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Fixed Point Theorems for Real- and Set-Valued Functions in Finite- and Infinite-Dimensional Spaces

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Abstract

Fixed point theorems play an important role in various branches of mathematics and have diverse applications to other fields. At its core, this thesis is devoted to the fixed point theorem of Brouwer which states that a continuous function on a nonempty, compact and convex subset of a finite-dimensional space must have a fixed point. Although the theorem can be proven analytically, this thesis follows a different approach: We use Sperner's lemma – an important result from combinatorial topology – and simplicial subdivisions to show that any continuous function mapping a simplex into itself must have a fixed point. We then extend the theorem to sets that are homeomorphic to simplices. The second part of the thesis is concerned with generalizations of Brouwer's fixed point theorem. On the one hand, the restriction to finite-dimensional spaces is relaxed. By introducing the concept of compact operators, Schauder's fixed point theorem is established – an analogue to Brouwer's theorem for infinite-dimensional spaces. On the other hand, the concept of point-to-point mappings (i.e. functions) is generalized and point-to-set mappings (so-called correspondences or set-valued functions) are introduced. These considerations lead to Kakutani's fixed point theorem, a result that has gained significant traction in applications such as economics or game theory. This is illustrated by using Kakutani's fixed point theorem to establish existence of pure-strategy Nash equilibria in a certain class of games. The first and foremost objective of this thesis is to provide an accessible and intuitive introduction to fixed point theorems and to pave the way for more advanced studies in this subject area.

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Acknowledgments

I would like to express my sincere gratitude to Prof. Dr. Delio Mugnolo and Dr. Matthias Täufer who served as the supervisors of this thesis. They were the ones who introduced me to the study of fixed point theorems in a seminar at the University of Hagen. The experiences I was able to gain there had a significant impact on the writing of this thesis. In fact, Dr. Täufer pointed out to me that the proof of Brouwer's fixed point theorem can be approached by means of Sperner's lemma in the first place. Prof. Dr. Mugnolo laid the foundation of basically all of my understanding of topics in real- and functional analysis by means of his excellent teaching of and support with the analytical courses at the Universtiy of Hagen. Moreover, I am very thankful for the approachability of Prof. Dr. Mugnolo and Dr. Täufer who always responded immediately to any inquires I had during the course of this thesis. Finally, I was very lucky to have been given a tremendous amount of freedom in the topic selection and in the general approach I took in this thesis. This enabled me to explore topics that go well beyond Brouwer's fixed point theorem and to shape my understanding of many interesting concepts in the subject area of analysis.

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1 Introduction

The study of fixed point theorems has not only become an integral part of many branches in mathematics but has gained significant traction in applications to other quantitative disciplines. One of the best-known and most fundamental results is the fixed point theorem of Banach according to which strictly contractive self-mappings on complete metric spaces must have a unique fixed point. On the one hand, this result is appealing since it does not only guarantee uniqueness of the fixed point but also provides a constructive method of how to find it. On the other hand, the required assumptions on the mapping are strong and could be difficult to verify. Another well-known result, Brouwer's fixed point theorem, is of a somewhat different nature: While its prerequisites on the self-mapping are relatively weak, it requires stronger assumptions on the underlying sets and spaces. More precisely, it states that any continuous function mapping a nonempty, compact and convex set into itself must have at least one fixed point. At its core, this thesis is devoted to the study of Brouwer's theorem. Instead of a pure analytical approach, we[1] use Sperner's lemma and simplicial subdivisions to offer a proof of Brouwer's fixed point theorem that is very accessible and only requires elementary tools from convex analysis. The aim of this thesis is twofold and some important notes should be made in this context:

1. My first and foremost objective is to provide an inherently accessible and intuitive approach to an important class of fixed point theorems. I have tried to aggregate insights from many different authors and sources and to present all concepts and proofs in a very detailed manner. On the one hand, this conflicts in a sense with the mathematical spirit of parsimony and conciseness. On the other hand, the thesis is self-contained and does not require repetitive references to result from other sources.

2. My second goal is to go beyond the fixed point theorem of Brouwer and to look at two important generalizations that derive from it: Schauder's fixed point theorem for infinite-dimensional spaces and Kakutani's fixed point theorem for set-valued functions.

Moreover, I have tried to offer many graphical illustrations in order to visualize important concepts and to provide additional perspectives to some of the results. The rest of this thesis proceeds as follows. Section 2 revises fundamental concepts from convex analysis and introduces Sperner's lemma. Section 3 constitutes the main part of this thesis. It

[1] Although this is a single-authored thesis, I will often follow the common convention and use "we" instead of "I". In my personal opinion, this is phonetically more appealing but can also be understood as "we the readers" as I am myself not an expert but rather a keen learner of the subject matter.

states and proves Brouwer's fixed point theorem for simplices and for sets that are homeomorphic to them. Section 4 deals with Schauder's fixed point theorem which generalizes the result of Brouwer to infinite-dimensional spaces. Section 5 offers a generalization to set-valued functions and introduces Kakutani's fixed point theorem while section 6 concludes. Appendix A illustrates the idea of an analytical proof of Brouwer's fixed point theorem while appendix B collects proofs of some auxiliary results. The following figure summarizes the pursued path of this thesis.

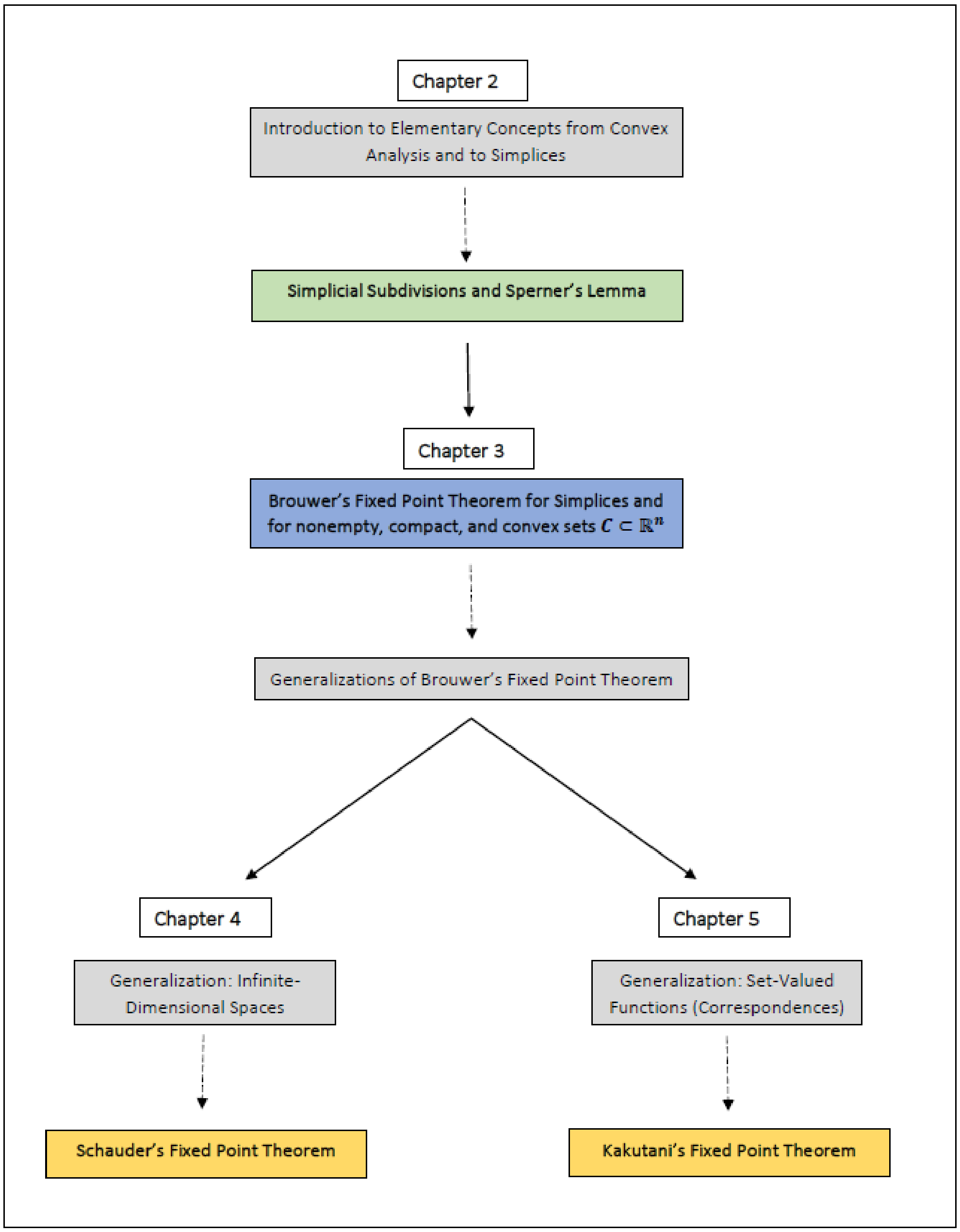


Figure 1: Brief overview of the main steps of this thesis.

2 Sperner's Lemma and Simplices

The core of this thesis is devoted to Brouwer's fixed point theorem, not least because many other fixed point results derive from it or constitute mere generalizations to other settings. Although the theorem can be proven in an analytical way, the proof is lengthy and requires a relatively sophisticated machinery of tools, some of which include topological no-retraction theorems and Gauss's divergence theorem.[2] By contrast, Sperner's lemma - an important result in combinatorial topology (Sperner, 1928) - allows to prove Brouwer's fixed point theorem by means of only elementary tools. There exist various (comparable) ways to prove Sperner's Lemma - some authors use a graph-theoretic approach (e.g. Border, 1985; Yuan, 2017) while others draw from tools that are more of combinatorial nature (e.g. Meister, 2018). Both approaches inherently rely on *simplices* and *simplicial subdivisions* among others. The following paragraph gathers important concepts that are needed to formulate and then to prove Sperner's lemma.

2.1 Fundamental definitions and terminology

Most of the required terminology describes basic concepts from convex analysis, the most fundamental of which is that of a convex set.

Definition 2.1 (Convex set) *[LL15; AY17]*
A subset $A \subset \mathbb{R}^n$ is ***convex*** *if and only if whenever x and y are two points from A, then the entire segment $[x, y]$ is a subset of A. Equivalently, A is convex if for all $x, y \in A$ and $\lambda \in [0, 1] \subset \mathbb{R}$, we have $\lambda x + (1 - \lambda)y \in A$.*

Intuitively, a convex set cannot contain any holes or bumps since the entire (line) segment connecting two points of the set must again be contained in it. For our purposes, $\mathbb{R}^n$ and any linear subspaces of $\mathbb{R}^n$ will constitute important convex sets.
Another important concept towards defining simplices is that of a *convex hull*. For a set $A \subset \mathbb{R}^n$, the convex hull of A is defined as the intersection of all convex subsets $K \subset \mathbb{R}^n$ with $A \subset K$ which is nothing but the smallest convex set that contains A (Zeidler, 1995). For our purposes, however, we will readily *define* convex hulls by means of finite convex combinations.

[2] I have presented an analytical proof of Brouwer's fixed point theorem in a seminar at the University of Hagen. Appendix A summarizes some of the fundamental steps in a figure. Detailed materials are available upon request.

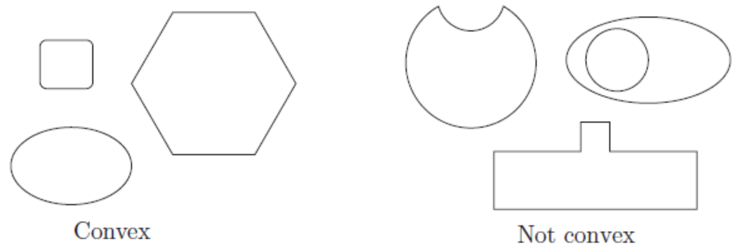


Figure 2: Examples (left) and counterexamples (right) of convex sets (Source: LL15).

Definition 2.2 (Convex combination) *[LL15; AY17]*
Let $x^1, ..., x^N \in \mathbb{R}^n$ and let $\lambda_1, ..., \lambda_N \in \mathbb{R}$. We call the linear combination

$$\sum_{i=1}^{N} \lambda_i x^i \;=\; \lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_k x^N$$

convex combination *if $\lambda_i \geq 0$ for $1 \leq i \leq N$ and $\sum_{i=1}^{N} \lambda_i = 1$.*

Definition 2.3 (Convex hull) *[KB85]*
For $A \subset \mathbb{R}^n$, the ***convex hull*** *of A, denoted by $co(A)$, is the set of all finite convex combinations of points in A, i.e.*

$$co(A) \;=\; \Big\{ \sum_{i=1}^{N} \lambda_i x^i \;\Big|\; x^i \in A, \lambda_i \geq 0 \;\; for\ 1 \leq i \leq N,\ \sum_{i=1}^{N} \lambda_i = 1 \Big\}.$$

A final notion that is required to define simplices is that of *affine independence*.

Definition 2.4 (Affine independence) *[KB85; EZ95]*
Let $\lambda_0, ..., \lambda_N \in \mathbb{R}$. The set $\{x^0, ..., x^N\} \subset \mathbb{R}^n$ is ***affine independent*** *if $\sum_{i=0}^{N} \lambda_i x^i = 0$ and $\sum_{i=0}^{N} \lambda_i = 0$ imply $\lambda_0 = \lambda_1 = ... = \lambda_N = 0$.*

Affine independence can be equivalently defined as $\{x^1 - x^0, ..., x^N - x^0\}$ to be linearly independent and this definition does not depend on the numbering of the points. Some authors also call an affine independent set to be in *general position* (Toenniessen, 2017; Rotman, 1988). Three points in $\mathbb{R}^2$ are affine independent if they form a triangle and hence do not lie on a straight line. We are now in a position to define the fundamental concept of N-simplices. Many authors follow Kuratowski (1972) and make simplices open sets (Border, 1985; Yuan, 2017); we will directly define simplices as to be closed sets.

Definition 2.5 (N-simplex and vertices) *[KJ08; EZ95]*
*Let $N \in \mathbb{N}$. An **N-simplex** $\mathcal{S}$ (or N-dimensional simplex) in $\mathbb{R}^n$ is the convex hull of an affine independent set of N+1 points $x^0, ..., x^N \in \mathbb{R}^n$. Formally,*

$$\mathcal{S} := co(\{x^0, ..., x^N\}) = \Big\{\sum_{i=0}^{N} \lambda_i x^i \;\Big|\; x^i \in \mathbb{R}^n, \lambda_i \geq 0 \;\; for\ 0 \leq i \leq N,\ \sum_{i=0}^{N} \lambda_i = 1\Big\}.$$

*The points $x^0, ..., x^N$ are called **vertices** of the simplex $\mathcal{S}$.*

Figure 3 below illustrates N-simplices for $N \in \{0, 1, 2, 3\}$. 2-simplices (i.e. triangles) will become particularly important as they provide an excellent starting point to graphically illustrate most of the concepts discussed in the following.

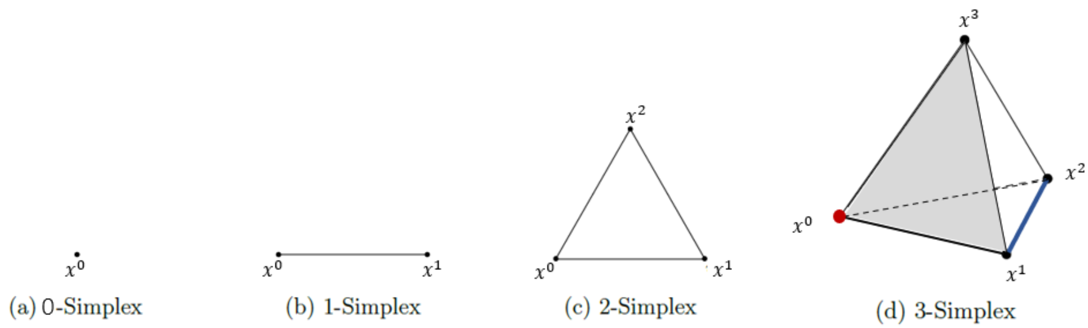


Figure 3: Different N-simplices (Source: KJ08; MB20).

Simplices can be regarded as a special case of convex hulls that require the points $x^0, ..., x^N \in \mathbb{R}^n$ to be affine independent. The so-called *standard n-simplex* is of particular interest in some of the subsequent proofs.

Definition 2.6 (Standard n-simplex in $\mathbb{R}^{n+1}$) *[KB85; HM18]*
*The **standard n-simplex** Δ^n is the subset of $\mathbb{R}^{n+1}$ that is spanned by the (n+1) basis vectors $e_1, ..., e_{n+1} \in \mathbb{R}^{n+1}$, i.e. by the vertices*

$$x^0 = (1, 0, ..., 0) \in \mathbb{R}^{n+1}$$
$$x^1 = (0, 1, ..., 0) \in \mathbb{R}^{n+1}$$
$$\vdots$$
$$x^n = (0, 0, ..., 1) \in \mathbb{R}^{n+1}.$$

It can be represented as

$$\Delta^n := co(\{x^0, ..., x^n\}) = \Big\{x = (x_0, ..., x_n) \in \mathbb{R}^{n+1} \;\Big|\; x_i \geq 0 \;\forall i \;\; and \;\; \sum_{i=0}^{n} x_i = 1\Big\}.$$

There is a canonical map from the standard n-simplex to any arbitrary n-simplex with vertices $(v^0, ..., v^n)$ *given by* $(x_0, ..., x_n) \in \Delta^n \mapsto \sum_{i=0}^n x_i v^i \in co(\{v^0, ..., v^n\})$.

We will be particularly interested in working with vertices, edges, and other "sides" of simplices. The following definition provides the necessary ground to do so.

Definition 2.7 (Face, k-face and boundary of a simplex) *[KJ08; JR88; EZ95]*
Let $\mathcal{S}$ *be an N-Simplex with vertices* $x^0, ..., x^N \in \mathbb{R}^n$*. A* ***face*** *of* $\mathcal{S}$ *is the convex hull of a (not necessarily proper) subset of* $\{x^0, ..., x^N\}$*. The* ***k-face*** *of* $\mathcal{S}$ *is the convex hull of* $k+1$ *distinct vertices of* $\mathcal{S}$ *where* $k = 0, 1, ..., N$*. A k-face of a simplex is also called a* ***k-dimensional subsimplex****. The* ***boundary*** $\partial\mathcal{S}$ *of* $\mathcal{S}$ *is the union of its (N-1)-faces.*

Figure 3(d) exemplifies a 0-face (red point), a 1-face or edge (blue line) and a 2-face (gray area) using a 3-simplex.[3] The following definition is fundamental for the rest of this section and in particular for Sperner's lemma and the proof of Brouwer's fixed point theorem in the next section. It describes a distinct way of how N-simplices can be "subdivided" into smaller N-subsimplices.

Definition 2.8 (Simplicial subdivision and mesh) *[KB85; ES28; EZ95]*
Let $\mathcal{S} := co(\{x^0, ..., x^N\})$ *be an N-simplex where* $N \geq 1$*. A* ***simplicial subdivision*** *(or triangulation) of* $\mathcal{S}$ *is a finite collection* $\mathcal{S}_1, \mathcal{S}_2, ..., \mathcal{S}_J$*,* $J \in \mathbb{N}$*, of N-subsimplices satisfying the following two conditions.*

1.) $\mathcal{S} = \bigcup_{j=1}^{J} \mathcal{S}_j$

2.) *For any* $j, k \in \{1, ..., J\}$ *with* $j \neq k$*, the intersection* $\mathcal{S}_j \cap \mathcal{S}_k$ *is either empty or equal to a common face.*

The ***mesh*** *of a simplicial subdivision is the diameter of the largest subsimplex where the diameter* $\delta(\mathcal{S})$ *of a simplex* $\mathcal{S} := co(\{x^0, ..., x^N\})$ *is defined by*

$$\delta(\mathcal{S}) := \sup_{j,k \in \{0,...,N\}} \|x^j - x^k\|$$

and is hence equal to the largest of its 1-faces.[4]

[3] In general, a k-simplex has ${}_{k+1}C_{s+1} := \binom{k+1}{s+1}$ s-dimensional faces (Nikaido, 1968). In the above example, the 3-simplex therefore has ${}_4C_3 := \binom{4}{3} := \frac{4!}{3!(4-3)!} = 4$ 2-faces which can be seen in Figure 3(d).
[4] Unless otherwise stated, we will use the Euclidean norm $\|\cdot\|_2$ for the remainder of this text.

Figure 4 below illustrates the concept of a simplicial subdivision for the case of a triangle (2-simplex). There are two particularly important examples of simplicial subdivisions: *equilateral* subdivisions and *barycentric* subdivisions, both of which can be used in proving Brouwer's fixed point theorem by means of Sperner's lemma.[5] Figure 5 uses a 2-simplex to illustrate different simplicial subdivisions.

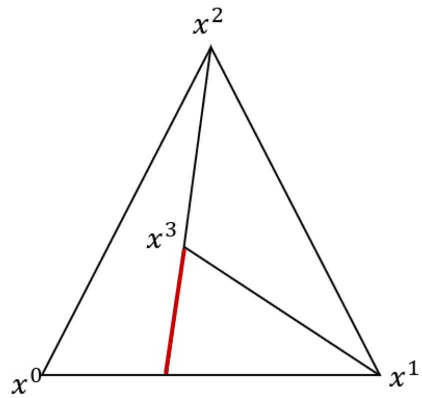


Figure 4: The left picture does *not* constitute a simplicial subdivison as it violates condition 2). The right picture shows a simplicial subdivison of a 2-simplex (Source: AY17).

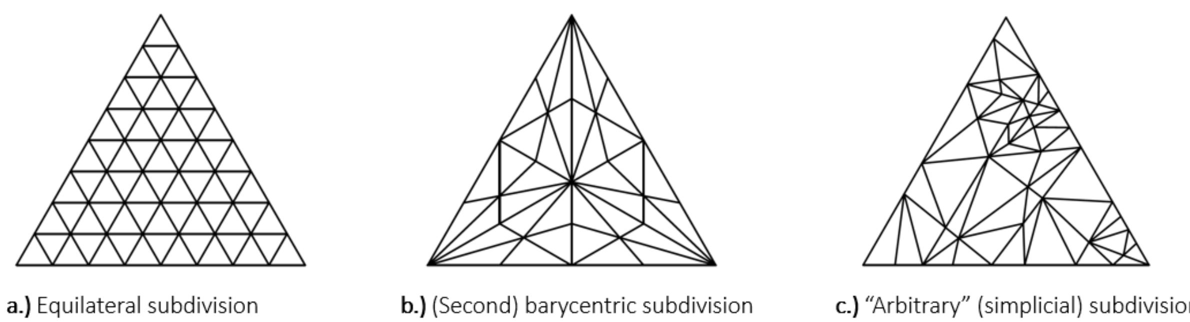


Figure 5: Different simplicial subdivisions of a 2-simplex (Source: MB20).

Given a simplicial subdivision of a simplex $\mathcal{S}$, we denote by V the set of all vertices of all subsimplices of the subdivision.[6] Sperner's lemma makes a statement about so-called *Sperner simplices* for simplicial subdivisions where V is labeled in a specific way.

[5]In fact, the proof only requires some simplicial subdivision with (arbitrarily) small mesh. By choosing the divisions fine enough, both equilateral and barycentric subdivisions fulfill this property (see section 3.1 for corresponding details on the barycentric subdivision and appendix B.1 for a detailed proof).

[6]This includes the vertices of the original simplex, vertices of subsimplices located on the boundary of the original simplex as well as vertices of subsimplices that are in the interior of the original simplex.

Definition 2.9 (Carrier) *[KB85; AY17]*
Let $\mathcal{P}(A)$ denote the power set of a set A and let y be contained in the convex hull of the vectors $x^0, ..., x^N \in A$, i.e. $y = \sum_{i=0}^N \lambda_i x^i$ with $\lambda_i \geq 0$ for $i \in \{0, ..., N\}$ and $\sum_{i=0}^N \lambda_i = 1$. We define the set-valued function $\chi\colon co(A) \to \mathcal{P}(\{0, ..., N\})$ by $\chi(y) = \{i \mid \lambda_i > 0\}$. It follows that if $\chi(y) = \{i_0, ..., i_k\}$, then $y \in co(\{x^{i_0}, ..., x^{i_k}\})$ and we call this face the ***carrier*** *of y.*

Definition 2.10 (Proper labeling and Sperner simplices) *[KB85; HM18; AY17]*
Let V denote the vertices of all subsimplices of a simplicial subdivision of an N-simplex $\mathcal{S} := co(\{x^0, ..., x^N\})$. Each function $f\colon V \to \{0, ..., N\}$ is called a labeling function. We call f a ***proper labeling*** *of the subdivision if $f(v) \in \chi(v)$ for all $v \in V$. An N-simplex is called* ***completely labeled*** *or a* ***Sperner simplex*** *if f takes on all values $0, ..., N$ on its vertices.*

Intuitively, a proper labeling function can only assign the "index values" to a vertex of a subsimplex of those vertices of the original simplex that were needed to "span" this given vertex of the subsimplex by a convex combination. To facilitate intuition further, we could equivalently define a proper labeling by the following condition which I will call the *Sperner labeling condition* in the following.[7]

Definition 2.11 (Sperner labeling condition) *[EZ95]*
Let $\mathcal{S}_1, ..., \mathcal{S}_J$, $J \in \mathbb{N}$, be a simplicial subdivision of the N-simplex $\mathcal{S}$. We say that the labeling function $f\colon V \to \{0, ..., N\}$ complies with the ***Sperner labeling condition*** *if each vertex of $\mathcal{S}_j$, $j \in \{1, ..., J\}$, is assigned a value $0, 1, ..., N$ such that the following condition holds: If*

$$v \in co(\{x^{i_0}, ..., x^{i_k}\}), \qquad k = 1, ..., N, \tag{2.1}$$

then one of the numbers $i_0, ..., i_k$ is associated with v, i.e. $f(v) \in \{i_0, ..., i_k\}$.

Note that condition (2.1) in particular implies that each vertex x^j, $j \in \{0, ..., N\}$, of the original N-simplex $\mathcal{S}$ carries the number j since $x^j \in co(\{x^j\})$.[8] Figure 6 below illustrates the Sperner labeling condition for a 2-simplex and an equilateral subdivision. For illustrative purposes, I follow Berger (2020) (and many other authors in the literature)

[7] Definitions 2.10 and 2.11 are not only equivalent but essentially *the same*. The latter merely concentrates explicitly on the vertices of the subsimplices $\mathcal{S}_1, ..., \mathcal{S}_J$ while the former "subsumes" all of those vertices within V.

[8] Also note that condition (2.1) uses double subscripts (i.e. $co(\{x^{i_0}, ..., x^{i_k}\})$ instead of $co(\{x^0, ..., x^k\})$. This is necessary since convex hulls do not have to be formed by the *first* k vertices.

and use colors instead of numbers as labels.[9] Subfigures 6b.) and 6c.) show that vertices on the edge of the original simplex (i.e. $v \in co\{x^0, x^1\}$, $v \in co\{x^1, x^2\}$ and $v \in co\{x^0, x^2\}$) can only be labeled with the colors of the lowest-dimensional faces of the original simplex $\mathcal{S}$ that contain the given vertices of a subsimplex. Subfigure 6d.) labels all vertices in the interior of the original simplex (i.e. all $v \in co\{x^0, x^1, x^2\}$) and shows that these points can be labeled arbitrarily by either red, blue, or orange since the Sperner labeling condition does not impose any further restrictions. Subfigure 6d.) also highlights all Sperner simplices of the simplicial subdivision (i.e. all subtriangles that are *completely labeled* with all three different colors on their vertices). In this example, the simplicial subdivision gives rise to a unique Sperner simplex. The question arises whether *every* labeling of a simplicial subdivision that adheres to the Sperner labeling condition gives rise to the existence of such Sperner simplices. Sperner's lemma proves that this is indeed the case.

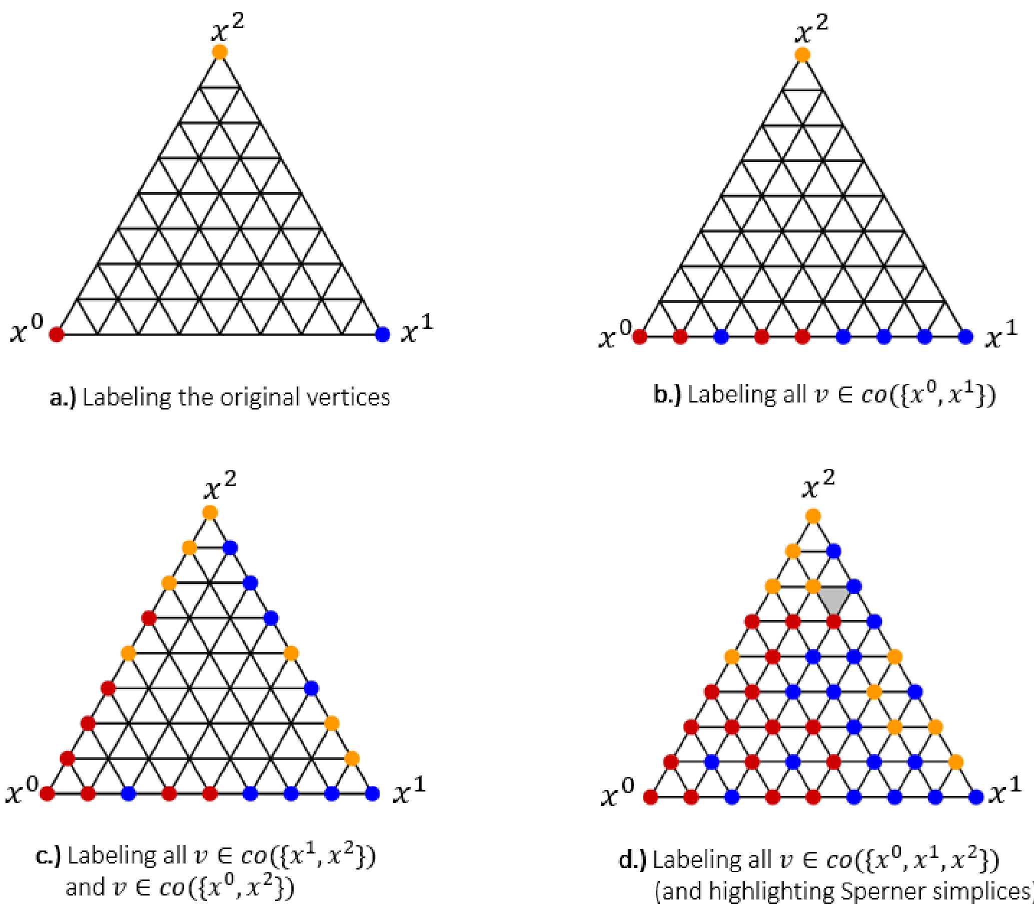


Figure 6: Illustration of the Sperner labeling condition (Source: MB20).

[9]In Figure 6, red is used for 0, blue is used for 1, and orange is used for 2.

2.2 Sperner's lemma

Sperner's lemma establishes that the number of completely-labeled N-subsimplices (i.e. Sperner simplices) for *any* properly-labeled simplicially-subdivided simplex is odd (Yuan, 2017). Since zero is an even number, *existence* of such simplices follows immediately.[10]

Theorem 2.12 (Sperner's lemma) *[MB20; JJ04; HM18; ES28; EZ95]*
Let $\mathcal{S}_1, ..., \mathcal{S}_J$, $J \in \mathbb{N}$, be a simplicial subdivision of the N-simplex $\mathcal{S}$ that is properly labeled by the function f (i.e. that fulfills the Sperner labeling condition (2.1)). Then the number of completely labeled (Sperner) N-subsimplices $\mathcal{S}_j$ is odd. In particular, there exists at least one such Sperner simplex.

Proof: By induction on N, the dimension of the simplex.

Step 1:
Base case: For $N = 0$, the 0-simplex $\mathcal{S}$ is a single point, i.e. $\mathcal{S} = co(\{x^0\}) = x^0$ and there do not exist other possibilities of simplicial subdivisions.[11] By the Sperner labeling condition (2.1), it must hold that $f(x^0) = 0$. There is thus only one completely labeled subsimplex, x^0 itself, and this is an odd number.

Step 2:
Although not strictly necessary, we look at $N = 1$ to provide some additional clarity. Each 1-subsimplex $\mathcal{S}_j$ is a line segment. For the remainder, we will call a (N-1)-face of $\mathcal{S}_j$ *distinguished* if and only if its vertices carry at least the numbers $\{0, ..., N-1\}$. For $N = 1$, this means that a 0-face (i.e. a vertex) of $\mathcal{S}_j$ is called distinguished if it is labeled by the number 0. For any 1-subsimplex $\mathcal{S}_j$, there are precisely two possibilities:

1. The line segment $\mathcal{S}_j$ has exactly one distinguished (N-1)-face (in which case it must be completely labeled, i.e. a Sperner simplex).

2. $\mathcal{S}_j$ has either two (both vertices labeled 0) or no (both vertices labeled 1) distinguished (N-1)-faces (in which case it is not a Sperner simplex).

Precisely one vertex of $\mathcal{S} = co(\{x^0, x^1\})$ is labeled with 0 (namely x^0) while the other is labeled with 1 (namely x^1). No matter how the interior points are labeled, one always has

[10] Although the basic idea of proving Sperner's lemma is always very similar, the nuances by which it is presented vary across different sources. I will focus on a proof that is more of combinatorial nature. By introducing elementary concepts from graph theory, the proof can be shortened and presented in a slightly different manner (see for instance Border, 1985; Yuan, 2017, among others).

[11] Some authors argue that for $N = 0$, there is no simplicial subdivision of a single point, but the *single* point is obviously an odd number (e.g. Meister, 2018).

to change numbers (from 0 to 1 or from 1 to 0) an odd number of times to get from the 0-corner to the 1-corner.[12] Thus, the number of Sperner simplices must be odd.

Step 3:

Induction hypothesis: Suppose the statement of Sperner's lemma holds for a fixed (but arbitrary) $N-1 \in \mathbb{N}$.

Step 4:

Induction step: We show that the statement also holds for N. To do so, we consider a properly labeled simplicial subdivision $\mathcal{S}_1, ..., \mathcal{S}_J$, $J \in \mathbb{N}$, of the N-simplex $\mathcal{S}$. Again, there are two possibilities for any N-subsimplex $\mathcal{S}_j$:

1. $\mathcal{S}_j$ has exactly one distinguished (N-1)-face (in which case it must be completely labeled, i.e. a Sperner simplex).

2. $\mathcal{S}_j$ has either two (all labels $\{0, ..., N-1\}$ appear at some of its vertices while exactly one label $k \in \{0, ..., N-1\}$ appears twice) or no distinguished (N-1)-faces (in which case it is not a Sperner simplex).

We introduce the following designations:[13]

→ **e** denotes the number of completely labeled simplices $\mathcal{S}_j$ (i.e. Sperner simplices). Our goal is to show that **e** is an odd number.

→ **f** denotes the number of *almost* completely labeled simplices $\mathcal{S}_j$, i.e. simplices whose vertices are labeled with the numbers $0, 1, ..., N-1$ but *not* with N, i.e. $f(\mathcal{S}_j) = \{0, 1, ...N-1\}$.

→ **g** denotes the number of distinguished (N-1)-faces of any $\mathcal{S}_j$ in the *interior* of $\mathcal{S}$.

→ **h** denotes the number of distinguished (N-1)-faces of any $\mathcal{S}_j$ on the *boundary* of $\mathcal{S}$.

We are now in a position to systematically count the total number of (N-1)-faces by considering each $\mathcal{S}_j$ separately:

[12] To see this, it is easiest to consider what happens to the number of Sperner simplices once changes to interior labels are made. Consider *any* three adjacent vertices of the simplicial subdivision. There are three possible cases which we illustrate based on a vertex that is labeled with 0 initially and is changed to 1. Case 1: A 0-vertex surrounded by two other 0-vertices is changed to 1, i.e. 0-0-0 becomes 0-1-0. This increases the number of Sperner simplices by 2. Case 2: A 0-vertex surrounded by two 1-vertices is changed to 1, i.e. 1-0-1 becomes 1-1-1. This decreases the number of Sperner simplices by 2. Case 3: A 0-vertex surrounded by one 0-vertex and one 1-vertex is changed to 1, i.e. 0-0-1 becomes 0-1-1. This merely changes the position of the Sperner simplex. In all cases, Sperner simplices are changed by an *even* number and since there must be at least one additional change from 0 to 1 from x^0 to x^1 somewhere, we must change numbers an *odd* number of times in total.

[13] I adhere to the original notation used by Sperner (1928).

1. Each completely labeled subsimplex S_j has precisely one distinguished (N-1)-face ($\Rightarrow 1 \cdot \mathbf{e}$). Each almost completely labeled subsimplex $\mathcal{S}_j$ has exactly two distinguished (N-1)-faces ($\Rightarrow 2 \cdot \mathbf{f}$).

2. By counting all subsimplices $\mathcal{S}_j$ in this way, we will count each distinguished (N-1)-face which is located in the interior of $\mathcal{S}$ *twice* since each (N-1)-face of a simplicial subdivision in the interior of $\mathcal{S}$ belongs to precisely *two* different $\mathcal{S}_j$ ($\Rightarrow 2 \cdot \mathbf{g}$). By contrast, all (N-1)-faces located on the boundary of $\mathcal{S}$ (and are hence a subset of $co(\{x^0, x^1, ..., x^{N-1}\})$) are counted only once ($\Rightarrow 1 \cdot \mathbf{h}$). This yields the following equation:

$$\mathbf{e} + 2\,\mathbf{f} \;=\; 2\,\mathbf{g} + \mathbf{h}. \tag{2.2}$$

3. Since the simplicial subdivision is properly labeled by f (i.e. fulfills the Sperner labeling condition (2.1)), we know that distinguished (N-1)-faces on the boundary of $\mathcal{S}$ can only appear on a *single* (N-1)-face of $\mathcal{S}$ - namely the one labeled with $\{0, 1, ..., N-1\}$.[14] We are hence in the $(N-1)$-dimensional case and know by our *induction hypothesis* that the number of Sperner simplices in this case is odd. In other words, $\mathbf{h}$ is odd. It follows immediately from equation (2.2) that $\mathbf{e}$ must also be odd. But $\mathbf{e}$ is precisely the number of Sperner simplices in the N-dimensional case. By induction, Sperner's lemma holds for any $N \in \mathbb{N}$ which completes the proof.

□

2.3 Relation to fixed point theory

The remainder of this thesis focuses exclusively on fixed point theorems for real- and set-valued functions. Given a set M and a mapping $f\colon M \to M$, $x \in M$ is called a fixed point of f if $f(x) = x$. At first sight it does not seem obvious how the existence of Sperner subsimplices in a properly labeled simplicial subdivision relates to fixed point theory at all. The first and foremost connection lies in the structural property of simplices. As will be shown in the subsequent chapter, simplices are homeomorphic to closed balls which in turn are homeomorphic to nonempty, compact, convex subsets of $\mathbb{R}^n$. The latter form the building block of Brouwer's fixed point theorem. Therefore, we can begin by proving existence of fixed points of functions that map a given simplex into itself. By means of simplicial subdivisions that allow for an arbitrarily small mesh, such fixed points can always

[14] This is because each vertex x^j of $\mathcal{S}$ is labeled by j, so there can only exist *one* such (N-1)-face labeled by $\{0, 1, ..., N-1\}$:

be found via Sperner simplices which must exist by Sperner's lemma.[15] The existence of such fixed points can then be "transferred" from simplices to homeomorphic sets which eventually proves Brouwer's fixed point theorem. Figure 7 illustrates the individual steps while the subsequent chapter formalizes these ideas.

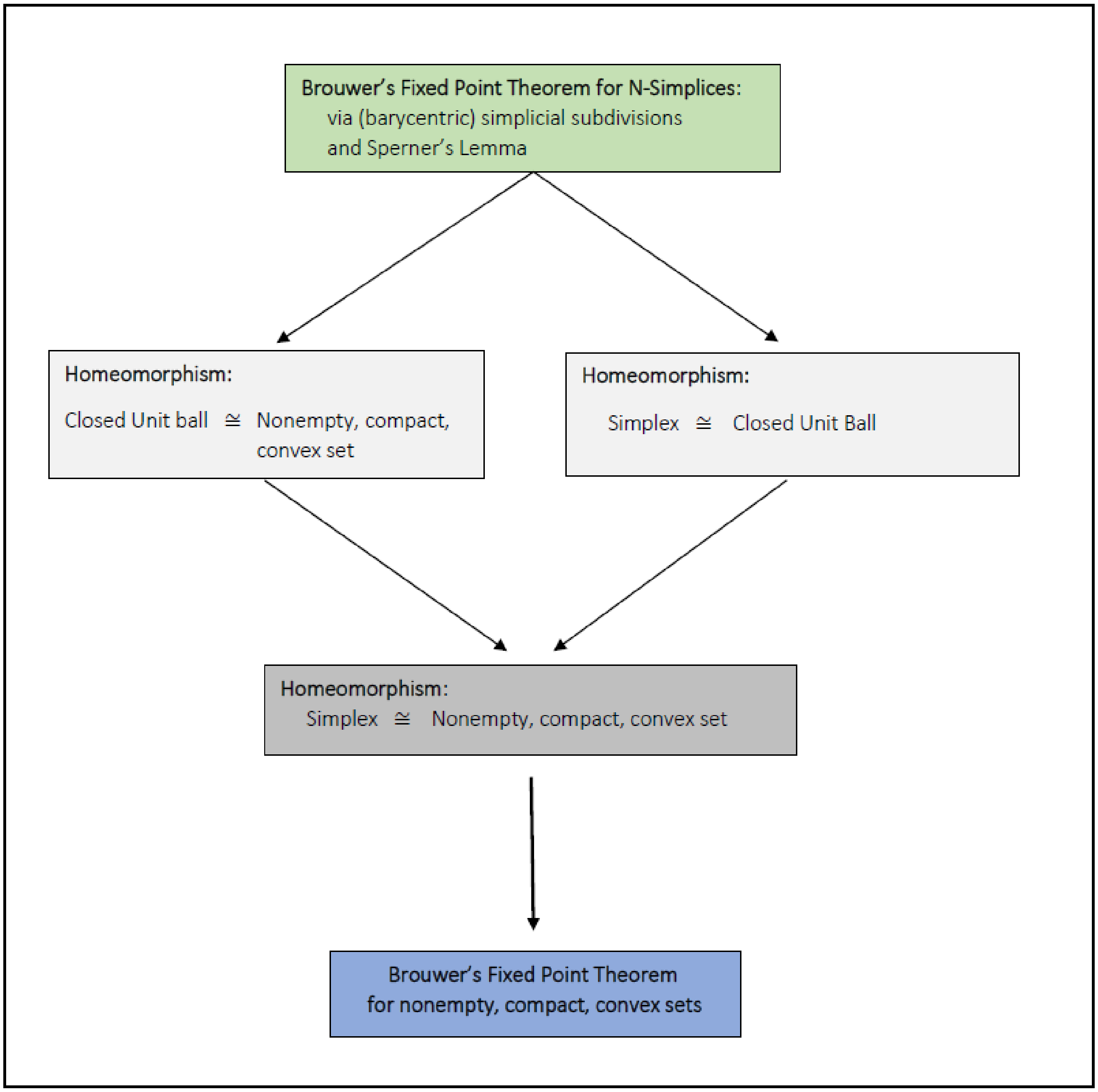


Figure 7: Illustration of the key steps to prove Brouwer's fixed point theorem via Sperner's lemma and simplicial subdivisions.

[15]Both equilateral and barycentric subdivisions have the property of being simplicial subdivisions whose mesh can be made arbitrarily small. Since most authors are only interested in using Sperner's lemma to prove fixed point theorems, they directly use these special subdivisions when formulating Sperner's lemma (e.g. Meister, 2018; Nikaido, 1968). Moreover, in all what follows, Sperner's lemma is solely needed to establish *existence* of completely labeled subsimplices - the 'odd number property' as such is never used (although it was essential to *establish* existence which immediately results from this property).

3 Brouwer's Fixed Point Theorem

One of the best-known fixed point theorems is doubtlessly the *Banach fixed point theorem*. It requires relatively few assumptions on the underlying space but fairly strong assumptions on the respective mapping.[16] *Brouwer's fixed point theorem*, by contrast, only requires the function to be continuous but puts stronger assumptions on the space (Brouwer, 1912). Albeit different versions of the theorem exist, the most general one that we are interested in is the following.

Theorem 3.1 (Brouwer's fixed point theorem) *[DW18]*
Let $C \subset \mathbb{R}^n$ be a nonempty, compact and convex set and let $f: C \to C$ be continuous. Then f has at least one fixed point, i.e. there exists $\xi \in C$ with $f(\xi) = \xi$.

Chronologically, Sperner's lemma has been proven years *after* Brouwer's fixed point theorem (Brouwer, 1912; Sperner, 1928) but it can significantly simplify the proof of the latter.[17] The next paragraph provides some necessary notions and properties of a particular simplicial subdivision before the proof of the above version of Brouwer's fixed point theorem will be successively developed.

3.1 The barycentric subdivision

The proof of Brouwer's fixed point theorem via Sperner's lemma relies on a simplicial subdivision whose mesh can be made (arbitrarily) small - the *barycentric subdivision* fulfills this property. We collect some necessary definitions before this can be shown formally.

Definition 3.2 (Barycentric coordinates and barycenter) *[ADQ12; FT17]*
Let $\{x^0, ..., x^N\} \subset \mathbb{R}^n$ be affine independent and let $x \in \mathcal{S} = co(\{x^0, ..., x^N\})$ with

$$x = \sum_{j=0}^{N} \lambda_j x^j \quad \text{and} \quad \sum_{j=0}^{N} \lambda_j = 1, \qquad \lambda_0, ..., \lambda_N \in \mathbb{R},\ \lambda_j \geq 0\ \forall j.$$

Then, the unique $\lambda_0, ..., \lambda_N$ are called the ***barycentric coordinates*** *of x. The* ***barycenter*** *of $\mathcal{S}$ is given by*

$$b(\mathcal{S}) := \frac{1}{N+1} \sum_{j=0}^{N} x^j.$$

[16]The space has to be a nonempty, complete metric space while the function has to be a strict contraction. The latter property could be particularly difficult to verify.

[17]See appendix A for an idea of how to prove Brouwer's fixed point theorem using analytical tools that does not rely on Sperner's lemma or simplicial subdivisions.

All barycentric coordinates are identical at the barycenter (i.e. $\lambda_0 = ... = \lambda_N = \frac{1}{1+N}$). The latter can hence be thought of as the *center of gravity* of the simplex (Rotman, 1988). Figure 8 illustrates this idea by means of different N-simplices.

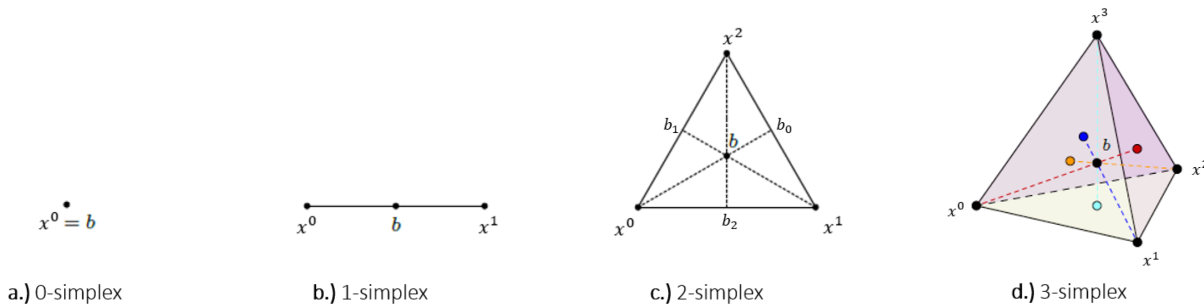


Figure 8: Illustration of the barycenter for different N-simplices (Source: MB20).

The concept of the barycenter can be used to inductively define the barycentric subdivision.

Definition 3.3 (Barycentric subdivision) *[JR88]*
The ***barycentric subdivision*** *$Sd(\mathcal{S})$ of an N-simplex $\mathcal{S}$ is a family of N-subsimplices defined inductively as follows:*

(i) The barycentric subdivision of a 0-simplex is the 0-simplex itself.

(ii) If $\phi_0, \phi_1, ..., \phi_N$ are the (N-1)-faces of the N-simplex $\mathcal{S}$ and if b is the barycenter of $\mathcal{S}$, then $Sd(\mathcal{S})$ consists of all N-subsimplices spanned by b and the (N-1)-subsimplices of the barycentric subdivisions $Sd(\phi_i)$, $i = 0, ..., N$.

Intuitively, an N-simplex can be barycentrically subdivided into (N+1)! smaller subsimplices (Toenniessen, 2017). To subdivide a 2-simplex (i.e. a triangle) $\mathcal{S} = co(\{x^0, x^1, x^2\})$, for instance, we start by subdividing 0- and 1-simplices first: The barycentric subdivisions of the 0-simplices (x^0, x^1, x^2) are the simplices themselves. The barycentric subdivision of the faces (1-simplices) each consists of two 1-subsimplices (line segments from the respective corners to the barycenters b_0, b_1, b_2). Finally, the barycentric subdivision of the triangle is obtained by combining its barycenter b with the barycenters b_0, b_1, b_2 of the 1-simplices and the barycenters x^0, x^1, x^2 of the 0-simplices. The result is the so-called *first barycentric subdivision* of the 2-simplex and is illustrated in figure 9a) below. Barycentrially subdividing the N-subsimplices of the barycentric subdivision again leads to the *second* barycentric subdivision (see figure 9b)). This process can be iterated which results in successively finer (in terms of the *mesh*) simplicial subdivisions of the original simplex. This observation is a key ingredient for the proof of Brouwer's fixed point theorem and is stated formally in the following proposition.[18] It suggests that the mesh can be made arbitrarily small by repeated barycentric subdivision. We call smaller N-subsimplices which result from a k-times application of the barycentric subdivision of $\mathcal{S}$ *derived subsimplices of order* k.

Proposition 3.4 (Mesh and iterated barycentric subdivision) *[ADQ12; HN68]*
Let $\mathcal{S} = co(\{x^0, x^1, ..., x^N\})$ *be an N-simplex with diameter* $\delta(\mathcal{S}) = \sup_{j,k\in\{0,...,N\}} \|x^j - x^k\|$ *and let* $\mathcal{S}^k$ *be any derived simplex of order* k *in the* k^{th} *barycentric subdivision of* $\mathcal{S}$*. Then the following inequality holds:*

$$\delta(\mathcal{S}^k) \;\leq\; \left(\frac{N}{N+1}\right)^k \delta(\mathcal{S}). \tag{3.1}$$

In particular, $\lim_{k\to\infty} \delta(\mathcal{S}^k) = 0$.

Proof: By induction on N, the dimension of the N-simplex.
$\rightarrow$ See appendix B.1.

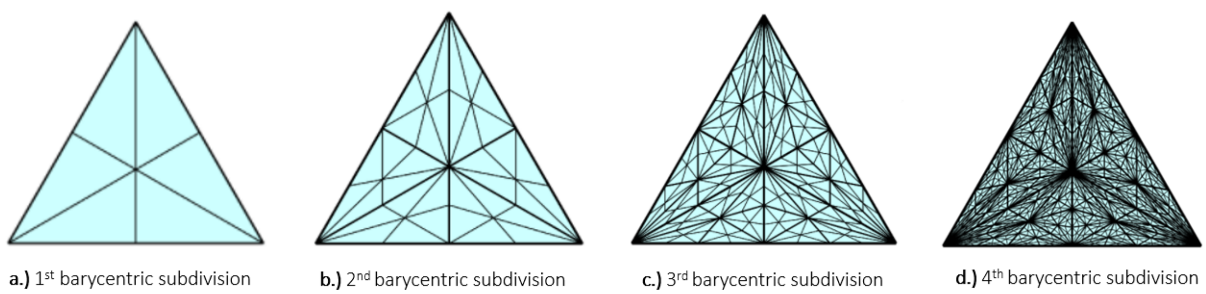


Figure 9: First four barycentric subdivisions of a 2-simplex
(Source: https://en.wikipedia.org/wiki/Barycentric_subdivision).

3.2 Brouwer's fixed point theorem for simplices

We are now in a position to develop the proof of Brouwer's fixed point theorem by means of barycentric subdivisions and Sperner's lemma. We will follow the individual steps as illustrated in figure 7 and begin by proving a fixed point theorem for N-simplices.

[18] Although the proof for this result can be found in some textbooks on algebraic topology, surprisingly few sources provide the proof in the context of Brouwer's fixed point theorem but simply take it as given.

Proposition 3.5 (Brouwer's fixed point theorem for N-simplices) *[MB20; HM18]* *Let $\mathcal{S} = co(\{x^0, ..., x^N\})$ denote an N-simplex. Any continuous map $f: \mathcal{S} \to \mathcal{S}$ has at least one fixed point, i.e. there exists $\xi \in \mathcal{S}$ with $f(\xi) = \xi$.*

Proof: By contradiction (assuming no fixed point would exist).

Step 1:

Initial observations: We can focus WLOG on the *standard* N-simplex as there is a canonical map from the standard N-simplex to an arbitrary N-simplex (see Def. 2.6). Therefore, let $x^0 = (1, 0, ..., 0), ..., x^N = (1, 0, ..., 0) \in \mathbb{R}^{N+1}$ and $\mathcal{S} = co(\{x^0, ..., x^N\})$. Further, let $f = (f_0, ..., f_N): \mathcal{S} \to \mathcal{S}$.

Step 2:

Construction of a sequence of simplicial subdivisions: We consider a sequence $(\mathcal{Z}_j)_{j\in\mathbb{N}}$ of simplicial subdivisions of $\mathcal{S}$ such that the sequence of the resulting meshs converges to zero, i.e. $(\delta(\mathcal{Z}_j))_{j\in N} \to 0$ as $j \to \infty$ where $\delta(\mathcal{Z}_j)$ denotes the maximum diameter of all subsimplices within the subdivision $\mathcal{Z}_j$. We know from proposition 3.4 that such a sequence exists: For every $\mathcal{Z}_{j+1}$, we can simply take the barycentric subdivisions of all subsimplices within $\mathcal{Z}_j$.

Step 3:

Construction of an appropriate labeling function: We denote by V the set of all vertices of all subsimplices of the simplicial subdivision $\mathcal{Z}_j$. For any $z = (z_0, ..., z_N) \in \mathcal{Z}_j$ we consider the labeling function $\lambda: V \to \{0, ..., N\}$ with $\lambda(z) := \min\{i \in \{0, ..., N\} \mid f_i(z) < z_i\}$. Each grid point of $\mathcal{Z}_j$ is hence assigned the smallest index i for which the i^{th} coordinate of $f(z) - z$ is smaller than 0. Note that λ is well-defined since for every $z \in \mathcal{S}$, the sum of its barycentric coordinates as well as the sum of the barycentric coordinates of $f(z)$ must always sum to 1 (this is because $f: \mathcal{S} \to \mathcal{S}$ and we consider the standard N-simplex), i.e.

$$1 \; = \; \sum_{i=0}^{N} z_i \; = \sum_{i=0}^{N} f_i(z) \; = 1 \qquad \forall z \in \mathcal{S}. \tag{3.2}$$

By our *contradiction assumption*, $f(z) \neq z$, so (3.2) implies that there will always exist at least one coordinate $i \in \{0, ..., N\}$ with $f_i(z) < z_i$ (and also at least one coordinate with $f_i(z) > z_i$). This makes λ well-defined.

Step 4:

Application of Sperner's lemma:

1. It holds that $\lambda(x^i) = i$, i.e. λ assigns to each vertex x^i of $\mathcal{S}$ the value i. This is because the i^{th} coordinate of $f_i(x^i) - x^i$ is the only possibility for $f_i(x^i) - x^i < 0$ to

hold (equality in every coordinate cannot hold as we would have found a fixed point otherwise).

2. Now let $x \in \mathcal{S}$ denote an arbitrary point on the edge of $\mathcal{S}$ opposite to x^i such that its i^{th} barycentric coordinate must be $x_i = 0$ (as x^i is not needed to 'span' x). As $f(x) \in \mathcal{S}$ (and hence $f_i(x) \geq 0$), it cannot be that $f_i(x) < x_i = 0$. Hence, $\lambda(x) \neq i$, i.e. λ can only assign the values to x of the vertices $x^0, ..., x^N$ of $\mathcal{S}$ that were needed to 'span' x. This is precisely the Sperner labeling condition (2.1).

Therefore, the simplicial subdivison of the N-simplex $\mathcal{S}$ is properly labeled by the function λ and the requirements for Sperner's lemma are fulfilled.

Step 5:

Convergent subsequences of vertices: By Sperner's lemma, there exists a completely labeled (Sperner) simplex in every simplicial subdivision $\mathcal{Z}_j$ of $\mathcal{S}$. We denote the vertices of such a Sperner simplex by $z^{(j,0)}, ..., z^{(j,N)}$. For each j, we next consider the sequence $(z^{(j,0)})_{j\in\mathbb{N}}$ which results from collecting the first coordinate of each Sperner simplex in the simplicial subdivision $\mathcal{Z}_j$. As $\mathcal{S}$ is bounded (it is even compact), there exists a convergent subsequence $(z^{(j_k,0)})_{k\in\mathbb{N}} \in \mathcal{S}$ by the Bolzano-Weierstraß theorem, i.e. $z^{(j_k,0)} \to z^* \in \mathcal{S}$. But since $(\delta(\mathcal{Z}_j))_{j\in N} \to 0$ as $j \to \infty$ (i.e. the subsimplices - including the considered Sperner simplices - become arbitrarily small), the other vertices $z^{(j_k,1)}, ..., z^{(j_k,N)}$ of the Sperner simplex must also converge *to the same* $z^* \in \mathcal{S}$, i.e.

$$\lim_{k\to\infty} z^{j_k,i} \;=\; z^* \in \mathcal{S} \qquad \forall i \in \{0, ..., N\}. \tag{3.3}$$

This is because the distance between $z^{(j_k,0)}$ and each vertex $z^{(j_k,i)}$, $i \in \{1, ..., N\}$, becomes arbitrarily small for $k \to \infty$.

Step 6:

Contradiction and existence of a fixed point: As $z^{(j,0)}, ..., z^{(j,N)}$ is a completely labeled Sperner simplex, it follows that $\lambda(z^{j,i}) = i$ for $i \in \{0, ..., N\}$. But by definition of λ, i.e. $\lambda(z) = \min\{i \in \{0, ..., N\} | f_i(z) < z_i\}$, this in turn implies that

$$f_i(z^{(j,i)}) < z_i^{(j,i)} \qquad \forall i \in \{0, ..., N\}. \tag{3.4}$$

As $z^{(j,i)}$ is a vertex of a Sperner simplex for all j, (3.4) holds for all j and in particular for the elements $(z^{(j_k,i)})_{k\in\mathbb{N}}$ of the respective convergent subsequences. Hence, by *continuity* *of* f, we get for all $i \in \{0, ..., N\}$

$$f_i(z^*) \;=\; f_i\big(\lim_{k\to\infty} z^{(j_k,i)}\big) \underbrace{=}_{\text{continuity}} \lim_{k\to\infty} f_i\big(z^{(j_k,i)}\big) \underbrace{\leq}_{(3.4)} \lim_{k\to\infty} z_i^{(j_k,i)} \underbrace{=}_{(3.3)} z_i^*. \tag{3.5}$$

But $f_i(z^*) \leq z_i^* \; \forall i \in \{0, ..., N\}$ means that *none* of the elements of $f(z^*) - z^*$ are positive. This directly contradicts our no-fixed-point-assumption $f(z) \neq z \; \forall z \in \mathcal{S}$ since we have argued by means of equation (3.2), i.e. $1 = \sum_{i=0}^{N} z_i = \sum_{i=0}^{N} f_i(z) = 1$, that there must be at least one coordinate $i \in \{0, ..., N\}$ with $f_i(z) > z_i$ for *all* $z \in \mathcal{S}$ and hence also for z^*. This contradiction shows that there must be at least one fixed point $\xi \in \mathcal{S}$ with $f(\xi) = \xi$.

□

3.3 Homeomorphisms between simplices, closed balls and compact, convex sets

The previous subsection presented a proof of Brouwer's fixed point theorem for N-simplices. To obtain a generalized version as in theorem 3.1, we proceed as in figure 7. We will first show homeomorphism results for closed unit balls in $\mathbb{R}^n$ and nonempty, compact and convex subsets of $\mathbb{R}^n$. Since the standard N-simplex also has these properties, it will also be homeomorphic to the closed unit ball in $\mathbb{R}^n$ as we will show thereafter. Finally, this will allow us to transfer the fixed point theorem for simplices to the desired more general environment.

Definition 3.6 (Homeomorphism) *[KB85]*
Two sets A and B are called ***homeomorphic*** *and we write $A \cong B$ if there exists a continuous bijection $f\colon A \to B$ such that f^{-1} is also continuous. We call such a function f a* ***homeomorphism****.*

To show that any nonempty, compact and convex subset $C \subset \mathbb{R}^n$ is homeomorphic to the closed unit ball $\overline{B}_1(0) := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \subset \mathbb{R}^n$, we proceed in two steps:

1. We will define a "helper function" $k\colon \mathbb{R}^n \setminus \{0\} \to C$ and prove that k is well-defined, bounded, and continuous.

2. We use k to explicitly construct a desired homeomorphism between C and the closed unit ball $\overline{B}_1(0)$.

Lemma 3.7 (Helper function k to construct homeomorphism) *[AY17; EZ95]*
*Let the set of points $r(x) := \{\alpha x \mid \alpha \in \mathbb{R}_+\}$ denote the **ray** of $x \in \mathbb{R}^n$ and let $C \subset \mathbb{R}^n$ denote a nonempty, compact and convex subset of $\mathbb{R}^n$. Then, the function $k\colon \mathbb{R}^n \setminus \{0\} \to C$ defined by*

$$k(x) = y \quad \textit{such that} \quad y \in r(x) \cap C, \quad \textit{but} \quad \forall \alpha > 1, \alpha y \notin C \tag{3.6}$$

is well-defined, bounded and continuous.

Proof: See appendix B.2.

Intuitively, for any $x \in \mathbb{R}^n$, the helper function k maps x to the unique point in C along the ray $r(x)$ that is farthest away from the origin (and hence on the boundary ∂C). Since it is without loss of generality to assume that the zero vector is contained in C (otherwise, "shift" C by applying a continuous translation), we can find points in any direction from 0 as we can always choose an arbitrarily small ϵ-ball $B_\epsilon(0) \subset C$. Figure 10 below provides some additional intuition on the helper function k.[19] Also note that k maps points that are already contained in C to the boundary of C.

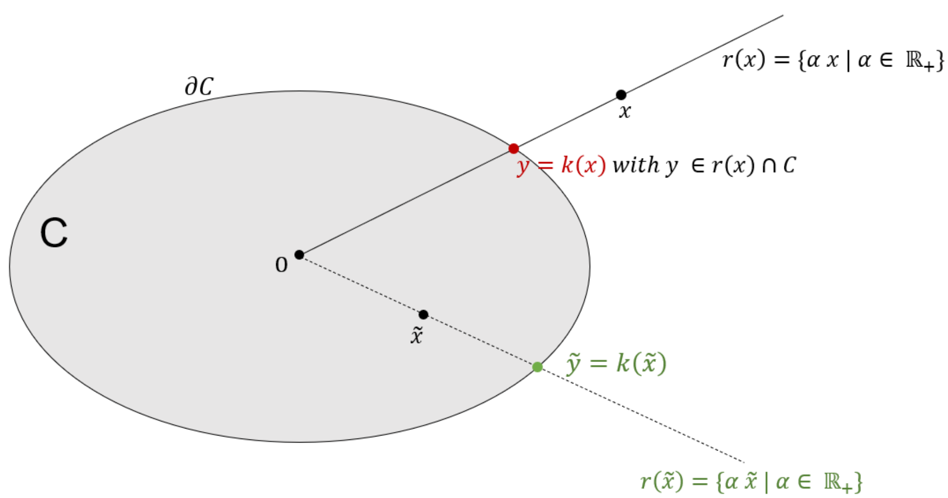


Figure 10: Illustration of the helper function k.

[19] The function k and its meaning for the homeomorphism in proposition (3.8) is inherently related to the so-called *Minkowski functional* $p\colon \mathbb{R}^n \to \mathbb{R}$ with $p(u) := \inf\{\lambda \mid \lambda^{-1}u \in C, \lambda > 0\}$. The helper function k directly provides the desired point $y \in C$ on the boundary of C while $p(u)$ provides the unique λ^* such that $p(u)^{-1}u = \lambda^{*-1}u$ yields this desired point and $\lambda^{-1}u \notin C$ for any $\lambda < \lambda^*$. In fact, many authors such as Werner (2009) or Zeidler (1995) use the Minkowski functional in the proofs. I use the helper function k as it "incorporates" the Minkowski functional already and has a slightly more intuitive appeal to me. Due to $k(u) = y = p(u)^{-1}u$, both approaches are of course equivalent.

By using the helper function k, we can now construct explicit homeomorphisms between the closed unit ball and nonempty, compact and convex sets.

Proposition 3.8 (Homeomorphism between $\overline{B}_1(0)$ and C) *[AY17; HM18; EZ95]*
Every nonempty, compact and convex subset $C \subset \mathbb{R}^n$ is homeomorphic to the closed unit ball $\overline{B}_1(0) := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \subset \mathbb{R}^n$.

Proof: By construction of an explicit homeomorphism via the helper function k.

Step 1:

Initial observations: We call two vectors $x, y \in \mathbb{R}^n$ *positively collinear* if $x = cy$ with $c \in \mathbb{R}_+$. We note that for any two collinear vectors, $k(x) = k(y)$ (where k is the helper function from lemma 3.7) as collinear vectors have the same direction and hence lie on the same ray: $r(x) = \{\alpha x \mid \alpha \in \mathbb{R}_+\} = r(y) = \{\alpha y \mid \alpha \in \mathbb{R}_+\}$.

Step 2:

Definition of two mappings by means of k: We first define two mappings $f\colon \mathbb{R}^n \to \mathbb{R}^n$ and $g\colon \mathbb{R}^n \to \mathbb{R}^n$ that will play a crucial role in the homeomorphism between C and $\overline{B}_1(0)$ that will be constructed in the following.

$$f(x) := \begin{cases} \frac{x}{\|k(x)\|} & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0 \end{cases} \tag{3.7}$$

$$g(x) := \begin{cases} x\|k(x)\| & \text{if } \; x \neq 0 \\ 0 & \text{if } \; x = 0 \end{cases} \tag{3.8}$$

Step 3:

Show that f is continuous: For $x \neq 0$, continuity is immediate since the identity function, the norm and the helper function k are all continuous. To establish continuity in $x = 0$, we have to show

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ s.t. } \forall y \in \mathbb{R}^n \text{ with } \|y - 0\| = \|y\| < \delta \text{ we have } \|f(y) - f(0)\| = \|f(y)\| < \varepsilon.$$

Let $\varepsilon > 0$. We are hence looking for a $\delta > 0$ s.t. $\|y\| < \delta$ implies $\left\| \frac{y}{\|k(y)\|} \right\| < \varepsilon$. The helper function k was constructed such that points $y \in \mathbb{R}^n \setminus \{0\}$ are mapped to $\partial C \subset C$. We have further established in the proof of Lemma 3.7 that C contains some open ball $B_{\varepsilon'}(0)$ in its interior such that $\|k(y)\| \geq \varepsilon' \; \forall y \in \mathbb{R}^n \setminus \{0\}$ (choose ε' small enough). Now consider

$\delta := \varepsilon \cdot \varepsilon'$ and a point $y \in \mathbb{R}^n \setminus \{0\}$ with $\|y\| < \delta = \varepsilon \cdot \varepsilon'$. We have

$$\|f(y)\| = \left\| \frac{y}{\|k(y)\|} \right\| = \frac{\|y\|}{\|k(y)\|} < \frac{\|y\|}{\varepsilon'} < \frac{\varepsilon \cdot \varepsilon'}{\varepsilon'} = \varepsilon.$$

This establishes continuity of f.

<u>*Step 4:*</u>
Show that g is continuous: Continuity for $x \neq 0$ follows analogously to that of f. To show continuity at $x = 0$, we recall the following: $k(y) \in C$ and hence $k(y)$ is bounded (lemma 3.7) so there exists an $M > 0$ such that $\|k(y)\| \leq M \; \forall y \in \mathbb{R}^n \setminus \{0\}$. Consider $\varepsilon > 0$ and define $\delta := \frac{\varepsilon}{M}$. For all $y \in \mathbb{R}^n$ such that $\|y\| < \frac{\varepsilon}{M} = \delta$ it holds that

$$\|g(y)\| = \| \, \|k(y)\| \, y \, \| = \|k(y)\| \, \|y\| < \|k(y)\| \, \frac{\varepsilon}{M} < M \, \frac{\varepsilon}{M} = \varepsilon.$$

This establishes continuity of g.

<u>*Step 5:*</u>
Show that f and g are inverses of each other, i.e. $f(g(x)) = x$ and $g(f(x)) = x \; \forall x \in \mathbb{R}^n$: For $x = 0$, this is immediate while for $x \neq 0$, we know that neither $f(x) = 0$ nor $g(x) = 0$ which follows by definition of the helper function k, in particular $k(x) \neq 0$. We have

$$f(g(x)) \underbrace{=}_{\text{def. of g}} f(\|k(x)\| \, x) \underbrace{=}_{\text{def. of f}} \frac{\|k(x)\| \, x}{\|k(\|k(x)\| \, x)\|}. \tag{3.9}$$

As $\|k(x)\| \in \mathbb{R}_+$ is a scalar, x and $\|k(x)\|x$ are positively collinear [see Step 1]. They lie on the same ray $r(x)$ and are thus mapped to the same point under $k(x)$, i.e.

$$k(x) = k(\|k(x)\| \, x) \quad \Leftrightarrow \quad \|k(x)\| = \|k(\|k(x)\| \, x)\|. \tag{3.10}$$

Using (3.10) in (3.9) yields $f(g(x)) = x$.

Similarly, we have

$$g(f(x)) \underbrace{=}_{\text{def. of f}} g\Big(\frac{x}{\|k(x)\|}\Big) \underbrace{=}_{\text{def. of g}} \left\| k\Big(\frac{x}{\|k(x)\|}\Big) \right\| \frac{x}{\|k(x)\|}. \tag{3.11}$$

and by analogous reasoning as before, x and $\frac{x}{\|k(x)\|}$ are also collinear, i.e.

$$k(x) = k\Big(\frac{x}{\|k(x)\|}\Big) \quad \Leftrightarrow \quad \|k(x)\| = \|k\Big(\frac{x}{\|k(x)\|}\Big)\|. \tag{3.12}$$

Using (3.12) in (3.11) yields $g(f(x)) = x$. Hence, f and g are inverse functions of one another. This will form the basis of using them to construct a homeomorphism in the following.

Step 6:

Establish a relation between $\overline{B}_1(0)$ and C by means of f and g: We want to establish a homeomorphism between the closed unit ball $\overline{B}_1(0) \subset \mathbb{R}^n$ and a nonempty, compact and convex subset $C \subset \mathbb{R}^n$.

Let $x \in C$ be arbitrary. We show that $f(x) \in \overline{B}_1(0)$:
Since $x \in C$, we must have that $k(x) = \alpha \cdot x$ with $\alpha \geq 1$ (this is because k maps points to the boundary of C and when $x \in C$, it must "scale up" x to do so). Consequently, $\frac{1}{\alpha} \in (0, 1]$. For this $x \in C$, we hence get:

$$f(x) \;=\; \frac{x}{\|k(x)\|} \;=\; \frac{x}{\alpha\|x\|} \;=\; \underbrace{\frac{1}{\alpha}}_{\leq 1}\, \underbrace{\frac{x}{\|x\|}}_{\text{length } 1} \;\in \overline{B}_1(0). \tag{3.13}$$

Now let $x \in \overline{B}_1(0)$ be arbitrary. We show that $g(x) \in C$:
For this $x \in \overline{B}_1(0)$, we get:

$$g(x) \;=\; \|k(x)\|x \;=\; \|k(x)\| \underbrace{\frac{x}{\|x\|}}_{=:u_x} \|x\| \;=\; \|k(x)\|\, \|x\|\, u_x, \tag{3.14}$$

where u_x is a vector that points into the same direction as x but has unit length. We know that both $0 \in C$ and $k(x) \in C$ and that $\|x\| \leq 1$ as $x \in \overline{B}_1(0)$. Further, by definition of k we know that $\|k(x)\| = \sup\{\|y\| \mid y \in C \cap r(x)\}$. But then, by convexity of C, any convex combination of 0 and $k(x)$ is also contained in C and thus $g(x) \in C$.

Step 7:

Construct the desired homeomorphism between $\overline{B}_1(0)$ and C: We have seen that f and g are continuous, inverses of each other and map elements of C to elements of $\overline{B}_1(0)$ and vice versa. Restricting the respective domains of f and g immediately yields the desired homeomorphism: Let $\hat{f}\colon C \to \overline{B}_1(0)$ and $\hat{g}\colon \overline{B}_1(0) \to C$ be constructed just as f and g above. Then, $\hat{f}$ and $\hat{g}$ are continuous and inverses of each other, i.e. $\hat{f}^{-1} = \hat{g}$. Hence, $\hat{f}$ is a continuous bijection that maps a nonempty, compact and convex set $C \subset \mathbb{R}^n$ into the closed unit ball $\overline{B}_1(0)$. As a consequence, $\overline{B}_1(0)$ and C are homeomorphic, i.e. $C \cong \overline{B}_1(0)$.

□

Proposition 3.8 provides the crucial steps in order to establish the desired version of Brouwer's fixed point theorem. Two corollaries follow immediately.

Corollary 3.9 (Homeomorphisms between compact and convex sets) *[HM18]*
Let C_1 and C_2 be any nonempty, compact and convex subsets of $\mathbb{R}^n$. Then C_1 is homeomorphic to C_2, i.e. $C_1 \cong C_2$.

Proof:
Proposition 3.8 has shown that there exist homeomorphisms $f_1 : C_1 \to \overline{B}_1(0)$ and $f_2 : \overline{B}_1(0) \to C_2$. As the composition $f_2 \circ f_1 : C_1 \to C_2$ is again a homeomorphism, the result follows immediately. □

Corollary 3.10 (Homeomorphism: Simplices and compact, convex sets) *[HM18]*
Each n-simplex $\mathcal{S} = co(\{x^0, ..., x^n\}) \subset \mathbb{R}^n$ is homeomorphic to the closed unit ball $\overline{B}_1(0) \subset \mathbb{R}^n$, i.e. $\mathcal{S} \cong \overline{B}_1(0)$. Furthermore, the n-simplex $\mathcal{S}$ is homeomorphic to any nonempty, compact and convex subset $C \subset \mathbb{R}^n$, i.e. $\mathcal{S} \cong C$.

Proof:
Note that the n-simplex $\mathcal{S}$ is spanned by n+1 affine independent vertices $x^0, ..., x^n \in \mathbb{R}^n$ and is hence a nonempty, compact and convex subset of $\mathbb{R}^n$. Proposition 3.8 implies that it is homeomorphic to $\overline{B}_1(0) \subset \mathbb{R}^n$ and by corollary 3.9, it is also homeomorphic to *any* nonempty, compact and convex subset of $\mathbb{R}^n$. □

Since simplices play a fundamental role during the entire course of this thesis, I provide an additional proof idea in appendix B.3 which constructs an explicit homeomorphism (very similar to the helper function k above) between a simplex and a closed ball which additionally maps the boundary $\partial\mathcal{S}$ to the $(n-1)$-dimensional sphere $S_1^{n-1}(0) = \{x \in \mathbb{R}^n \mid x = 1\}$. Spoken visually, this homeomorphism between the simplex and the closed ball does the following (compare Berger, 2020; Ossa, 1992): For a given simplex, a closed ball is put around the simplex. Then, the simplex is "stretched out" such that its boundary is mapped to the boundary of the closed ball. Intuitively, one could think of this simplex as to be composed of an elastic material, just like a balloon, and air is blown into the simplex until it has the form of a ball. Figure 11 provides further intuition in this regard by using a continuous rotation to exemplify that a 2-simplex (i.e. a triangle) is homeomorphic to a 2-ball (i.e. a disc).

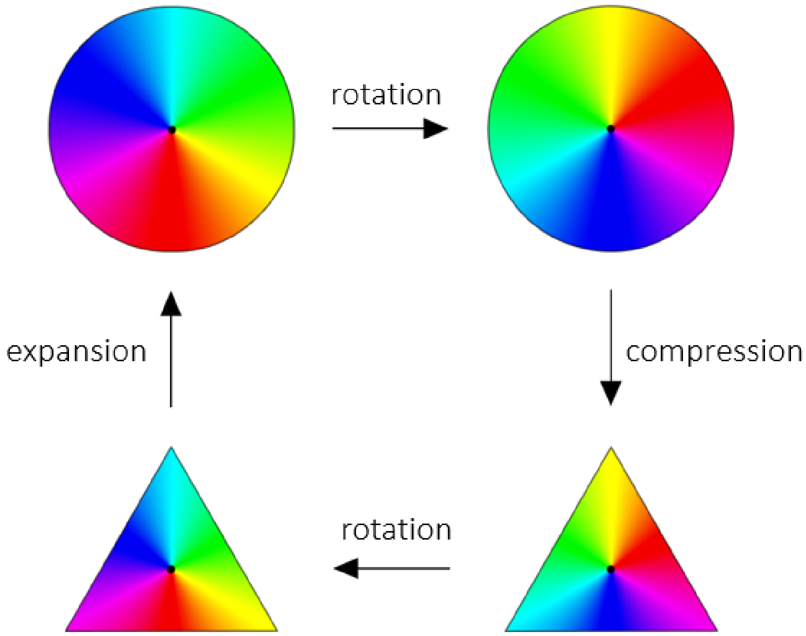


Figure 11: Illustration of the homeomorphic nature between a 2-simplex (triangle) and a disc by means of a continuous rotation (Source: MB20).

3.4 Brouwer's fixed point theorem for compact, convex sets

Proposition 3.5 established a fixed-point theorem for simplices. By proving in the previous section that simplices are in fact homeomorphic to nonempty, compact and convex sets, we have shown that the special case of simplices does not limit the generality of our fixed point result. This leads immediately to the desired version of Brouwer's fixed point theorem.

Theorem 3.11 (Brouwer's fixed point theorem) *[DW18]*
Let $C \subset \mathbb{R}^n$ be a nonempty, compact and convex set and let $f\colon C \to C$ be continuous. Then f has at least one fixed point, i.e. there exists $\xi \in C$ with $f(\xi) = \xi$.

Proof:
We know from proposition 3.8 and corollaries 3.9 and 3.10 that there exists a homeomorphism $h\colon C \to \mathcal{S}$ between the nonempty, compact and convex set C and an N-simplex $\mathcal{S}$. We further know that the composition of homeomorphisms is again a homeomorphism. We define $g := h \circ f \circ h^{-1}$ and see that

$$g := \underbrace{\underbrace{h}_{C\to\mathcal{S}} \circ \underbrace{f}_{C\to C} \circ \underbrace{h^{-1}}_{\mathcal{S}\to C}}_{\mathcal{S}\to\mathcal{S}}, \tag{3.15}$$

i.e. g maps the simplex into itself. By proposition 3.5, there exists a fixed point $x^* \in \mathcal{S}$ with $g(x^*) = x^*$. Since h is a homeomorphism, h^{-1} is also a continuous bijection. Applying it to both sides of (3.15) yields

$$\begin{aligned} g(x^*) = h(f(h^{-1}(x^*))) \quad &\Leftrightarrow \quad h^{-1}(\underbrace{g(x^*)}_{\overset{!}{=}x^*}) = f(h^{-1}(x^*)) \\ &\Leftrightarrow \quad \underbrace{h^{-1}(x^*)}_{=:\xi\in C} = f(\underbrace{h^{-1}(x^*)}_{=:\xi\in C}) \\ &\Leftrightarrow \quad \xi = f(\xi). \end{aligned} \tag{3.16}$$

Therefore, (3.16) shows that $\xi := h^{-1}(x^*) \in C$ is a fixed point of $f\colon C \to C$ which proves the theorem.

□

Theorem 3.11 is the most-often-cited version of Brouwer's fixed point theorem and is also sufficient for many applications.[20] However, from the extensive discussion about homeomorphisms in the last section, it follows immediately that the fixed point theorem is *even more general* and that focusing on compact and convex sets is also just a special case:

Corollary 3.12 (General version of Brouwer's fixed point theorem) *[DW09; MR20]* *Let M be* any *set that is homeomorphic to the closed unit ball $\overline{B}_1(0) \subset \mathbb{R}^n$ (and hence homeomorphic to an N-simplex). Then, any continuous mapping $f\colon M \to M$ has a fixed point, i.e. there exists a $\xi \in M$ with $f(\xi) = \xi$.*

Proof:
We only replace the nonempty, compact and convex set C in the proof of Brouwer's fixed point theorem above (i.e. theorem 3.11) by the more general set M. Other than that, the proof is *identical*.

□

It is important to note that Brouwer's result as stated in theorem 3.11 provides *sufficient conditions* for a fixed point to exist. The conditions are *not necessary* so that a function

[20] In the economic theory of general equilibrium, for instance, the price simplex $P := \{p \in \mathbb{R}^L_+ \mid \sum_{l=1}^{L} p_l = 1\}$ is a nonempty, compact and convex set. A continuous price adjustment function $T\colon P \to P$ therefore has to have a fixed point p^* which can be shown to possess the properties of a competitive equilibrium price vector, i.e. to "clear the market" such that excess demand is zero.

$f\colon X \to X$ can still exhibit fixed points although some of the conditions on f or X are violated.

Example 3.13 (Brouwer's conditions are not necessary)
Consider $X = (0, 3)$ and the function $f\colon X \to X$ with

$$f(x) = \begin{cases} x & \text{if} \quad x \in (0, 2] \\ 2x - 3 & \text{if} \quad x \in (2, 3). \end{cases}$$

Then X is not compact and f is not continuous but f still has an infinite number of fixed points: $x^* \in (0, 2]$.

Nevertheless, the sufficient conditions of Brouwer's fixed point theorem are still "tight" – once a single condition fails, the theorem does not have to hold anymore, i.e. there do not have to exist fixed points. This is illustrated by the following examples.

Example 3.14 (Violations of Brouwer's sufficient conditions)

(a) X ***not closed, hence not compact:***

$$X = (0, 1) \text{ ✗} \qquad\qquad f(x) = x^2 \text{ ✓}$$

The only fixed points of f are $x_1 = 0$ and $x_2 = 1$, but $0 \notin X$ and $1 \notin X$.

(b) X ***not bounded, hence not compact:***

$$X = \mathbb{R} \text{ ✗} \qquad\qquad f(x) = x + 1 \text{ ✓}$$

f is a self-mapping since $X = \mathbb{R}$, but f cannot have a fixed point since $f(x)$ has a constant vertical distance of 1 from the $45°$-line.

(c) X ***not convex:***

$$X = [-2, 1] \cup [1, 2] \text{ ✗} \qquad\qquad f(x) = -x \text{ ✓}$$

The only possible fixed point of f is $x = 0$, but $0 \notin X$.

(d) f ***is not continuous:***

$$X = [0, 3] \text{ ✓} \qquad\qquad f(x) = \begin{cases} x + 1 & \text{if} \quad x \in [0, 2] \\ 3 - x & \text{if} \quad x \in (2, 3] \end{cases} \qquad \text{✗}$$

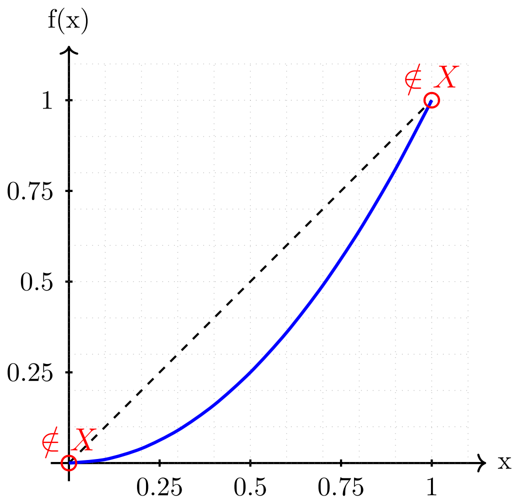
Due to a discontinuity at $x = 2$, $f(x)$ does not cross the 45°-line.

(e) f ***is not a self-mapping:***

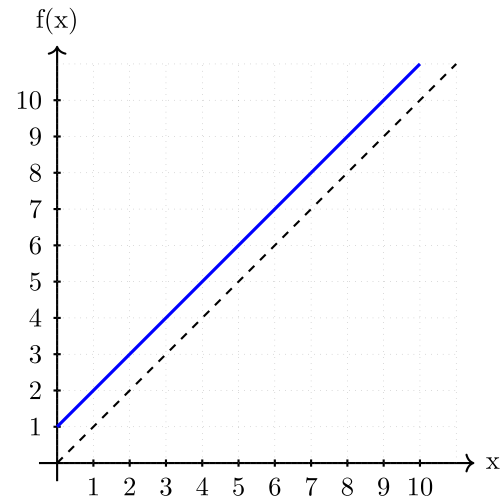
$$X = [0, 2],\ Y = [1, 3]\ \checkmark \qquad f\colon X \to Y \ \textit{ with } \ f(x) = x + 1\ ✗$$

f does not have a fixed point as it is not a self-mapping: $X \neq Y$.

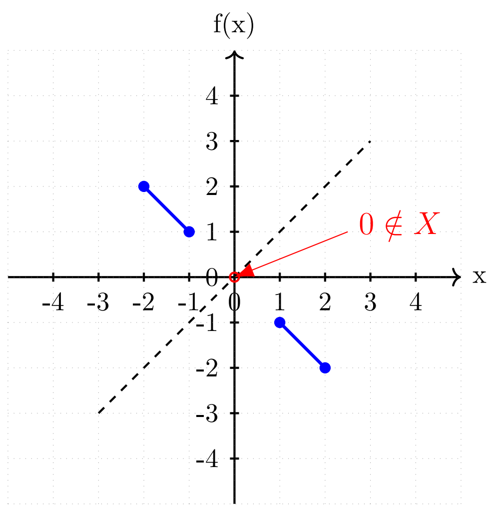
(f) $X = \emptyset$ ***(trivial) or*** X ***is infinite-dimensional (see proposition 4.3).***



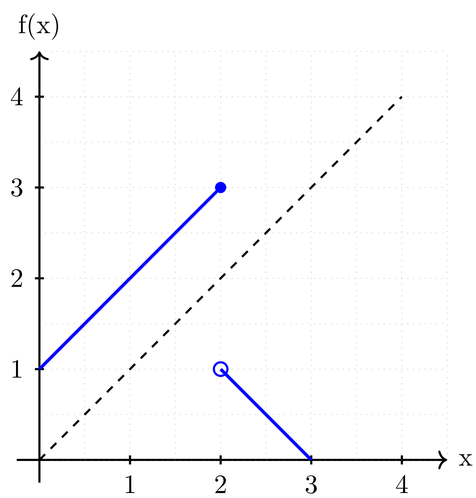
(a) X not closed



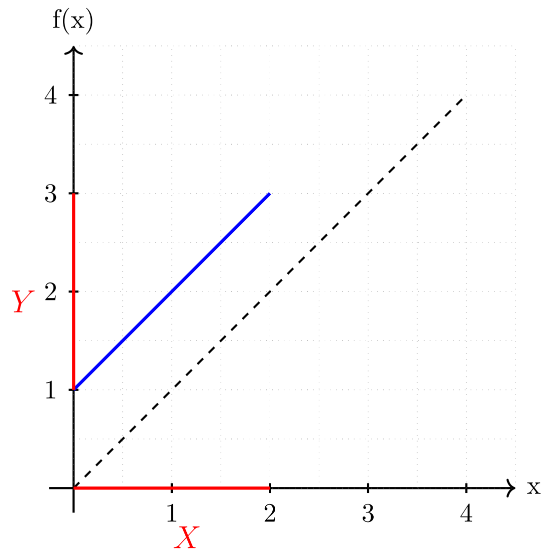
(b) X not bounded



(c) X not convex



(d) f not continuous



(e) f not a self-mapping

$X = \emptyset$ is a trivial example.

Infinite-dimensional X with a continuous function f also does not have to admit fixed points (see proposition 4.3). This observation will lead to **Schauder's fixed point theorem**.

(f) $X = \emptyset$ or X infinite-dimensional

Figure 12: Violations of the sufficient conditions of Brouwer's fixed point theorem.

While example 3.14 illustrates that none of the sufficient conditions of Brouwer's fixed point theorem can be dropped without invalidating the theorem, example 3.13 illustrates another aspect worth highlighting: The theorem does *not* make a statement about the *number* of fixed points. In particular, the theorem does not guarantee uniqueness. This is illustrated in the following example which shows that once all sufficient conditions of Brouwer's fixed point theorem are fulfilled, there could be one, several or even an infinite number of fixed points.

Example 3.15 (Brouwer's FPT: Number of fixed points)

(a) Unique fixed point: $X = \{(x,y)^T \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\},\ f(x)\colon (x,y)^T \mapsto (-y,x)^T$.[21]

f describes the continuous transformation which rotates every vector from X by $90°$ (counterclockwise) around the origin. The unique fixed point of f is $(x,y)^T = (0,0)^T$.

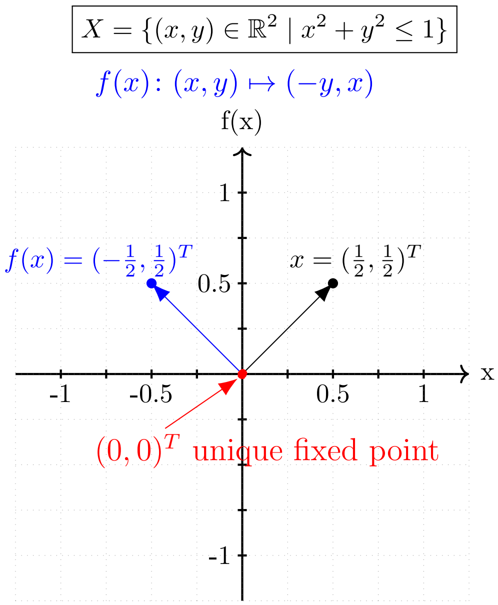
(b) $n \in \mathbb{N}$ ***(here:*** $n = 2$***) fixed points:*** $X = [0,1],\ f(x) = x^2$.

f has two fixed points on X: $x_1 = 0$ and $x_2 = 1$.

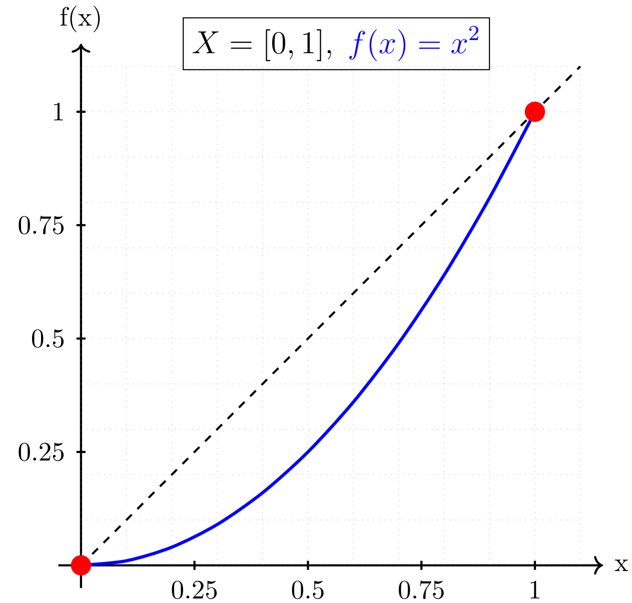
(c) An infinite number of fixed points: $X = [0,1],\ f(x) = x$.

All $x \in X$ are fixed points of f since f coincides with the $45°$-line on X.

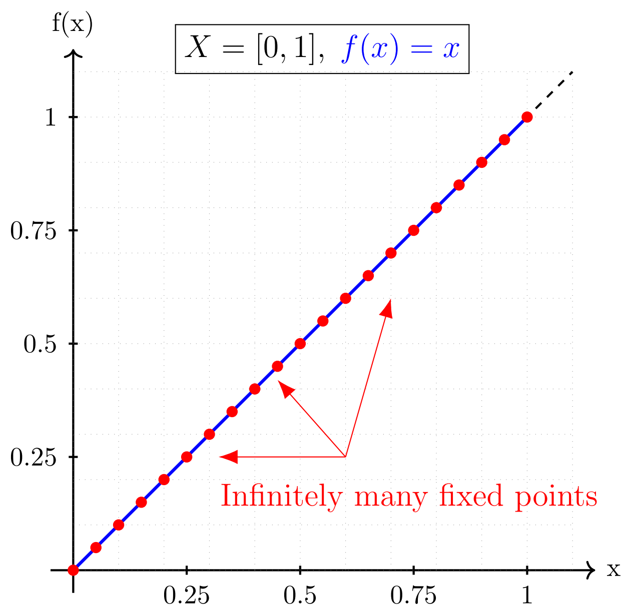
[21]Source of this example: Hammond et al. (2008).



(a) Unique fixed point



(b) $n \in \mathbb{N}$ (here $n = 2$) fixed points



(c) Infinitely many fixed points

Figure 13: Examples of Brouwer's fixed point theorem and the number of fixed points.

Although Brouwer's fixed point theorem is very useful in applications and in proofs of other (fixed point) theorems, it should again be highlighted that it is a mere *existence result*. It does neither establish uniqueness nor does is provide a constructive way of how to obtain a fixed point. The following two observations provide further room for generalizations:

1. Brouwer's fixed point theorem in the above formulation inherently relies on the *finite-dimensionality* of the respective space (we considered subsets $C \subset \mathbb{R}^n, n < \infty$).

2. Brouwer's fixed point theorem is a result for *functions* $f : C \to C$. In many applications, however, more general mappings are required. A prominent generalization of the theorem considers so-called *correspondences* or *set-valued mappings*.

Both aspects should be addressed in the subsequent chapters of this thesis. While the first point will lead us to infinite-dimensional spaces and *Schauder's fixed point theorem*, the second results in *Kakutani's fixed point theorem* for set-valued mappings. The following figure provides an idea of the remaining course of this thesis.

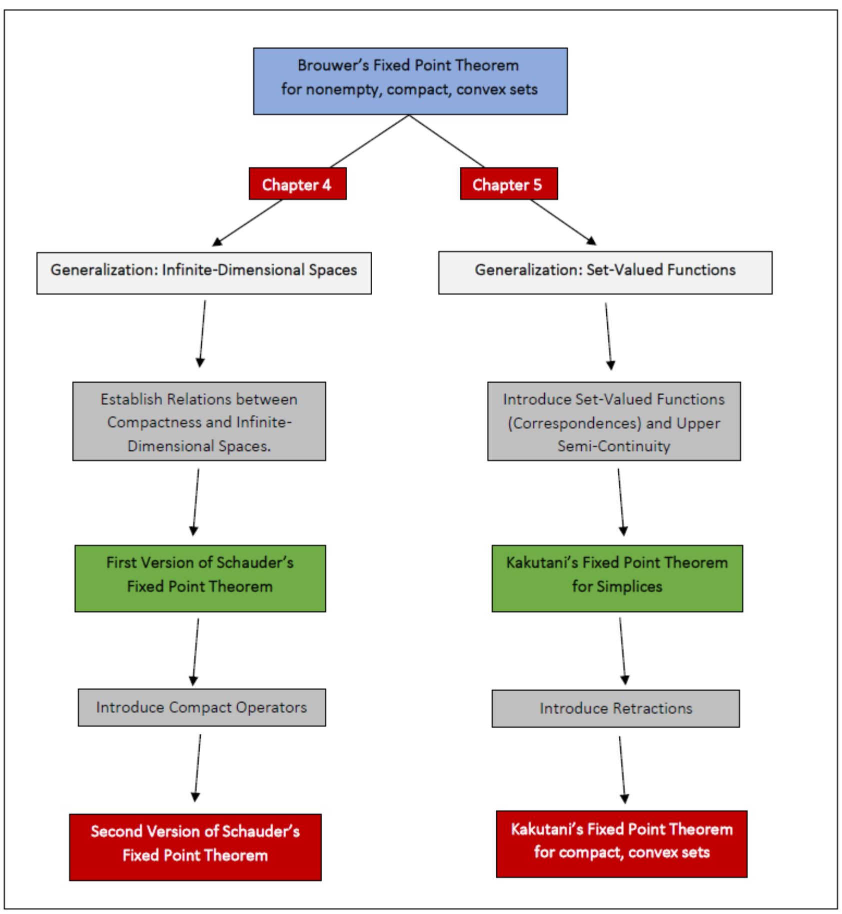


Figure 14: Illustration of the main steps to generalize Brouwer's fixed point theorem.

4 Infinite-Dimensional Spaces and Schauder's Fixed Point Theorem

Although we have extended Brouwer's fixed point theorem to fairly general subsets of $\mathbb{R}^n$, our results so far are still restricted to *finite-dimensional* spaces. In the preceding proofs, we have made use of simplices which are convex hulls of *finitely* many vectors.[22] What is more, we have formulated Brouwer's fixed point theorem for *compact* subsets $M \subset \mathbb{R}^n$ – in fact, our argumentation only required M to be closed and bounded. While this is equivalent to compactness for subsets in $\mathbb{R}^n$ by the theorem of Heine-Borel, analogous reasoning does *no longer* hold in infinite dimensions. This chapter therefore sheds some light on generalizing Brouwer's fixed point theorem to infinite-dimensional vector spaces resulting in two versions of the *Schauder fixed point theorem*.[23] In order to do so, the following subsection provides the necessary machinery.[24]

4.1 Infinite dimensions and compactness

We first revise the concept of *compactness* which we define using finite subcovers.

Definition 4.1 (Open covers and compactness) *[HH04]*
*Let $M \subset X$ be a nonempty subset of some normed space X. We call the collection $\mathcal{F}$ of open subsets of X an **open cover** of M if $M \subseteq \bigcup_{Q \in \mathcal{F}} Q := \{x \in X \mid \exists\, Q \in \mathcal{F} :\ x \in Q\}$, i.e. every $x \in M$ is contained in at least one $Q \in \mathcal{F}$. The set M is **compact** if every open cover of M contains a finite subcover.*

One could equivalently define compactness of a set M by means of *sequential compactness*: Every sequence in M contains a convergent subsequences with limit in M. In all of our observations so far (e.g. when considering simplices or closed unit balls), we did not have to make use of these definitions. Instead, we have implicitly used the *theorem of Heine-Borel* stating that a subset $M \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded (compare e.g. Heusser, 2004). For infinite-dimensional spaces, this result does no longer

[22] The analytical proof of Brouwer's fixed point theorem (illustrated in appendix A) is also inherently dependent on the finite dimensionality of the space. One reason is that it uses the approximation theorem of Weierstrass (which makes a statement for a compact set $K \subset \mathbb{R}^n$) to approximate "smooth" C^2-functions by polynomials.

[23] Schauder's fixed point theorem constitutes the center of attention of this section which is why some auxiliary results will only be stated without detailed proofs.

[24] Although many of the presented definitions apply to more general settings (such as topological spaces), we will mostly limit attention to normed vector spaces or even Banach spaces since this suffices for the considerations of this thesis.

hold true. We exemplify this observation and its impact on fixed point theorems by means of the following two results, the first of which we state without proof.

Proposition 4.2 (Riesz' theorem on closed unit balls) *[DW09; DW18]*
Let X be a normed space. Then the following statements are equivalent:

(i) dim$(X) < \infty$.

(ii) The closed unit ball $\overline{B}_1(0) := \{x \in X \mid \|x\| \leq 1\}$ is compact.

This observation leads to the following negative result which shows that Brouwer's fixed point theorem for closed unit balls cannot be generalized without further ado to infinite dimensions.

Proposition 4.3 (Kakutani's negative result) *[MR20; SK43]*
Let H be an infinite-dimensional, separable Hilbert space.[25] *Then there exists a continuous mapping $A\colon H \to H$ which maps the closed unit ball into itself but which does* ***not*** *contain a fixed point.*

Proof: By constructing an explicit mapping $A\colon \overline{B}_1(0) \to \overline{B}_1(0)$ which has no fixed point.

Step 1:
Define a mapping $U\colon H \to H$ by means of an orthonormal basis of H: We denote by $B := \overline{B}_1(0)$ the closed unit ball in H and let $(y_z)_{z\in\mathbb{Z}}$ be a complete orthonormal system (i.e. an *orthonormal basis*) of H.[26] For all $m, n \in \mathbb{Z}$, it hence holds that all basis vectors have length 1 and are mutually orthogonal, i.e.

$$\langle y_n, y_m\rangle = \delta_{nm} := \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m, \end{cases}$$

where δ_{mn} is the Kronecker delta. Next, we define the mapping $U\colon H \to H$ by

$$U(y_z) = y_{z+1} \quad \forall z \in \mathbb{Z}.$$

[25] A *Hilbert space* is a vector space with a scalar product $\langle\cdot,\cdot\rangle$ which is complete with respect to the norm induced by the scalar product. A metric space is called *separable* if it contains a countable, dense subset.

[26] Although the proof is omitted here, one can show that every Hilbert space admits an orthonormal basis. Since H is an *infinite-dimensional* space here, $(y_z)_{z\in\mathbb{Z}}$ is *not* a Hamel basis, so elements of H cannot necessarily be represented as a *finite* – but rather as a countably infinite – linear combination of the basis vectors $(y_z)_{z\in\mathbb{Z}}$ (i.e. as an *unconditionally convergent series*).

It is sufficient to prescribe how U transforms the orthonormal basis vectors (here: U maps any basis vector to the "subsequent" basis vector). This is because any $x \in H$ can be represented by means of the basis vectors and a Fourier expansion as[27]

$$x = \sum_{i=-\infty}^{\infty} \underbrace{\langle x, y_i \rangle}_{=:\alpha_i} y_i = \sum_{i=-\infty}^{\infty} \alpha_i y_i \qquad \text{with} \qquad \sum_{i=-\infty}^{\infty} |\alpha_i|^2 < \infty. \tag{4.1}$$

Hence U can be extended via (4.1) to any $x \in H$ by

$$U(x) = \sum_{i=-\infty}^{\infty} \alpha_i y_{i+1}. \tag{4.2}$$

As U is linear and bounded, it is a *continuous* operator.

Step 2:
Show that U is a unitary transformation of H onto itself: We next show that $U\colon H \to H$ preserves the inner product (and hence the (induced) norm $\|\cdot\|_H$). To do so, we define the set $S_r := \{x \in H \mid \|x\|_H = r\}$ for $0 < r < \infty$ and show that $U(x) \in S_r$ for all $x \in S_r$ (in other words, U does not change the length of its input). But this can be seen immediately from (4.2) as U is linear, the $\alpha_i, i \in \mathbb{Z}$, stay unaffected and all $y_z, z \in \mathbb{Z}$, have unit length.

Step 3:
Use U to define a continuous mapping $\varphi\colon B \to B$ which will fulfill the desired properties: We define

$$\varphi(x) := \frac{1}{2}(1 - \|x\|_H) y_0 + U(x). \tag{4.3}$$

φ is continuous as the sum of continuous functions. Further, for all $x \in H$ with $\|x\|_H \leq 1$ (i.e. for all $x \in B$), we have

$$\begin{aligned}
\|\varphi(x)\|_H &\leq \frac{1}{2}(1 - \|x\|_H) \underbrace{\|y_0\|_H}_{=1} + \underbrace{\|U(x)\|_H}_{=\|x\|_H} \\
&\leq \frac{1}{2}(1 - \|x\|_H) + \|x\|_H \\
&= \frac{1}{2} + \frac{1}{2} \underbrace{\|x\|_H}_{\leq 1} \leq 1.
\end{aligned}$$

[27] We note that since H is a Hilbert space and $(y_z)_{z\in\mathbb{Z}}$ is an orthonormal basis, *Parseval's identity* holds for all $x \in H$, i.e. $\|x\|^2 = \langle x, x \rangle = \sum_{i=-\infty}^{\infty} |\langle x, y_i \rangle|^2 = \sum_{i=-\infty}^{\infty} |\alpha_i|^2$.

Hence $\varphi\colon B \to B$ maps the closed unit ball onto itself.

Step 4:

Show that φ cannot have a fixed point: By contradiction, assume that there does exist a fixed point, i.e. $x_0 \in B$ with $\varphi(x_0) = x_0$. Using the definition (4.3) of φ, we can express this as

$$\begin{aligned} \varphi(x_0) = x_0 \quad &\Leftrightarrow \quad \frac{1}{2}(1 - \|x\|_H)y_0 + U(x_0) = x_0 \\ &\Leftrightarrow \quad x_0 - U(x_0) = \frac{1}{2}(1 - \|x_0\|_H)y_0 \end{aligned} \tag{4.4}$$

In other words, if φ has a fixed point, then (4.4) must hold. We next show that there *cannot* exist a fixed point by systematically ruling out all possible cases.

Case 1: Center of the closed unit ball B:

Let $x_0 = 0$. By (4.4), we have

$$0 - U(0) = \frac{1}{2}(1-0)y_0 \quad \Leftrightarrow \quad 0 = \frac{1}{2}y_0.$$

This is a *contradiction* since y_0 is a basis vector. Hence, the center of B (i.e. the origin of H) cannot be a fixed point of φ.

Case 2: Boundary of the closed unit ball B:

Let x_0 s.t. $\|x_0\|_H = 1$. Using (4.4) again yields

$$x_0 - U(x_0) = \underbrace{\frac{1}{2}(1 - \underbrace{\|x_0\|_H}_{=1})y_0}_{=0} \; = \; 0 \quad \Leftrightarrow \quad x_0 = U(x_0). \tag{4.5}$$

We next represent x_0 (using Parseval's identity and the orthonormal basis) as

$$x_0 = \sum_{i\in\mathbb{Z}} \alpha_i y_i \quad \text{with} \quad \|x_0\|_H = \sum_{i\in\mathbb{Z}} |\alpha_i|^2 = 1. \tag{4.6}$$

Using (4.2), i.e. $U(x) = \sum_{i=-\infty}^{\infty} \alpha_i y_{i+1}$, as well as (4.5) and (4.6), we get

$$x_0 = U(x_0) \quad \Leftrightarrow \quad \sum_{i=-\infty}^{\infty} \alpha_i y_i \; = \; \sum_{i=-\infty}^{\infty} \alpha_i y_{i+1}.$$

We next consider any basis vector $y_j, j \in \mathbb{Z}$, and take the scalar product with y_j on both sides. Since $\langle y_i, y_j\rangle = \delta_{ij}$, this yields $\alpha_j = \alpha_{j-1}$ for all $j \in \mathbb{Z}$ (since $y_j \in \mathbb{Z}$ was arbitrary).

Therefore, all a_j must be identical constants. But then $\|x\|_H = \sum_{i\in\mathbb{Z}} |a_i|^2 = \infty \neq 1$ which contradicts $x_0 \in B$. Hence, points on the boundary of B cannot be fixed points of φ.

Case 3: Interior of the closed unit ball B:
Let x_0 s.t. $0 < \|x_0\|_H < 1$. We then have

$$x_0 = \sum_{i\in\mathbb{Z}} \alpha_i y_i \quad \text{with} \quad \|x_0\|_H = \sum_{i\in\mathbb{Z}} |\alpha_i|^2 < 1$$

and using this representation in equation (4.4), i.e. in the necessary condition for a fixed point, we get

$$\begin{aligned}
& x_0 - U(x_0) = \frac{1}{2}(1 - \|x_0\|_H) y_0 \\
\Leftrightarrow \quad & \sum_{i\in\mathbb{Z}} \alpha_i y_i - \sum_{i\in\mathbb{Z}} \alpha_i y_{i+1} = \frac{1}{2}(1 - \|x_0\|_H) y_0 \\
\Leftrightarrow \quad & \sum_{i\in\mathbb{Z}} \alpha_i y_i - \sum_{i\in\mathbb{Z}} \alpha_{i-1} y_i = \frac{1}{2}(1 - \|x_0\|_H) y_0 \\
\Leftrightarrow \quad & \sum_{i\in\mathbb{Z}} (\alpha_i - \alpha_{i-1}) y_i = \frac{1}{2}(1 - \|x_0\|_H) y_0 \\
\Leftrightarrow \quad & (\alpha_0 - \alpha_{-1}) y_0 + \sum_{\substack{i\in\mathbb{Z} \\ i\neq 0}} (\alpha_i - \alpha_{i-1}) y_i = \frac{1}{2}(1 - \|x_0\|_H) y_0.
\end{aligned}$$

Taking the scalar product with y_0 and with $y_j, j \in \mathbb{Z} \setminus \{0\}$, on both sides results in the two equations:

$$\alpha_0 - \alpha_{-1} = \frac{1}{2}(1 - \underbrace{\|x_0\|_H}_{<1}) > 0 \quad \Leftrightarrow \quad \alpha_0 > \alpha_{-1} \qquad \text{for } i = 0 \tag{4.7}$$

$$(\alpha_i - \alpha_{i-1}) = 0 \quad \Leftrightarrow \quad \alpha_i = \alpha_{i-1} \qquad \forall i \in \mathbb{Z} \setminus \{0\}. \tag{4.8}$$

These equations yield

$$\underbrace{\ldots = \alpha_{-3} = \alpha_{-2} = \alpha_{-1}}_{(4.8)} \underbrace{<}_{(4.7)} \underbrace{\alpha_0 = \alpha_1 = \alpha_2 = \alpha_3 = \ldots}_{(4.8)}, \tag{4.9}$$

which immediately implies $\|x_0\|_H^2 = \sum_{i\in\mathbb{Z}} |\alpha_i|^2 = \infty > 1$ which contradicts $x_0 \in B$. Hence, points in the interior of B cannot be fixed points of φ.

Since cases 1-3 exhaust all possibilities of potential fixed points $x_0 \in B$, we have lead the fixed-point assumption to a contradiction. Hence, $\varphi\colon B \to B$ is a continuous self-mapping of the closed unit ball into itself but which does *not* have a fixed point. This proves the result.

□

These results show that continuous mappings in infinite-dimensional Banach spaces need not necessarily admit fixed points. In order to establish results comparable to Brouwer's fixed point theorem, we therefore need some further assumptions on the functions involved. This will lead to so-called *compact operators*. Those operators can be approximated (arbitrarily well) by "finite-dimensional operators" and we can try to apply Brouwer's fixed point theorems to these operators as a consequence. Ruzicka (2020) directly introduces such compact operators to establish both versions of *Schauder's fixed point theorem*. As the first version strictly speaking does not yet require the notion of compact operators[28], I want to pursue a slightly different path and mainly follow Heusser (2004) and Werner (2018) at first. This will allow us to prove the first version of Schauder's fixed point theorem by only using compactness arguments of the set. I will thereafter introduce compact operators to prove the second (more often used) version of Schauder's fixed point theorem. For these considerations, we need one final introductory concept: *Relative compactness.*

Definition 4.4 (Relative compactness) *[HH04; MR20]*
*Let $M \subset X$ be a nonempty subset of some normed space X. M is called **relatively compact** if the closure $\overline{M}$ is compact. Equivalently, M is called relatively compact if every sequence in M contains a convergent subsequence (whose limit point does **not** necessarily have to be in M).*

4.2 Schauder's fixed point theorem

The first version of Schauder's fixed point theorem for *compact* and convex subsets is an immediate generalization of Brouwer's fixed point result (theorem 3.11). However, we again stress that compactness as introduced in definition 4.1 is used (instead of mere closedness + boundedness). Before we state the theorem, we collect two auxiliary lemmas.[29]

[28] The first version of the theorem uses continuous operators on compact sets. Such operators are necessarily compact as well, so we *do* use compact operators *implicitly*. However, we do not yet have to use the notion or concrete properties of compact operators in the first version explicitly.

[29] See for instance Heusser (2004) for detailed proofs.

Lemma 4.5 (Compactness and convex hulls) *[HH04]*
The convex hull $K := co(\{x^1, ..., x^m\})$ of finitely many elements of a normed space is compact.

Proof Idea:
Use the concept of sequential compactness, consider an arbitrary sequence (y_n) from $K = co(\{x^1, ..., x^m\})$ and show that this sequence contains a convergent subsequent with limit point $y_0 \in K$ using the theorem of Bolzano-Weierstrass. □

The next lemma provides a fixed point result for subsets of linear spans of finitely many vectors. We define the linear span of the elements $x^1, ..., x^m$ of some vector space X as to be the set

$$span(\{x^1, ..., x^m\}) := \Big\{ \sum_{i=1}^{m} \lambda_i x^i \;\Big|\; m \in \mathbb{N},\; \lambda_i \in \mathbb{R} \text{ for } i = 1, ..., m \Big\}. \tag{4.10}$$

Lemma 4.6 (Fixed point result for subsets of a linear span) *[HH04]*
Let $x^1, ..., x^m$ be elements of a normed space X and let C be a nonempty, compact and convex subset of $span(\{x^1, ..., x^m\})$. Then, every continuous function $f : C \to C$ has at least one fixed point.

Proof idea:
Follows from the previous lemma and an application of Brouwer's fixed point theorem for nonempty, compact and convex sets. □

We can now state and prove the first version of Schauder's fixed point theorem for infinite-dimensional spaces, compact and convex subsets and continuous maps.

Theorem 4.7 (Schauder's fixed point theorem - 1st version) *[HH04; DW18]*
Let C be a nonempty, compact and convex subset of a (infinite-dimensional) normed space X and let $f : C \to C$ be continuous. Then f has at least one fixed point, i.e. $\exists\, x_0 \in C$ with $f(x_0) = x_0$.

Proof: Showing existence of a fixed point via Brouwer's fixed point theorem.
Step 1:
Use compactness of C to form an ε-net: Let $\varepsilon > 0$ be arbitrary. We surround every point $x \in C$ with an open ε-neighborhood $U_\varepsilon(x)$ with radius ε. By the assumed compactness of C, there exists a finite subset $M := \{x^1, ..., x^m\} \subset C$ such that the ε-neighborhoods

$U_\varepsilon(x^1), ..., U_\varepsilon(x^m)$ of the points of M already cover C. More precisely:

$$\forall x \in C \;\; \exists\, x_j \in M \quad \text{with} \quad \|x - x_j\| < \varepsilon. \tag{4.11}$$

We could equivalently express this idea by stating that the set C can be covered by a "*finite ε-net*":

$$C \;\subset\; \bigcup_{j=1}^{m} U_\varepsilon(x_j)$$

<u>*Step 2:*</u>
Define functions ϕ_j to create a convex combination: For all $j \in \{1, ..., m\}$ we next define the functions $\phi_j \colon C \to \mathbb{R}$ by

$$\phi_j(x) := \begin{cases} 0, & \text{if} \quad \|x - x_j\| \geq \varepsilon \\ \varepsilon - \|x - x_j\| & \text{if} \quad \|x - x_j\| < \varepsilon. \end{cases} \tag{4.12}$$

It follows immediately that all ϕ_j are nonnegative and continuous. Furthermore, (4.11) implies that the sum $\sum_{j=1}^m \phi_j$ is positive. Therefore, the following expressions are well-defined. Let

$$\lambda_j(x) := \frac{\phi_j(x)}{\sum_{j=1}^m \phi_j} \qquad \forall x \in C \quad \text{and} \quad \forall j \in \{1, ..., m\} \tag{4.13}$$

and thereby obtain for all $x \in C$ and $j \in \{1, ..., m\}$:

$$\lambda_j(x) \geq 0 \qquad \text{and} \qquad \sum_{j=1}^{m} \lambda_j(x) = 1.$$

As a consequence,

$$g(x) := \sum_{j=1}^{m} \lambda_j(x)\, x_j \tag{4.14}$$

defines a continuous function $g \colon C \to K_0$ where $K_0 := co(\{x^1, ..., x^m\})$ is the convex hull of $x^1, ..., x^m$.

Step 3:

Show that $\|g(x) - x\| < \varepsilon$: We next consider an arbitrary $x \in C$ and have:

$$g(x) - x \;=\; \sum_{j=1}^{m} \lambda_j(x)\, x_j \;-x \;=\; \sum_{j=1}^{m} \lambda_j(x)\, x_j \;-\; \underbrace{\sum_{j=1}^{m} \lambda_j(x)}_{=1}\; x \;=\; \sum_{j=1}^{m} \lambda_j(x)[x_j - x]$$

and thereby

$$\|g(x) - x\| \;=\; \left\| \sum_{j=1}^{m} \lambda_j(x)[x_j - x] \right\| \;\leq\; \sum_{j=1}^{m} \lambda_j(x) \|x_j - x\|. \tag{4.15}$$

From the definition of $\lambda_j(x)$ in (4.13) which uses the functions ϕ_j defined in (4.12), we know that $\lambda_j(x) = 0$ whenever $\|x_j - x\| \geq \epsilon$. We hence only have to sum over all j with $\|x_j - x\| < \epsilon$ in (4.15) and obtain:

$$\sum_{j=1}^{m} \lambda_j(x) \|x_j - x\| \;<\; \varepsilon \underbrace{\sum_{j=1}^{m} \lambda_j(x)}_{=1} \;= \varepsilon. \tag{4.16}$$

In total, we have established that $\|g(x) - x\| < \varepsilon$ for an arbitrary $x \in C$ and hence $\forall x \in C$.

Step 4:

Define the composition $h := g \circ f$ *and use the preparatory lemmas*: We define the function $h := g \circ f \colon C \to K_0 := co(\{x^1, ..., x^m\}) \subset C$. Since g and f are both continuous, so is the composition h. Considering the restriction $\tilde{h}$ of h on the convex hull K_0, it follows that $\tilde{h} := h|_{K_0}$ is a continuous self-map from the convex hull K_0 into itself. K_0 is nonempty, convex and compact ($\rightarrow$ Lemma 4.5) and it is a subset of the normed space $span(\{x^1, ..., x^m\})$. By Lemma 4.6 (and hence by Brouwer's fixed point theorem), $\tilde{h}$ must have a fixed point, i.e.

$$\exists\, z \in K_0 \quad \text{with} \quad \tilde{h}(z) = z \quad \Leftrightarrow \quad g(f(z)) = z. \tag{4.17}$$

Next, we note that $f \colon C \to C$, so $f(z) \in C$. Combining this fact with $\|g(x) - x\| < \varepsilon\ \forall x \in C$ as established above, we get

$$\|f(z) - z\| \underbrace{=}_{(4.17)} \|f(z) - g(f(z))\| \underbrace{<}_{f(z) \in C} \varepsilon. \tag{4.18}$$

Step 5:
Establish existence of a fixed point $x_0 \in C$ of f: So far, we have established the following:

$$\forall\, \varepsilon > 0 \;\; \exists\, z = z(\varepsilon) \in C \quad \text{with} \quad \|f(z) - z\| < \varepsilon. \tag{4.19}$$

Since this is true for *any* $\varepsilon > 0$, we can always find a $z_n \in C$ for any $n \in \mathbb{N}$ such that

$$\|f(z_n) - z_n\| < \frac{1}{n}. \tag{4.20}$$

As C is compact and $f(z_n) \in C$, we can then find a convergent subsequence $(z_{n_k})_{k\in\mathbb{N}}$ and a limit point $x_0 \in C$ with

$$\lim_{k\to\infty} f(z_{n_k}) = x_0 \in C. \tag{4.21}$$

By (4.20) we must also have $\lim_{k\to\infty} z_{n_k} = x_0$. Using this fact together with the continuity of f implies

$$\lim_{k\to\infty} f(z_{n_k}) \underbrace{=}_{\text{cont.}} f(\lim_{k\to\infty} z_{n_k}) \;=\; f(x_0). \tag{4.22}$$

By combining (4.21) and (4.22), we have finally established that $x_0 = f(x_0)$. In other words, $x_0 \in C$ is a fixed point of f which proves the theorem.

□

Compactness of the set C has been a crucial assumption in the above version of Schauder's fixed point theorem. We have emphasized that infinite-dimensional spaces require a clear differentiation between compactness and closedness + boundedness. While the former always implies the latter, the converse is not true. Therefore, Schauder's fixed point theorem as presented above does **not** necessarily hold for closed and bounded (instead of compact) sets. The following example taken from Werner (2018) illustrates this point.

Example 4.8 (Schauder and closed & bounded sets) *[DW18]*
Let C be the closed and bounded unit ball in $(l^2, \|\cdot\|)$, i.e. the Banach space of all bounded sequences $(x_n)_{n\in\mathbb{N}}$ with respect to the l^2-norm: $\|(x_n)_{n\in\mathbb{N}}\|_{l^2} := \left(\sum_{n=1}^{\infty} |x_n|^2\right)^{\frac{1}{2}}$. By the theorem of Riesz (see proposition 4.2), we know that this closed unit ball is ***not*** *compact.*

Next, consider the continuous mapping $F\colon C \to C$ with

$$F\colon x = (x_n)_{n\in\mathbb{N}} \mapsto \Big(\sqrt{1 - \|x\|_{l^2}^2}, x_1, x_2, ...\Big). \tag{4.23}$$

By definition of the l^2-norm and the fact that $x = (x_n)_{n\in\mathbb{N}} \in C$, we always have

$$\begin{aligned}\|F(x)\|_{l^2}^2 &= 1 - \|x\|_{l^2}^2 + |x_1|^2 + |x_2|^2 + ... \\ &= 1 - \sum_{n=1}^{\infty} |x_n|^2 + |x_1|^2 + |x_2|^2 + ... = 1\end{aligned}$$

*so F indeed maps C into itself. If F **had** a fixed point $\xi = (\xi_n)_{n\in\mathbb{N}}$, we would necessarily require all elements of ξ to coincide: $\xi_1 = \xi_2 = ...$, i.e. ξ would have to be a constant sequence (follows immediately from the definition of F: F "shifts" all sequence elements to the right, so in order to be a fixed point, the first element must be equal to the second which must be equal to the third, etc.). But the only constant sequence in l^2 is the sequence consisting of zeros: $\mathcal{O} := (y_n)_{n\in N}$ with $y_n = 0\ \forall n \in \mathbb{N}$, but $F(\mathcal{O}) \neq \mathcal{O}$. Hence, the mapping F does not have a fixed point.* □

It is possible to establish a version of Schauder's fixed point theorem for closed and bounded (instead of compact) sets. To do so, however, additional assumption have to be imposed on the operators.

Definition 4.9 (Compact operators and finite rank) *[MR20; DW18]*
Let X and Y be (infinite-dimensional) normed vector spaces and let $M \subset X$ be some subset. We consider the continuous operator $T\colon M \subset X \to Y$ and define the following concepts:

*(i) The operator T is called **compact** if and only if T maps bounded subsets $B \subset M$ into relatively compact sets, i.e. $\overline{T(B)}$ is compact.*

*(ii) The operator T is said to be of **finite rank** if and only if the range $\mathcal{R}(T)$ of T is contained in a finite-dimensional subspace of Y.*

The idea of the following version of Schauder's fixed point theorem is to approximate the respective mappings by means of so-called *Schauder-operators* and to apply Brouwer's fixed point theorems to these "finite-dimensional operators" in order to establish the existence of a fixed point. *Compactness* of the considered mappings is crucial in order for this approach to be feasible. The following proposition formalizes this idea.[30]

[30]Since I want to mainly focus on Schauder's fixed point theorem itself, I skip the details of the proof of this result. An in-depth exposition can be found in Ruzicka (2020), p.23ff.

Proposition 4.10 (Approximation of compact operators) *[MR20]*
Let X and Y be (infinite-dimensional) Banach spaces and let $M \subset X$ be a nonempty and bounded subset of X. Further let $T\colon M \subset X \to Y$ be an operator. Then the following two statements are equivalent:

(i) T is a compact operator.

(ii) There exist compact ***Schauder-operators*** *$P_n\colon M \subset X \to Y$, $n \in \mathbb{N}$, of finite rank which can uniformly approximate the operator T on the set M. In other words, for all $x \in M$ and for all $n \in \mathbb{N}$:*

$$\|P_n(x) - T(x)\| < \frac{1}{n}. \tag{4.24}$$

Proof idea:
$(ii) \Rightarrow (i)$: First show continuity of T by exploiting that P_n is continuous for a fixed (but arbitrary) $n \in \mathbb{N}$. To establish the relative compactness of $T(M)$ (and hence to get compactness of T), show the existence of an $\frac{\varepsilon}{3}$-net of $T(M)$.

$(i) \Rightarrow (ii)$: Note that relative compactness of $T(M)$ implies the existence of an $\frac{\varepsilon}{n}$-net:

$$\exists x_i \in M, i \in \{1, ..., N\}, \quad \text{with} \quad T(M) \subset \bigcup_{i=1}^{N} B_{\frac{1}{n}}(x_i).$$

Next, define $y_i := Tx_i$ for all $i \in \{1, ..., N\}$ and construct a partition of unity with respect to the cover $B_{\frac{1}{n}}(Tx_i)$. Thereafter, consider the convex hull $M_n := co(\{y_1, ..., y_N\})$ and explicitly define the Schauder-operators $P_n\colon M \to M_n$. Finally, show that the Schauder-operators fulfill all required properties (finite rank, compactness, uniform approximation of T on M). □

Proposition (4.10) can also be used to establish a slightly different proof of the first version of Schauder's fixed point theorem from the one presented above (see for instance Ruzicka, 2020). The main purpose of introducing the proposition here, however, was to emphasize how and why stronger assumptions (namely the compactness of operators) are needed to establish Brouwer's fixed point theorem for infinite-dimensional spaces. Before we state and prove an alternative version of Schauder's fixed point theorem, we provide one additional lemma without proof which is due to Mazur.

Lemma 4.11 (Lemma of Mazur) *[MR20]*
Let X be a (infinite-dimensional) Banach space and let $M \subset X$ be a relatively compact subset of X. Then, the convex hull $co(M)$ of M is also relatively compact.

Proof idea:
Use the relative compactness of M to establish the existence of a finite $\frac{\varepsilon}{2}$-net. Then show that $co(M)$ does also have a finite ε-net which implies that $co(M)$ is relatively compact.

□

We are now in a position to state and prove an alternative version of Schauder's fixed point theorem. As discussed, it requires compactness of operators but is otherwise in the spirit of a natural generalization of Brouwer's fixed point theorem to infinite-dimensional Banach spaces.

Theorem 4.12 (Schauder's fixed point theorem - 2$^{\text{nd}}$ version) *[MR20; DW18]*
Let X be a (infinite-dimensional) Banach space and let $M \subset X$ be a nonempty, closed, bounded and convex subset of X. Further let $T: M \subset X \to M$ be a compact operator. Then, T has a fixed point, i.e. there exists $x \in M$ with $Tx = x$.

Proof:
We fist let $N := \overline{co(T(M))} \subset M$. Since T is a compact operator, $T(M)$ is relatively compact by definition. The lemma of Mazur (Lemma 4.11) implies that $co(T(M))$ is relatively compact, so N (i.e. its closure) is compact, convex and nonempty. Since T is a compact operator, it is in particular continuous. Finally, T maps the set N into itself. This follows from the following chain of arguments:

$$N \subset M \quad \Rightarrow \quad T(N) \subset T(M) \quad \Rightarrow \quad T(N) \subset \overline{co(T(N))} \subset \overline{co(T(M))} =: N, \tag{4.25}$$

so in total $T(N) \subset N$. Therefore, T is a continuous self-map on a nonempty, compact and convex subset $M \subset X$. By the first version of Schauder's fixed point theorem (theorem 4.7), T has at least one fixed point. This proves the result.

□

5 Set-Valued Functions and Kakutani's Fixed Point Theorem

This section looks at a generalization of Brouwer's fixed point theorem from a different angle. Rather than formulating Brouwer's ideas for more general sets or spaces, we follow Kakutani (1941) and focus on *correspondences* (or "point-to-set mappings") which generalize the notion of conventional functions (or "point-to-point mappings"). We first introduce necessary concepts such as new notions of continuity and of fixed points which arise when dealing with correspondences. We then state and prove *Kakutani's fixed point theorem* for simplices and finally for nonempty, compact and convex sets.

5.1 Correspondences, upper semi-continuity and fixed points

The most fundamental notion of this section is that of a set-valued function.

Definition 5.1 (Set-valued function) *[HH04; KT84]*
Let X, Y be two sets and let $\mathcal{P}(Y)$ denote the power set of Y (i.e. the set of all subsets of Y). We call every function

$$f\colon X \to \mathcal{P}(Y) \quad \text{with} \quad f(x) \neq \emptyset \;\; \forall x \in X$$

a ***set-valued function*** *or* ***correspondence****. We thus have $f(x) \subseteq Y \; \forall x \in X$.*

Many authors use double-arrow notation $f\colon X \twoheadrightarrow Y$ to distinguish a correspondence from a conventional point-valued function $g\colon X \to Y$. However, it is immediate that the latter can be regarded as a special case of correspondences when identifying the values of $g(x)$ with the singleton sets $\{g(x)\}$ in Y. Figure 15 illustrates a set-valued function from $\mathbb{R}^3$ to $\mathbb{R}^2$. The next definition introduces a concept of continuity for correspondences that is closely inspired by the conventional definition of continuity for point-valued maps.

Definition 5.2 (Upper semi-continuity via open neighborhoods) *[HH04]*
The set-valued function $f\colon X \to \mathcal{P}(Y)$ is called ***upper semi-continuous at*** *$x \in X$ if for every open neighborhood V of the set $f(x)$ there exists an open neighborhood U of the point $x \in X$ such that $f(u) \subset V \;\forall u \in U$. The correspondence is called* ***upper semi-continuous on*** *X if it is upper semi-continuous at every point $x \in X$.*

We will follow most of the literature on Kakutani's fixed point theorem and focus exclusively on compact domains and *compact-valued* correspondences for which $f(x)$ is a compact set for every $x \in X$. In this case, we can equivalently *define* upper semi-continuity in

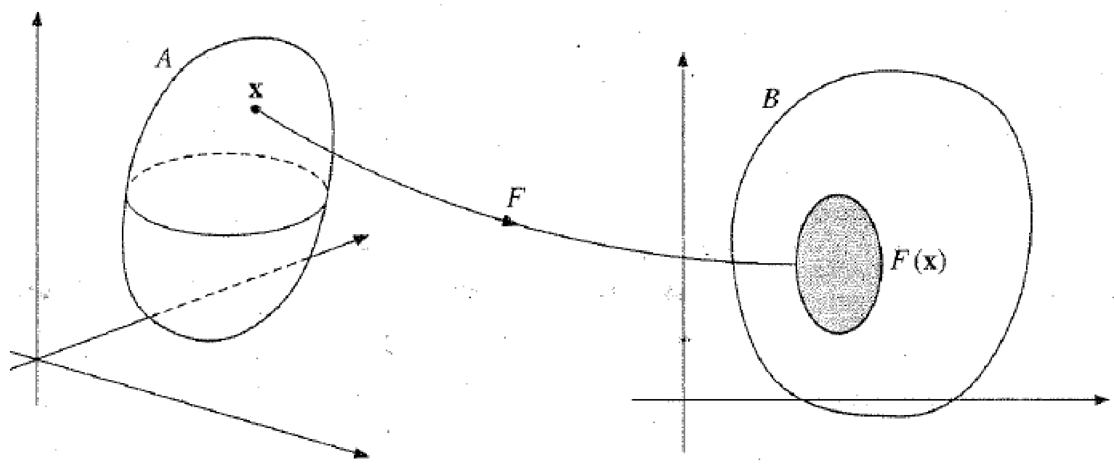


Figure 15: Illustration of a correspondence F from $A \subseteq \mathbb{R}^3$ to $B \subseteq \mathbb{R}^2$ (Source: HS08).

terms of convergent sequences.[31] Although not strictly required for defining "sequential upper semi-continuity", we follow Kakutani (1941) and directly include notions such as the convexity of the underlying set into the definition.

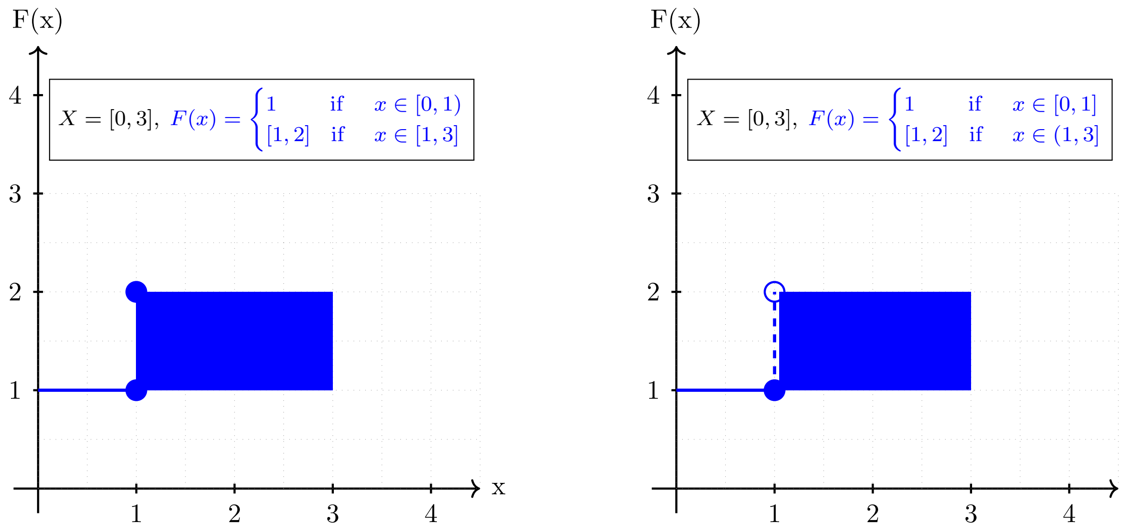
Definition 5.3 (Upper semi-continuity via sequences) *[HH04; SK41; AY17]*
Let $\mathbf{P}(X)$ denote the family of all nonempty, closed and convex subsets of X. If S is a nonempty, compact and convex set, then the set-valued function $\Phi\colon S \to \mathbf{P}(S)$[32] is called ***upper semi-continuous*** *if for arbitrary sequences (x_n), (y_n) in S, we have that $x_n \to x_0$, $y_n \to y_0$ and $y_n \in \Phi(x_n)$ for all $n \in \mathbb{N}$ imply $y_0 \in \Phi(x_0)$.*

Intuitively, if a correspondence f is upper semi-continuous at a point x^0 of its domain, then $f(x)$ cannot "explode" as x moves slightly away from x^0 (compare to Hammond et al., 2008). Figure 16 provides an exemplary illustration of a set-valued mapping that fulfills upper semi-continuity and one that does not. Finally, we introduce a natural generalization of the concept of fixed points to set-valued functions.

Definition 5.4 (Fixed points for correspondences) *[HH04; SK41]*
$x^* \in S$ is a ***fixed point of the set-valued function*** *$\Phi\colon S \to \mathbf{P}(S)$ if $x^* \in \Phi(x^*)$.*

[31] This is very similar to the equivalence of continuity and sequential continuity for point-valued functions in metric spaces. We further highlight that many authors use the term *upper hemi-continuity* instead of upper semi-continuity but essentially mean the same thing. Klein and Thompson (1984) emphasize that although there are no agreed-upon conventions in general, upper semi-continuity is often used when X and Y are topological spaces while upper hemi-continuity is used when Y is additionally a metric space. Since we focus on metric spaces exclusively, the two notions are equivalent for our purposes and we will use the term upper semi-continuity throughout the rest of this thesis.

[32] Since this implies that the image of Φ is always a convex set, we also say that Φ is *convex-valued*.



(a) Upper semi-continuous correspondence (b) **Non**-upper semi-continuous correspondence

Figure 16: Illustration of an example and counterexample of upper semi-continuity.

5.2 Kakutani's fixed point theorem

We established Brouwer's fixed point result (theorem 3.11) by first proving a corresponding statement for self-mappings on simplices. We take a very similar approach in order to prove Kakutani's fixed point theorem.

Proposition 5.5 (Kakutani's fixed point theorem for simplices) *[SK41; AY17]* *Let $\mathcal{S}$ be an r-dimensional closed simplex in a Euclidean space and let $\Phi\colon \mathcal{S} \to \mathbf{P}(\mathcal{S})$ be an upper semi-continuous, set-valued function. Then Φ has a fixed point, i.e. there exists $x_0 \in \mathcal{S}$ such that $x_0 \in \Phi(x_0)$.*

Proof: Proving existence of a fixed point x_0 via Brouwer's fixed point theorem.

Step 1:

Focus on the barycentric simplicial subdivision: We follow Kakutani (1941) and immediately consider the n^{th} barycentric simplicial subdivision (see definition 3.3) of $\mathcal{S}$ which we call $\mathcal{S}_n$ (any simplicial subdivision whose mesh can be made arbitrarily small would suffice). Further, we denote by V the set of all vertices of all subsimplices of $\mathcal{S}_n$.

Step 2:

Define an appropriate linear point-to-point mapping: We associate with each $x^n \in V$ an arbitrary point $y^n \in \Phi(x^n)$, i.e. we establish a mapping ϕ_n with $x^n \mapsto y^n$. Next, we extend this mapping linearly inside each subsimplex of $\mathcal{S}_n$ as follows:

Let $co(\{x^{i_0}, ..., x^{i_k}\})$, $k \leq n$, be a subsimplex for which each vertex x^{i_j} is assigned a point $y^{i_j} \in \Phi(x^{i_j})$ $\forall j \in \{0, ..., k\}$, i.e. $y^{i_j} = \phi_n(x^{i_j}) \in \Phi(x^{i_j})$. Since we can represent any $x \in co(\{x^{i_0}, ..., x^{i_k}\})$ by means of its barycentric coordinates within this subsimplex, i.e. $x = \sum_{j=0}^{k} \lambda_j x^{i_j}$, we can apply the mapping ϕ_n to this x and obtain

$$\phi_n(x) \;=\; \phi_n\Big(\sum_{j=0}^{k} \lambda_j x^{i_j}\Big) \;=\; \sum_{j=0}^{k} \lambda_j y^{i_j}.$$

In other words, the resulting linear map $\phi_n \colon \mathcal{S} \to \mathcal{S}$ is a point-to-point mapping (i.e. a conventional function) with $x \in \mathcal{S} \mapsto \phi(x) \in \mathcal{S}$. Moreover, the function is continuous as $\mathcal{S}$ is a convex set and convex combinations (recall that $\sum_{j=0}^{k} \lambda_j = 1$) are continuous.

Step 3:

Application of Brouwer's fixed point theorem: We have just established that ϕ_n is a continuous function mapping the simplex $\mathcal{S}$ into itself. By Brouwer's fixed point theorem (i.e. proposition 3.5 for simplices), ϕ_n has a fixed point: $\exists\, x^n \in \mathcal{S}$ s.t. $\phi_n(x^n) = x^n$.

Step 4:

Consider the sequence $(\phi_i)_{i\in\mathbb{N}}$ and the resulting fixed points: The above argumentation applies to any order $n \in \mathbb{N}$ of the barycentric subdivision. We hence consider the sequence $(\phi_i)_{i\in\mathbb{N}}$ and the corresponding sequence of fixed points $(x_i)_{i\in\mathbb{N}}$. The latter is a sequence on the compact simplex $\mathcal{S}$ and therefore possesses a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ by the Bolzano-Weierstrass theorem with limit point in $\mathcal{S}$: $\lim_{k\to\infty} x_{n_k} = x_0 \in \mathcal{S}$. We will next show that this limit point is the fixed point of the correspondence Φ, i.e. $x_0 \in \Phi(x_0)$.

Step 5:

Using subsimplices and a decreasing mesh of the subdivision: We consider an r-dimensional subsimplex of $\mathcal{S}_n$ containing the fixed point x_n which we call $\mathcal{R}_n := co(\{z_n^0, ..., z_n^r\})$ where $z_n^0, ..., z_n^r$ are its vertices. Next, we take a very similar approach to the proof of proposition 3.5, i.e. Brouwer's fixed point theorem for simplices: We increase n (the granularity of the barycentric subdivision) and focus on the sequences of the individual vertices z_n^i. We can again find convergent subsequences $(z_{n_k}^i)_{k\in\mathbb{N}}$ for each z_n^i, $i \in \{0, ..., r\}$ and we know from proposition 3.4 that the mesh of the barycentric subdivision approaches zero. This implies that the vertices of the subsimplex $\mathcal{R}_n$ (which contains the fixed point x_n) come closer and closer together. Since x_n approaches x_0, it follows that all convergent subsequences of the individual vertices $(z_{n_k}^i)_{k\in\mathbb{N}}$, $i \in \{0, ..., r\}$, all have to converge to x_0 as well.

Step 6:

Look at the corresponding $y_n^i = \phi_n(z_n^i)$: We next define $y_n^i := \phi_n(z_n^i)$ $\forall i \in \{0, ..., r\}$ as the images of the vertices of $\mathcal{R}_n$ under ϕ_n. By definition of ϕ_n, we know that $y_n^i = \phi_n(z_n^i) \in \Phi(z_n^i)\,\forall i \in \{0,...,r\}$ and $\forall n \in \mathbb{N}$. As $x_n \in \mathcal{R}_n$ by our premise, we can represent the fixed point x_n by its barycentric coordinates as a convex combination of the vertices of $\mathcal{R}_n := co(\{z_n^0,...,z_n^r\})$, i.e. $x_n = \sum_{i=0}^r \lambda_n^i z_n^i$ with $\sum_{i=0}^r \lambda_n^i = 1$. By linearity of ϕ_n,

$$x_n \underbrace{=}_{\text{fixed point}} \phi_n(x_n) = \phi_n\left(\sum_{i=0}^r \lambda_n^i z_n^i\right) \underbrace{=}_{\text{linearity}} \sum_{i=0}^r \lambda_n^i \phi_n(z_n^i) \underbrace{=}_{\text{def of } y_n^i} \sum_{i=0}^r \lambda_n^i y_n^i. \tag{5.1}$$

Note that $\lambda_n^i \in [0,1]$ and $y_n^i \in \mathcal{S}$ for all $n \in \mathbb{N}$ and for all $i \in \{0,...,r\}$ with $\mathcal{S}$ being compact. By the Bolzano-Weierstrass theorem, we can thus again find convergent subsequences $(n'_k)_{k\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$ (this is the (n_k) used above) such that

$$\lim_{k\to\infty} \lambda_{n'_k}^i = \lambda_0^i \qquad \text{and} \qquad \lim_{k\to\infty} y_{n'_k}^i = y_0^i \qquad \forall i \in \{0,...,r\}.$$

We know that $\lambda_n^i \geq 0\; \forall i \in \{0,...,1\}$ and $\sum_{i=0}^r \lambda_n^i = 1$ while equation (5.1) established that we can represent x_n as $x_n = \sum_{i=0}^r \lambda_n^i y_n^i$. Since these properties are naturally preserved under continuity, it follows that $\lambda_0^i \geq 0\;\, \forall i \in \{0,...,1\}$, that $\sum_{i=0}^r \lambda_0^i = 1$ and most importantly that $x_0 = \sum_{i=0}^r \lambda_0^i y_0^i$.

Note in particular that we have established that subsequencs $(z_{n_k}^i)_{k\in\mathbb{N}}$ of the vertices of $\mathcal{R}_n$ converge to x_0. But this implies in particular that $\lim_{k\to\infty} z_{n'_k}^i = x_0$ for all $i \in \{0,...,r\}$ as $(n'_k)_{k\in\mathbb{N}}$ is merely a subsequence of $(n_k)_{k\in\mathbb{N}}$ and hence subsequences $(z_{n'_k}^i)_{k\in\mathbb{N}}$ of $(z_{n_k}^i)_{n\in\mathbb{N}}$ keep the same limit point (here: x_0).

Step 7:

Use upper semi-continuity to establish that x_0 is a fixed point of Φ: By definition of ϕ_n, we have that $\phi_{n'_k}(z_{n'_k}^i) = y_{n'_k}^i \in \Phi(z_{n'_k}^i)$ for all $i \in \{0,...,r\}$. In total, we therefore have

$$\lim_{k\to\infty} z_{n'_k}^i = x_0 \qquad \text{and} \qquad \phi_{n'_k} = y_{n'_k}^i \in \Phi(z_{n'_k}^i) \quad \forall i \in \{0,...,r\}. \tag{5.2}$$

Since Φ is upper semi-continuous (see definition 5.3), equation (5.2) implies that

$$\lim_{k\to\infty} y_{n'_k}^i = y_0^i \in \Phi(\lim_{k\to\infty} z_{n'_k}^i) = \Phi(x_0). \tag{5.3}$$

As Φ is convex-valued by assumption (i.e. Φ maps $\mathcal{S}$ into nonempty, closed and *convex* subsets $\mathbf{P}(\mathcal{S})$), $\Phi(x_0)$ is a convex set and therefore contains the convex combination $x_0 = \sum_{i=0}^r \lambda_0^i y_0^i$. But this means precisely that $x_0 \in \Phi(x_0)$ and hence x_0 is a fixed point of Φ.

□

Just as (point-valued) functions are a special case of correspondences, it should become immediate that Brouwer's fixed point theorem is a special case of Kakutani's fixed point theorem when each $\Phi(x)$ consists only of a single point $\phi(x)$ (compare to Kakutani, 1941). We finally generalize the above theorem to arbitrary bounded, closed and convex subsets in the Euclidean space. To do so, we require a preparatory lemma and the concept of *retractions.*

Lemma 5.6 (Compositions and upper semi-continuity) *[SK41; AY17]*
Let S and S' be two sets and let $f\colon S \to S'$ be a continuous, point-valued function. Further let $g\colon S' \to \mathbf{P}(S)$ be an upper semi-continuous correspondence such that $g \circ f\colon S \to \mathbf{P}(S)$. Then, the correspondence $g \circ f$ is also upper semi-continuous.

Proof:

We consider two convergent sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in S with $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} y_n = y_0$, and $y_n \in g(f(x_n))\ \forall n \in \mathbb{N}$. To show that $g \circ f$ is indeed upper semi-continuous, we have to show that $y_0 \in (g \circ f)(x_0) = g(f(x_0))$.

Since f is continuous (and hence sequentially continuous), we know that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to a point in S', i.e. $\lim_{n\to\infty}(f(x_n))_{n\in\mathbb{N}} = f(x_0) \in S'$. But since the domain of g is precisely S' and since g is upper semi-continuous by our premise, it follows that $y_0 \in g(f(x_0))$ which proves the desired result.

□

Definition 5.7 (Retraction and retracting property) *[HH04; SK41; AY17]*
Let X, Y be two sets with $Y \subseteq X$. A function $f\colon X \to Y$ is called a ***retraction*** *– or is said to possess the* ***retracting property*** *– if $f(y) = y$ for all $y \in Y$.*

Intuitively, a function $f\colon X \to Y$ with $Y \subset X$ that has the retracting property does not impose any restrictions on the mapping of points $x \in X \setminus Y$, but it acts as the identity map $f|_Y = id_Y$ for all $y \in Y$. In other words, all $y \in Y$ are fixed points of f.
We can now state and prove Kakutani's fixed point theorem in the most general version desired for this thesis.

Theorem 5.8 (Kakutani's fixed point theorem) *[SK41; AY17]*
Let S be any nonempty, compact and convex set in a Euclidean space and let $\Phi\colon S \to \mathbf{P}(S)$ be an upper semi-continuous correspondence.[33] *Then Φ has a fixed point, i.e. there exists $x_0 \in S$ with $x_0 \in \Phi(x_0)$.*

[33] Recall that $\mathbf{P}(S)$ is the set of all nonempty, closed and convex subsets of S. Since $\Phi\colon S \to \mathbf{P}(S)$, $\Phi(x)$ is readily nonempty, closed and convex for any $x \in S$ and we do not have to state this as an additional requirement separately. By contrast, some authors consider a correspondence $\hat{\Phi}\colon S \to \mathcal{P}(S)$ (instead of $\mathbf{P}(S)$) but then have to state these additional requirements.

Proof: Using a continuous retraction to establish existence of a fixed point of Φ.

Step 1:

Introduce a simplex $\mathcal{S}'$ with $S \subset \mathcal{S}'$: We first consider an arbitrary closed simplex $\mathcal{S}'$ which contains the set S, i.e. the domain of the correspondence $\Phi\colon S \to \mathbf{P}(S)$. Such a simplex $\mathcal{S}'$ must exist since S is a compact and convex set so we can always "choose the simplex large enough" so that $S \subset \mathcal{S}'$.

Step 2:

Construction of a continuous retraction and a composition: We construct a continuous retraction ϕ by considering the mapping $\phi\colon \mathcal{S}' \to S$ with $S \subset \mathcal{S}'$ which leaves all elements of S unchanged (i.e. $\phi(s) = s \ \forall s \in S$) and which maps elements of $\mathcal{S}' \setminus S$ to the boundary ∂S of the closed set S.

Next, we consider the composition $\Phi \circ \phi\colon \mathcal{S}' \to \mathbf{P}(S)$. Since $S \subset \mathcal{S}'$ (by step 1), it is immediate that $\mathbf{P}(S) \subset \mathbf{P}(\mathcal{S}')$ (recall that $\mathbf{P}(X)$ denotes the family of all nonempty, closed and convex subsets of a set X). Therefore, $\Phi \circ \phi$ maps elements from $\mathcal{S}'$ to subsets of $\mathbf{P}(\mathcal{S}')$ and by our premise, Φ is an upper semi-continuous correspondence while ϕ is a continuous function. Therefore, all conditions of Lemma 5.6 are fulfilled and it follows that $\Phi \circ \phi$ is upper semi-continuous.

Step 3:

Apply Kakutani's fixed point theorem for simplices: We have established that $\Phi \circ \phi$ is an upper semi-continuous correspondence which maps elements from a simplex $\mathcal{S}'$ to $\mathbf{P}(\mathcal{S}')$, i.e. to nonempty, compact and convex subsets of that simplex. By *Kakutani's fixed point theorem for simplices* (see proposition 5.5), $\Phi \circ \phi$ must have a fixed point x_0, i.e.

$$\exists\, x_0 \in \mathcal{S}' \quad \text{with} \quad x_0 \in (\Phi \circ \phi)(x_0) = \Phi(\phi(x_0)). \tag{5.4}$$

Step 4:

Show that x_0 is also a fixed point of Φ: Since $\Phi\colon S \to \mathbf{P}(S)$, we know that

$$\Phi(\phi(x_0)) \subset S \tag{5.5}$$

and since $x_0 \in \Phi(\phi(x_0))$, we particularly have that $x_0 \in S$.

Recall that $\phi\colon \mathcal{S}' \to S$, $S \subset \mathcal{S}'$, was defined to be a retraction and hence leaves elements of S unchanged. Since x_0 is such an element, it follows that $\phi(x_0) = x_0$.

Using equation (5.5) then immediately implies that

$$x_0 \in \Phi(\underbrace{\phi(x_0)}_{=x_0}) \qquad \Rightarrow \qquad x_0 \in \Phi(x_0). \tag{5.6}$$

Hence, x_0 is also a fixed point of Φ which proves the result.

□

Just as in Brouwer's fixed point theorem 3.11, the sufficient conditions on X and Φ of Kakutani's fixed point theorem are "tight" in the sense that relaxing one of them can invalidate the theorem. This is illustrated in the following examples which look at each condition on Φ separately.[34]

Example 5.9 (Kakutani's sufficient conditions on the correspondence)

(a) $F\colon X \to \mathcal{P}(X)$ ***is not convex-valued, i.e.*** $\mathcal{P}(X)$ ***is not convex:***

$$X = [0,1] \ \checkmark \qquad\qquad F(x) = \begin{cases} 1 & \textit{if} \quad x \in [0, 0.5) \\ \{0,1\} & \textit{if} \quad x = 0.5 \\ 0 & \textit{if} \quad x \in (0.5, 1] \end{cases} \qquad ✗$$

X is compact and convex and F is upper semi-continuous (i.e. has a closed graph), but F is not convex-valued. It does not have a fixed point since $F(x)$ does not intersect the $45°$-line.

(b) F ***is not upper semi-continuous (i.e. does not have a closed graph):***

$$X = [0,1] \ \checkmark \qquad\qquad F(x) = \begin{cases} [0.6, 0.8] & \textit{if} \quad x \in [0, 0.5) \\ [0.2, 0.4] & \textit{if} \quad x \in [0.5, 1] \end{cases} \qquad ✗$$

F is nonempty and convex-valued[35] *for all $x \in X$, but F is not upper semi-continuous*

[34] Technically, example 3.14 can also serve as an example in this regard since functions are just special cases of correspondences. For illustrative purposes, however, different examples with point-to-set mappings should be provided here.

[35] Note that $F(X)$ does not have to be a convex set for the conditions of Kakutani's FPT to be fulfilled. Rather, $F(x)$ has to be a convex set *for each* $x \in X$. This is clearly fulfilled in this example as $[0.6, 0.8]$ and $[0.2, 0.4]$ are both convex sets.

and does not contain a fixed point. To see this, consider the two sequences

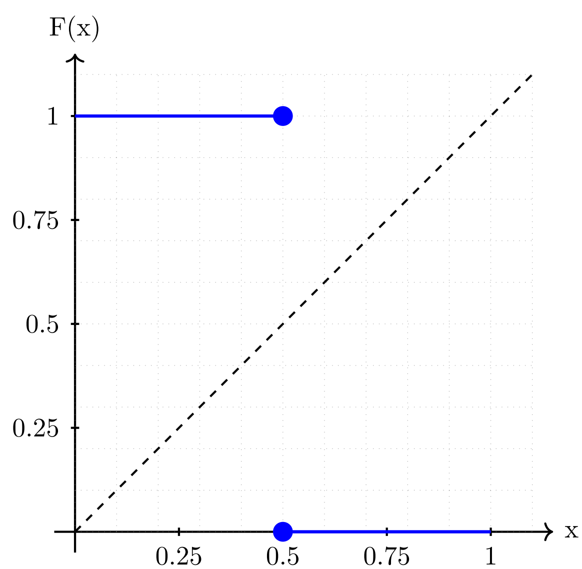
$$(x_n)_{n\in\mathbb{N}} \;\; \textit{with} \;\; x_n := 0.5 - \frac{1}{n} \qquad \textit{and} \qquad x_n \longrightarrow x_0 = 0.5,$$
$$(y_n)_{n\in\mathbb{N}} \;\; \textit{with} \;\; y_n := 0.7 \qquad \textit{and} \qquad y_n \longrightarrow y_0 = 0.7.$$

We then have $x_n < 0.5$ *and hence* $F(x_n) = [0.6, 0.8]\ \forall n \in \mathbb{N}$. *Therefore* $y_n = 0.7 \in F(x_n)\ \forall n \in \mathbb{N}$, *but* $F(x_0) = F(0.5) = [0.2, 0.4]$ *and* $y_0 = 0.7 \notin F(x_0)$. *Hence,* F *is not upper semi-continuous.*

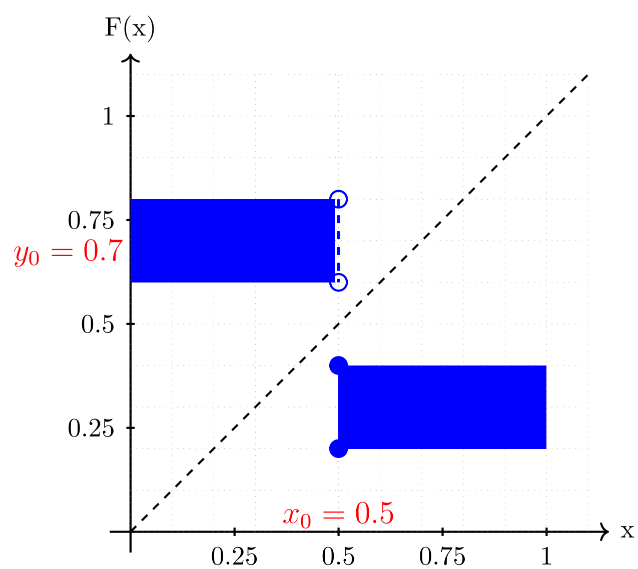
(c) F ***is empty for some*** $x \in X$***:***

$$X = [0,1] \;\; \checkmark \qquad\qquad F(x) = \begin{cases} [0.6, 0.8] & \textit{if} \quad x \in [0, 0.5] \\ \quad \emptyset & \textit{if} \quad x \in (0.5, 0.7) \\ [0.2, 0.4] & \textit{if} \quad x \in [0.7, 1] \end{cases} \;\; ✗$$

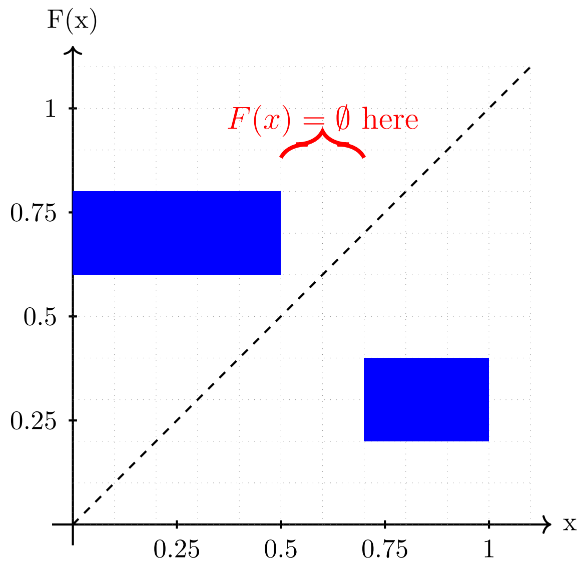
F *is empty for* $x \in (0.5, 0.7)$. *It does not intersect the* $45°$*-line and hence does not have a fixed point.*



(a) F not convex-valued



(b) F not upper semi-continuous



(c) $F(x)$ is empty for some $x \in X$.

Figure 17: Violations of the sufficient conditions of Kakutani's fixed point theorem.

Moreover (and also in congruence to Brouwer's fixed point theorem), Kakutani's fixed point theorem does not make a statement about the number of fixed points. When all sufficient conditions are fulfilled, the fixed point could be unique, but there can also be several (and even infinitely many) fixed points. This is illustrated with the following example.

Example 5.10 (Kakutani's FPT: Number of fixed points)

(a) Unique fixed point:

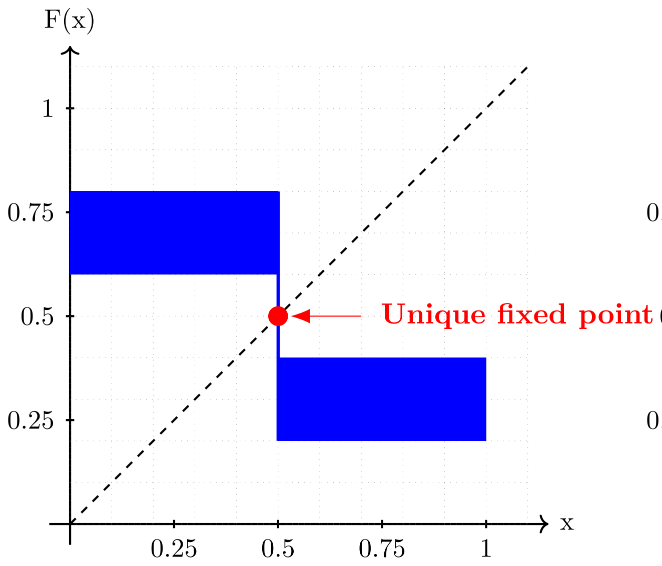
$$X = [0,1], \quad F(x) = \begin{cases} [0.6, 0.8] & \textit{if} \quad x \in [0, 0.5) \\ [0.2, 0.8] & \textit{if} \quad x = 0.5 \\ [0.2, 0.4] & \textit{if} \quad x \in (0.5, 1] \end{cases}$$

X and F fulfill all sufficient conditions of Kakutani's fixed point theorem. The fixed point in this case is unique and is given by $x^* = 0.5$.

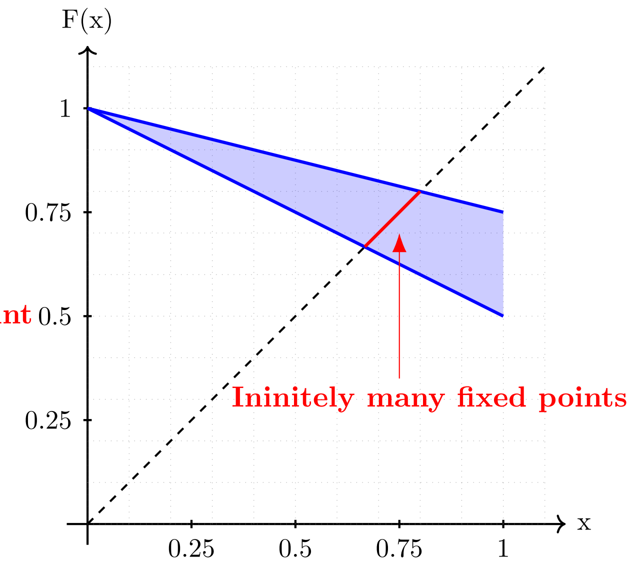
(b) Infinitely many fixed points:[36] $X = [0,1]$ $F(x) = \left[1 - \frac{x}{2}, 1 - \frac{x}{4}\right]$.

X and F fulfill all sufficient conditions of Kakutani's fixed point theorem. In this case, there is a continuum of fixed points: $x^* \in \left[\frac{2}{3}, \frac{4}{5}\right]$.

[36]Source of this example: https://en.wikipedia.org/wiki/Kakutani_fixed-point_theorem)



(a) Unique fixed point



(b) Infinitely many fixed points

Figure 18: Examples of Kakutani's fixed point theorem and the number of fixed points.

5.3 An application to game theory

The fixed point theorems discussed so far have important applications in many different fields. One discipline that has made extensive use of these results is that of mathematical economics. Footnote (20) on page 28 discusses an application in *general equilibrium theory* where Brouwer's fixed point theorem helps to establish the existence of a competitive equilibrium price vector at which total supply in the economy precisely equals total demand (and hence "clears the market"). Another field of mathematical economics in which fixed point theorems prove useful is *game theory*. Game theory can be understood as the mathematical analysis of rational decision-making by agents (or 'players') in strategic interactions (compare e.g. Meister, 2018; Osborne and Rubinstein, 2011). This section is devoted to prove an important game-theoretical result, namely the existence of a so-called *pure-strategy Nash equilibrium* in a certain class of games. The proof relies almost exclusively on Kakutani's fixed point theorem and should hence serve as a "practical application" of the theoretical results covered so far. In order to establish the proof, we first need to introduce some important concepts, the most basic of which is that of a *rational preference relation*.

Definition 5.11 (Continuous, quasi-concave preference relation) *[MWG95; OR11]*
A ***preference relation*** $\succsim_i$ *of agent* i *on some set* C *(usually a set of possible conse-*

quences) is a complete, transitive and reflexive binary relation.[37]
*The preference relation $\succsim_i$ is said to be **continuous** if for all sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} a_k = a_0$ and $\lim_{k\to\infty} b_k = b_0$ for which $a_k \succsim_i b_k\ \forall k$, we have $a_0 \succsim_i b_0$.*
*The preference relation $\succsim_i$ is called **quasi-concave** on $C := \mathbb{R}^n$ if for every $b \in C$ the "upper-contour set" $\{a \in C \mid a \succsim_i b\}$ is a convex set.*

Generally speaking, a *strategic game* is a model of interactive and interdependent decision-making among agents who choose their actions simultaneously and evaluate outcomes according to their preferences. Formally:

Definition 5.12 (Strategic game) *[OR11]*
*A **strategic game** is a triple $\langle N, (A_i), (\succsim_i)\rangle$ consisting of*

- *a finite set N (the set of **players**)*
- *for each player $i \in N$, a nonempty set A_i (the set of **actions** available to player i)*
- *for each player $i \in N$ a preference relation $\succsim_i$ on $A := \bigtimes_{j\in N} A_j$ (the **preference relation** of player i).*

*If A_i is finite $\forall i \in N$, we say that the game is **finite**.*

It is essential to highlight that the preference relation $\succsim_i$ of player i is defined on $A = \bigtimes_{j\in N} A_j$ and **not** on A_i. This is precisely what distinguishes an isolated decision problem of a single agent from *interdependent* decision-making in games where players *do* care about other players' actions as those might influence player i's final payoff. This observation leads to the notion of a *profile*.

Definition 5.13 (Profile) *[OR11; AY17]*
*A **profile** $\mathbf{x} = (x_i)_{i\in N}$ is a collection of values (one for each player $i \in N$) of some variable x_i. Given a profile $\mathbf{x}$ and some player i, we denote by $x_{-i} = (x_j)_{j\in N\setminus\{i\}}$ all elements of the profile $\mathbf{x}$ except for x_i. We then have $\mathbf{x} = (x_{-i}, x_i)$.*

Game theorists are interested in what action profile will be "played" by rational decision-makers who always choose the best (i.e. most preferred) actions available given their (correct) expectations about the other players' behavior (compare to Osborne, 2011; Osborne

[37]**Completeness**: $\forall a, b \in C$, we have $a \succsim_i b$ or $b \succsim_i a$ or both (denoted by $a \sim_i b$).
Reflexivity: $\forall a \in C$, we have $a \succsim_i a$.
Transitivity: $\forall a, b, c \in C\colon a \succsim_i b \ \wedge \ b \succsim_i c \ \Rightarrow a \succsim_i c$.

In the context of economic decision making, we usually say that agent i *weakly prefers a to b* (or regards a to be *at least as good as* b) if $a \succsim_i b$. We further write $a \succ_i b$ if agent i *strictly prefers a to b*. Finally, we write $a \sim_i b$ if the agent is *indifferent between a and b*.

and Rubinstein, 2011). The best-known *solution concept* in game theory to determine the strategy profile and outcome of a game is that of a pure-strategy[38] Nash equilibrium.

Definition 5.14 (Nash equilibrium via preference relations) *[OR11]*
A ***pure-strategy Nash equilibrium*** *of a strategic game* $\langle N, (A_i), (\succsim_i)\rangle$ *is a profile* $\mathbf{a}^* \in A$ *of actions such that for all players* $i \in N$ *we have*

$$(a^*_{-i}, a^*_i) \succeq_i (a^*_{-i}, a_i) \quad \textit{for all } a_i \in A_i. \tag{5.7}$$

A Nash equilibrium captures a certain notion of optimality: *Given the equilibrium actions* a^*_{-i} *of the other players*, the choice a^*_i of player i results in an outcome that she (weakly) prefers over all other outcomes that would have resulted had she chosen some other action $a_i \in A_i$. In other words, a^*_i is a *best response* to (a^*_{-i}) for all $i \in N$ and no player has an incentive to *deviate unilaterally* from the Nash equilibrium profile. The following (equivalent) restatement of this definition provides further intuition and will prove handier in the existence proof.

Definition 5.15 (Nash equilibrium via best-response correspondences) *[OR11]*
For any $a_{-i} \in A_{-i} := \bigtimes_{j \in N\setminus\{i\}} A_j$*, we denote by* $B_i \colon A_{-i} \to \mathcal{P}(A_i)$ *the* ***best-response correspondence of player i****.* $B_i(a_{-i})$ *is the set of player i's optimal actions given the action choices* a_{-i} *of the other players, i.e.*

$$B_i(a_{-i}) := \{a_i \in A_i \mid (a_{-i}, a_i) \succsim_i (a_{-i}, a'_i) \; \forall a'_i \in A_i\}. \tag{5.8}$$

A ***pure-strategy Nash equilibrium*** *of a strategic game* $\langle N, (A_i), (\succsim_i)\rangle$ *is a profile* $\mathbf{a}^* \in A$ *such that*

$$a^*_i \in B_i(a^*_{-i}) \quad \forall i \in N. \tag{5.9}$$

Hence, Nash equilibrium actions of the players are *mutually best responses* to each other. The aim of this section is to show that there will always exist pure-strategy Nash equilibria under certain conditions. The following observation shows how Kakutani's fixed point theorem can help with that.

[38] Pure-strategy profiles refer to *deterministic* strategies in which each player chooses precisely one action (with certainty). In many games (and real-world situations), however, choices are non-deterministic and strategies can become *mixed*. An illustrative example is the game of *rock-paper-scissors*. In mixed-strategy action profiles, players randomize across possible actions and their preferences are defined on the set of probability distributions over A. We will restrict attention to pure-strategy equilibria since the application of Kakutani's fixed point theorem already comes into effect in full in this environment.

Observation 5.16 (Kakutani's fixed point theorem and Nash equilibria)
For any $\mathbf{a} \in A = \bigtimes_{i\in N} A_i$ *we denote by* $B\colon A \to \mathcal{P}(A)$ *with*

$$B(\mathbf{a}) \;=\; \bigtimes_{i\in N} B_i(a_{-i}) \;=\; \bigtimes_{i\in N}\{a_i \in A_i \mid (a_{-i}, a_i) \;\succsim_i\; (a_{-i}, a_i') \;\forall a_i' \in A_i\} \tag{5.10}$$

the ***best-response correspondence*** *of the strategic game* $\langle N, (A_i), (\succsim_i)\rangle$*. We have argued in (5.9) that an action profile* $\mathbf{a}^*$ *is a pure-strategy Nash equilibrium if* $a_i^* \in B_i(a_{-i}^*)$ *for all* $i \in N$*. By means of the set-valued function* B*, we can write (5.9) in vector form as* $\mathbf{a}^* \in B(\mathbf{a}^*)$*. In other words, Nash equilibrium profiles* $\mathbf{a}^*$ *are* ***fixed points of the best-response correspondence*** B*. Once* A *and* B *fulfill the conditions of Kakutani's fixed point theorem, the latter guarantees that the strategic game has at least one Nash equilibrium profile.*

Before stating and proving the main result of this section, we need as a final notion the concept of a *utility function*. For any given action of the other players a_{-i} a utility function of player i associates a numerical value in $\mathbb{R}$ (sometimes called "utils") with every possible action choice a_i such that actions yielding higher utils are preferred. More formally:

Definition 5.17 (Utility function for a strategic game) *[OR11; MWG95]*
Let $\langle N, (A_i), (\succsim_i)\rangle$ *be a strategic game. The function* $u_i\colon A_i \to \mathbb{R}$ *is called a* ***utility function*** *of the strategic game if for all* $a_i, a_i' \in A_i$ *and for all (fixed) actions of the other players* $a_{-i} \in A_{-i} = \bigtimes_{j\in N\setminus\{i\}} A_j$*, we have*

$$u_i(a_i) \;\geq\; u_i(a_i') \quad\Leftrightarrow\quad (a_{-i}, a_i) \;\succsim_i\; (a_{-i}, a_i'). \tag{5.11}$$

We also say that the utility function ***represents*** *the preferences of the agent.*

The following proposition goes back to Debreu (1959) and justifies the use of utility functions (in lieu of preference relations).[39]

Proposition 5.18 (Debreu's representation theorem) *[GD59; MWG95; RS11]*
Let A_i *be a connected set, completely preordered by the preference relation* $\succsim_i$*. If for every* $a_i' \in A_i$*, the upper and lower contour sets* $\{a_i \mid a_i \succsim_i a_i'\}$ *and* $\{a_i \mid a_i' \succsim_i a_i\}$ *are closed, there exists a* ***continuous utility function*** $u_i\colon A_i \to \mathbb{R}$ *that represents* $\succsim_i$*.*

[39] Since the focus of this section is the application of Kakutani's fixed point theorem, we only provide a brief idea of the proof of this proposition. The original proof can be found in Debreu (1959). Adjusted (but slightly more accessible) versions which use slightly stronger assumptions can be found in Mas-Colell et al. (1995) or Starr (2011).

In particular, such a utility function exists if the strategy set A_i is convex and the preference relation $\succsim_i$ is continuous and quasi-concave.

Proof idea **(for continuous $\succsim_i$ and convex $A_i \subset \mathbb{R}^L$):**

→ The idea is that we can always establish indifference between any $x \in \mathbb{R}^L$ and the "balanced choice" αe with $e = (1, 1, ..., 1)$. The scalar $\alpha(x) \in \mathbb{R}$ will then be shown to act as the asked for utility function.

→ Continuity of $\succsim_i$ is used to show that the contour sets $A_i^+ := \{\alpha \in \mathbb{R}_+ \mid \alpha e \succsim_i x\}$ and $A_i^- := \{\alpha \in \mathbb{R}_+ \mid x \succsim_i \alpha e\}$ are closed. Connectedness is used to show that $A_i^+ \cap A_i^- \neq \emptyset$. Hence there exists a value $\alpha(x)$ s.t. $\alpha(x)e \sim_i x$.

→ As a final step, uniqueness is shown, i.e. $A_i^+ \cap A_i^- = \{x^*\}$ and $\alpha(x)e \sim_i x^*$. For any $x \in A_i$, $u_i(x)\colon A_i \to \mathbb{R}$ with $u_i(x) := \alpha(x) = \|x^*\|$ then serves as a utility function.

□

Using continuous utility functions representing preferences and the best-response correspondence $B(\mathbf{a})$ from observation (5.16), we are now in a position to use Kakutani's fixed point theorem to prove the main result of this section.

Theorem 5.19 (Existence of a pure-strategy Nash equilibrium) *[OR11; AY17]*
Any strategic game $\langle N, (A_i), (\succsim_i)\rangle$ has a pure-strategy Nash equilibrium $\mathbf{a}^* \in A$ if the following conditions are satisfied for all players $i \in N$:

1. *The action sets A_i are nonempty, compact and convex subsets of a Euclidean space.*
2. *$\succsim_i$ is continuous and quasi-concave on A_i.*[40]

Proof: Show that the set $A = \bigtimes_{i\in N} A_i$ and the correspondence $B\colon A \to \mathcal{P}(A)$ with $B(\mathbf{a}) = \bigtimes_{i\in N} B_i(a_{-i})$ fulfill all requirements of Kakutani's fixed point theorem.

Step 1:
Show that $A = \bigtimes_{i\in N} A_i$ is nonempty: A_i is nonempty, compact and convex for all $i \in N$ by assumption. Since these properties are "preserved" under the Cartesian product, it immediately follows that A fulfills these required properties.

Step 2:
Show that $B_i(a_{-i})$ is nonempty by using Debreu's representation theorem and Weierstrass

[40] In the context of a strategic game, the preference relation $\succsim_i$ over A is said to be *quasi-concave* on A_i if for every $\mathbf{a} \in A$, the set $\{a_i' \in A_i \mid (a_{-i}, a_i') \succeq_i (a_{-i}, a_i) = \mathbf{a}\}$ is convex (Osborne, 2011).

extreme value theorem: Consider a fixed (but arbitrary) player $i \in N$ and some fixed (but arbitrary) action $a_{-i} \in A_{-i} = \bigtimes_{j \in N \setminus \{i\}} A_j$ of the other players. In order to show that $B_i(a_{-i})$ is nonempty[41], we first invoke *Debreu's representation theorem* 5.18 by which agent i's preferences $\succsim_i$ can be represented by a continuous utility function $u_i \colon A_i \to \mathbb{R}$. A_i is compact by assumption while u_i is continuous. By *Weierstrass' extreme value theorem*, u_i attains its maximum on A_i, i.e. $\exists a_i^* \in A_i$ with $u_i(a_i^*) \geq u_i(a_i)\ \forall a_i \in A_i$. Since u_i represents $\succsim_i$, we have by definition (5.11) that $(a_{-i}, a_i^*) \ \succsim_i \ (a_{-i}, a_i')\ \forall a_i \in A_i$. It hence follows from definition (5.8) of the best-response correspondence B_i of player i that $a_i^* \in B_i(a_{-i})$. Therefore, $B_i(a_{-i})$ is nonempty.

<u>*Step 3:*</u>
Show that $B_i(a_{-i})$ is closed: We first consider an arbitrary point $p \in \overline{B_i(a_{-i})}$. Since the closure is a closed set, there must exist a sequence in $B_i(a_{-i})$ converging to p, i.e. $\exists (p^k)_{k \in \mathbb{N}}$ with $p^k \in B_i(a_{-i})\ \forall k \in \mathbb{N}$ and $\lim_{k \to \infty} p^k = p$. Since $p^k \in B_i(a_{-i})\ \forall k \in \mathbb{N}$, we know by definition (5.8) that $(a_{-i}, p^k) \succsim_i (a_{-i}, a_i)\ \forall a_i \in A_i$. For each $a_i \in A_i$ we can then always construct two convergent sequences:

1. *A sequence using the p^k from above*: $\big((a_{-i}, p^k)\big)_{k \in N}$ with $\lim_{k \to \infty}(a_{-i}, p^k) = (a_{-i}, p)$.
2. *A constant sequence with the fixed a_i*: $\big((a_{-i}, a_i)\big)_{k \in N}$ with $\lim_{k \to \infty}(a_{-i}, a_i) = (a_{-i}, a_i)$.

We know from above that $(a_{-i}, p^k) \succsim_i (a_{-i}, a_i)\ \forall a_i \in A_i$ and by *continuity of the preference relation*, this also holds true for the limit, i.e. $(a_{-i}, p) \succsim_i (a_{-i}, a_i)\ \forall a_i \in A_i$. It follows that also $p \in B_i(a_{-i})$. But this immediately implies that $B_i(a_{-i})$ is closed since we have considered an arbitrary p (from $B_i(a_{-i})$ as it turned out) and a sequence $(p^k)_{k \in \mathbb{N}}$ with $p_k \in B_i(a_{-i})$ converging to p.

<u>*Step 4:*</u>
Show that $B_i(a_{-i})$ is convex: Consider some $a_i \in B_i(a_{-i})$ (which exists as $B_i(a_{-i})$ is nonempty by Step 2). Since $\succsim_i$ is *quasi-concave* on A_i by assumption, we know from the definition of quasi-concavity (see Footnote (40) on p.61) that the "upper contour set" with respect to a_i, i.e.

$$S_{a_i} := \{a_i' \in A_i \mid (a_{-i}, a_i') \succsim_i (a_{-i}, a_i)\}, \tag{5.12}$$

is convex. We will next show that $S_{a_i} \overset{!}{=} B_i(a_{-i})$ which implies that $B_i(a_{-i})$ is also convex. $S_{a_i} \subset B_i(a_{-i})$: Since $a_i \in B_i(a_{-i})$, i.e. a_i is a best response to the action profile a_{-i} of the other players, it follows from definition (5.8) that *no* a_i' can be *strictly* preferred to a_i. In

[41] I.e. economically speaking to show that there does always exist at least one best response action to *any* action profile of the other players.

other words, all other potential best responses $a_i' \in A_i$ can only be weakly preferred to a_i, i.e. $(a_{-i}, a_i') \succsim_i (a_{-i}, a_i)$. Hence $S_{a_i} \subset B_i(a_{-i})$.

$B_i(a_{-i}) \subset S_{a_i}$: Consider an arbitrary additional best response $a_i^* \in B_i(a_{-i})$. a_i^* must also be at least as good as a_i (otherwise, a_i would be strictly preferred over a_i^* contradicting a_i^* to be a best response). In other words, we must have $(a_{-i}, a_i^*) \succsim_i (a_{-i}, a_i)$ for any additional best response a_i^*. But this means precisely that $a_i^* \in S_{a_i}$. Hence $B_i(a_{-i}) \subset S_{a_i}$.

In total, $S_{a_i} \subset B_i(a_{-i}) \; \wedge B_i(a_{-i}) \subset S_{a_i} \;\Rightarrow\; S_{a_i} \overset{!}{=} B_i(a_{-i})$. Therefore, $B_i(a_{-i})$ is convex.

<u>*Step 5:*</u>

Show that the properties proven for $B_i(a_{-i})$ also hold for $B\colon A \to \mathcal{P}(A)$: We have shown that for all $i \in N$ and for any arbitrary action profile $a_{-i} \in A_{-i} = \bigtimes_{j \in N \setminus \{i\}} A_j$ of the other players, the best response correspondence $B_i(a_{-i})$ is nonempty, closed and convex. Since these properties are preserved under the Cartesian product, it follows that

$$B(\mathbf{a}) \;=\; \bigtimes_{i \in N} B_i(a_{-i}) \;=\; \bigtimes_{i \in N} \{a_i \in A_i \mid (a_{-i}, a_i) \;\succsim_i\; (a_{-i}, a_i') \;\forall a_i' \in A_i\} \tag{5.13}$$

must also be nonempty, closed and convex. In other words, it holds that $B\colon A \to \mathbf{P}(A)$.[42]

<u>*Step 6:*</u>

Show that B is upper semi-continuous (see definition (5.3)): Consider arbitrary sequences $(\mathbf{x}^k)_{k \in N}$, $(\mathbf{y}^k)_{k \in N} \in A$ with $\lim_{k\to\infty} \mathbf{x}^k = \mathbf{x}^0$ and $\lim_{k\to\infty} \mathbf{y}^k = \mathbf{y}^0$ and where $\mathbf{y}^k \in B(\mathbf{x}^k)$ $\forall k \in N$.[43] Since $\mathbf{y}^k = (y_1^k, y_2^k, ..., y_N^k) \in B(\mathbf{x}^k)$ $\forall k \in \mathbb{N}$, this holds in particular for every individual player, i.e. $y_i^k \in B_i(x_{-i}^k)$ $\forall i \in N$ and $\forall k \in \mathbb{N}$. Therefore, consider a fixed (but arbitrary) player $i \in N$. By definition (5.8) of B_i, we know that it must hold that y_i^k is at least weakly preferred to any other action $a_i \in A_i$ given that the other players choose x_{-i}, i.e. $(x_{-i}^k, y_i^k) \;\succsim_i (x_{-i}^k, a_i)$ $\forall a_i \in A_i$ and $\forall k \in \mathbb{N}$. Next, fix one (arbitrary) $a_i \in A_i$. Since A_i and A_{-i} are both closed, we can proceed similarly as in Step 3 and construct two convergent sequences:

1. *A sequence using the y_i^k from above*: $\big((x_{-i}^k, y_i^k)\big)_{k \in N}$ with $\lim_{k\to\infty}(x_{-i}^k, y_i^k) = (x_{-i}^0, y_i^0)$.
2. *A constant sequence with the fixed a_i*: $\big((x_{-i}^k, a_i)\big)_{k \in N}$ with $\lim_{k\to\infty}(x_{-i}^k, a_i) = (x_{-i}^k, a_i)$.

As $(x_{-i}^k, y_i^k) \;\succsim_i (x_{-i}^k, a_i)$ $\forall a_i \in A_i$, we know from *continuity of the preference relation* $\succsim_i$ that this must also hold in the limit, i.e. $(x_{-i}^0, y_i^0) \;\succsim_i (x_{-i}^0, a_i)$ $\forall a_i \in A_i$. By definition of B_i, this means $y_i^0 \in B_i(x_{-i}^0)$ $\forall i \in N$. Since this is true for all individual players $i \in N$,

[42] Recall that $\mathbf{P}(A)$ denotes the set of all nonempty, closed and convex subsets of A.

[43] Possible since A is nonempty, closed and convex. Also recall that $\mathbf{x} = (x_1, x_2, ..., x_N)$.

it is also true for the best-response correspondence B. In other words, $\mathbf{y}^0 \in B(\mathbf{x}^0)$ which shows that $B\colon A \to \mathbf{P}(A)$ is upper semi-continuous.

Step 7:

Application of Kakutani's fixed point theorem: We have shown that A is nonempty, compact and convex and that $B\colon A \to \mathbf{P}(A)$ is upper semi-continuous. Hence, all requirements of *Kakutani's fixed point theorem* (5.8) are fulfilled. It follows that B must have a fixed point, i.e. $\exists\, \mathbf{a}^* \in A$ with $\mathbf{a}^* \in B(\mathbf{a}^*)$. By observation 5.16, this precisely means that $\mathbf{a}^*$ is a Nash equilibrium profile of the strategic game $\langle N, (A_i), (\succsim_i)\rangle$ which was to be shown.

□

Theorem 5.19 illustrates an important practical application of Kakutani's fixed point theorem in the field of economics. From a pure game-theoretical perspective, however, the theorem only applies to a small subset of games as it requires fairly strong conditions which might *not* be fulfilled in many 'real-world' examples. To illustrate this point, recall that the requirement of A_i being a convex set critically entered the proof of theorem 5.19. But this implies immediately that the theorem *cannot* be used to establish the existence of pure-strategy Nash equilibria in *finite games* (i.e. in games where the action sets A_i are finite for all players $i \in N$). Nevertheless, by introducing (non-deterministic) mixed strategies, one can show that the theorem *does* generalize and that every finite strategic game has a mixed-strategy Nash equilibrium (see for instance Mas-Colell et al., 1995; Osborne and Rubinstein, 2011). Since the proof of this result requires several additional concepts from game theory and does not use Kakutani's fixed point theorem differently than theorem 5.19 above, we will not present it here. In total, it should have become clear that albeit fixed point theorems are essential from a purely mathematical perspective, they have become indispensable for various practical application and are hence an important tool in many applied fields.

6 Conclusion

The objective of this thesis was to provide an intuitive and self-contained introduction to Brouwer's fixed point theorem and important generalizations thereof. Instead of using a relatively complicated machinery of analytical tools, Brouwer's fixed point theorem was proven by means of Sperner's lemma which required only elementary concepts from convex analysis. Simplices and simplicial subdivisions have played a crucial role throughout the course of this thesis: The fixed point theorems of Brouwer and Kakutani were proven for simplices at first before they were extended to nonempty, compact and convex sets with the help of homeomorphisms. The ideas of Brouwer's fixed point theorem have formed the foundation of this thesis. Yet, the required assumptions of the theorem are not always fulfilled – especially in concrete applications. Therefore, I have focused on two important strands of generalization: Schauder's fixed point theorem extended the fixed point results to infinite-dimensional spaces. A special emphasize was put on the fact that the equivalence of compactness and closedness + boundedness does not extend to infinite-dimensional spaces. Extending Brouwer's fixed point theorem hence required additional assumptions which is why compact operators have been introduced in this regard. Kakutani's fixed point theorem leaves the sets and spaces unchanged but uses correspondences (i.e. set-valued functions) in order to provide a generalization of Brouwer's fixed point theorem that has become crucial in economic applications and in the study of game theory. The latter point was illustrated by showing how Kakutani's fixed point theorem can be used to establish the existence of pure-strategy Nash equilibria in certain classes of games. It should once more be highlighted that all of the fixed point theorems discussed in this thesis are mere existence results and do neither establish uniqueness of the fixed points nor a constructive way to find them. Moreover, it goes without saying that various further generalizations of the presented fixed point theorems exist and that this thesis has only shed light on a small subset of important results. Notwithstanding, I hope that the detailed step-by-step approach and the focus on intuition and graphical arguments can pave the way for further studies of more advanced issues in fixed point theory.

A Idea of an Analytical Proof of Brouwer's Fixed Point Theorem

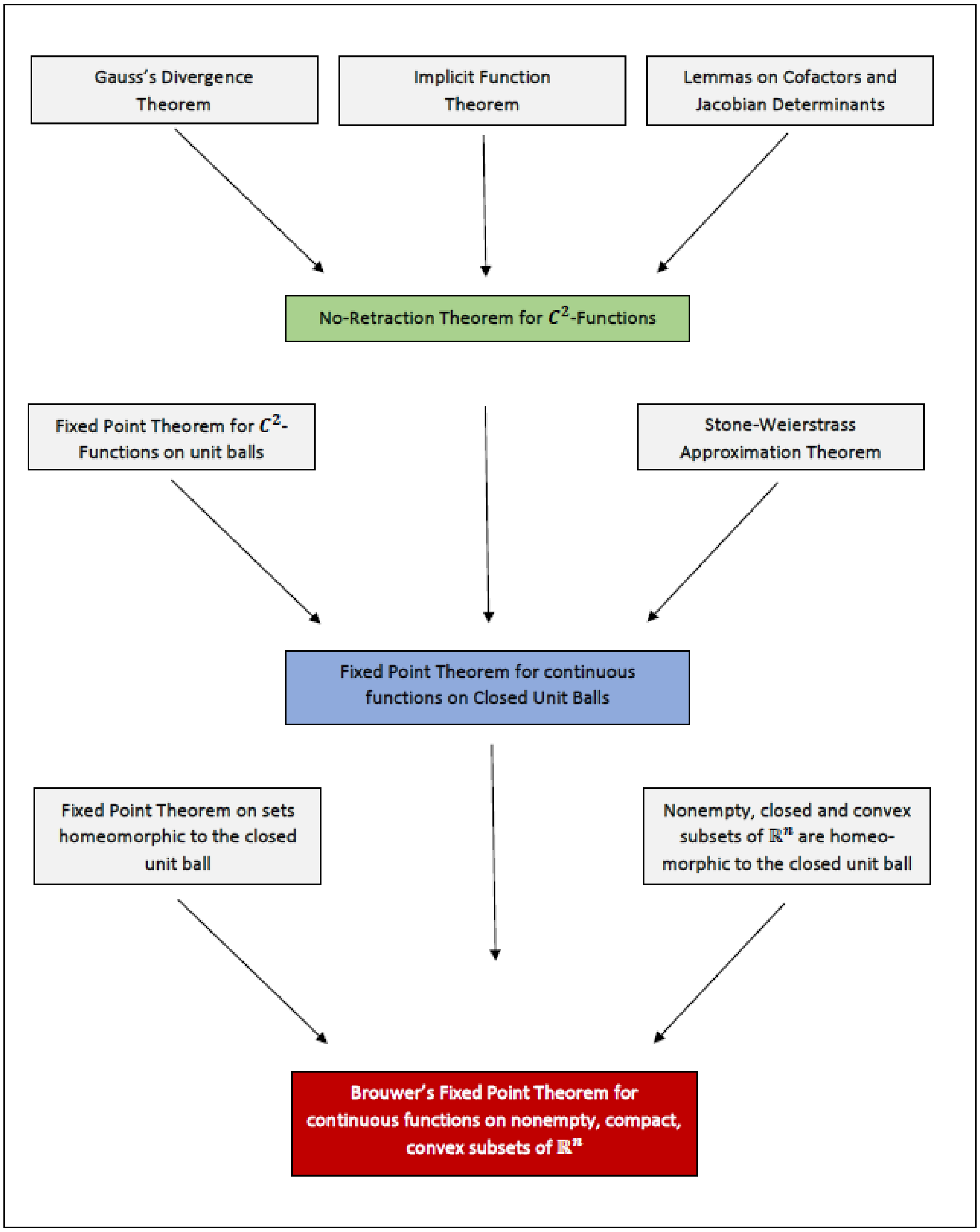


Figure 19: Key steps to prove Brouwer's fixed point theorem with analytical tools.

B Proofs of Some Auxiliary Results

B.1 Proof of Proposition 3.4

Proposition B.1 (Mesh and iterated barycentric subdivision) *[ADQ12; HN68]* *Let $\mathcal{S} = co(\{x^0, x^1, ..., x^N\})$ be an N-simplex with diameter $\delta(\mathcal{S}) = \sup_{j,k\in\{0,...,N\}} \|x^j - x^k\|$ and let $\mathcal{S}^k$ be any derived simplex of order k in the k^{th} barycentric subdivision of $\mathcal{S}$. Then the following inequality holds:*

$$\delta(\mathcal{S}^k) \;\leq\; \left(\frac{N}{N+1}\right)^k \delta(\mathcal{S}). \tag{B.1}$$

In particular, $\lim_{k\to\infty} \delta(\mathcal{S}^k) = 0$.

Proof: By induction on N, the dimension of the N-simplex.

Step 1:

Initial observation: It is sufficient to prove the result for $k = 1$ as an iterative application to the resulting inequality readily leads to the result for an arbitrary k. Furthermore, we can focus on a fixed (but arbitrary) dimension N of the simplex $\mathcal{S}$. For the remainder of the proof, let T denote an arbitrary N-dimensional derived subsimplex of $\mathcal{S}$. We thus have to show that

$$\delta(T^1) \equiv \delta(T) \;\leq\; \frac{N}{N+1}\delta(\mathcal{S}). \tag{B.2}$$

Step 2:

Base case: For N=0, the 0-simplex is given by a single point $\mathcal{S} = co(x^0) = x^0$. By definition 3.3 of the barycentric subdivision, $Sd(\mathcal{S}) = \mathcal{S}$. Therefore, $\delta(T) = \delta(\mathcal{S}) = 0$.

Step 3:

Induction hypothesis: Assume (B.2) to hold for a fixed (but arbitrary) $N-1$. We will now show the result for N.

Step 4:

Induction step: The arbitrary derived subsimplex is given by $T = co(\{y^0, ..., y^N\})$. Since we can focus WLOG on the case $k = 1$ (i.e. the *first* barycentric subdivision), we may assume that y^k is the barycenter of the original simplex $\mathcal{S}$ (since the barycenter of $\mathcal{S}$ is a vertex of *every* derived subsimplex for $k = 1$). Therefore, y^k is given by $y^k = \frac{1}{N+1}\sum_{i=0}^N x^i$ by definition of the barycenter of $\mathcal{S}$. We next look at *any* $x \in \mathcal{S}$ which can be represented by its barycentric coordinates as $x = \sum_{i=0}^N \lambda_i x^i$ with $\sum_{i=0}^N \lambda_i = 1$ and $\lambda_i \geq 0\ \forall i$. We first consider a vertex x^j of $\mathcal{S}$. We have

$$\begin{aligned}
\|x^j - y^k\| &= \left\| x^j - \frac{1}{N+1}\sum_{i=0}^{N} x^i \right\| \\
&= \left\| \frac{N+1}{N+1}x^j - \frac{1}{N+1}x^j - \frac{1}{N+1}\sum_{i\neq j}^{N} x^i \right\| \\
&= \left\| \frac{1}{N+1}\sum_{i\neq j}^{N}(x^j - x^i) \right\| && (\Delta - \text{inequ.}) \\
&\leq \frac{1}{N+1}\sum_{i\neq j}^{N}\|x^j - x^i\| && (\text{Sum of N terms}) \\
&\leq \frac{N}{N+1}\sup_{j,i\in\{0,\ldots,N\}}\|x^j - x^i\| && (\text{Def. of } \delta(\mathcal{S})) \\
&= \frac{N}{N+1}\,\delta(\mathcal{S}). && \text{(B.3)}
\end{aligned}$$

Next, we use (B.3) and consider the arbitrary $x \in \mathcal{S}$:

$$\begin{aligned}
\|x - y^k\| &= \left\| \sum_{i=0}^{N}\lambda_i x^i - 1\cdot y^k \right\| = \left\| \sum_{i=0}^{N}\lambda_i x^i - \sum_{i=0}^{N}\lambda_i \cdot y^k \right\| \\
&= \left\| \sum_{i=0}^{N}\lambda_i (x^i - y^k) \right\| \\
&\leq \sum_{i=0}^{N}\lambda_i \underbrace{\|x^i - y^k\|}_{\text{use } (B.3)} \\
&\leq \frac{N}{N+1}\,\delta(\mathcal{S}). && \text{(B.4)}
\end{aligned}$$

Since (B.4) holds for *any* $x \in \mathcal{S}$, it holds in particular for $x \in \{y^0, ..., y^{k-1}\}$, i.e. x being one of the simplices (except y^k) of our derived N-simplex T. Hence, it holds for the distance of y^k to any of the vertices of T (and hence for the maximum):

$$\sup_{0\leq i\leq k-1}\|y^i - y^k\| \leq \frac{N}{N+1}\,\delta(\mathcal{S}).$$

But by the *induction hypothesis*, we have

$$\begin{aligned}\sup_{0\leq i\leq k-1} \|y^i - y^k\| \;&=\; \delta(co(\{y^0, ..., y^{k-1}\})) \\ &\overset{IH}{\leq}\; \frac{N-1}{(N-1)+1}\,\delta(\mathcal{R}) \;=\; \underbrace{\frac{N-1}{N}}_{\leq \frac{N}{N+1}}\,\underbrace{\delta(\mathcal{R})}_{\leq \delta(\mathcal{S})} \\ &\leq\; \frac{N}{N+1}\,\delta(\mathcal{S}),\end{aligned} \tag{B.5}$$

where $\mathcal{R}$ is the (N-1)-dimensional face of the original simplex $\mathcal{S}$ in whose barycentric subdivision the derived simplex $co(\{y^0, ..., y^{k-1}\})$ has emerged. We have thus shown (B.2) for N, i.e.

$$\delta(T) = \sup_{j,i\in\{0,...,N\}} \|y^i - y^j\| \leq \frac{N}{N+1}\,\delta(\mathcal{S}).$$

With the introductory remarks in *Step 1*, the desired result (B.1) follows for all $N \in \mathbb{N}$ by induction.

Finally, since $\left(\frac{N}{N+1}\right) < 1$ and $\delta(\mathcal{S}) < \infty$, $\lim\limits_{k\to\infty} \delta(\mathcal{S}^k) = 0$ follows immediately for any derived simplex $\mathcal{S}^k$ of order k in the k^{th} barycentric subdivision of the N-simplex $\mathcal{S}$. This shows that the mesh of the barycentric subdivision of an N-simplex can be made arbitrarily small by choosing the order of the barycentric subdivision k large enough.

□

B.2 Proof of Lemma 3.7

Lemma B.2 (Helper function k to construct homeomorphism) *[AY17; EZ95]*
*Let the set of points $r(x) := \{\alpha x \mid \alpha \in \mathbb{R}_+\}$ denote the **ray** of $x \in \mathbb{R}^n$ and let $C \subset \mathbb{R}^n$ denote a nonempty, compact and convex subset of $\mathbb{R}^n$. Then, the function $k\colon \mathbb{R}^n \setminus \{0\} \to C$ defined by*

$$k(x) = y \quad \textit{such that} \quad y \in r(x) \cap C, \quad \textit{but} \quad \forall \alpha > 1, \alpha y \notin C \tag{B.6}$$

is well-defined, bounded and continuous.

Proof:

Step 1:
Show that k is well-defined: We want to show that an arbitrary $x \in \mathbb{R}^n \setminus \{0\}$ is assigned a unique $y \in r(x) \cap C$ by k. To do so, we first consider the set $N := \{\|y\| \mid y \in r(x) \cap C\}$ which consists of the norms of all y that lie on the ray $r(x)$ and are contained in C. As C is compact and has a nonempty interior by assumption, the same is true for $r(x) \cap C$ and thereby especially for N. Consequently, $r_x := \sup(N)$ must exist and $r_x \in \overline{N}$. By a standard result on suprema, there exists a sequence $(y_n)_{n\in\mathbb{N}}$, $y_n \in r(x) \cap C \ \forall n \in \mathbb{N}$, such that $\lim_{n\to\infty}\|y_n\| = r_x$, i.e the norms of the sequence elements must converge to the supremum r_x of N. Further note that all elements of the sequence $(y_n)_{n\in\mathbb{N}}$ are contained in the compact set $r(x) \cap C$, so by the Bolzano-Weierstrass theorem there exists a convergent subsequence $(y_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} y_{n_k} = y \in r(x) \cap C$. As the norms of $(y_{n_k})_{k\in\mathbb{N}}$ converge to r_x as just established, it must also hold for the limit point y that $\|y\| = r_x = \sup(N)$. As C is convex by assumption and $0 \in C$, we know that $\lambda y \in C \ \forall y \in [0, 1]$. Assume there would exist a $y' \in r(x) \cap C$ such that $y' = \alpha y$ with $\alpha > 1$. This would mean that $\|y'\| = \|\alpha y\| = |\alpha|\|y\| = \alpha\|y\| > \|y\|$ which is a contradiction to $\|y\| = r_x = \sup(N)$. This shows that the assignment $y = k(x)$ is unique and well-defined.

Step 2:
Show that k is bounded: This follows immediately from the definition of k as k is bounded below by 0 and bounded above by the compact (hence bounded) set K.

Step 3:
Show that k is continuous: (By contradiction, via the definition of k)
We know that WLOG $0 \in C$, so there exists an ε' s.t. $B_{\varepsilon'}(0) \subset C$ (choose ε' small enough). We define two sets which are needed in the proof:

1. $K := co(B_{\varepsilon'}(0) \cup k(x))$

Since $B_{\varepsilon'}(0) \subset C$ and also $k(x) \in C$ [by def. of k], it follows that $K \subset C$ since K is by definition the smallest convex set that contains $B_{\varepsilon'}(0)$ and $k(x)$.

2. $T := K \cap \partial B(k(x), \varepsilon)$
   We can always choose ε small enough such that $0 \notin B(k(x), \varepsilon)$. This ε will become important in the following. We also note that $T \subset C$.

We proceed *by contradiction*, assuming that k is *not continuous* at a point $x \in \mathbb{R}^n \setminus \{0\}$. In that case, there is an $\varepsilon > 0$ such that for all $\delta > 0$, there exists a $y_\delta \in \mathbb{R}^n \setminus \{0\}$ such that

$$\|x - y_\delta\| < \delta \quad \text{but} \quad \|k(x) - k(y_\delta)\| \geq \varepsilon. \tag{B.7}$$

Since (B.7) holds for *all* $\delta > 0$, let $\delta = \frac{1}{n}$ in the following. We now construct a sequence $(y_n)_{n \in \mathbb{N}}$ with $y_n = y_\delta$, where y_δ is precisely the point in (B.7) for which continuity of k at x is *not* fulfilled.

Next, we collect some important observations on the properties of this sequence:

1. $\lim_{n\to\infty} y_n = x$. This follows immediately as $\|x - y_n\| < \delta \; \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \delta = 0$. In other words, y_n approaches x *arbitrarily* close (this will become important shortly).

2. (B.7) readily implies that $k(y_n) \notin B_\varepsilon(x) \; \forall n \in N$.

3. Choosing n sufficiently large (i.e. $n \geq N$ where N is some threshold), we know that the intersection of the ray $r(y_n) = \{\alpha y_n \mid \alpha \in \mathbb{R}_+\}$ and the set T is nonempty. This follows since y_n approaches x for n sufficiently large and T is defined via $k(x)$. Hence, for $n \geq N$, there exists a point $z(y_n) \in r(y_n) \cap T$.

4. For $n \geq N$, the distance from the origin to $k(y_n)$ has to be larger than the distance from the origin to $B_\varepsilon(k(x))$.

Property (4) follows again by contradiction:

$$\text{Assume } k(y_n) \text{ was closer to the origin than } B_\varepsilon(k(x)). \tag{B.8}$$

Since $z(y_n) \in T$ and hence $z(y_n) \in \partial B_\varepsilon(k(x))$, (B.8) implies in particular that $k(y_n)$ would be closer to the origin than $z(y_n)$. Furthermore, we know that $k(y_n)$ [by def. of k] as well as $z(y_n)$ [by def. of $z(y_n)$] are located on the ray $r(y_n) = \{\alpha y_n \mid \alpha \in \mathbb{R}_+\}$ (i.e. they point in the *same* directions). We can hence always find two scalars $\phi_1, \phi_2 \in \mathbb{R}_+$ such that

$$k(y_n) = \phi_1 y_n \qquad \text{and} \qquad z(y_n) = \phi_2 y_n. \tag{B.9}$$

Therefore,

$$\|k(y_n)\| \underbrace{<}_{(B.8)} \|z(y_n)\| \quad \Leftrightarrow \quad \|\phi_1 y_n\| \underbrace{<}_{(B.9)} \|\phi_2 z(y_n)\| \quad \Leftrightarrow \quad \phi_1 < \phi_2. \tag{B.10}$$

But then, defining $\tilde{\alpha} := \frac{\phi_2}{\phi_1}$ yields $\tilde{\alpha} > 1$ and at the same time

$$\tilde{\alpha} k(y_n) \;=\; \frac{\phi_2}{\phi_1}\, \phi_1 y_n \;=\; \phi_2 y_n \;=\; z(y_n). \tag{B.11}$$

But this is a direct contradiction to the defining property of k as for all $x \in \mathbb{R}^n \setminus \{0\}$, $k(x) = y$ such that precisely $\alpha y \;\notin C$ for $\alpha > 1$. Therefore, our temporary assumption (B.8) must have been wrong. This establishes property (4) for the sequence $(y_n)_{n\in\mathbb{N}}$. In other words: $k(y_n)$ must have a larger distance from the origin than $B_\varepsilon(k(x))$.

Next, we look at the sequence $(k(y_n))_{n\in\mathbb{N}}$. Since $k(y_n) \in C$ and C is compact, the Bolzano-Weierstrass theorem guarantees the existence of a convergent subsequence with limit point $y \in C$. As $k(y_n) \in r(y_n)\ \forall n \in N$ and since $\lim_{n\to\infty} y_n = x$ [property (1) above], it must also hold that the difference in the *directions* of the rays $r(x)$ and $r(y_n)$ becomes arbitrarily small. The limit point y of the sequence $(k(y_n))_{n\in\mathbb{N}}$ must therefore fulfill $y \in r(x)$. Since $k(x) \in r(x)$ as well, this means that y and $k(x)$ point in the same direction. Normalizing both vectors to unit length therefore implies the following equalities.

$$\frac{y}{\|y\|} \;=\; \frac{k(x)}{\|k(x)\|} \quad \Leftrightarrow \quad y \;=\; \underbrace{\frac{\|y\|}{\|k(x)\|}}_{=:\,\beta}\, k(x) \;=\; \beta k(x). \tag{B.12}$$

By property (4) of $(y_n)_{n\in\mathbb{N}}$ above, each element of the sequence (and hence the limit point y) must have a larger distance from the origin than $B_\varepsilon(k(x))$ (and hence in particular than $k(x)$), i.e. $\|y\| > \|k(x)\|$. But then (B.12) yields $y \in C$ with $y = \beta k(x)$ with $\beta > 1$. This is again a direct contradiction to the definition of the function k.

This contradiction implies that our assumption (B.7) must have been wrong. In other words, there cannot exist an $\varepsilon > 0$ s.t. $\|x - y_n\| < \delta$ but $\|k(x) - k(y_n)\| \geq \varepsilon$. This proves that k must be continuous for all $x \in \mathbb{R}^n \setminus \{0\}$.

$\square$

B.3 Homeomorphism between a simplex and a closed ball

In order to gain some further intuition, we provide an idea of the proof that an N-simplex is homeomorphic to a closed unit ball. Furthermore, the constructed homeomorphism has an additional property - it maps the boundary of the simplex to the boundary of the ball. Figure 20 below illustrates one key ingredient for the proof of this result - we see that the principle idea is very similar to the helper function k introduced in section 3.3.

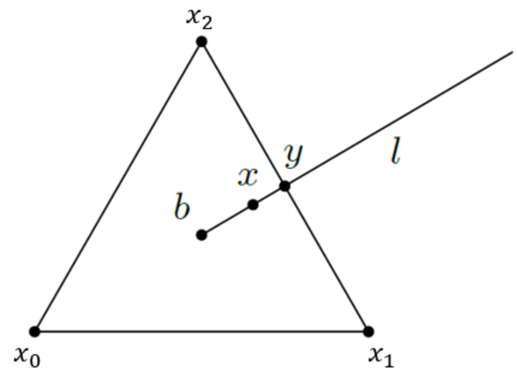


Figure 20: Idea used in the proof of proposition B.3 (Source: MB20).

Proposition B.3 (Homeomorphism between simplex and closed unit ball) *[EO92]* *Let $S = co(x^0, ..., x^N) \subset \mathbb{R}^n$ be an N-simplex and let $\overline{B}_1(0) = \{x \in \mathbb{R}^N \mid \|x\| \leq 1\} \subset \mathbb{R}^N$ denote the N-dimensional closed unit ball. Then, there exists a homeomorphism between $\mathcal{S}$ and $\overline{B}_1(0)$, i.e. $S \cong \overline{B}_1(0)$, such that ∂S is mapped to the $(N-1)$-dimensional sphere $S^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$.*

Proof:
Let $N > 0$ and denote by $x^0, ..., x^N$ the vertices of $\mathcal{S}$. We further denote by $b \in \mathcal{S} \setminus \partial\mathcal{S}$ an interior point of the simplex, for instance the barycenter of $\mathcal{S}$ (see definition 3.2), i.e. $b := \frac{1}{N+1} \sum_{i=0}^{N} x^i$. Note that we can always assume that the ball is centered at b (otherwise apply a continuous translation to the ball). We can always represent any point $x \in \mathcal{S} \setminus b$ uniquely by a convex combination of b and y where $y \in \partial S$ represents a point on the boundary of $\mathcal{S}$ (compare to figure 20 above):

$$x \;=\; b + \lambda(y - b) \;=\; (1-\lambda)b + \lambda y, \qquad \lambda \in [0, 1],$$

as the simplex is a convex set. Next, we define the line $l(x)$ which starts at b and passes through a given $x \in \mathcal{S} \setminus \{b\}$:

$$l(x) := \{b + \mu(x - b) \mid \mu \geq 0\} \;=\; \{(1 - \mu)b + \mu x \mid \mu \geq 0\}.$$

As the simplex $\mathcal{S}$ is bounded, $l(x)$ will eventually cross $\partial\mathcal{S}$ and we denote this unique intersection point by $y \in \partial\mathcal{S}$ (recall that the simplex is compact, hence especially closed). We can then explicitly construct the desired homeomorphism $f\colon \mathcal{S} \to \overline{B}_1(0)$ by connecting the barycenter b with the points y on the boundary of $\mathcal{S}$ with line segments and projecting the line segments by scaling them with the appropriate radius of the ball $\overline{B}_1(0)$. In other words:

$$f(x) := \underbrace{b}_{=0 \text{ WLOG}} + \frac{1}{\|y - b\|}\,(y - b) \;=\; \frac{1}{\|y - b\|}\,(y - b) \quad \forall x \in \mathcal{S}. \tag{B.13}$$

and $f(b) = 0$. Note that this is very similar to what we have done in lemma 3.7 by means of the helper function k. We have also proved there that k - and thereby the constructed homeomorphisms - fulfill several important properties, e.g. continuity. Finally note that we do not have to restrict ourselves to *unit* balls, as analogous reasoning and the adjusted homeomorphism

$$\hat{f}(x) := \frac{\beta}{\|y - b\|}\,(y - b) \quad \forall x \in \mathcal{S}.$$

and $\hat{f}(b) = 0$ would also hold for a closed ball with radius β.

□

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Fabian Smetak

(May 2021)

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