/2$.

The postulate of *Free-Formula Independence* says that consistent formulae do not add any inconsistency to the overall inconsistency [HK06, HK08, HK10, Thi09, GH11]. This means removing them from the knowledge base should not change the inconsistency value of the new knowledge base according to the postulate. Again, the inconsistency measure $\mathcal{I}_{\mathsf{MI}}$ satisfies this postulate, while the measure $\mathcal{I}_{mv}$ does not satisfy *Free-Formula Independence* [HK08, Thi17b].

Postulate (Free-Formula Independence). *If* $\alpha \in \mathsf{Free}(\mathcal{K})$ *then* $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a, b\}$. $\mathsf{Free}(\mathcal{K}) = \{b\}$. $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = 1$ *and* $\mathcal{I}_{\mathsf{MI}}(\mathcal{K} \setminus \{b\}) = 1$. $\mathcal{I}_{mv}(\mathcal{K}) = 1/2$ *and* $\mathcal{I}_{mv}(\mathcal{K} \setminus \{b\}) = 1$.

Dominance says that the substitution of a consistent formula by a weaker formula does not lead to an increase of the inconsistency value [HK06, HK08, HK10]. $\mathcal{I}_c$ is an exemplary inconsistency measure that satisfies *Dominance*, while $\mathcal{I}_{\mathsf{MI}}$ does not satisfy this postulate [MLJB11, Thi17b].

Postulate (Dominance). *If* $\alpha \not\models \perp$ *and* $\alpha \models \beta$ *then* $\mathcal{I}(\mathcal{K} \cup \{\alpha\}) \geq \mathcal{I}(\mathcal{K} \cup \{\beta\})$.

Example. *Let* $\mathcal{K} = \{a, \neg a\}$, $\alpha = \{a\}$ *and* $\beta = \{a \wedge a\}$. $\mathcal{I}_c(\mathcal{K} \cup \{\alpha\}) = 1$ *and* $\mathcal{I}_c(\mathcal{K} \cup \{\beta\}) = 1$. $\mathcal{I}_{\mathsf{MI}}(\mathcal{K} \cup \{\alpha\}) = 1$, *but* $\mathcal{I}_{\mathsf{MI}}(\mathcal{K} \cup \{\beta\}) = 2$.

The next three postulates are originally proposed by Thimm [Thi09]. The postulate of *Super-Additivity* states that the inconsistency sum of two disjoint knowledge bases is not greater than the inconsistency of the joint knowledge base. Again, $\mathcal{I}_{\mathsf{MI}}$ is an exemplary inconsistency measure that satisfies *Super-Additivity*, while I_{mv} does not satisfy this postulate [Thi13, Thi17b].

Postulate (Super-Additivity). *If* $\mathcal{K} \cap \mathcal{K}' = \emptyset$ *then* $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') \geq \mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}')$

Example. *Let* $\mathcal{K} = \{a \wedge \neg a, c\}$ *and* $\mathcal{K}' = \{b \wedge \neg c, d\}$. $\mathcal{I}_{\mathsf{MI}}(\mathcal{K} \cup \mathcal{K}') = 2 > \mathcal{I}_{\mathsf{MI}}(\mathcal{K}) + \mathcal{I}_{\mathsf{MI}}(\mathcal{K}') = 1$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a, c\}$ *and* $\mathcal{K}' = \{b \wedge \neg b, c\}$. $\mathcal{I}_{mv}(\mathcal{K} \cup \mathcal{K}') = 2/3 < \mathcal{I}_{mv}(\mathcal{K}) + \mathcal{I}_{mv}(\mathcal{K}') = 1$.

Safe-Formula Independence says that removing a safe formula, meaning a consistent formula which signature is disjoint from the signature in the knowledge base, does not change the inconsistency value. This is a weaker requirement than *Free-Formula Independence*, as every safe formula is, by definition, also a free formula. Here, $\mathcal{I}_{\mathsf{MI}}$ is an exemplary inconsistency measure that satisfies *Safe-Formula Independence*, while I_{mv} does not satisfy this postulate [Thi13, Thi17b].

Postulate (Safe-Formula Independence). *If* $\alpha \in \mathsf{Safe}(\mathcal{K})$ *then* $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a, c\}$. $\mathsf{Safe}(\mathcal{K}) = \{c\}$. $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = \mathcal{I}_{\mathsf{MI}}(\mathcal{K} \setminus \{c\}) = 1$, *but* $\mathcal{I}_{mv}(\mathcal{K}) = 1/2$ *and* $\mathcal{I}_{mv}(\mathcal{K} \setminus \{c\}) = 1$.

The *Penalty* postulate states that adding an inconsistent formula to a knowledge base increases the inconsistency of the knowledge base. Again, $\mathcal{I}_{\mathsf{MI}}$ is an exemplary inconsistency measure that satisfies *Penalty*, while I_{mv} does not satisfy this postulate [Thi13, Thi17b].

Postulate (Penalty). *If* $\alpha \notin \mathsf{Free}(\mathcal{K})$ *then* $\mathcal{I}(\mathcal{K}) > \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a, c \wedge \neg c\}$ *and* $\alpha = \{c \wedge \neg c\}$. $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = 2 > \mathcal{I}_{\mathsf{MI}}(\mathcal{K} \setminus \{\alpha\}) = 1$, *but* $\mathcal{I}_{mv}(\mathcal{K}) = \mathcal{I}_{mv}(\mathcal{K} \setminus \{\alpha\}) = 1$.

The following five postulates focus especially on measures that use minimal inconsistent subsets. *MI-Separability* states that the sum of the inconsistency values of two knowledge bases that have "non-interfering" sets of minimal inconsistent subsets is equal to the inconsistency value of their union [HK10]. Once again, $\mathcal{I}_{\mathsf{MI}}$ is an exemplary inconsistency measure that satisfies *MI-Separability*, while I_{mv} does not satisfy this postulate [HK10, Thi17b].

Postulate (MI-Separability). *If* $\mathsf{MI}(\mathcal{K} \cup \mathcal{K}') = \mathsf{MI}(\mathcal{K}) \cup \mathsf{MI}(\mathcal{K}')$ *and* $\mathsf{MI}(\mathcal{K} \cap \mathcal{K}') = \emptyset$ *then* $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}')$.

Example. *Let* $\mathcal{K} = \{a, \neg a\}$ *and* $\mathcal{K}' = \{b, \neg b\}$. $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = 1$, $\mathcal{I}_{\mathsf{MI}}(\mathcal{K}') = 1$. $\mathcal{I}_{\mathsf{MI}}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}_{\mathsf{MI}}(\mathcal{K}) + \mathcal{I}_{\mathsf{MI}}(\mathcal{K}') = 2$. $\mathcal{I}_{mv}(\mathcal{K}) = 1$ *and* $\mathcal{I}_{mv}(\mathcal{K}') = 1$, *but* $\mathcal{I}_{mv}(\mathcal{K} \cup \mathcal{K}') = 1$.

MI-Normalization states that the inconsistency value of a minimal inconsistent subset should be one [HK10]. It thus represents an atomic unit of inconsistency measurement [Thi17b, Thi18]. As an example, the measure $\mathcal{I}_{\mathsf{MI}}$ satisfies the postulate, whereas the measure $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ does not satisfy *MI-Normalization* [HK10, Thi17b].

Postulate (MI-Normalization). *If* $M \in \mathsf{MI}(\mathcal{K})$ *then* $\mathcal{I}(M) = 1$.

Example. *Let* $\mathcal{K} = \{a, \neg a, b\}$. $\mathsf{MI}(\mathcal{K}) = \{a, \neg a\}$. *Then,* $\mathcal{I}_{\mathsf{MI}}(\{a, \neg a\}) = 1$, *but* $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(\{a, \neg a\}) = 1/2$.

The postulate of *Attenuation* requires smaller minimal inconsistent sets to have larger inconsistency values than larger inconsistent sets [MLJ11]. As examples, the measure $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ satisfies *Attenuation*, while $\mathcal{I}_{\mathsf{MI}}$ does not satisfy the postulate [Thi17b, Thi18].

Postulate (Attenuation). $M, M' \in \mathsf{MI}(\mathcal{K})$ *and* $|M| > |M'|$ *implies* $\mathcal{I}(M) < \mathcal{I}(M')$.

Example. *Let* $\mathcal{K} = \{a, \neg a, b \wedge \neg b\}$. $M = \{a, \neg a\}$, $M' = \{b \wedge \neg b\}$. $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(M) = 1/2$ *and* $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}(M') = 1$, *but* $\mathcal{I}_{\mathsf{MI}}(M) = \mathcal{I}_{\mathsf{MI}}(M') = 1$

The counterpart of the *Attenuation* postulate is *Equal Conflict*. It states that minimal inconsistent subsets with the same size should have the same inconsistency value [MLJ11]. Again, the measure $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$ satisfies *Equal Conflict*, while $\mathcal{I}_{\mathsf{MI}}$ does not satisfy the postulate [Thi17b, Thi18]. Examples for these measures are given above for *Attenuation*.

Postulate (Equal Conflict). $M, M' \in \mathsf{MI}(\mathcal{K})$ *and* $|M| = |M'|$ *implies* $\mathcal{I}(M) = \mathcal{I}(M')$.

Almost Consistency states that, when the number of elements in a minimal inconsistent subset becomes infinitely large, the inconsistency value reaches zero in the limit [MLJ11]. As examples again, the measure $\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}$ satisfies *Almost Consistency*, while $\mathcal{I}_{\mathsf{MI}}$ does not satisfy the postulate [Thi17b, Thi18]. The definitions of both inconsistency measures highlight the difference: While $\mathcal{I}_{\mathsf{MI}}$ considers the number of minimal inconsistent subsets independently from its size, the measure $\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}$ takes the size into account with the result that infinitely large minimal inconsistent subsets reach inconsistency value zero in the limit.

Postulate (Almost Consistency). *Let $M_1, M_2, ...$ be a sequence of minimal inconsistent sets M_i with $lim_{i\to\infty}|M_i| = \infty$, then $lim_{i\to\infty}\mathcal{I}(M_i) = 0$.*

Example. *Let $\mathcal{K} = \{a_1, ..., a_i, \neg(a_1 \wedge ... \wedge a_i)\}$. $\mathcal{K}$ is minimally inconsistent. Then, $lim_{i\to\infty}\mathcal{I}_{\mathsf{MI}^{\mathsf{C}}}(\mathcal{K}) = 0$, while $lim_{i\to\infty}\mathcal{I}_{\mathsf{MI}}(\mathcal{K}) = \infty$.*

The following postulates are not restricted to measures that use minimal inconsistent subsets. The postulate of *Contradiction* is based upon *Normalization*. *Contradiction* requires the inconsistency value to be maximal, meaning to be one, if all subsets of the given knowledge base are inconsistent [MLJB11]. As examples, the measure $\mathcal{I}_{D_f}$ satisfies *Contradiction*, while the measure $\mathcal{I}_{mv}$ does not satisfy this postulate [MLJB11, Thi17b].

Postulate (Contradiction). *$\mathcal{I}(\mathcal{K}) = 1$ iff for all $\emptyset \neq \mathcal{K}' \subseteq \mathcal{K}, \mathcal{K}' \models \perp$.*

Example. *Let $\mathcal{K} = \{a, \neg a\}$. $\mathcal{I}_{D_f}(\mathcal{K}) = 1/2$. $\mathcal{I}_{mv}(\mathcal{K}) = 1$, but $\{a\} \not\models \perp$.*

Free-Formula Dilution is a weaker form of *Free-Formula Independence* and is originally based upon *Normalization*. It does not require the inconsistency value to stay equal when removing free formulae like *Free-Formula Independence*, but the value can decrease [MLJB11]. As examples, $\mathcal{I}_{\mathsf{MI}}$ satisfies this postulate, while I_{mv} does not satisfy *Free-Formula Dilution* [Thi17b]. An example of how these two measures behave with regard to free formulae is given above with the introduction of *Free-Formula Independence*.

Postulate (Free-Formula Dilution). *If $\alpha \in \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) \geq \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$.*

Having two given knowledge bases with pairwise equivalent formulae, the postulate of *Irrelevance of Syntax* requires them to have the same inconsistency value [GH11, Thi11]. Exemplary measures are $\mathcal{I}_{\mathsf{MI}}$ which satisfies this postulate, while the measure I_{mv} does not satisfy *Irrelevance of Syntax* [Thi17b].

Postulate (Irrelevance of Syntax). *If $\mathcal{K} \equiv_b \mathcal{K}'$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K}')$.*

Example. *Let $\mathcal{K} = \{\neg a, \neg b, a\}$ and $\mathcal{K}' = \{\neg a, \neg b, a \wedge (b \vee \neg b)\}$. $I_{\mathsf{MI}}(\mathcal{K}) = I_{\mathsf{MI}}(\mathcal{K}') = 1$. $I_{mv}(\mathcal{K}) = 1/2$, but $I_{mv}(\mathcal{K}') = 1$.*

The *Exchange* postulate states that switching consistent parts of the knowledge base with equivalent ones the inconsistency value does not change [Bes14]. The measure $\mathcal{I}_c$ is an exemplary measure that satisfies *Exchange*, while $\mathcal{I}_{mv}$ does not satisfy this postulate [Thi17b].

Postulate (Exchange). *If* $\mathcal{K}' \nvDash \perp$ *and* $\mathcal{K}' \equiv_b \mathcal{K}''$ *then* $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}(\mathcal{K}' \cup \mathcal{K}'')$.

Example. *As in the previous example, let* $\mathcal{K} = \{\neg a, \neg b, a\}$ *and* $\mathcal{K}' = \{\neg a, \neg b, a \wedge (b \vee \neg b)\}$. $I_c(\mathcal{K}) = I_c(\mathcal{K}') = 1$. $I_{mv}(\mathcal{K}) = 1/2$, *but* $I_{mv}(\mathcal{K}') = 1$.

Adjunction Invariance requires that the set notation of a knowledge base, compared to the conjunction notation of its formulae, has the same inconsistency value [Bes14]. Again, the measure $\mathcal{I}_c$ is an exemplary measure that satisfies *Adjunction Invariance*, while $\mathcal{I}_{\mathsf{MI}}$ does not satify this postulate [Thi17b].

Postulate (Adjunction Invariance). $\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) = \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$.

Example. *Let* $\mathcal{K} = \{a, a \wedge a, \neg a\}$ *and* $\mathcal{K}' = \{a \wedge a \wedge a, \neg a\}$. *Then,* $I_c(\mathcal{K}) = I_c(\mathcal{K}') = 1$. $I_{\mathsf{MI}}(\mathcal{K}) = 2$, *but* $I_{\mathsf{MI}}(\mathcal{K}') = 1$.

A publication by Besnard and Grant focused on the difference between relative and absolute measures which they define, on the one hand, as measuring the total amount of inconsistency in a knowledge base, and, on the other hand, measuring the proportion of a knowledge base that is inconsistent [BG20]. That is, why relative measures build upon the postulate of *Normalization*. For relative measures they defined two postulates.

First, *Free-Formula Reduction* is a postulate which states that adding a free formula to an inconsistent knowledge base decreases the inconsistency value of the knowledge base. The D_f inconsistency measure satisfies *Free-Formula Reduction*, while the measure I_{mv} does not satisfy this postulate [BG20].

Postulate (Free-Formula Reduction). *For* $\alpha \notin \mathcal{K}$, α *is free for* $\mathcal{K}$, *and* $\mathcal{I}(\mathcal{K}) \neq 0$, *then* $\mathcal{I}(\mathcal{K} \cup \{\alpha\}) < \mathcal{I}(\mathcal{K})$.

Example. *Let* $\mathcal{K} = \{a \wedge \neg a\}$ *and* $\alpha = \{a \vee a\}$. $I_{D_f}(\mathcal{K}) = 1$ *and* $I_{D_f}(\mathcal{K} \cup \{\alpha\}) = 1/2$, *but* $I_{mv}(\mathcal{K}) = I_{mv}(\mathcal{K} \cup \{\alpha\}) = 1$.

Second, *Relative Separability* describes how inconsistency values behave when splitting a knowledge base into two language-disjoint parts with different ratios. The proportion of inconsistency for the whole knowledge base should be in between the proportion of the two parts. The mv-inconsistency measure satisfies *Relative Separability*, while the drastic measure I_d does not satisfy this postulate [BG20].

Postulate (Relative Separability). *If* $\mathcal{I}(\mathcal{K}) \lessapprox \mathcal{I}(\mathcal{K}')$ *and* $\mathsf{At}(\mathcal{K}) \cap \mathsf{At}(\mathcal{K}') = \emptyset$, *then* $\mathcal{I}(\mathcal{K}) \lessapprox \mathcal{I}(\mathcal{K} \cup \mathcal{K}') \lessapprox \mathcal{I}(\mathcal{K}')$ *where either* $\lessapprox$ *is* $<$ *in every instance or* $\lessapprox$ *is* $=$ *in every instance.*

Example. *Let* $\mathcal{K} = \{a \wedge \neg a\}$ *and* $\mathcal{K}' = \{b\}$. *Then,* $I_{mv}(\mathcal{K}) = 1$, $I_{mv}(\mathcal{K}') = 0$, *and* $I_{mv}(\mathcal{K} \cup \mathcal{K}') = 1/2$. $I_d(\mathcal{K}) = 1$, $I_d(\mathcal{K}') = 0$, *but* $I_d(\mathcal{K} \cup \mathcal{K}') = 1$.

Some of these postulates are incompatible and other postulates imply further postulates [Thi17b, Thi18]. This means that any inconsistency measure cannot fulfill all of the described postulates. Furthermore, for each postulate, we can find an inconsistency measure that satisfies this postulate and contrary another inconsistency measure, that does not satisfy this postulate. Examples have been given for each postulate. An overview of the compliance of a large number of inconsistency measures is given by Thimm [Thi17b, Thi18].

3.2.2 Expressivity

Expressivity is an evaluation method for inconsistency measures that quantifies to which extent a given inconsistency measure is capable to distinguish between different inconsistent knowledge bases [Thi16a]. This method has been introduced by Thimm [Thi16a] to extend and complement the evaluation of inconsistency measures beyond the described rationality postulates. The reason for this is that single rationality postulates commonly focus on a single characteristic of inconsistency [Thi16a]. Furthermore, the rationality of the proposed postulates has been discussed by researchers, see for example, publications by Besnard [Bes14, Bes17].

Four expressivity characteristics have been proposed by Thimm [Thi16a]. They evaluate how many different values an inconsistency measure can attain, with the differentiation between four different dimensions of sub-classes of knowledge bases, each focusing on a different characteristic of the size of the knowledge base.

Definition 16. *The four subclasses for the set of all knowledge bases are defined as*

$$\begin{aligned}\mathbb{K}^v(n) &= \{\mathcal{K} \in \mathbb{K} \mid |\mathsf{At}(\mathcal{K})| \leq n\} \\ \mathbb{K}^f(n) &= \{\mathcal{K} \in \mathbb{K} \mid |\mathcal{K}| \leq n\ \} \\ \mathbb{K}^l(n) &= \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : \mathsf{len}(\phi) \leq n\} \\ \mathbb{K}^p(n) &= \{\mathcal{K} \in \mathbb{K} \mid \forall \phi \in \mathcal{K} : |\mathsf{At}(\phi)| \leq n\}.\end{aligned}$$

These four subclasses are the set of all knowledge bases that mention at most a certain number of different atoms $\mathbb{K}^v(n)$, the set of all knowledge bases that contain at most a defined number of formulae $\mathbb{K}^f(n)$, the set of all knowledge bases that contain only formulae with a maximal defined length $\mathbb{K}^l(n)$, and the set of all knowledge bases that contain only formulae that mention a certain number of different atoms each $\mathbb{K}^p(n)$. The expressivity characteristics are defined in the following [Thi16a].

Definition 17. *Let $\mathcal{I}$ be an inconsistency measure and $n > 0$. Let $\alpha \in \{v, f, l, p\}$. The α-characteristic $\mathcal{C}^\alpha(\mathcal{I}, n)$ of $\mathcal{I}$ wrt. n is defined as $\mathcal{C}^\alpha(\mathcal{I}, n) = |\{\mathcal{I}(\mathcal{K}) \mid \mathcal{K} \in \mathbb{K}^\alpha(n)\}|$.*

As examples, find the results for α-characteristics of the already introduced inconsistency measures in the following Table 1. Results are taken from Thimm [Thi16a].

With regard to expressivity, inconsistency measures should consider the size of the knowledge base in a way, that larger knowledge bases with regard to these four subclasses provide a larger variety of inconsistency values compared to the given knowledge base. For example, results of the drastic measure I_d with regard to expressivity characteristics are not favorable, as it can only take two values, regardless

	$\mathcal{C}^v(\mathcal{I}, n)$	$\mathcal{C}^f(\mathcal{I}, n)$	$\mathcal{C}^l(\mathcal{I}, n)$	$\mathcal{C}^p(\mathcal{I}, n)$
I_d	2	2	2^*	2
I_{MI}	∞	$\binom{n}{\lfloor n/2 \rfloor} + 1$	∞^*	∞
$I_{\mathsf{MI}^\mathsf{C}}$	∞	$\leq \Psi(n)^{**}$	∞^*	∞
I_c	$n+1$	∞	∞^*	∞
I_{D_f}	∞	$\leq \Psi(n)^{**}$	∞^*	∞
I_{mv}	$n+1$	∞^*	∞^*	∞

Table 1: α-characteristics for some inconsistency measures for $n \geq 1$ (*for $n > 1$, ** $\Psi(n)$ is the number of monotone Boolean functions of n variables. Results are taken from Thimm [Thi16a].

of the size of the knowledge base for all four α-characteristics. Contrary, all the other measures, that are shown in Table 1, have maximal expressivity with regard to $\mathcal{C}^l(\mathcal{I}, n)$ and $\mathcal{C}^p(\mathcal{I}, n)$ which is ideal.

With the definition of expressivity, it is possible to compare inconsistency measures quantitatively in a form of an order relationship which is an essential part of the described method [Thi16a]. Basis for a comparative analysis is given by the following definition of the expressivity order [Thi16a].

Definition 18. *An inconsistency measure $\mathcal{I}$ is at least as expressive as an inconsistency measure $\mathcal{I}'$ wrt. a characteristic $\mathcal{C}^\alpha(\alpha \in \{f, v, l, p\})$, denoted by $\mathcal{I} \succeq_\alpha \mathcal{I}'$, if there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $\mathcal{C}^\alpha(\mathcal{I}, n) \geq \mathcal{C}^\alpha(\mathcal{I}', n)$.*

Two inconsistency measures $\mathcal{I}$ and $\mathcal{I}'$ are equally expressive wrt. $\mathcal{C}^\alpha$ if both $\mathcal{I} \succeq_\alpha \mathcal{I}'$ and $\mathcal{I}' \succeq_\alpha \mathcal{I}$ and we write $\mathcal{I} \sim_\alpha \mathcal{I}$. If $\mathcal{I} \succeq_\alpha \mathcal{I}'$ but not $\mathcal{I} \sim_\alpha \mathcal{I}$, we write $\mathcal{I} \succ_\alpha \mathcal{I}'$. It means that $\mathcal{I}$ is strictly more expressive than $\mathcal{I}'$.

So, regarding the number of atoms within different knowledge bases, the following example shows the expressivity order. Here, inconsistency measures within the same set have the same expressivity. This means, $\mathcal{I}_c$ and $\mathcal{I}_{mv}$ have the same expressivity, which in turn is higher than the expressivity of the drastic measure and lower than the expressivity of the measures $\mathcal{I}_{\mathsf{MI}}$, $\mathcal{I}_{\mathsf{MI}^\mathsf{C}}$, and $\mathcal{I}_{D_f}$.

Example. $\{\mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{\mathsf{MI}^\mathsf{C}}, \mathcal{I}_{D_f}\} \succ_v \{\mathcal{I}_c, \mathcal{I}_{mv}\} \succ_v \{\mathcal{I}_d\}$.

An overview of expressivity results for a large number of inconsistency measures is given by Thimm [Thi16a, Thi17a, Thi18]. Within this overview the drastic measure has least expressivity. This is not surprising as it can only take two values for all four α-characteristics. The overview also shows that a large number of inconsistency measures have maximal expressivity with regard to $\mathcal{C}^l(\mathcal{I}, n)$ and $\mathcal{C}^p(\mathcal{I}, n)$.

However, expressivity should not be the only evaluation for inconsistency measures, but should rather serve as an additional means to characterize inconsistency measures besides rationality postulates [Thi16a].

3.2.3 Complexity

Research on computational complexity of different inconsistency measures is relatively sparse compared to the number of inconsistency measures that have been proposed, but has gained some attention in recent years. With regard to the existing literature, the following three decision problems are considered for the investigation of the computational complexity of inconsistency measurement [MQX+10, XM12, TW19, PG23a]. The first one is $\textsc{Exact}_{\mathcal{I}}$ which is about the decision whether a given value x is the inconsistency value of a given knowledge base. $\textsc{Upper}_{\mathcal{I}}$ and $\textsc{Lower}_{\mathcal{I}}$ are the problems of deciding whether a given value x is the upper or lower bound of the inconsistency value of a given knowledge base. Furthermore, the functional problem of determining the actual inconsistency value, $\textsc{Value}_{\mathcal{I}}$, is considered as well for the analysis.

$\textsc{Exact}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}, x \in \mathbb{R}^{\infty}_{\geq 0}$
	Output:	TRUE iff $\mathcal{I}(K) = x$
$\textsc{Upper}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}, x \in \mathbb{R}^{\infty}_{\geq 0}$
	Output:	TRUE iff $\mathcal{I}(K) \leq x$
$\textsc{Lower}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}, x \in \mathbb{R}^{\infty}_{\geq 0} \setminus \{0\}$
	Output:	TRUE iff $\mathcal{I}(K) \geq x$
$\textsc{Value}_{\mathcal{I}}$	**Input**:	$\mathcal{K} \in \mathbb{K}$
	Output:	The value of $\mathcal{I}(K)$

Ma et al. analyze the complexity of an inconsistency measure based on 4-valued logic which is a variant of the contension inconsistency measure [MQX+10]. They prove that $\textsc{Lower}_{\mathcal{I}}$ is coNP-complete, while $\textsc{Upper}_{\mathcal{I}}$ is NP-complete, and $\textsc{Exact}_{\mathcal{I}}$ is D^p_1-complete. Furthermore, $\textsc{Value}_{\mathcal{I}}$ is $\mathsf{FP}^{\mathsf{NP}[\log n]}$-complete.

Xiao and Ma propose the mv-inconsistency measure and also analyze the complexity of it [XM12]. They show that $\textsc{Lower}_{\mathcal{I}}$ is Σ^p_2-complete, while $\textsc{Upper}_{\mathcal{I}}$ is Π^p_2-complete, and $\textsc{Exact}_{\mathcal{I}}$ is D^p_2-complete. They further show that $\textsc{Value}_{\mathcal{I}}$ is in $\mathsf{FP}^{\Sigma^p_2[\log n]}$.

Thimm and Wallner analyse and give a comprehensive overview on complexity of several inconsistency measures [TW19]. An overview of some of the introduced inconsistency measures with results of $\textsc{Lower}_{\mathcal{I}}$, $\textsc{Upper}_{\mathcal{I}}$, $\textsc{Exact}_{\mathcal{I}}$, and $\textsc{Value}_{\mathcal{I}}$ is given in Table 2. Overall, the measures I_d and I_c are within the first level of the polynomial hierarchy, the measure I_{mv} is on the second level of the polynomial hierarchy, and the measures I_{MI} and $I_{\mathsf{MI^C}}$ are beyond the polynomial hierarchy. This shows that there can be considerable differences in computational complexity with regard to different inconsistency measures.

In recent times, there have been more publications that analyze computational complexity of inconsistency measures. Parisi and Grant investigate several inconsis-

	$\text{EXACT}_\mathcal{I}$	$\text{UPPER}_\mathcal{I}$	$\text{LOWER}_\mathcal{I}$	$\text{VALUE}_\mathcal{I}$
I_d	$\mathsf{D}_1^p \cap \mathsf{coD}_1^p$	NP-c	coNP-c	FNP
I_{MI}	$\mathsf{C_{=}NP}$-h	CNP-c	CNP-c	#·coNP-c
$I_{\mathsf{MI^C}}$	$\mathsf{C_{=}NP}$-h	CNP-h	CNP-h	$\mathsf{FP}^{\#\cdot\mathsf{coNP}}$
I_c	D_1^p-c	NP-c	coNP-c	$\mathsf{FP}^{\mathsf{NP}[\log n]}$-c
I_{mv}	D_2^p-c	Π_2^p-c	Σ_2^p-c	$\mathsf{FP}^{\Sigma_2^p[\log n]}$

Table 2: Computational complexity of the introduced inconsistency measures. The ending "-c" means completeness. Results are taken from [MQX+10, XM12, TW19].

tency measures for databases and also give complexity results for these measures [PG20, PG23a, PG23b].

4 Definition and Evaluation of a new Inconsistency Measure

First, this section gives a formal definition of the new inconsistency measure. Based on the state of research on evaluation methods for inconsistency measurement, the new inconsistency measure is then analyzed with regard to the 20 described rationality postulates. After that, the analysis focuses on expressivity and gives results for the α-characteristics. Finally, the computational complexity is analyzed for the three decision problems of, whether a given value is the lower bound, upper bound, or the actual value of the inconsistency value of a given knowledge base and for the functional problem of determining the actual inconsistency value.

4.1 Definition

The new inconsistency measure multiplies, for each atom a of the underlying language, the number of subsets from which a can be entailed with the number of subsets from which the negation of a can be entailed. The inconsistency measure is then the sum of these products for all atoms a of the underlying language. A definition is given in the following. We call this measure the entailment inconsistency measure I_e.

Definition 19. *For each $a \in \mathsf{At}$ let $M_a(\mathcal{K}) = \{M \subseteq \mathcal{K} \mid M \models a\}$ and $M_{\neg a}(\mathcal{K}) = \{M \subseteq \mathcal{K} \mid M \models \neg a\}$.*

The entailment inconsistency measure is defined as $\mathcal{I}_e(\mathcal{K}) = \sum_{a \in \mathsf{At}} |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})|$.

Example. *Let $\mathcal{K} = \{a \wedge \neg a, b\}$. Then $M_a(\mathcal{K}) = \{\{a \wedge \neg a\}, \{a \wedge \neg a, b\}\}$, $M_{\neg a}(\mathcal{K}) = \{\{a \wedge \neg a\}, \{a \wedge \neg a, b\}\}$ and $M_b(\mathcal{K}) = \{\{b\}, \{a \wedge \neg a\}, \{a \wedge \neg a, b\}\}$. Thus, $\mathcal{I}_e(\mathcal{K}) = 4$.*

With this definition the new measure I_e does not only consider inconsistent formulae or minimal inconsistent sets, but focuses on which atoms cause the inconsistency in the knowledge base. Thus, the measure is also able to take into account hidden conflicts, which can arise in formulae, and which are not part of a minimal inconsistent set. These are described by De Bona and Hunter as so-called "iceberg inconsistencies" [DH17]. The following example is taken from their publication [DH17].

Example. *Let* $\mathcal{K} = \{\neg s \wedge \neg g, (s \vee m) \wedge g, \neg m\}$. $\mathsf{MI}(\mathcal{K}) = \{\{\neg s \wedge \neg g, (s \vee m) \wedge g\}\}$. *Thus,* $\mathsf{Free}(\mathcal{K}) = \{\neg m\}$.

The example shows that $\neg m$ does not belong to any minimal inconsistent subset. However, we can argue that this formula of the knowledge base takes part in the inconsistencies as well. That is because both m and $\neg m$ can be entailed with the formulae given. In contrast to that, with the inconsistency measure I_e, the part of this atom to the overall inconsistency is taken into account.

Example. *Again, let* $\mathcal{K} = \{\neg s \wedge \neg g, (s \vee m) \wedge g, \neg m\}$.
$M_m(\mathcal{K}) = \{\{\neg s \wedge \neg g, (s \vee m) \wedge g\}, \{\neg s \wedge \neg g, (s \vee m) \wedge g, \neg m\}\}$, *so* $|M_m(\mathcal{K})| = 2$.
$M_{\neg m}(\mathcal{K}) = \{\{\neg m\}, \{\neg m, (s \vee m) \wedge g\}, \{\neg m, \neg s \wedge \neg g\}, \{\neg s \wedge \neg g, (s \vee m) \wedge g, \neg m\}\}$, *so* $|M_{\neg m}(\mathcal{K})| = 4$.

Having defined the entailment inconsistency measure I_e, the next section evaluates this new inconsistency measure.

4.2 Evaluation

Results of the evaluation of the entailment inconsistency measure with regard to rationality postulates, expressivity and computational complexity are given in this section.

4.2.1 Rationality Postulates

The following subsection provides results for the compliance of the proposed inconsistency measure I_e with regard to the 20 introduced rationality postulates. For reasons of better readability, the definition of each postulate is given again, directly followed by the result of the entailment inconsistency measure for this postulate. An overview of all results is given in Table 3.

CO	NO	MO	IN	DO	SA	SI	PY	MI	MN
✓	x	✓	x	x	✓	x	✓	x	x
AT	EC	AC	CD	FD	SY	EX	AI	FR	RS
x	x	x	x	✓	x	x	x	x	x

Table 3: Compliance of the inconsistency measure with the rationality postulates (✓reads "holds" and x reads "fails").

Postulate (Consistency, CO). $\mathcal{I}(\mathcal{K}) = 0$ *iff* $\mathcal{K}$ *is consistent.*

Proposition. *The entailment inconsistency measure satisfies* Consistency*.*

Proof. *Either* $|M_a(\mathcal{K})| = 0$ *or* $|M_{\neg a}(\mathcal{K})| = 0$ $\forall a \in \mathsf{At}$ *iff* $\mathcal{K}$ *is consistent. Then* $\mathcal{I}_e(\mathcal{K}) = 0$*.*

Postulate (Normalization, NO)**.** $0 \leq \mathcal{I}(\mathcal{K}) \leq 1$.

Proposition. *The entailment inconsistency measure does not satisfy* Normalization*.*

Example. $\mathcal{K} = \{a, \neg a\}$*.* $M_a = \{\{a\}, \{a, \neg a\}\}, M_{\neg a} = \{\{\neg a\}, \{a, \neg a\}\}$*,* $|M_a| = 2, |M_{\neg a}| = 2$*. So* $\mathcal{I}_e(\mathcal{K}) = 4$*.*

Postulate (Monotony, MO)**.** *If* $\mathcal{K} \subseteq \mathcal{K}'$ *then* $\mathcal{I}(\mathcal{K}) \leq \mathcal{I}(\mathcal{K}')$*.*

Proposition. *The entailment inconsistency measure satisfies* Monotony*.*

Proof. *This is shown further below with the proof of Super-Additivity. Super-Additivity implies Monotony [Thi17b]. Thus, as* I_e *satisfies Monotony, Super-Additivity is satisfied by the entailment inconsistency measure.*

Postulate (Free-Formula Independence, IN)**.** *If* $\alpha \in \mathsf{Free}(\mathcal{K})$ *then* $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$*.*

Proposition. *The entailment inconsistency measure does not satisfy* Free-Formula Independence*.*

Example. $\mathcal{K} = \{a \wedge \neg a, a\}$*.* $a \in \mathsf{Free}(\mathcal{K})$*.* $M_a = \{\{a \wedge \neg a\}, \{a\}, \{a \wedge \neg a, a\}\}$*,* $M_{\neg a} = \{\{a \wedge \neg a\}, \{a \wedge \neg a, a\}\}$*.* $\mathcal{I}_e(\mathcal{K}) = 6$ *but* $\mathcal{I}_e(\mathcal{K} \setminus \{a\}) = 1$*.*

Postulate (Dominance, DO)**.** *If* $\alpha \nvDash \perp$ *and* $\alpha \vDash \beta$ *then* $\mathcal{I}(\mathcal{K} \cup \{\alpha\}) \geq \mathcal{I}(\mathcal{K} \cup \{\beta\})$*.*

Proposition. *The entailment inconsistency measure does not satisfy* Dominance*.*

Example. $\mathcal{K} = \{a, \neg a\}, \alpha = \{a\}, \beta = \{a \wedge a\}$*.* $\alpha \nvDash \perp$ *and* $\alpha \vDash \beta$ *but* $\mathcal{I}_e(\mathcal{K} \cup \{\alpha\}) = 4$ *and* $\mathcal{I}_e(\mathcal{K} \cup \{\beta\}) = 24$*.*

Postulate (Super-Additivity, SA)**.** *If* $\mathcal{K} \cap \mathcal{K}' = \emptyset$ *then* $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') \geq \mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}')$

Proposition. *The entailment inconsistency measure satisfies the postulate of* Super-Additivity*.*

Proof. *By monotonicity of classical entailment, it follows that if* $M \vDash a$ *then* $M \cup M' \vDash a$ *for any* M'*. As a consequence, the number of subsets from which a can be concluded cannot decrease in a joint knowledge base for any two knowledge bases* $\mathcal{K}$ *and* $\mathcal{K}'$ *with* $\mathcal{K} \cap \mathcal{K}' = \emptyset$*, thus* $|M_a(\mathcal{K})| + |M_a(\mathcal{K}')| \leq |M_a(\mathcal{K} \cup \mathcal{K}')|$ $\forall a \in \mathsf{At}(\mathcal{K} \cup \mathcal{K}')$*.*

$$\begin{aligned}
\mathcal{I}_e(\mathcal{K}) + \mathcal{I}_e(\mathcal{K}') &= \sum_{a \in \mathsf{At}(\mathcal{K})} |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})| + \sum_{a \in \mathsf{At}(\mathcal{K}')} |M_a(\mathcal{K}')| \cdot |M_{\neg a}(\mathcal{K}')| \\
&\leq \sum_{a \in \mathsf{At}(\mathcal{K}) \cup \mathsf{At}(\mathcal{K}')} (|M_a(\mathcal{K})| + |M_a(\mathcal{K}')|) \cdot (|M_{\neg a}(\mathcal{K})| + |M_{\neg a}(\mathcal{K}')|) \\
&\leq \sum_{a \in \mathsf{At}(\mathcal{K}) \cup \mathsf{At}(\mathcal{K}')} |M_a(\mathcal{K} \cup \mathcal{K}')| \cdot |M_{\neg a}(\mathcal{K} \cup \mathcal{K}')| \\
&= \mathcal{I}_e(\mathcal{K} \cup \mathcal{K})
\end{aligned}$$

Note that from the first to the second line of the proof, the following inequality is used:

$$\begin{aligned}
&|M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})| + |M_a(\mathcal{K}')| \cdot |M_{\neg a}(\mathcal{K}')| \\
&\leq (|M_a(\mathcal{K})| + |M_a(\mathcal{K}')|) \cdot (|M_{\neg a}(\mathcal{K})| + |M_{\neg a}(\mathcal{K}')|) \\
&= |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})| + |M_a(\mathcal{K}')| \cdot |M_{\neg a}(\mathcal{K}')| + |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K}')| + \\
&\quad |M_a(\mathcal{K}')| \cdot |M_{\neg a}(\mathcal{K})|
\end{aligned}$$

Postulate (Safe-Formula Independence, SI). *If* $\alpha \in \mathsf{Safe}(\mathcal{K})$ *then* $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$.

Proposition. *The entailment inconsistency measure does not satisfy* Safe-Formula Independence.

Example. $\mathcal{K} = \{a \wedge \neg a, b\}$. $\mathsf{Safe}(\mathcal{K}) = \{b\}$. $\mathcal{I}_e(\mathcal{K}) = 10$, *but* $\mathcal{I}_e(\mathcal{K} \setminus \{b\}) = 1$.

Postulate (Penalty, PY). *If* $\alpha \notin \mathsf{Free}(\mathcal{K})$ *then* $\mathcal{I}(\mathcal{K}) > \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$.

Proposition. *The entailment inconsistency measure satisfies* Penalty.

Proof. $\alpha \notin \mathsf{Free}(\mathcal{K})$, *so* $\exists a \in \mathsf{At}(\alpha)$ *with* $|M_a(\alpha)| > 0$ *and* $|M_{\neg a}(\alpha)| > 0$. *This is because every formula, which is not a free formula, causes some kind of conflict, as it is part of at least one minimal inconsistent set. With that* $\mathcal{I}_e(\mathcal{K} \cup \alpha) > \mathcal{I}_e(\mathcal{K})$ *for any* $\mathcal{K}$.

Postulate (MI-Separability, MI). *If* $\mathsf{MI}(\mathcal{K} \cup \mathcal{K}') = \mathsf{MI}(\mathcal{K}) \cup \mathsf{MI}(\mathcal{K}')$ *and* $\mathsf{MI}(\mathcal{K} \cap \mathcal{K}') = \emptyset$ *then* $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}')$.

Proposition. *The entailment inconsistency measure does not satisfy* MI-Separability.

Example. $\mathcal{K} = \{a \wedge \neg a \wedge c\}$ *and* $\mathcal{K}' = \{b \wedge \neg b \wedge \neg c\}$. $\mathsf{MI}(\mathcal{K} \cup \mathcal{K}') = \{\mathcal{K}, \mathcal{K}'\} = \{a \wedge \neg a \wedge c, b \wedge \neg b \wedge \neg c\}$. $\mathsf{MI}(\mathcal{K}) = \mathcal{K}$, $\mathsf{MI}(\mathcal{K}') = \mathcal{K}'$ *but* $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = 27$ *and* $\mathcal{I}(\mathcal{K}) + \mathcal{I}(\mathcal{K}') = 4$.

Postulate (MI-Normalization, MN). *If* $M \in \mathsf{MI}(\mathcal{K})$ *then* $\mathcal{I}(M) = 1$.

Proposition. *The entailment inconsistency measure does not satisfy* MI-Normalization.

Example. $\mathcal{K} = \{a, \neg a\}$. $\mathcal{K}$ *is minimally inconsistent.* $|M_a| = 2, |M_{\neg a}| = 2$. *So* $\mathcal{I}_e(\mathcal{K}) = 4$.

Postulate (Attenuation, AT). $M, M' \in \mathsf{MI}(\mathcal{K})$ *and* $|M| > |M'|$ *implies* $\mathcal{I}(M) < \mathcal{I}(M')$.

Proposition. *The entailment inconsistency measure does not satisfy* Attenuation.

Example. $\mathcal{K} = \{a, \neg a\}$ *and* $\mathcal{K}' = \{a \wedge \neg a\}$. $\mathcal{K}$ *and* $\mathcal{K}'$ *are minimally inconsistent.* $|\mathcal{K}| > |\mathcal{K}'|$ *but* $\mathcal{I}_e(\mathcal{K}) = 4$ *and* $\mathcal{I}_e(\mathcal{K}') = 1$.

Postulate (Equal Conflict, EC). $M, M' \in \mathsf{MI}(\mathcal{K})$ *and* $|M| = |M'|$ *implies* $\mathcal{I}(M) = \mathcal{I}(M')$.

Proposition. *The entailment inconsistency measure does not satisfy* Equal Conflict.

Example. $\mathcal{K} = \{a \wedge \neg a\}$ *and* $\mathcal{K}' = \{a \wedge \neg a \wedge b \wedge \neg b\}$. $\mathcal{K}$ *and* $\mathcal{K}'$ *are minimally inconsistent.* $|\mathcal{K}| = |\mathcal{K}'| = 1$ *but* $\mathcal{I}_e(\mathcal{K}) = 1$ *and* $\mathcal{I}_e(\mathcal{K}') = 2$.

Postulate (Almost Consistency, AC). *Let $M_1, M_2, ...$ be a sequence of minimal inconsistent sets M_i with $\lim\limits_{i\to\infty} |M_i| = \infty$, then $\lim\limits_{i\to\infty} \mathcal{I}(M_i) = 0$.*

Proposition. *The entailment inconsistency measure does not satisfy* Almost Consistency.

Example. *$\mathcal{K} = \{a_1, ..., a_i, \neg(a_1 \wedge ... \wedge a_i)\}$, $\mathcal{K}$ is minimally inconsistent. $\lim\limits_{i\to\infty} \mathcal{I}_e(\mathcal{K}) = \lim\limits_{i\to\infty} \sum_{a_i} |M_{a_i}(\mathcal{K})| \cdot |M_{\neg a_i}(\mathcal{K})| = \lim\limits_{i\to\infty} \sum_{a_i} 2^{2i} = \infty$.*

Postulate (Contradiction, CD). *$\mathcal{I}(\mathcal{K}) = 1$ iff for all $\emptyset \neq \mathcal{K}' \subseteq \mathcal{K}, \mathcal{K}' \models \perp$.*

Proposition. *The entailment inconsistency measure does not satisfy* Contradiction.

Example. *$\mathcal{K} = \{a \wedge \neg a \wedge b \wedge \neg b\}$. With that $\mathcal{I}_e(\mathcal{K}) = 2$ but for all $\emptyset \neq \mathcal{K}' \subseteq \mathcal{K}, \mathcal{K}' \models \perp$.*

Postulate (Free-Formula Dilution, FD). *If $\alpha \in \mathsf{Free}(\mathcal{K})$ then $\mathcal{I}(\mathcal{K}) \geq \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$*

Proposition. *The entailment inconsistency measure satisfies* Free-Formula Dilution.

Proof. *I_e satisfies* Monotony. Monotony *implies* Free-Formula Dilution *[Thi17b]. Thus, I_e satisfies* Free-Formula Dilution.

Postulate (Irrelevance of Syntax, SY). *If $\mathcal{K} \equiv_b \mathcal{K}'$ then $\mathcal{I}(\mathcal{K}) = \mathcal{I}(\mathcal{K}')$.*

Proposition. *The entailment inconsistency measure does not satisfy* Irrelevance of Syntax.

Example. *$\mathcal{K} = \{a \wedge \neg a\}$ and $\mathcal{K}' = \{a \wedge \neg a \wedge b \wedge \neg b\}$. Then $\mathcal{K} \equiv_b \mathcal{K}'$ but $\mathcal{I}_e(\mathcal{K}) = 1$ and $\mathcal{I}_e(\mathcal{K}') = 2$.*

Postulate (Exchange, EX). *If $\mathcal{K}' \not\models \perp$ and $\mathcal{K}' \equiv_b \mathcal{K}''$ then $\mathcal{I}(\mathcal{K} \cup \mathcal{K}') = \mathcal{I}(\mathcal{K}' \cup \mathcal{K}'')$.*

Proposition. *The entailment inconsistency measure does not satisfy* Exchange.

Example. *$\mathcal{K} = \{\neg a\}$, $\mathcal{K}' = \{a\}$, and $\mathcal{K}'' = \{a, a \wedge a\}$. Then $\mathcal{K}' \not\models \perp$ and $\mathcal{K}' \equiv_b \mathcal{K}''$ but $\mathcal{I}_e(\mathcal{K} \cup \mathcal{K}') = 4$ and $\mathcal{I}_e(\mathcal{K} \cup \mathcal{K}'') = 24$.*

Postulate (Adjunction Invariance, AI). *$\mathcal{I}(\mathcal{K} \cup \{\alpha, \beta\}) = \mathcal{I}(\mathcal{K} \cup \{\alpha \wedge \beta\})$.*

Proposition. *The entailment inconsistency measure does not satisfy* Adjunction Invariance.

Example. *$\mathcal{K} = \{\neg a, a\}$ and $\mathcal{K}' = \{\neg a \wedge a\}$. Then $\mathcal{I}_e(\mathcal{K}) = 4$ but $\mathcal{I}_e(\mathcal{K}') = 1$.*

Postulate (Free-Formula Reduction, FR). *For $\alpha \notin \mathcal{K}$, α is free for $\mathcal{K}$, and $\mathcal{I}(\mathcal{K}) \neq 0$, then $\mathcal{I}(\mathcal{K} \cup \{\alpha\}) < \mathcal{I}(\mathcal{K})$.*

Proposition. *The entailment inconsistency measure does not satisfy* Free-Formula Reduction.

Example. *$\mathcal{K} = \{\neg a \wedge a\}$ and $\alpha = \{b\}$. With that $\mathcal{I}_e(\mathcal{K}) = 1$ and $\mathcal{I}_e(\mathcal{K} \cup \alpha) = 4$.*

Postulate (Relative Separability, RS). *If* $\mathcal{I}(\mathcal{K}) \precapprox \mathcal{I}(\mathcal{K}')$ *and* $\mathsf{At}(\mathcal{K}) \cap \mathsf{At}(\mathcal{K}') = \emptyset$*, then* $\mathcal{I}(\mathcal{K}) \precapprox \mathcal{I}(\mathcal{K} \cup \mathcal{K}') \precapprox \mathcal{I}(\mathcal{K}')$ *where either* $\precapprox$ *is* $<$ *in every instance or* $\precapprox$ *is* $=$ *in every instance.*

Proposition. *The entailment inconsistency measure does not satisfy* Relative Separability.

Example. $\mathcal{K} = \{\neg a \wedge a\}$ *and* $\mathcal{K}' = \{b, \neg b\}$*. With that* $\mathcal{I}_e(\mathcal{K}) = 1$, $\mathcal{I}_e(\mathcal{K}') = 4$ *and* $\mathcal{I}_e(\mathcal{K} \cup \mathcal{K}') = 16 + 36 = 52$.

Overall, the newly proposed entailment inconsistency measure I_e satisfies five of the 20 described rationality postulates. This number seems to be quite low. E.g., the measure $\mathcal{I}_{\mathsf{MI}}$ satisfies 12 of 18 rationality postulates (excluding the two last rationality postulates *Free-Formula Reduction* and *Relative Separability*). However, it must be noted that rationality postulates might be not an appropriate method to characterize inconsistency measures alone which has been discussed in publications, e.g., by Besnard and Thimm [Bes14, Thi18]. For example, the drastic inconsistency measure also satisfies 12 of the 20 postulates. This has been shown for 18 rationality postulates by Thimm [Thi17b] and can be easily verified for the remaining two postulates of *Free-Formula Reduction* and *Relative Separability*. These results of the drastic measure seems "better" compared to the new inconsistency measure I_e, but the drastic measure itself is obviously not highly meaningful as it differentiates only between consistent and inconsistent knowledge bases.

Nevertheless, there are some reasons why the inconsistency measures perform relatively poor with regard to rationality postulates. First, the entailment inconsistency measure is not built upon minimal inconsistent sets, which is something that some rationality postulates, like *MI-Separability*, *MI-Normalization*, *Attenuation*, *Equal Conflict*, or *Almost Consistency*, implicitly require. As a consequence, these postulates are not satisfied by the entailment inconsistency measure I_e.

Besides that, the inconsistency measure is not based upon *Normalization*, as it can take values greater than one. This is as well a prerequisite for a number other rationality postulates, like *Contradiction*, *Free-Formula Reduction*, and *Relative Separability*. Nevertheless, the entailment inconsistency measure satisfies *Free-Formula Dilution*, which, originally, also was defined based on normalized measures. But looking at the definition, we can see that it is well applicable for measures that are not normalized.

Thus, it seems that the existing rationality postulates lack the ability to entirely describe the characteristics of the newly proposed inconsistency measure. With these results the question arises, if there are possible new rationality postulates that capture the characteristic of the proposed inconsistency measure better than the already described rationality postulates.

We could think of a weaker form of *Safe-Formula Independence*. Like the weaker form of *Free-Formula Independence*, which is *Free-Formula Dilution*, a rationality postulate called *Safe-Formula Dilution* could be introduced. This postulate does not require that the inconsistency value stays the same when removing safe formulae, but allows the inconsistency value to decrease. As every safe formula is also a free formula, an inconsistency measure which satisfies *Free Formula-Dilution*, also satisfies *Safe-Formula Dilution*. This can be applied to the entailment inconsistency measure I_e as well, but in the following an alternative proof is given for the satisfaction of this postulate.

Postulate (Safe-Formula Dilution, SD). *If* $\alpha \in \mathsf{Safe}(\mathcal{K})$ *then* $\mathcal{I}(\mathcal{K}) \geq \mathcal{I}(\mathcal{K} \setminus \{\alpha\})$.

Proposition. *The entailment inconsistency measure satisfies* Safe-Formula Dilution.

Proof. *Let* $\mathcal{K}$ *be a knowledge base and* $\alpha \in \mathsf{Safe}(\mathcal{K})$. $\forall a \in \mathsf{At}(\mathcal{K} \setminus \{\alpha\})$, *we have* $M_a(\mathcal{K}) \geq M_a(\mathcal{K} \setminus \{\alpha\})$ *because of monotonicity of classical entailment.*

$$\begin{aligned}
I_e(\mathcal{K}) &= \sum_{a \in \mathsf{At}(\mathcal{K})} |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})| \\
&= \sum_{a \in \mathsf{At}(\mathcal{K} \setminus \{\alpha\})} |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})| + \sum_{a \in \mathsf{At}(\alpha)} |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})| \\
&\geq \sum_{a \in \mathsf{At}(\mathcal{K} \setminus \{\alpha\})} |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})| \\
&\geq \sum_{a \in \mathsf{At}(\mathcal{K} \setminus \{\alpha\})} |M_a(\mathcal{K} \setminus \{\alpha\})| \cdot |M_{\neg a}(\mathcal{K} \setminus \{\alpha\})| \\
&= I_e(\mathcal{K} \setminus \{\alpha\})
\end{aligned}$$

This postulate captures a certain characteristic of the entailment inconsistency measure, as safe formulae increase the inconsistency value by monotonicity of classical entailment and on the contrary by removing them decrease the inconsistency value.

A further characteristic, that is worth being analyzed in more detail, is the behavior of the entailment inconsistency measure towards *Adjunction Invariance*. It is arguable, if it should be a favorable feature that an inconsistency measure recognizes the difference between $\{a, b\}$ and $\{a \wedge b\}$. There are inconsistency measures which do not recognize the difference at all (by that satisfying *Adjunction Invariance*) and others that show an incoherent behavior with regard to this difference [Thi16a]. An example for the latter are the measures I_{MI}, $I_{\mathsf{MI^C}}$, $I_{\mathsf{MI^C}}$, and I_{D_f}. In the publication by Thimm, the different behavior is described as $\wedge$-*Indifference*, $\wedge$-*Penalty* and $\wedge$-*Mitigation* with $\wedge$-*Indifference* being the same as *Adjunction Invariance*, $\wedge$-*Penalty* meaning $\mathcal{I}(\mathcal{K} \cup \{a, b\}) \leq \mathcal{I}(\mathcal{K} \cup \{a \wedge b\})$, and $\wedge$-*Mitigation* meaning $\mathcal{I}(\mathcal{K} \cup \{a, b\}) \geq \mathcal{I}(\mathcal{K} \cup \{a \wedge b\})$. With these definitions, the question arises how the entailment inconsistency measure behaves with regard to $\wedge$-*Penalty* and $\wedge$-*Mitigation* as it does not satisfy $\wedge$-*Indifference*. The following example, taken from Thimm [Thi16a], shows that, like the before-mentioned inconsistency measures, the entailment inconsistency measure does not satisfy neither $\wedge$-*Penalty* nor $\wedge$-*Mitigation*. Adding the conjunction $\{a \wedge b\}$ to the knowledge base leads to a higher value compared to separate formulae $\{a, b\}$. For adding $\{a \wedge \neg a, \neg\neg a\}$ and $\{a \wedge \neg a \wedge \neg\neg a\}$, it is the other way round: adding separate formulae leads here to a higher value.

Example. *Let* $\mathcal{K} = \{a, \neg a\}$*, then* $\mathcal{I}_e(\mathcal{K}) = 4$*,*
$\mathcal{I}_e(\mathcal{K} \cup \{a, b\}) = 16$*, but* $\mathcal{I}_e(\mathcal{K} \cup \{a \wedge b\}) = 24$*,*
$\mathcal{I}_e(\mathcal{K} \cup \{a \wedge \neg a, \neg\neg a\}) = 168$*, but* $\mathcal{I}_e(\mathcal{K} \cup \{a \wedge \neg a \wedge \neg\neg a\}) = 36$*.*

As a final remark regarding rationality postulates, it should be mentioned that a further analysis might include a normalized entailment inconsistency measure. As proposed by Besnard and Grant [BG20], measures that do not originally satisfy *Normalization* can be normalized so that they are relative measures. This could be done with the proposed measure as well, see the following definition of a relative entailment inconsistency measure $\mathcal{I}_e^r$. The upper bound of the measure is defined by the number of all subsets of the knowledge base from which both each atom and its negation could be entailed.

Definition 20. $\mathcal{I}_e^r(\mathcal{K}) = \frac{\sum_{a \in \mathsf{At}(\mathcal{K})} |M_a(\mathcal{K})| \cdot |M_{\neg a}(\mathcal{K})|}{|\mathsf{At}(\mathcal{K})|(2^{|\mathcal{K}|}-1)^2}$

Such a normalized measure satisfies *Normalization* and might also satisfy those rationality postulates that are based upon *Normalization*. A detailed analysis of satisfying rationality postulates goes beyond scope of this bachelor thesis and might be a topic for future research.

4.2.2 Expressivity

The results regarding α-characteristics for the new inconsistency measure are shown in Table 4. Proofs for the results are given in the following.

$\mathcal{C}^v(\mathcal{I}, n)$	$\mathcal{C}^f(\mathcal{I}, n)$	$\mathcal{C}^l(\mathcal{I}, n)$	$\mathcal{C}^p(\mathcal{I}, n)$
∞	∞	∞^*	∞

Table 4: α-characteristics of the new inconsistency measure for $n \geq 1$ (*for $n > 1$).

Proposition. $\mathcal{C}^v(\mathcal{I}, n) = \infty$.

Proof. *For* $|\mathsf{At}(\mathcal{K})| = 1$ *consider the knowledge bases* $\mathcal{K}_i = \{\neg a, a, a \wedge a, ..., \bigwedge_{j=1}^{i} a\}$ *for* $i \in \mathbb{N}$*.*

The subsets that entail $\neg a$ *are all subsets of* $\mathcal{K} \setminus \{\neg a\}$*, each combined by union with* $\{\neg a\}$*. Thus,* $|M_{\neg a}(\mathcal{K}_i)| = 2^i$*.*

The subsets that entail a *are all subsets of the knowledge base excluding the empty set and* $\{\neg a\}$*. Thus,* $|M_a(\mathcal{K}_i)| = 2^{i+1} - 2$*.*

With this $\mathcal{I}(\mathcal{K}_i) = |M_a| \cdot |M_{\neg a}| = 2^{2i+1} - 2^{i+1}$ *and* $\lim_{i \to \infty} \mathcal{I}(\mathcal{K}_i) = \infty$*. So,* $\mathcal{C}^v(\mathcal{I}, n) = \infty$*.*

Proposition. $\mathcal{C}^f(\mathcal{I}, n) = \infty$.

Proof. *For* $|\mathcal{K}| = 1$ *consider the knowledge bases* $\mathcal{K}_i = \{\neg a_1 \wedge a_1 \wedge ... \wedge \neg a_i \wedge a_i\}$ *for* $i \in \mathbb{N}$*.*

Then $|M_{\neg a_i}(\mathcal{K}_i)| = 1$ *and* $|M_{a_i}(\mathcal{K}_i)| = 1$*. With this* $\mathcal{I}(\mathcal{K}_i) = \sum_i |M_{a_i}| \cdot |M_{\neg a_i}| = \sum_i 1 = i$ *and* $\lim_{i \to \infty} \mathcal{I}(\mathcal{K}_i) = \infty$*. Thus* $\mathcal{C}^f(\mathcal{I}, n) = \infty$*.*

Proposition. $\mathcal{C}^l(\mathcal{I}, n) = \infty$ *for* $n > 1$.

Proof. $\mathcal{C}^l(\mathcal{I}, 1) = 1$ *is trivial.*

For $\mathsf{len}(\phi) > 1$ *consider the knowledge bases* $\mathcal{K}_i = \{a_1, \neg a_1, ..., a_i, \neg a_i\}$ *for* $i \in \mathbb{N}$.

The subsets that entail $\neg a$ *are all subsets of the knowledge base* $\mathcal{K}_i \setminus \{\neg a_i\}$*, which number is* 2^{2i-1}*, combined with* $\{\neg a_i\}$*. Then* $|M_{\neg a_i}(\mathcal{K}_i)| = 2^{2i-1}$.

The combination of subsets which entail a_i *are calculated in analogy to the subsets that entail* $\neg a_i$*. Thus,* $|M_{a_i}(\mathcal{K}_i)| = 2^{2i-1}$.

With this $\mathcal{I}(\mathcal{K}_i) = 2^{4i-2}i$ *and* $\lim_{i \to \infty} \mathcal{I}(\mathcal{K}_i) = \infty$*. Thus* $\mathcal{C}^l(\mathcal{I}, n) = \infty$.

Proposition. $\mathcal{C}^p(\mathcal{I}, n) = \infty$.

Proof. *For* $|\mathsf{At}(\phi)| = 1$ *consider the same knowledge bases as before* $\mathcal{K}_i = \{a_1, \neg a_1, ..., a_i, \neg a_i\}$ *for* $i \in \mathbb{N}$*. Then,* $|M_{\neg a_i}(\mathcal{K}_i)| = 2^{2i-1}$ *and* $|M_{a_i}(\mathcal{K}_i)| = 2^{2i-1}$*. With this* $\mathcal{I}(\mathcal{K}_i) = 2^{4i-2}i$ *and* $\lim_{i \to \infty} \mathcal{I}(\mathcal{K}_i) = \infty$*. Thus* $\mathcal{C}^p(\mathcal{I}, n) = \infty$.

The results show that the proposed inconsistency measure I_e has maximal expressivity values with regard to all four expressivity characteristics. This means, that this newly proposed inconsistency measure is able to distinguish very well inconsistency of different knowledge bases with regard to their size(s). With these results, the entailment inconsistency measure is comparable to the measure $\mathcal{I}^{\Sigma}_{dalal}$ because this measure also has maximal expressivity with regard to all four α-characteristics [Thi16a].

The following example shows the expressivity order the introduced inconsistency measures as well as the entailment inconsistency measure. Inconsistency measures in the same set have the same expressivity.

Example. *Expressivity orders for the introduced inconsistency measures:*

$\{\mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{\mathsf{MI}^\mathsf{C}}, \mathcal{I}_{D_f}, \mathcal{I}_e\} \succ_v \{\mathcal{I}_{mv}, \mathcal{I}_c\} \succ_v \{\mathcal{I}_d\}$.

$\{\mathcal{I}_{mv}, \mathcal{I}_c, \mathcal{I}_e\} \succ_f \{\mathcal{I}_{\mathsf{MI}}\} \succ_f \{\mathcal{I}_d\}$.

$\{\mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{\mathsf{MI}^\mathsf{C}}, \mathcal{I}_{D_f}, \mathcal{I}_{mv}, \mathcal{I}_c, \mathcal{I}_e\} \succ_l \{\mathcal{I}_d\}$.

$\{\mathcal{I}_{\mathsf{MI}}, \mathcal{I}_{\mathsf{MI}^\mathsf{C}}, \mathcal{I}_{D_f}, \mathcal{I}_{mv}, \mathcal{I}_c, \mathcal{I}_e\} \succ_p \{\mathcal{I}_d\}$.

For all four expressivity characteristics, the entailment inconsistency measure is within the set with the highest expressivity. Thus, it performs better than the other shown measures in the example because each of these measures do not have maximal expressivity for at least one α- characteristic.

4.2.3 Complexity

The results of the computational complexity analysis is shown in Table 5. Proofs for the results are given in the following.

Complexity is shown for the problem variant of determining the subsets from which an atom can be entailed. The reason for this is, that for the problem of determining the inconsistency value of a knowledge base, this variant problem has to be solved twice for each atom. So, the overhead compared to the variant problem is linear, and is therefore omitted in the further analysis.

$\textsc{Exact}_{\mathcal{I}}$	$\textsc{Upper}_{\mathcal{I}}$	$\textsc{Lower}_{\mathcal{I}}$	$\textsc{Value}_{\mathcal{I}}$
$\mathsf{C_{=}NP}$	CNP	CNP	#·coNP

Table 5: Computational complexity of the proposed inconsistency measure. All statements are membership statements.

Proposition. $\textsc{Value}_{\mathcal{I}}$ *is in* #·coNP.

Proof. *We recall, #·$\mathcal{C}$ is the class of counting problems with witness function w, which meets the following conditions: for every input string x, every $y \in w(x)$ is polynomially bounded by x and the decision problem of deciding $y \in w(x)$ for given strings x and y is in $\mathcal{C}$. In this case, the input x is the given knowledge base $\mathcal{K}$ and the witness function w is $M_a(\mathcal{K}) = \{M \subseteq \ M \vDash a\}$. The size of all subsets M, so especially $M \in M_a(\mathcal{K})$, is polynomially bounded by the size of $\mathcal{K}$ as each formula of the knowledge base can appear maximally once in a subset:* $\mathsf{len}(M) \leq \mathsf{len}(\mathcal{K})$. *Deciding $M \in M_a$ means deciding $M \vDash a$, which is equivalent to deciding whether $M \cup \neg a$ is unsatisfiable for a given subset M. This problems corresponds to* UNSAT *which is in* coNP. *So, the underlying decision problem is in* coNP. *As a result,* $\textsc{Value}_{\mathcal{I}}$ *is in* #·coNP.

Proposition. $\textsc{Upper}_{\mathcal{I}}$ *and* $\textsc{Lower}_{\mathcal{I}}$ *are in* CNP, $\textsc{Exact}_{\mathcal{I}}$ *is in* $\mathsf{C_{=}NP}$.

Proof. *Membership follows from the fact, that there is a problem in $B \in$ coNP s.t. $(\mathcal{K}, M)$ is a yes instance iff $M \vDash a$ as this is equivalent to $M \cup \neg a$, with $M \in M_a(\mathcal{K})$ and $M_a(\mathcal{K}) = \{M \subseteq \mathcal{K} \mid M \vDash a\}$. For a given instance of* $\textsc{Lower}_{\mathcal{I}}$*, i.e., a knowledge base $\mathcal{K}$ and an integer k, (K, k) is a yes instance of* $\textsc{Lower}_{\mathcal{I}}$ *(*$\textsc{Exact}_{\mathcal{I}}$ *), iff there are at least (exactly) k subsets $M \in M_a(\mathcal{K})$ iff $\mathsf{C}^k_M(\mathcal{K}, M) \in B$. The size of any subset $M \in M_a$ is polynomially bounded by the size of $\mathcal{K}$ as each formula of the knowledge base can appear maximally once in a subset:* $\mathsf{len}(M) \leq \mathsf{len}(\mathcal{K})$. *Thus,* $\textsc{Lower}_{\mathcal{I}}$ *is in* CNP *and* $\textsc{Exact}_{\mathcal{I}}$ *is in* $\mathsf{C_{=}NP}$.

Membership for $\textsc{Upper}_{\mathcal{I}}$ *follows from the fact, that an instance $(\mathcal{K}, k)$ is a yes instance of* $\textsc{Upper}_{\mathcal{I}}$ *iff $(\mathcal{K}, k+1)$ is a no instance of* $\textsc{Lower}_{\mathcal{I}}$*. This, in turn, can be characterized as a no instance of a problem in* CNP. CNP *is closed under complement, i.e.,* coCNP = CNP.

The author of this text was not able to develop a substractive reduction using a problem already known to be #·coNP-complete for the proof of hardness and thus completeness for the complexity results. Thus, this remains an open issue which should be addressed in future work.

The results show that the computational complexity of the proposed inconsistency measure is likely beyond the polynomial hierarchy. Although, the definition of the new measure seems "simple", the computational complexity is very high. A reason might be that the measure involves counting certain structures, and that the number of these structures is exponential to the size of the knowledge base, see also Section 4.2.2.

The new inconsistency measure is, with regard to computational complexity, comparable to the measure I_{MI}. This measure is beyond the polynomial hierarchy as well and shows the same complexity membership results for the three decision problems and the functional problem [TW19]. Also, the results support the observation about the tendency of measures with high expressivity values towards higher computational complexity [TW19].

5 Summary and Outlook

The objective of this bachelor thesis was to define a new inconsistency measure that is based on inferences in subsets and evaluate that new inconsistency measure with regard to rationality postulates, expressivity and computational complexity.

The new entailment inconsistency has been defined. This measure quantifies inconsistency by multiplying the subsets from which an atom and its negation can be entailed. The product is summed up for all atoms in the knowledge base. The advantage of this measure is that "hidden" conflicts, which are conflicts, that are not part of a minimal inconsistent subsets, can be measured.

The evaluation of the entailment inconsistency measure with regard to rationality postulates shows that only five of in total 20 rationality postulates can be satisfied. These results are inferior compared to other inconsistency measures. The reasons have been discussed: Many rationality postulates implicitly require *Normalization* or taking minimal inconsistent subsets into account. Not being a normalized measure and not building upon minimal inconsistent subsets are thus reasons that lead to the result of not complying with these postulates. Future work could include an analysis of a normalized measure built upon the entailment inconsistency measure which might satisfy a higher number of rationality postulates.

Although performing relatively poor with regard to rationality postulates, the entailment inconsistency measure shows very good results with regard to expressivity: For all four α-characteristics the expressivity is maximal. So, the entailment inconsistency measure is very well able to distinguish between knowledge bases of different sizes. With these results of maximal expressivity, it is superior to many other inconsistency measures.

Regarding computational complexity the results show that the entailment inconsistency measure is computationally demanding. Complexity of the measure is beyond the third level of the polynomial hierarchy with regard of the decision problems of the upper/lower and exact bound and the functional problem of computing the actual inconsistency value. One reason for this is the underlying problem of counting certain structures in knowledge bases. The result of high computational complexity leads to difficulties in using the entailment measure in practical use cases. Proofs for hardness and completeness are still an open issue which should be addressed in future work.

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