Random Schrödinger operators with a background potential

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1 Notations, assumptions

We consider Schrödinger operators on $L^2(\mathbb{R})$ of the form

$$H_\omega \;=\; -\frac{d^2}{dx^2} + U + V_{per} + V_\omega. \tag{1}$$

We assume that the background potential U belongs to the space of real valued uniformly square integrable functions

$$L^2_{\text{loc, unif}} \;=\; \{F:\mathbb{R}\to\mathbb{R} \mid \sup_{x\in\mathbb{R}} \int_{x-1}^{x+1} |F(x)|^2\,dx < \infty\} \tag{2}$$

and

$$U(x) \;\to\; a^- \quad \text{as} \quad x\to-\infty, \qquad U(x) \;\to\; a^+ \quad \text{as} \quad x\to+\infty. \tag{3}$$

Moreover, V_{per} is a 1-periodic real valued function in $L^2_{\text{loc, unif}}$.
V_ω is a random alloy-type potential of the form

$$V_\omega(x) = \sum_{k=-\infty}^{\infty} q_k(\omega) f(x-k) \quad (x\in\mathbb{R}), \tag{4}$$

where q_k are independent random random variables with a common distribution P_0.
We suppose that f, called the single site potential, is a real valued function satisfying

$$|f(x)| \leqslant C\,(1+|x|)^{-\gamma} \quad (x\in\mathbb{R}) \tag{5}$$

for some $\gamma > 1$.
We assume for simplicity that supp P_0 is a compact subset of $\mathbb{R}$. We remark that it would be sufficient that enough moments of P_0 exist. Moreover, f may have local singularities.

Under the above assumptions the potentials U, V_{per}, V_{ω} and there sums belong to $L^2_{\text{loc, unif}}$, hence they are H_0-bounded by [8], Theorem XIII.96 and all operators are essentially self adjoint on $C_0^{\infty}(\mathbb{R})$.
We introduce the following notations:

$$H_0 = -\frac{d^2}{dx^2} \quad \text{(the free Hamiltonian)}, \tag{6}$$

$$H_U = H_0 + U \tag{7}$$

$$H_{per} = H_0 + V_{per}, \tag{8}$$

$$H_{U,per} = H_0 + U + V_{per}. \tag{9}$$

2 The essential spectra of H_{U+V} and $H_{U,per}$

One of the main observations of this section is the following result.

Theorem 2.1. *Let $U_1, U_2, V : \mathbb{R} \to \mathbb{R}$ be H_0-bounded measurable functions and*

$$U_j(x) \xrightarrow[x \to -\infty]{} a^-, \quad U_j(x) \xrightarrow[x \to \infty]{} a^+ \quad (j = 1, 2)$$

for some $a^{\pm} \in \mathbb{R}$. Then

$$\sigma_{ess}(H_{U_1+V}) = \sigma_{ess}(H_{U_2+V}).$$

Proof. We need to prove that

$$\sigma_{ess}(H_{U_1+V}) \subset \sigma_{ess}(H_{U_2+V}),$$

$$\sigma_{ess}(H_{U_2+V}) \subset \sigma_{ess}(H_{U_1+V}).$$

We'll prove the first inclusion (the proof of the second one is similar). Let

$$\lambda \in \sigma_{ess}(H_{U_1+V}).$$

By Weyl's criterion and Theorem 3.11 in [3] we conclude that there is a Weyl sequence of functions $\varphi_n \in C_0^{\infty}(\mathbb{R})$ such that

$$\|\varphi_n\|_2 = 1 \quad (n \in \mathbb{N}),$$

$$\|(H_{U_1+V} - \lambda I)\varphi_n\|_2 \to 0 \tag{10}$$

such that either

$$\text{supp } \varphi_n \subset (-\infty, n) \quad \text{for all } n \tag{11}$$

or

$$\text{supp } \varphi_n \subset (n, \infty) \quad \text{for all } n \tag{12}$$

holds. Assume (11) is true, then

$$\|(H_{U_1+V}-\lambda I)\varphi_n\|_2-\left\|\left(H_V-\left(\lambda-a^-\right)I\right)\varphi_n\right\|_2\to 0,$$

$$\|(H_{U_2+V}-\lambda I)\varphi_n\|_2-\left\|\left(H_V-\left(\lambda-a^-\right)I\right)\varphi_n\right\|_2\to 0$$

and hence

$$\|(H_{U_1+V}-\lambda I)\varphi_n\|_2-\|(H_{U_2+V}-\lambda I)\varphi_n\|_2\to 0.$$

From this and (10) we obtain

$$\|(H_{U_2+V}-\lambda I)\varphi_n\|_2\to 0,$$

therefore

$$\lambda\in\sigma_{ess}(H_{U_2+V}).$$

□

As a corollary to the proof of Theorem 2.1 we get

Corollary 2.2. *Let $U,V:\mathbb{R}\to\mathbb{R}$ be measurable, H_0-bounded and*

$$U(x)\xrightarrow[x\to-\infty]{}a^-,\quad U(x)\xrightarrow[x\to\infty]{}a^+$$

(in the usual sense), where $a^\pm\in\mathbb{R}$. Then

$$\sigma_{ess}(H_{U+V})\subset\left(a^-+\sigma_{ess}(H_V)\right)\cup\left(a^++\sigma_{ess}(H_V)\right), \tag{13}$$

Remark 2.3. The previous theorem shows that the knowledge of $V,a^\pm$ is sufficient for unique determination of $\sigma_{ess}(H_{U+V})$. In fact,

$$\sigma_{ess}(H_{U+V})=\sigma_{ess}(H_{U_c+V}),$$

where

$$U_c=a^-\chi_{(-\infty,0]}+a^+\chi_{(0,\infty)}.$$

In general equality in (13) does not hold. However, for the case of periodic potentials we have:

Theorem 2.4. *Let $U:\mathbb{R}\to\mathbb{R}$ be measurable, H_0-bounded and satisfy the conditions*

$$U(x)\xrightarrow[x\to-\infty]{}a^-,\quad U(x)\xrightarrow[x\to\infty]{}a^+$$

and let W be a H_0-bounded periodic potential, then

$$\sigma_{ess}(H_0+U+W)=\left(a^-+\sigma_{ess}(H_0+W)\right)\cup\left(a^++\sigma_{ess}(H_0+W)\right).$$

Remark 2.5. It is well known that under the above assumptions on W we have $\sigma_{ess}(H_0+W)=\sigma(H_0+W)$. See [8].

Proof. In the view of Corollary 2.2, we need to prove that

$$a^- + \sigma_{ess}(H_0 + W) \subset \sigma_{ess}(H_0 + U + W), \tag{14}$$

$$a^+ + \sigma_{ess}(H_0 + W) \subset \sigma_{ess}(H_0 + U + W). \tag{15}$$

We'll prove (14) (the proof of (15) is similar). Let

$$\lambda \in a^- + \sigma_{ess}(H_0 + W),$$

i.e. $\lambda - a^- \in \sigma_{ess}(H_0 + W)$.
Then there is a Weyl sequence $\varphi_n \in C_0^\infty(\mathbb{R})$ with

1. $\|\varphi_n^-\|_2 = 1 \quad (n \in \mathbb{N})$,
2. $\|(H_0 + W - (\lambda - a^-)I)\,\varphi_n^-\|_2 \to 0$,

Since W is periodic any shift of φ_n by an integer times the period of W is also a Weyl sequence for $H_0 + W + a^-$. Thus we may assume that $\operatorname{supp}\varphi_n \subset (-\infty, -n)$. As in the previous proofs one easily sees that this sequence is also a Weyl sequence for $H_0 + U + W$. □

3 The essential spectrum of H_ω

We turn to the spectrum of H_ω. To do so, we first describe the spectrum of $H_0 + V_\omega$, i.e. the case $U = 0$.
We follow the investigation in [4].

Definition 3.1. A potential $W(x) = \sum_{k\in\mathbb{Z}} \rho_k f(x-k)$ is called *admissible*, if $\rho_k \in \operatorname{supp} P_0$ for all k. Let us denote by $\mathscr{P}$ the set of all admissible potentials, generated by ℓ-periodic ρ_k for some $\ell \in \mathbb{N}$.

Theorem 3.2. *The spectrum $\sigma(H_0 + V_\omega)$ is independent of ω almost surely and is given (almost surely) by*

$$\Sigma := \sigma(H_0 + V_\omega) = \overline{\bigcup_{W\in\mathscr{P}} \sigma(H_0 + W)} \tag{16}$$

For a proof we refer to [4].
In particular, the following result was proved in [4].

Lemma 3.3. *If W is a periodic admissible potential and $\lambda \in \sigma(H_0 + W)$ then there are sequences $\varphi_n^+, \varphi_n^- \in L^2(\mathbb{R})$ in the domain of $H_0 + W$, such that*

1. $\|\varphi_n^+\| = \|\varphi_n^-\| = 1$
2. *The supports of φ_n^+ and φ_n^- are compact and satisfy*

   $\operatorname{supp}\varphi_n^+ \subset [n, \infty)$ and $\operatorname{supp}\varphi_n^- \subset (-\infty, -n]$

3. *For almost all* ω

$$\|(H_0+V_\omega-\lambda)\,\varphi_n^+\| \;\to\; 0 \textit{ and } \|(H_0+V_\omega-\lambda)\,\varphi_n^-\| \;\to\; 0$$

From this we conclude

Theorem 3.4. *Almost surely*

$$\sigma(H_\omega) \;=\; \sigma\left(H_0+V_\omega+a^-\right)\cup\sigma\left(H_0+V_\omega+a^+\right) \tag{17}$$

Proof. By Corollary 2.2 we know that

$$\sigma(H_\omega) \;\subset\; \sigma\left(H_0+V_\omega+a^-\right)\cup\sigma\left(H_0+V_\omega+a^+\right). \tag{18}$$

To prove the converse we observe that for any $W\in\mathscr{P}$

$$\sigma\left(H_0+W+a^\pm\right) \;\subset\; \sigma_{ess}(H+U+W) \tag{19}$$

by Theorem 2.4. It is easy to see (e. g. as in [4]) that almost surely for $W\in\mathscr{P}$

$$\sigma_{ess}(H+U+W) \;\subset\; \sigma_{ess}(H+U+V_\omega). \tag{20}$$

We conclude that

$$\bigcup_{W\in\mathscr{P}}\sigma\left(H_0+W+a^+\right)\cup\bigcup_{W\in\mathscr{P}}\sigma\left(H_0+W+a^-\right) \;\subset\; \sigma_{ess}(H+U+V_\omega). \tag{21}$$

Since the righthand side is a closed set we infer from Theorem 3.2 that almost surely

$$\sigma\left(H_0+V_\omega+a^-\right)\cup\sigma\left(H_0+V_\omega+a^+\right) \;\subset\; \sigma(H_\omega). \tag{22}$$

□

4 The Integrated Density of States

In this section we investigate the integrated density of states of the operators H_ω.

Definition 4.1. Let A be a self adjoint operator bounded below and with (possibly infinite) purely discrete spectrum $\lambda_1(A)\le\lambda_2(A)\le\lambda_3(A)\le\dots$ where we count eigenvalues according to their multiplicities. Then we set

$$N(A,E) \;:=\; \#\{j\,|\,\lambda_j(A)\le E\}. \tag{23}$$

For $H=H_0+W$ (with $W\in L^2_{\text{loc, unif}}$) and $a,b\in\mathbb{R}$, $a<b$ we define $H^D_{a,b}$ to be the operator H restricted to $L^2([a,b])$ with Dirichlet boundary conditions both at a and b. Similarly, $H^N_{a,b}$ has Neumann

boundary conditions at a and b, $H_{a,b}^{D,N}$ has Dirichlet boundary condition at a and Neumann boundary condition at b, $H_{a,b}^{N,D}$ has Neumann boundary condition at a and Dirichlet one at b.

If for $H = H_0 + W$ the limit

$$\mathscr{N}(E) = \mathscr{N}(H,E) \; := \; \lim_{L\to\infty} \frac{1}{2L} N\left(H_{-L,L}^{D}, E\right) \tag{24}$$

exists for all but countably many E, we call $\mathscr{N}(E)$ the *integrated density of states* for H.

It is well known that under our assumptions the integrated density of states for $H = H_0 + V_\omega$ exists, more precisely:

Theorem 4.2. *If V_ω satisfies the assumptions of Section 1, then the integrated density of states for $\mathscr{N}(H,E)$ exists and for all but countably many E the following equalities hold:*

$$\mathscr{N}(H,E) = \lim_{L\to\infty} \frac{N\left(H_{-L,L}^{N}(E)\right)}{2L} = \lim_{L\to\infty} \frac{\mathbb{E}\left(N\left(H_{-L,L}^{D}(E)\right)\right)}{2L} = \lim_{L\to\infty} \frac{\mathbb{E}\left(N\left(H_{-L,L}^{N}(E)\right)\right)}{2L}. \tag{25}$$

($\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$.)

For a proof see [5]. The proof there uses the method of Dirichlet-Neumann bracketing (see [8]), in particular it is used:

Theorem 4.3. *If $a < c < b$ and $X,Y \in \{D,N\}$, then*

$$N\left(H_{a,c}^{X,D}, E\right) + N\left(H_{c,b}^{D,Y}, E\right) \;\le\; N\left(H_{a,c}^{X,Y}, E\right) \;\le\; N\left(H_{a,c}^{X,N}, E\right) + N\left(H_{c,b}^{N,Y}, E\right). \tag{26}$$

For the integrated density of states of the operator $H_\omega = H_0 + U + V_{per} + V_\omega$ we have the following result.

Theorem 4.4. *The integrated density of states $\mathscr{N}(H_\omega, E)$ exists and can be expressed in terms of $\mathscr{N}_0(E)$, the integrated density of states of $H_0 + V_\omega$ by:*

$$\mathscr{N}(H_\omega, E) \;=\; \frac{1}{2}\mathscr{N}_0(E - a^-) \;+\; \frac{1}{2}\mathscr{N}_0(E - a^+) \tag{27}$$

To prove this result we need the following lemma:

Lemma 4.5. *For the integrated density of states $\mathscr{N}_0$ of $H_0 + V_\omega$ we have for any fixed M with $M < L$ and any $X,Y \in \{D,N\}$:*

$$\mathscr{N}_0(E) \;=\; \lim_{L\to\infty} \frac{1}{L}\,\mathbb{E}\left(N\left((H_0 + V_\omega)_{M,L}^{X,Y}\right)\right) \tag{28}$$

$$= \lim_{L\to\infty} \frac{1}{L}\,\mathbb{E}\left(N\left((H_0 + V_\omega)_{-L,-M}^{X,Y}\right)\right) \tag{29}$$

Proof. By the stationarity of the potential we have

$$\mathbb{E}\left(N\left((H_0 + V_\omega)_{M,L}^{X,Y}\right)\right) \;=\; \mathbb{E}\left(N\left((H_0 + V_\omega)_{-(L-M)/2,(L-M)/2}^{X,Y}\right)\right). \tag{30}$$

Thus, the lemma follows from Theorem 4.2 □

We know prove Theorem 4.4.

Proof.

$$\mathbb{E}\left(N\left((H_0+U+V_\omega)^{X,Y}_{-L,L}\right)\right) \leq \mathbb{E}\left(N\left((H_0+U+V_\omega)^{X,N}_{-L,-M}\right)\right) + \\ + \mathbb{E}\left(N\left((H_0+U+V_\omega)^{N,N}_{-M,M}\right)\right) + \mathbb{E}\left(N\left((H_0+U+V_\omega)^{N,Y}_{M,L}\right)\right). \quad (31)$$

We take $M > 0$ so large that $|U(x) - a^-| < \varepsilon/2$ for $x \leq -M$ and $|U(x) - a^+| < \varepsilon/2$ for $x \geq M$. Let us divide inequality (31) by $2L$. Then the middle term goes to zero as $L \to \infty$. Moreover in the limit the first term on the right hand side can be bounded by $\frac{1}{2}\mathcal{N}_0(E - a_-) + \varepsilon/2$. Similarly the third term can be bounded by $\frac{1}{2}\mathcal{N}_0(E - a_+) + \varepsilon/2$. Since $\varepsilon > 0$ was arbitrary we proved

$$\mathbb{E}\left(N\left((H_0+U+V_\omega)^{X,Y}_{-L,L}\right)\right) \leq \frac{1}{2}\mathcal{N}_0(E - a^-) + \frac{1}{2}\mathcal{N}_0(E - a^+). \quad (32)$$

The inverse inequality follows if we use Dirichlet, instead of Neumann boundary conditions for the inequalities (31). □

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