Shallow water equations in channel networks

Maya Briani

Istituto per le Applicazioni del Calcolo "M. Picone"
Consiglio Nazionale delle Ricerche

m.briani@iac.cnr.it
http://www.iac.rm.cnr.it/∼briani

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Plan of the talk

1. The Shallow Water Equations
2. The Riemann Problem
3. Water Flow in Canal Network
4. The Junction Riemann Problem
5. Fluvial to torrential transition
6. Conclusions and Open Problems

The Shallow Water Equations

The one-dimensional shallow water equations describe the water propagation in a canal with rectangular cross-section and constant slope:

$$\begin{cases} \partial_t h + \partial_x (hv) = 0 & \text{conservation of mass} \\ \partial_t (hv) + \partial_x (hv^2 + \frac{1}{2} g h^2) = 0 & \text{conservation of momentum} \end{cases} \quad (1)$$

- $h(x,t)$ the water height
- $v(x,t)$ the water velocity at time t and location x along the canal
- g the gravity constant

For the purpose of this talk, we have assumed a steady state friction on all canals and horizontal canals with zero slope.

The Shallow Water Equations

We reformulate system (1) in vector form as

$$\partial_t u + \partial_x f(u) = 0 \tag{2}$$

where

$$u = \begin{pmatrix} h \\ q \end{pmatrix} \quad f(u) = \begin{pmatrix} hv \\ hv^2 + \frac{1}{2}gh^2 \end{pmatrix} \tag{3}$$

and $q = hv$ (*discharge*, it measures the flow rate of water past a point).

The Riemann Problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ \\ u(x,0) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0. \end{cases} \end{cases} \tag{4}$$

Here $u(x,0) = (h(x,0), q(x,0))$ and $u_l = (h_l, q_l)$ and $u_r = (h_r, q_r)$.

The Riemann Problem

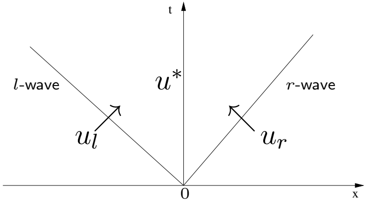


Figure: The solution to the Riemann problem. The intermediate state u^* is constant in the region delimited by l-wave and r-wave. l- and r-waves are shocks or rarefactions.

- The solution to this Riemann problem consists of the l-wave and the r-wave separated by an intermediate state $u^* = (h^*, q^*)$.
- This intermediate state is connected to $u_l = (h_l, q_l)$ through a physically correct l-waves, and to $u_r = (h_r, q_r)$ through a physically correct r-wave.

The Shallow Water Equations - The Riemann Problem

For smooth solution, system (2) can equivalently be written in the quasilinear form

$$\partial_t u + A(u)\partial_x u = 0$$

where the Jacobian matrix $A(u) = f'(u)$ is

$$A(u) = \begin{pmatrix} 0 & 1 \\ -v^2 + gh & 2v \end{pmatrix}$$

The eigenvalues of the matrix $A(u)$ are

$$\lambda_1(u) = v - \sqrt{gh}, \quad \lambda_2(u) = v + \sqrt{gh}$$

with the corresponding eigenvectors $r_1(u) = (1, v + \sqrt{gh})^T$ and $r_2(u) = (1, v + \sqrt{gh})$.

The Riemann Problem

- The shallow water equations are *genuinely nonlinear* ($\nabla \lambda_j(u) \cdot r_j(u) \neq 0$, $j = 1, 2$) and so the Riemann problem always consists of two waves, each of which is a shock or rarefaction.
- The left and right characteristics are associated to λ_1 and λ_2 respectively.
- $\lambda_1 = v - \sqrt{gh}$ and $\lambda_2 = v + \sqrt{gh}$ can be of either sign.
- The ratio $Fr = |v|/\sqrt{gh}$ is called the Froude number.
- When $v = q/h$ is smaller than the speed $\sqrt{gh}$ of the gravity waves:

  $$|v| < \sqrt{gh} \text{ or } Fr < 1$$

  the fluid is said to be fluvial or subcritical.

  If $|v| > \sqrt{gh}$ the fluid is said to be torrential or supercritical.
- Under the *fluvial regime*

  $$\lambda_1 < 0 \quad \lambda_2 > 0$$

  and there will be one left (with negative speed) and one right (with positive speed) going wave.

The solution always consists of two waves, each of which is a shock or rarefaction:

(R) Centered Rarefaction Waves. Assume u^+ lies on the positive i-rarefaction curve through u^-, then we get

$$u(x,t) = \begin{cases} u^- & \text{for } x < \lambda_i(u^-)t, \\ R_i(x/t; u^-) & \text{for } \lambda_i(u^-)t \leq x \leq \lambda_i(u^+)t, \\ u^+ & \text{for } x > \lambda_i(u^+)t, \end{cases}$$

(S) Shocks. Assume that the state u^+ is connected to the right of u^- by an i-shock, then calling $\lambda = \lambda_i(u^+, u^-)$ the Rankine-Hugoniot speed of the shock, the function

$$u(x,t) = \begin{cases} u^- & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases}$$

provides a the solution to the Riemann problem. For strictly hyperbolic systems, we have that

$$\lambda_i(u^+) < \lambda(u^-, u^+) < \lambda_i(u^-), \quad \lambda(u^-, u^+) = \frac{q^+ - q^-}{h^+ - h^-}.$$

The Riemann Problem - Lax curves

To find the intermediate state u^* in general we can define two functions ϕ_l and ϕ_r by

$$\phi_l(h) = \begin{cases} v_l - 2(\sqrt{gh} - \sqrt{gh_l}) & \text{if } h < h_l \text{ (rarefaction)} \\ v_l - (h - h_l)\sqrt{g\frac{h+h_l}{2hh_l}} & \text{if } h > h_l \text{ (shock wave),} \end{cases}$$

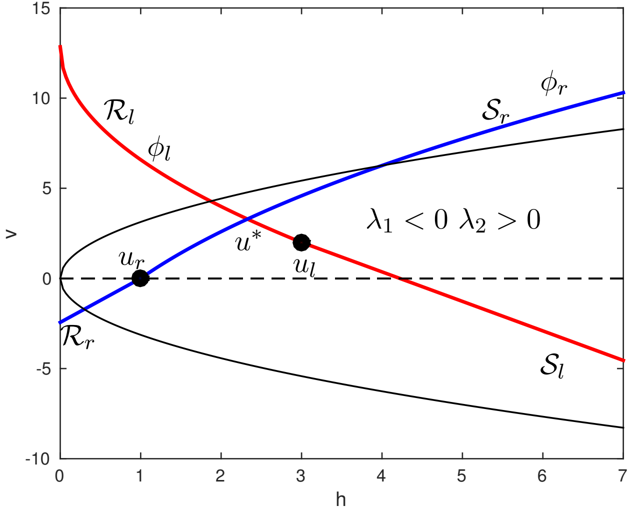
and

$$\phi_r(h) = \begin{cases} v_r + 2(\sqrt{gh} - \sqrt{gh_r}) & \text{if } h < h_r \text{ (rarefaction)} \\ v_r + (h - h_r)\sqrt{g\frac{h+h_r}{2hh_r}} & \text{if } h > h_r \text{ (shock wave).} \end{cases}$$

For a given state h

- the function $\phi_l(h)$ returns the value of v such that (h, hv) can be connected to u_l by a physically correct l-wave
- the function $\phi_r(h)$ returns the value of v such that (h, hv) can be connected to u_r by a physically correct r-wave.
- So, h^* is such that $\phi_l(h^*) = \phi_r(h^*)$

The Riemann Problem

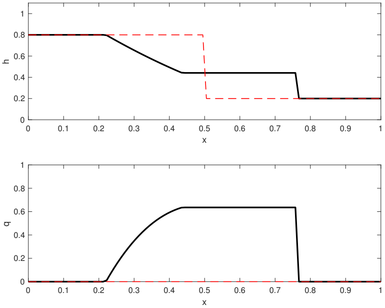


Example: Dam-Break and Riemann Problem

Consider the Riemann problem with

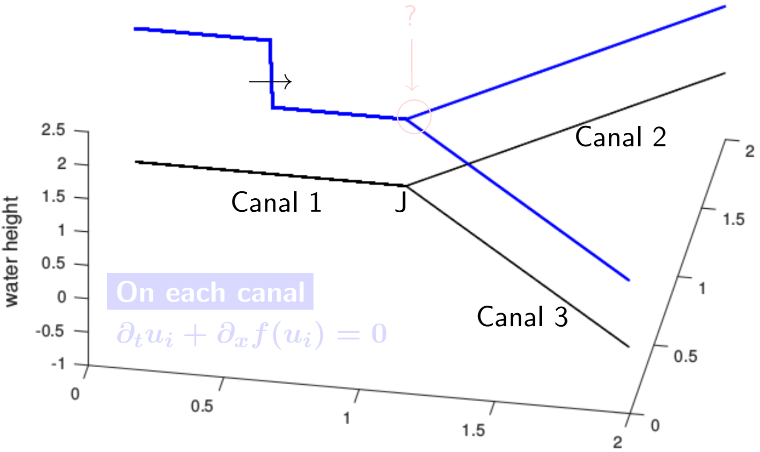
$$u_l = \begin{pmatrix} h_l \\ q_l \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u_r = \begin{pmatrix} h_r \\ q_r \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}.$$

- $h_l > h_r$ and $q_l = q_r = 0$. This Riemann problem models what happens in a dam separating two levels of water breaks at time $t = 0$
- The solution consists of a l-rarefaction and a r-shock



Water Flow in a Canal Network

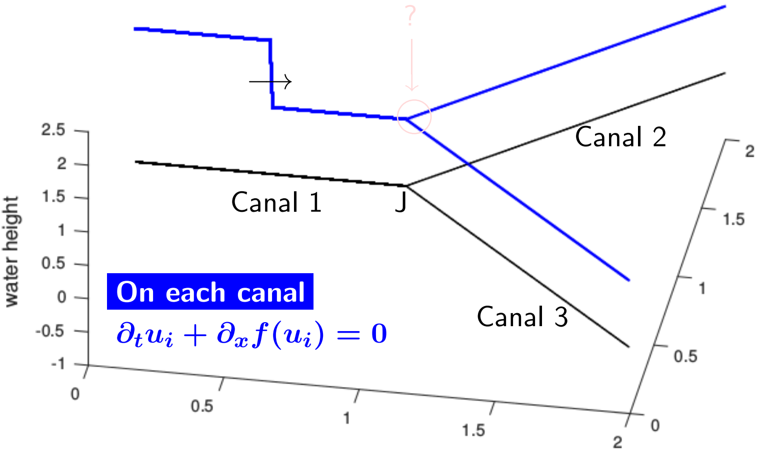
Water Flow in a Channel Network



On each canal

$\partial_t u_i + \partial_x f(u_i) = 0$

Water Flow in a Channel Network



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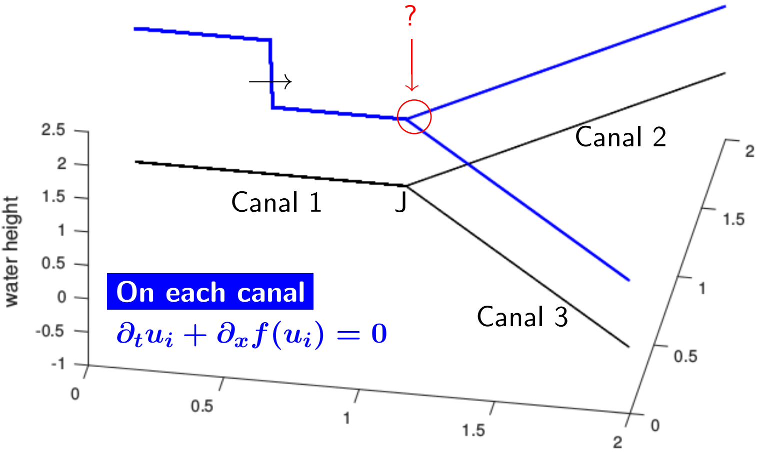
Water Flow in a Channel Network



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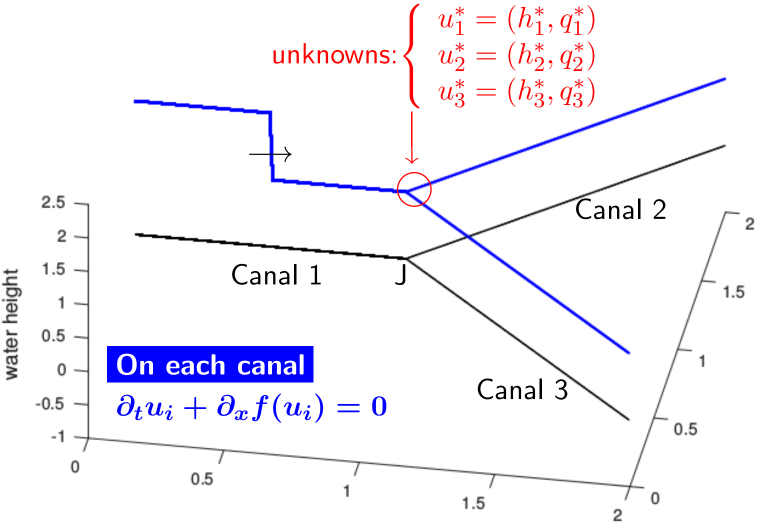
Water Flow in a Channel Network



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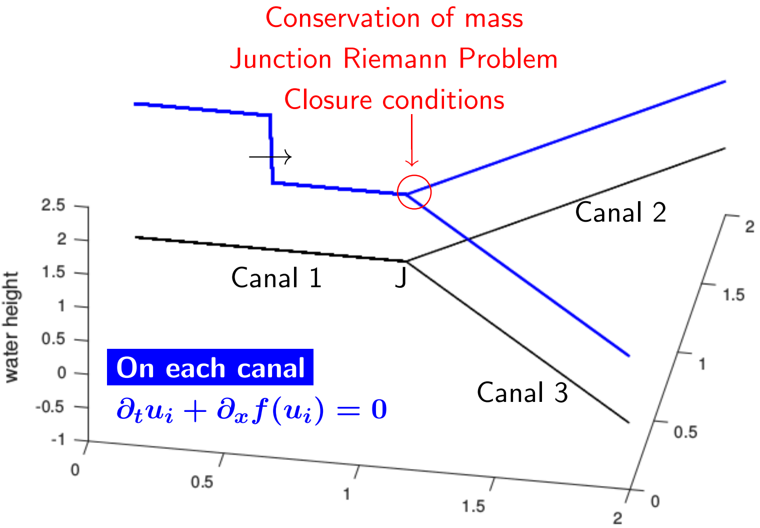
Water Flow in a Channel Network



Water Flow in Canal Network: 1-to-2 Junction

Assuming that the three canals are connected at $x = 0$:

Canal 1 $(x < 0)$

$$\partial_t u_1 + \partial_x f(u_1) = 0$$

Canal 2 and 3 $(x > 0)$

$$\partial_t u_2 + \partial_x f(u_2) = 0$$
$$\partial_t u_3 + \partial_x f(u_3) = 0$$

Assuming the **conservation of mass**

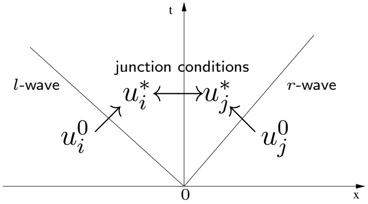
$$q_1^*(0^-, t) = q_2^*(0^+, t) + q_3^*(0^+, t)$$

To get a well-posed problem we need 5 additional conditions

The Junction Riemann Problem

The solution is determined once one assigns a **Riemann Solver** at the junction. Considering only **subcritical** states

- given constant initial conditions (u_i^0, u_j^0) (i ranges over incoming canals, j over outgoing ones);
- the Junction Riemann solution consists of intermediate states (u_i^*, u_j^*) satisfying some other junction conditions

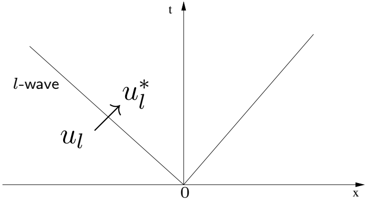


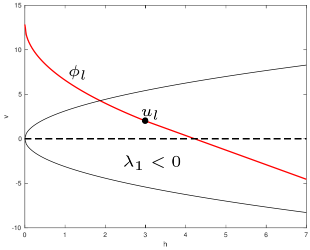
Left-half Riemann Problem (the case of an incoming canal)

We fix a left state and we look for the right states attainable by waves of negative speed. $\Rightarrow$ Fix $u_l = (h_l, q_l)$, we look for the set of points $u_l^* = (h_l^*, q_l^*)$ such that the solution to the Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(x,0) = \begin{cases} u_l & \text{if } x < 0 \\ u_l^* & \text{if } x > 0 \end{cases} \end{cases}$$

contains only waves with negative speed ($\lambda_1(u_l^*) < 0$).



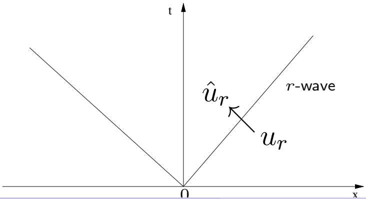


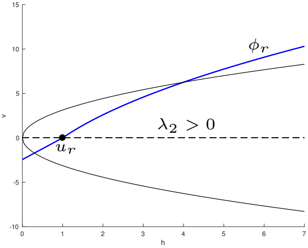
Right-half Riemann problem (the case of an outgoing canal)

We fix a right state and we look for the left states attainable by waves of positive speed. $\Rightarrow$ Fix $u_r = (h_r, q_r)$, we look for the set of points $u_r^* = (h_r^*, q_r^*)$ such that the solution to the Riemann problem

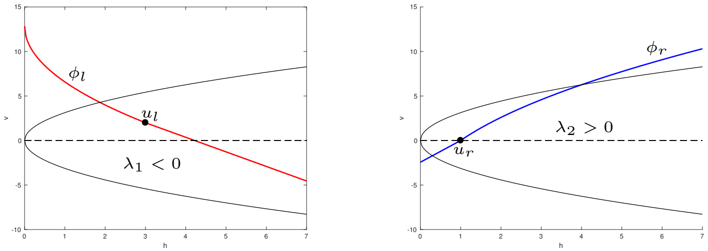
$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(x,0) = \begin{cases} u_r^* & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases} \end{cases}$$

contains only waves with positive speed ($\lambda_2(u_r^*)) > 0$).





The Junction Riemann Problem



$$v_l^* = \phi_l(h_l^*; h_l, v_l) \qquad v_r^* = \phi_r(h_r^*; h_r, v_r)$$

$$|v_{l,r}^*| < \sqrt{g h_{l,r}^*}$$

Water Flow in a Canal Network - Junction Conditions

We have so far set 4 conditions:

$$q_1^* = q_2^* + q_3^* \quad \text{and}$$

$$v_1^* = \phi_l(h^*; u_1^0) \quad v_2^* = \phi_r(h^*; u_2^0) \quad v_3^* = \phi_r(h^*; u_3^0)$$

We need 2 additional conditions:

- Physical reasons motivate different choices of conditions that are originally derived by engineers
- Which conditions are used often depends on if the flow is subcritical or supercritical

Water Flow in Canal Network - Junction Conditions

The conservation of mass is usually coupled with the following:

- Equal water pressure (equal water heights)

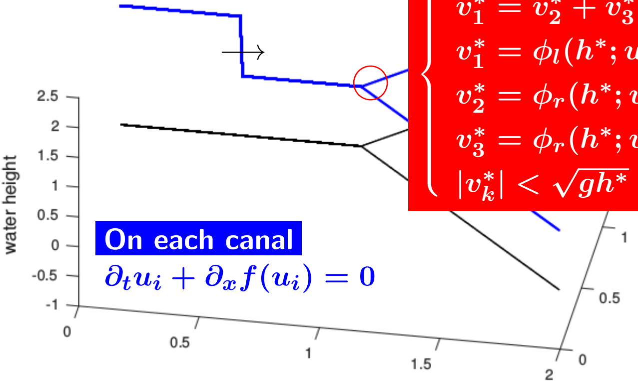
$$\frac{1}{2}gh_k^2 = \frac{1}{2}gh_l^2 \quad \forall t > 0$$

- Energy continuity (equal of energy levels)

$$h_k + \frac{v_k^2}{2g} = h_l + \frac{v_l^2}{2g} \quad \forall t > 0$$

Other conditions which depend on the geometry:
▶ **Preprint 2021** M. Briani, G. Puppo, M. Ribot, *Angle dependence in coupling conditions for shallow water equations at canal junctions*

Water Flow in a Channel Network



$$\begin{cases} h_1^* = h_2^* = h_3^* = h^*, \\ v_1^* = v_2^* + v_3^*, \\ v_1^* = \phi_l(h^*; u_1^0), \\ v_2^* = \phi_r(h^*; u_2^0), \\ v_3^* = \phi_r(h^*; u_3^0), \\ |v_k^*| < \sqrt{gh^*} \end{cases}$$

On each canal

$$\partial_t u_i + \partial_x f(u_i) = 0$$

starting by Riemann data

$$u_1(J^-) = u_1^0, \quad u_2(J^+) = u_2^0, \quad u_3(J^+) = u_3^0.$$

The Junction Riemann Problem

The solution at the Junction then consists on **solving the non-linear system**

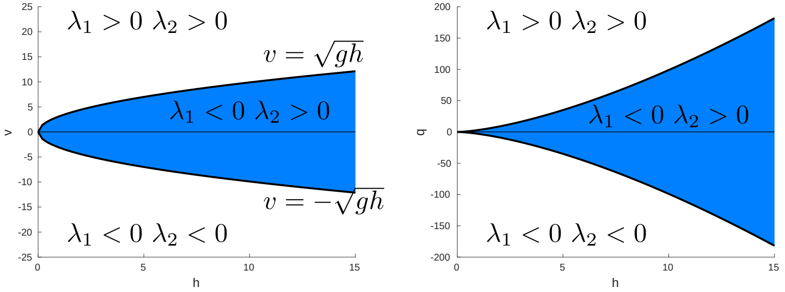
$$\begin{cases} h_1^* = h_2^* = h_3^* = h^*, \\ v_1^* = v_2^* + v_3^*, \\ v_1^* = \phi_l(h^*; u_1^0), \\ v_2^* = \phi_r(h^*; u_2^0), \\ v_3^* = \phi_r(h^*; u_3^0), \end{cases} \tag{5}$$

with $h^* > 0$ and the *subcritical assumption* $|v_k^*| < \sqrt{gh^*}$, $k = 1, 2, 3$

- the system admits a unique solution (see for instance Marigo 2010) **... but** the solution not always verifies the subcritical condition
- suitable initial data have to be given to ensure the fluvial regime to the problem.

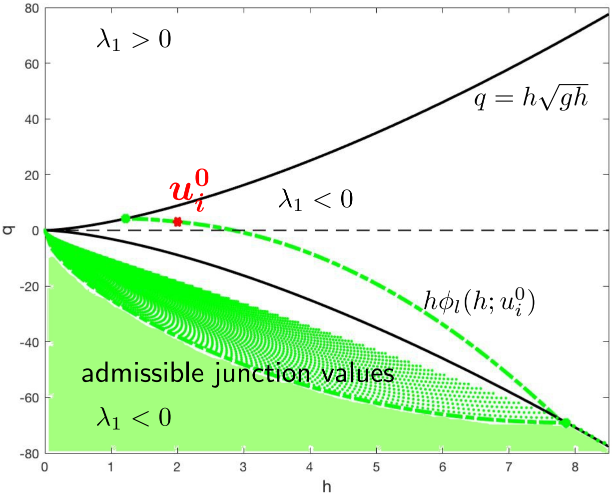
Fluvial and Torrential regime

What happens if we expand the domain to include the torrential regime?

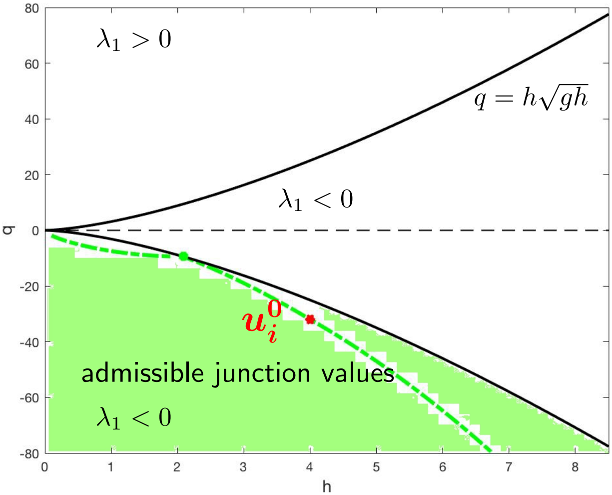


What happens if one of the states is in the torrential regime?

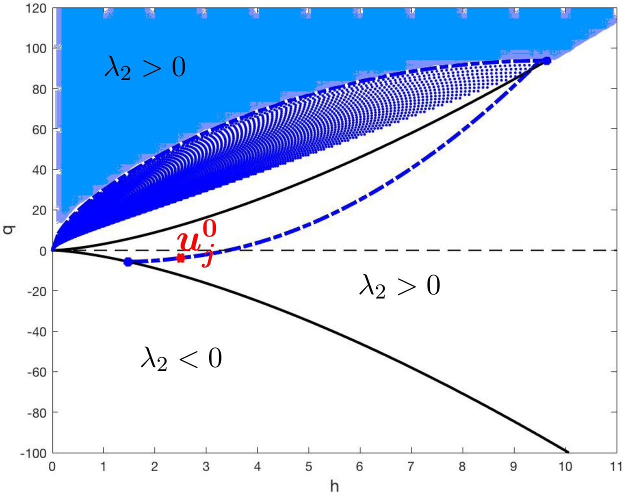
Fluvial and Torrential regime: the case of an incoming canal



Fluvial and Torrential regime: the case of an incoming canal

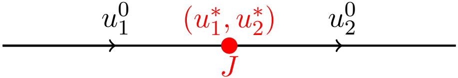


Fluvial and Torrential regime: outgoing canal



The case study of a simple network

We consider a fictitious network formed by two canals intersecting at one single point, which artificially represents the junction.



- Conservation of Mass $q_1^* = q_2^*$
- Junction Riemann Problem: $u_1^* \in \mathcal{N}(u_1^0)$, $u_2^* \in \mathcal{P}(u_2^0)$
- we need ? additional conditions

The case study of a simple network: $1 \to 1$ Junction

- Conservation of Mass $q_1^* = q_2^*$
- Junction Riemann Problem: $u_1^* \in \mathcal{N}(u_1^0)$, $u_2^* \in \mathcal{P}(u_2^0)$
- we need additional conditions ...

In this simple junction, the natural assumption (consistent with the dynamic of shallow-water equations on a single canal) should be to assume the **conservation of the momentum**:

$$\frac{(q_1^*)^2}{h_1^*} + \frac{1}{2}g(h_1^*)^2 = \frac{(q_2^*)^2}{h_2^*} + \frac{1}{2}g(h_1^*)^2.$$

The case study of a simple network: $1 \to 1$ Junction

From the conservation of the momentum and $q_1^* = q_2^*$

$$\left(\frac{h_2}{h_1}\right)^3 - \left(2\mathcal{F}_1^2 + 1\right)\left(\frac{h_2}{h_1}\right) + 2\mathcal{F}_1^2 = 0$$

and we have two possible relations for the heights values at the junction:

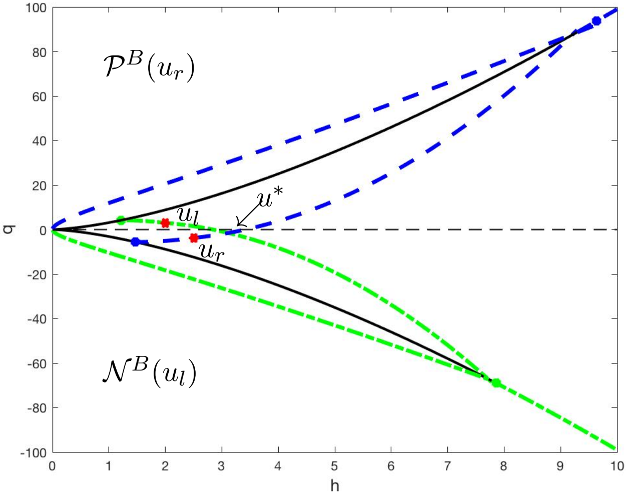
$h_1^* = h_2^*$ (equal heigths) or $\frac{h_2^*}{h_1^*} = \frac{1}{2}\left(-1 + \sqrt{1 + 8\mathcal{F}_1^2}\right)$

Let us assume equal water heights, no jump at the junction

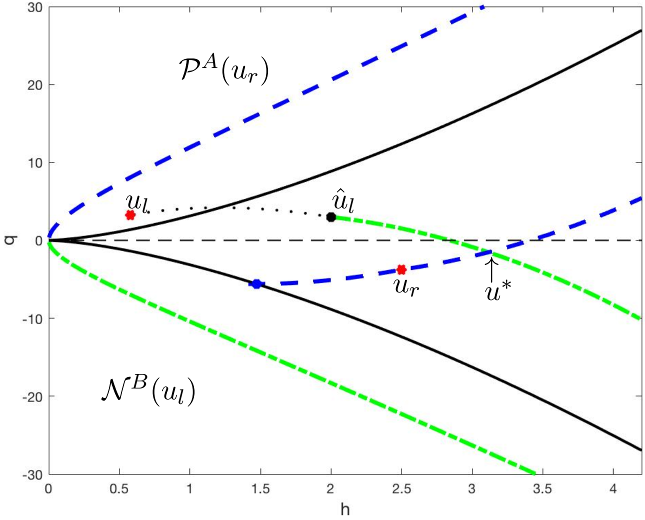
$$h_1^* = h_2^*$$

Then $u_1^* = u_2^* = u^*$ and the solution is identified (if exists) by the intersection between the two admissible regions $\mathcal{N}(u_1^0)$ and $\mathcal{P}(u_2^0)$!

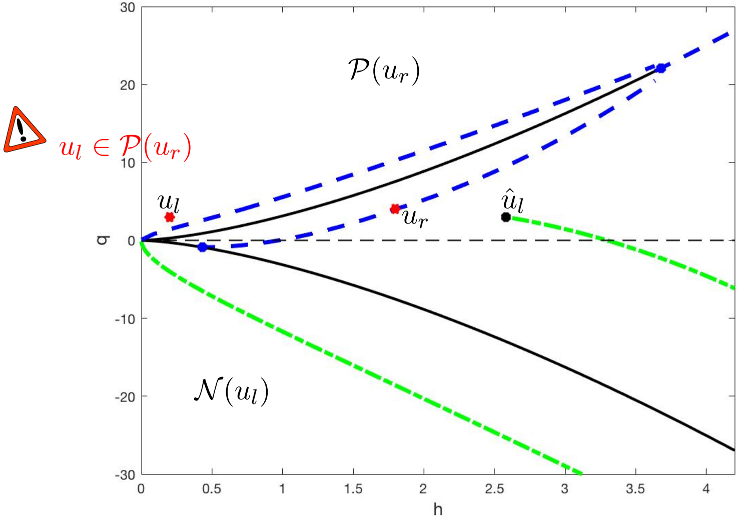
$1 \to 1$ Junction: Fluvial $\to$ Fluvial



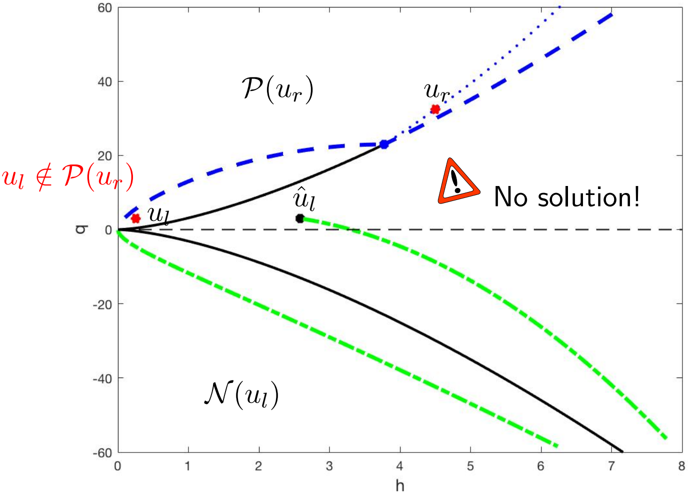
$1 \to 1$ Junction: Torrential $\to$ Fluvial



$1 \to 1$ Junction: Torrential $\to$ Fluvial



1 → 1 Junction: Torrential → Torrential

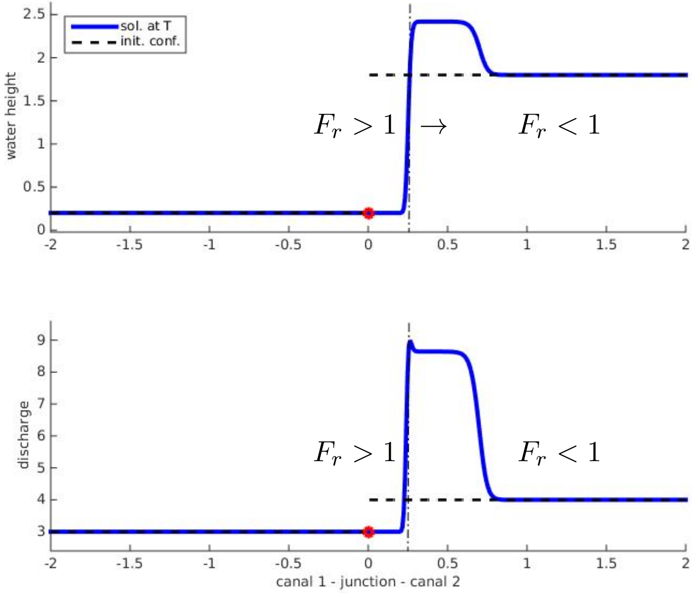


The case study of a simple network: $1 \to 1$ Junction

Assuming $h_1^* = h_2^*$ the solution does not always exist ...

- Assuming $h_1^* = h_2^*$ the solution does not always exist ...
- For $h_1^* \neq h_2^*$ we get new possible solutions at the junction ... the cases Torrential$\to$ Fluvial and case Torrential$\to$ Torrential may admit solution even if their admissible regions have empty intersection in the subcritical region.

Consistency with the case of a single canal:
for appropriate values of (h_l, q_l), for Torrential$\to$ Fluvial we get the same solution considering our simple network as a simple canal, i.e. we get a stationary shock at the virtual junction called *hydraulic jump* characterized indeed by the conservation of the momentum in the transition from a supercritical to subcritical flow.


sol. at T
init. conf.
water height
$F_r > 1$ $\rightarrow$ $F_r < 1$
discharge
$F_r > 1$ $F_r < 1$
canal 1 - junction - canal 2

The extension to more complex network is still an Open Problem!

Conclusion

- Two regimes exist for this hyperbolic system of balance laws: the fluvial, corresponding to eigenvalues with different sign, and the torrential, corresponding to both positive eigenvalues
- After analyzing the Lax curves for incoming and outgoing canals, we provide admissibility conditions for Riemann solvers, describing possible solutions for constant initial data on each canal.
- The simple case of one incoming and outgoing canal is treated showing that, already in this simple example, regimes transitions appear naturally at junctions.

Further work

- Open canals flow with fluvial to torrential phase transitions on more complex networks
- Condition at the node depending on the geometry of the network and comparison with 2D simulations (joint work with M. Ribot and G. Puppo)
- Problems with non constant bottom level and non constant width channels on networks

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Thank you very much for your attention