An Invitation to
Random Schrödinger operators

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Abstract

This review is an extended version of my mini course at the États de la recherche: Opérateurs de Schrödinger aléatoires at the Université Paris 13 in June 2002, a summer school organized by Frédéric Klopp.

These lecture notes try to give some of the basics of random Schrödinger operators. They are meant for nonspecialists and require only minor previous knowledge about functional analysis and probability theory. Nevertheless this survey includes complete proofs of Lifshitz tails and Anderson localization.

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1. Preface

In these lecture notes I try to give an introduction to (some part of) the basic theory of random Schrödinger operators. I intend to present the field in a rather self contained and elementary way. It is my hope that the text will serve as an introduction to random Schrödinger operators for students, graduate students and researchers who have not studied this topic before. If some scholars who are already acquainted with random Schrödinger operators might find the text useful as well I will be even more satisfied.

Only a basic knowledge in Hilbert space theory and some basics from probability theory are required to understand the text (see the Notes below). I have restricted the considerations in this text almost exclusively to the Anderson model, i.e. to random operators on the Hilbert space $\ell^2(\mathbb{Z}^d)$. By doing so I tried to avoid many of the technical difficulties that are necessary to deal with in the continuous case (i.e. on $L^2(\mathbb{R}^d)$). Through such technical problems sometimes the main ideas become obscured and less transparent.

The theory I present is still not exactly easy staff. Following Einstein's advice, I tried to make things as easy as possible, but not easier.

The author has to thank many persons. The number of colleagues and friends I have learned from about mathematical physics and especially disordered systems is so large that it is impossible to mention a few without doing injustice to many others. A lot of the names can be found as authors in the list of references. Without these persons the writing of this review would have been impossible.

A colleague and friend I *have* to mention though is Frédéric Klopp who organized a summer school on Random Schrödinger operators in Paris in 2002. My lectures there were the starting point for this review. I have to thank Frédéric especially for his enormous patience when I did not obey the third, forth, ..., deadline for delivering the manuscript.

It is a great pleasure to thank Bernd Metzger for his advice, for many helpful discussions, for proofreading the manuscript, for helping me with the text and especially with the references and for many other things.

Last, not least I would like to thank Jessica Langner, Riccardo Catalano and Hendrik Meier for the skillful typing of the manuscript, for proofreading and for their patience with the author.

Notes and Remarks

For the spectral theory needed in this work we recommend [117] or [141]. We will also need the min-max theorem (see [115]).

The probabilistic background we need can be found e.g. in [95] and [96].

For further reading on random Schrödinger operators we recommend [78] for the state of the art in multiscale analysis. We also recommend the textbook [128]. A modern survey on the density of states is [67].

2. Introduction: Why random Schrödinger operators ?

2.1. The setting of quantum mechanics.

A quantum mechanical particle moving in d-dimensional space is described by a vector ψ in the Hilbert space $L^2(\mathbb{R}^d)$. The time evolution of the state ψ is determined by the *Schrödinger operator*

$$H = H_0 + V \tag{2.1}$$

acting on $L^2(\mathbb{R}^d)$. The operator H_0 is called the *free operator*. It represents the kinetic energy of the particle. In the absence of magnetic fields it is given by the Laplacian

$$H_0 = -\frac{\hbar^2}{2m}\Delta = -\frac{\hbar^2}{2m}\sum_{\nu=1}^{d}\frac{\partial^2}{\partial x_\nu{}^2}\,. \tag{2.2}$$

The physics of the system is encoded in the potential V which is the multiplication operator with the function $V(x)$ in the Hilbert space $L^2(\mathbb{R}^d)$. The function $V(x)$ is the (classical) potential energy. Consequently, the forces are given by

$$F(x) = -\nabla V(x)\,.$$

In the following we choose physical units in such a way that $\frac{\hbar^2}{2m} = 1$ since we are not interested in the explicit dependence of quantities on $\hbar$ or m. The time evolution of the state ψ is obtained from the time dependent Schrödinger equation

$$i\,\frac{\partial}{\partial t}\,\psi \;=\; H\,\psi\,. \tag{2.3}$$

By the spectral theorem for self adjoint operators equation (2.3) can be solved by

$$\psi(t) = e^{-itH}\psi_0 \tag{2.4}$$

where ψ_0 is the state of the system at time $t = 0$.

To extract valuable information from (2.4) we have to know as much as possible about the spectral theory of the operator H and this is what we try to do in this text.

2.2. Random Potentials.

In this review we are interested in random Schrödinger operators. These operators model disordered solids. Solids occur in nature in various forms. Sometimes they are (almost) totally ordered. In crystals the atoms or nuclei are distributed on a periodic lattice (say the lattice $\mathbb{Z}^d$ for simplicity) in a completely regular way. Let us assume that a particle (electron) at the point $x \in \mathbb{R}^d$ feels a potential of the form $q\,f(x - i)$ due to an atom (or ion or nucleus) located at the point $i \in \mathbb{Z}^d$. Here, the constant q, the charge or coupling constant in physical terms, could be absorbed into the function f. However, since we are going to vary this quantity from atom to atom later on, it is useful to write the potential in the above way. Then, in a regular crystal our particle is exposed to a total potential

$$V(x) = \sum_{i \in \mathbb{Z}^d} q \, f(x - i) \,. \tag{2.5}$$

We call the function f the *single site potential* to distinguish it from the total potential V. The potential V in (2.5) is periodic with respect to the lattice $\mathbb{Z}^d$, i. e. $V(x - i) = V(x)$ for all $x \in \mathbb{R}^d$ and $i \in \mathbb{Z}^d$. The mathematical theory of Schrödinger operators with periodic potentials is well developed (see e.g. [**41**], [**115**]). It is based on a thorough analysis of the symmetry properties of periodic operators. For example, it is known that such operators have a spectrum with band structure, i.e. $\sigma(H) = \bigcup_{n=0}^{\infty} [a_n, b_n]$ with $a_n < b_n \leq a_{n+1}$. This spectrum is also known to be absolutely continuous.

Most solids do not constitute ideal crystals. The positions of the atoms may deviate from the ideal lattice positions in a non regular way due to imperfections in the crystallization process. Or the positions of the atoms may be completely disordered as is the case in amorphous or glassy materials. The solid may also be a mixture of various materials which is the case for example for alloys or doped semiconductors. In all these cases it seems reasonable to look upon the potential as a random quantity.

For example, if the material is a pure one, but the positions of the atoms deviate from the ideal lattice positions randomly, we may consider a random potential of the form

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q \, f(x - i - \xi_i(\omega)) \,. \tag{2.6}$$

Here the ξ_i are random variables which describe the deviation of the 'ith' atom from the lattice position i. One may, for example assume that the random variables ξ_i are independent and identically distributed. We have added a subscript ω to the potential V to make clear that V_ω depends on (unknown) random parameters.

To model an amorphous material like glass or rubber we assume that the atoms of the material are located at completely random points η_i in space. Such a random potential may formally be written as

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q \, f(x - \eta_i). \tag{2.7}$$

To write the potential (2.7) as a sum over the lattice $\mathbb{Z}^d$ is somewhat misleading, since there is, in general, no natural association of the η_i with a lattice point i. It is more appropriate to think of a collection of random points in $\mathbb{R}^d$ as a random point measure. This representation emphasizes that any ordering of the η_i is completely artificial.

A *counting measure* is a Borel measure on $\mathbb{R}^d$ of the form $\nu = \sum_{x \in M} \delta_x$ with a countable set M without (finite) accumulation points. By a *random point measure* we mean a mapping $\omega \mapsto \mu_\omega$, such that μ_ω is a counting measure with the property that the function $\omega \mapsto \mu_\omega(A)$ is measurable for any bounded Borel set A. If $\nu = \nu_\omega$

is the random point measure $\nu = \sum_i \delta_{\eta_i}$ then (2.7) can be written as

$$V_\omega(x) = \int_{\mathbb{R}^d} q\, f(x - \eta)\; d\nu(\eta)\,. \tag{2.8}$$

The most frequently used example of a random point measure and the most important one is the *Poisson random measure* μ_ω. Let us set $n_A = \mu_\omega(A)$, the number of random points in the set A. The Poisson random measure can be characterized by the following specifications

- The random variables n_A and n_B are independent for disjoint (measurable) sets A and B.
- The probability that $n_A = k$ is equal to $\frac{|A|^k}{k!}\; e^{-|A|}$, where $|A|$ is the Lebesgue measure of A.

A random potential of the form (2.8) with the Poisson random measure is called the *Poisson model* .

The most popular model of a disordered solid and the best understood one as well is the *alloy-type potential* (see (2.9) below). It models an unordered alloy, i.e. a mixture of several materials the atoms of which are located at lattice positions. The type of atom at the lattice point i is assumed to be random. In the model we consider here the different materials are described by different charges (or coupling constants) q_i. The total potential V is then given by

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q_i(\omega)\, f(x - i)\,. \tag{2.9}$$

The q_i are random variables which we assume to be independent and identically distributed. Their range describes the possible values the coupling constant can assume in the considered alloy. The physical model suggests that there are only finitely many values the random variables can assume. However, in the proofs of some results we have to assume that the distribution of the random variables q_i is continuous (even absolutely continuous) due to limitations of the mathematical techniques. One might argue that such an assumption is acceptable as a purely technical one. On the other hand one could say we have not understood the problem as long as we can not handle the physically relevant cases.

For a given ω the potential $V_\omega(x)$ is a pretty complicated 'normal' function. So, one may ask: What is the advantage of 'making it random'?

With the introduction of random variables we implicitly change our point of view. From now on we are hardly interested in properties of H_ω for a single given ω. Rather, we look at 'typical' properties of H_ω. In mathematical terms, we are interested in results of the form: The set of all ω such that H_ω has the property $\mathcal{P}$ has probability one. In short: $\mathcal{P}$ holds for $\mathbb{P}$-almost all ω (or $\mathbb{P}$-almost surely). Here $\mathbb{P}$ is the probability measure on the underlying probability space.

In this course we will encounter a number of such properties. For example we will see that (under weak assumptions on V_ω), there is a closed, nonrandom (!) subset Σ of the real line such that $\Sigma = \sigma(H_\omega)$, the spectrum of the operator H_ω, $\mathbb{P}$-almost surely.

This and many other results can be proven for various types of random Schrödinger operators. In this lecture we will restrict ourselves to a relatively simple system known as the Anderson model. Here the Hilbert space is the sequence space $\ell^2(\mathbb{Z}^d)$ instead of $L^2(\mathbb{R}^d)$ and the free operator H_0 is a finite-difference operator rather than the Laplacian. We will call this setting the *discrete case* in contrast to Schrödinger operators on $L^2(\mathbb{R}^d)$ which we refer to as the *continuous case* . In the references the reader may find papers which extend results we prove here to the continuous setting.

2.3. The one body approximation.

In the above setting we have implicitly assumed that we describe a single particle moving in a static exterior potential. This is at best a caricature of what we find in nature. First of all there are many electrons moving in a solid and they interact with each other. The exterior potential originates in nuclei or ions which are themselves influenced both by the other nuclei and by the electrons. In the above discussion we have also implicitly assumed that the solid we consider extends to infinity in all directions, (i.e. fills the universe). Consequently, we ought to consider infinitely many interacting particles. It is obvious that such a task is out of range of the methods available today. As a first approximation it seems quite reasonable to separate the motion of the nuclei from the system and to take the nuclei into account only via an exterior potential. Indeed, the masses of the nuclei are much larger than those of the electrons.

The second approximation is to neglect the electron-electron interaction. It is *not at all clear* that this approximation gives a qualitatively correct picture. In fact, there is physical evidence that the interaction between the electrons is fundamental for a number of phenomena.

Interacting particle systems in condensed matter are an object of intensive research in theoretical physics. In mathematics, however, this field of research is still in its infancy despite of an increasing interest in the subject.

If we neglect the interactions between the electrons we are left with a system of noninteracting electrons in an exterior potential. It is not hard to see that such a system (and the corresponding Hamiltonian) separates, i.e. the eigenvalues are just sums of the one-body eigenvalues and the eigenfunctions have product form. So, if $\psi_1, \psi_2, \ldots, \psi_N$ are eigenfunctions of the one-body system corresponding to eigenvalues $E_1, E_2, \ldots, E_N$ respectively, then

$$\Psi(x_1, x_2, \ldots, x_N) = \psi_1(x_1) \cdot \psi_2(x_2) \cdot \ldots \cdot \psi_N(x_N) \ . \qquad (2.10)$$

is an eigenfunction of the full system with eigenvalue $E_1 + E_2 + \ldots + E_N$.

However, there is a subtlety to obey here, which is typical to many particle Quantum Mechanics. The electrons in the solid are indistinguishable, since we are unable to 'follow their trajectories'. The corresponding Hamiltonian is invariant under permutation of the particles. As a consequence, the N-particle Hilbert space consists either of totally *symmetric* or of totally *antisymmetric* functions of the particle positions $x_1, x_2, \ldots, x_N$. It turns out that for particles with integer spin the

symmetric subspace is the correct one. Such particles, like photons, phonons or mesons, are called Bosons.

Electrons, like protons and neutrons, are Fermions, particles with half integer spin. The Hilbert space for Fermions consists of totally antisymmetric functions, i.e.: if $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$ are the coordinates of N electrons, then any state ψ of the system satisfies $\psi(x_1, x_2, x_3, \ldots, x_N) = -\psi(x_2, x_1, x_3, \ldots, x_N)$ and similarly for interchanging any other pair of particles.

It follows, that the product in (2.10) is *not* a vector of the (correct) Hilbert space (of antisymmetric functions). Only its anti-symmetrization is

$$\Psi_f(x_1, x_2, \ldots, x_N) := \sum_{\pi \in S_N} (-1)^\pi \, \psi_1(x_{\pi_1}) \psi_2(x_{\pi_2}) \ldots, \psi_N(x_{\pi_N}) \, . \quad (2.11)$$

Here, the symbol S_N stands for the permutation group and $(-1)^\pi$ equals 1 for even permutations (i.e. products of an even number of exchanges), it equals -1 for odd permutations.

The anti-symmetrization (2.11) is non zero only if the functions ψ_j are pairwise different. Consequently, the eigenvalues of the multi-particle system are given as sums $E_1 + E_2 + \ldots + E_N$ of the eigenvalues E_j of the one-particle system where the eigenvalues E_j are all different. (We count multiplicity, i.e. an eigenvalue of multiplicity two may occur twice in the above sum). This rule is known as the Pauli-principle.

The ground state energy of a system of N identical, *noninteracting* Fermions is therefore given by

$$E_1 + E_2 + \ldots + E_N$$

where the E_n are the eigenvalues of the single particle system in increasing order, $E_1 \leq E_2 \leq \ldots$ counted according to multiplicity.

It is not at all obvious how we can implement the above rules for the systems considered here. Their spectra tend to consist of whole intervals rather than being discrete, moreover, since the systems extend to infinity they ought to have infinitely many electrons.

To circumvent this difficulty we will introduce a procedure known as the 'thermodynamic limit': We first restrict the system to a finite but large box (of length L say), then we define quantities of interest in this system, for example the number of states in a given energy region *per unit volume*. Finally, we let the box grow indefinitely (i.e. send L to infinity) and hope (or better prove) that the quantity under consideration has a limit as L goes to infinity. In the case of the number of states per unit volume this limit, in deed, exists. It is called the density of states measure and will play a major role in what follows. We will discuss this issue in detail in chapter 5.

Notes and Remarks

Standard references for mathematical methods of quantum mechanics are [**57**], [**117**], [**114**], [**116**], [**115**] and [**141**], [**30**].

Most of the necessary prerequisites from spectral theory can be found in [**117**] or [**141**]. A good source for the probabilistic background is [**95**] and [**96**].

The physical theory of random Schrödinger operators is described in [**9**], [**99**], [**135**] and [**136**]. References for the mathematical approach to random Schrödinger operators are [**23**], [**30**], [**97**], [**58**], [**112**] and [**128**].

3. Setup: The Anderson model

3.1. Discrete Schrödinger operators.

In the Anderson model the Hilbert space $L^2(\mathbb{R}^d)$ is replaced by the sequence space

$$\ell^2(\mathbb{Z}^d) = \{(u_i)_{i \in \mathbb{Z}^d} \,|\, \sum_{i \in \mathbb{Z}^d} |u_i|^2 < \infty\} \tag{3.1}$$

$$= \{u : \mathbb{Z}^d \to \mathbb{C} \,|\, \sum_{n \in \mathbb{Z}^d} |u(n)|^2 < \infty\} \,. \tag{3.2}$$

We denote the norm on $\ell^2(\mathbb{Z}^d)$ by

$$\| \, u \, \| \; = \; \Big(\sum_{n \in \mathbb{Z}^d} |u(n)|^2 \Big)^{\frac{1}{2}} \tag{3.3}$$

Here, we think of a particle moving on the lattice $\mathbb{Z}^d$, so that in the case $\|u\| = 1$ the probability to find the particle at the point $n \in \mathbb{Z}^d$ is given by $|u(n)|^2$. Note, that we may think of u either as a function $u(n)$ on $\mathbb{Z}^d$ or as a sequence u_n indexed by $\mathbb{Z}^d$.

It will be convenient to equip $\mathbb{Z}^d$ with two different norms. The first one is

$$\| \, n \, \|_\infty := \sup_{\nu=1,\dots,d} | \, n_\nu | \,. \tag{3.4}$$

This norm respects the cubic structure of the lattice $\mathbb{Z}^d$. For example, it is convenient to define the cubes ($n_0 \in \mathbb{Z}^d$, $L \in \mathbb{N}$)

$$\Lambda_L(n_0) := \{n \in \mathbb{Z}^d; \| \, n - n_0\|_\infty \le L\} \,. \tag{3.5}$$

$\Lambda_L(n_0)$ is the cube of side length $2L + 1$ centered at n_0. It contains $|\Lambda_L(n_0)| :=$ $(2L + 1)^d$ points. Sometimes we call $|\Lambda_L(n_0)|$ the volume of $\Lambda_L(n_0)$. In general, we denote by $|A|$ the number of elements of the set A. To shorten notation we write Λ_L for $\Lambda_L(0)$. The other norm we use on $\mathbb{Z}^d$ is

$$\| \, n \, \|_1 := \sum_{\nu=1}^{d} | \, n_\nu | \,. \tag{3.6}$$

This norm reflects the graph structure of $\mathbb{Z}^d$. Two vertices n and m of the graph $\mathbb{Z}^d$ are connected by an edge, if they are nearest neighbors, i. e. if $\| \, n - m\|_1 = 1$. For arbitrary $n, m \in \mathbb{Z}^d$ the norm $\| \, n - m\|_1$ gives the length of the shortest path between n and m.

The kinetic energy operator H_0 is a discrete analogue of the (negative) Laplacian, namely

$$(H_0\, u)(n) = -\sum_{\|\,m-n\|_1=1} (u(m) - u(n)) \ . \tag{3.7}$$

This operator is also known as the *graph Laplacian* for the graph $\mathbb{Z}^d$ or the *discrete Laplacian*. Its quadratic form is given by

$$\langle u, H_0\, v \rangle = \frac{1}{2} \sum_{n\in\mathbb{Z}^d} \sum_{\|\,m-n\|_1=1} \overline{(u\,(n) - u\,(m))}\,(v(n) - v(m)) \ . \tag{3.8}$$

We call this sesquilinear form a Dirichlet form because of its similarity to the classical Dirichlet form

$$\langle u, -\triangle\, v \rangle = \int_{\mathbb{R}^d} \overline{\nabla u(x)} \cdot \nabla v(x)\, dx \ .$$

The operator H_0 is easily seen to be symmetric and bounded, in fact

$$\|H_0\, u\| = \Big(\sum_{n\in\mathbb{Z}^d} \Big(\sum_{\|\,j\,\|_1=1} \big(u(n+j) - u(n)\big) \Big)^2 \Big)^{\frac{1}{2}} \tag{3.9}$$

$$\leq \sum_{\|\,j\,\|_1=1} \Big(\sum_{n\in\mathbb{Z}^d} |\,u(n+j) - u(n)\,|^2 \Big)^{\frac{1}{2}} \tag{3.10}$$

$$\leq \sum_{\|\,j\|_1=1} \Big(\sum_{n\in\mathbb{Z}^d} \big(|\,u(n+j)\,| + |\,u(n)\,|\big)^2 \Big)^{\frac{1}{2}} \tag{3.11}$$

$$\leq \sum_{\|\,j\|_1=1} \Big(\Big(\sum_{n\in\mathbb{Z}^d} |\,u(n+j)\,|^2 \Big)^{\frac{1}{2}} + \Big(\sum_{n\in\mathbb{Z}^d} |\,u(n)\,|^2 \Big)^{\frac{1}{2}} \Big) \tag{3.12}$$

$$\leq \quad 4\,d\, \|\,u\,\| \ . \tag{3.13}$$

From line (3.9) to (3.10) we applied the triangle inequality for ℓ^2 to the functions $\sum f_j(n)$ with $f_j(n) = u(n+j) - u(n)$. In (3.12) and (3.13) we used the triangle inequality and the fact that any lattice point in $\mathbb{Z}^d$ has $2d$ neighbors.

Let us define the Fourier transform from $\ell^2(\mathbb{Z}^d)$ to $L^2([0, 2\pi]^d)$ by

$$(\mathcal{F}u)(k) = \hat{u}(k) = \sum_n u_n\, e^{-i\,n\cdot k} \ . \tag{3.14}$$

$\mathcal{F}$ is a unitary operator. Under $\mathcal{F}$ the discrete Laplacian H_0 transforms to the multiplication operator with the function $h_0(k) = 2 \sum_{\nu=1}^d (1 - \cos(k_\nu))$, i.e. $\mathcal{F}H_0\mathcal{F}^{-1}$ is the multiplication operator on $L^2([0, 2\pi]^d)$ with the function h_0. This shows that the spectrum $\sigma(H_0)$ equals $[0, 4d]$ (the range of the function h_0) and that H_0 has purely absolutely continuous spectrum.

It is very convenient that the 'discrete Dirac function' δ_i defined by $(\delta_i)_j = 0$ for $i \neq j$ and $(\delta_i)_i = 1$ is an 'honest' ℓ^2-vector, in fact the collection $\{\delta_i\}_{i\in\mathbb{Z}^d}$ is an

orthonormal basis of $\ell^2(\mathbb{Z}^d)$. This allows us to define *matrix entries* or a *kernel* for every (say bounded) operator A on $\ell^2(\mathbb{Z}^d)$ by

$$A(i,j) = \langle \delta_i, A\delta_j \rangle \ . \tag{3.15}$$

We have $(Au)(i) = \sum_{j \in \mathbb{Z}^d} A(i,j)u(j)$. So, the $A(i,j)$ define the operator A uniquely.

In this representation the multiplication operator V is diagonal, while

$$H_0(i,j) = \begin{cases} -1 & \text{if} \quad \|\,i-j\,\|_1 = 1, \\ 2d & \text{if} \quad i=j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

In many texts the diagonal term in H_0 is dropped and absorbed into the potential V. Moreover, one can also neglect the $-$-sign in the offdiagonal terms of (3.16). The corresponding operator is up to a constant equivalent to H_0 and has spectrum $[-2d, 2d]$.

In this setting the potential V is a multiplication operator with a function $V(n)$ on $\mathbb{Z}^d$. The simplest form to make this random is to take $V(n) = V_\omega(n)$ itself as independent, identically distributed random variables (see Section 3.4), so we have

$$H_\omega = H_0 + V_\omega \ .$$

We call this random operator the *Anderson model* . For most of this course we will be concerned with this operator.

3.2. Spectral calculus.

One of the most important tools of spectral theory is the functional calculus (spectral theorem) for self adjoint operators. We discuss this topic here by giving a brief sketch of the theory and establish notations. Details about functional calculus can be found in [**117**]. (For an alternative approach see [**34**]).

Throughout this section let A denote a self adjoint operator with domain $D(A)$ on a (separable) Hilbert space $\mathcal{H}$. We will try to define functions $f(A)$ of A for a huge class of functions f. Some elementary functions of A can be defined in an obvious way. One example is the resolvent which we consider first.

For any $z \in \mathbb{C}$ the operator $A - z = A - z\,\text{id}$ is defined by $(A-z)\varphi = A\varphi - z\varphi$. The resolvent set $\rho(A)$ of A is the set of all $z \in \mathbb{C}$ for which $A - z$ is a bijective mapping from $D(A)$ to $\mathcal{H}$. The spectrum $\sigma(A)$ of A is defined by $\sigma(A) = \mathbb{C} \setminus \rho(A)$. For self adjoint A we have $\sigma(A) \subset \mathbb{R}$. The spectrum is always a closed set. If A is bounded, $\sigma(A)$ is compact.

For $z \in \rho(A)$ we can invert $A - z$. The inverse operator $(A - z)^{-1}$ is called the resolvent of A. For self adjoint A the $(A - z)^{-1}$ is a bounded operator for all $z \in \rho(A)$.

Resolvents observe the following important identities, known as the *resolvent equations*

$$(A - z_1)^{-1} - (A - z_2)^{-1} = (z_1 - z_2)(A - z_1)^{-1}(A - z_2)^{-1} \tag{3.17}$$

$$= (z_1 - z_2)(A - z_2)^{-1}(A - z_1)^{-1} \tag{3.18}$$

and, if $D(A) = D(B)$,

$$(A - z)^{-1} - (B - z)^{-1} = (A - z)^{-1}(B - A)(B - z)^{-1} \tag{3.19}$$

$$= (B - z)^{-1}(B - A)(A - z)^{-1} \tag{3.20}$$

For $z \in \mathbb{C}$ and $M \subset \mathbb{C}$ we define

$$\text{dist}(z, M) = \inf\{|z - \zeta|; \zeta \in M\} \tag{3.21}$$

It is not hard to see that for any self adjoint operator A and any $z \in \rho(A)$ the operator norm $\|(A - z)^{-1}\|$ of the resolvent is given by

$$\|(A - z)^{-1}\| = \frac{1}{\text{dist}(z, \sigma(A))} . \tag{3.22}$$

In particular, for a self adjoint operator A and $z \in \mathbb{C} \setminus \mathbb{R}$

$$\|(A - z)^{-1}\| \leq \frac{1}{\text{Im} \, z} . \tag{3.23}$$

For the rest of this section we assume that the operator A is bounded. In this case, polynomials of the operator A can be defined straightforwardly

$$A^2 \varphi = A(A(\varphi)) \tag{3.24}$$

$$A^3 \varphi = A\Big(A\big(A(A\varphi)\big)\Big) \quad \text{etc.} \tag{3.25}$$

More generally, if P is a complex valued polynomial in one real variable, $P(\lambda) = \sum_{j=0}^{n} a_n \lambda^j$ then

$$P(A) = \sum_{j=0}^{n} a_n A^j . \tag{3.26}$$

It is a key observation that

$$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)| \tag{3.27}$$

Let now f be a function in $C(\sigma(A))$ the complex-valued continuous functions on (the compact set) $\sigma(A)$. The Weierstraß approximation theorem tells us, that on $\sigma(A)$ the function f can be uniformly approximated by polynomials. Thus using (3.27) we can define the operator $f(A)$ as a norm limit of polynomials $P_n(A)$. These operators satisfy

$$(\alpha f + \beta g)\,(A) \;=\; \alpha f(A) + \beta g(A) \tag{3.28}$$

$$f \cdot g\,(A) \;=\; f(A)\,g(A) \tag{3.29}$$

$$\overline{f}\,(A) \;=\; f(A)^* \tag{3.30}$$

$$\text{If } f \geq 0 \quad \text{then} \quad \langle \varphi, f(A)\varphi \rangle \geq 0 \;\text{ for all }\; \varphi \in \mathcal{H} \tag{3.31}$$

By the Riesz-representation theorem it follows, that for each $\varphi \in \mathcal{H}$ there is a positive and bounded measure $\mu_{\varphi,\varphi}$ on $\sigma(A)$ such that for all $f \in C\big(\sigma(A)\big)$

$$\langle \varphi, f(A)\varphi \rangle \;=\; \int f(\lambda)\, d\mu_{\varphi,\varphi}(\lambda)\,. \tag{3.32}$$

For $\varphi, \psi \in \mathcal{H}$, using the polarization identity, we find complex-valued measures $\mu_{\varphi,\psi}$ such that

$$\langle \varphi, f(A)\varphi \rangle \;=\; \int f(\lambda)\, d\mu_{\varphi,\psi}(\lambda)\,. \tag{3.33}$$

Equation (3.33) can be used to define the operator $f(A)$ for bounded *measurable* functions. The operators $f(A)$, $g(A)$ satisfy (3.28)–(3.31) for bounded measurable functions as well, moreover we have:

$$\|f(A)\| \;\leq\; \sup_{\lambda \in \sigma(A)} |f(\lambda)| \tag{3.34}$$

with equality for continuous f.

For any Borel set $M \subset \mathbb{R}$ we denote by χ_M the *characteristic function* of M defined by

$$\chi_M(\lambda) \;=\; \begin{cases} 1 & \text{if } \lambda \in M \\ 0 & \text{otherwise.} \end{cases} \tag{3.35}$$

The operators $\mu(A) = \chi_M(A)$ play a special role. It is not hard to check that they satisfy the following conditions:

$$\mu(A) \quad \text{is an orthogonal projection.} \tag{3.36}$$

$$\mu(\emptyset) \;=\; 0 \quad \text{and} \quad \mu\big(\sigma(A)\big) \;=\; 1 \tag{3.37}$$

$$\mu(M \cap N) \;=\; \mu(M)\,\mu(N) \tag{3.38}$$

If the Borel sets M_n are pairwise disjoint, then for each $\varphi \in \mathcal{H}$

$$\mu\Big(\bigcup_{n=1}^{\infty} M_n \Big)\varphi \;=\; \sum_{n=1}^{\infty} \mu(M_n)\,\varphi \tag{3.39}$$

Since $\mu(M) = \chi_M(A)$ satisfies (3.36)–(3.39) it is called the *projection valued measure* associated to the operator A or the *projection valued spectral measure* of A. We have

$$\langle \varphi, \mu(M)\psi \rangle \; = \; \mu_{\varphi,\psi}(M) \tag{3.40}$$

The functional calculus can be implemented for *unbounded self adjoint* operators as well. For such operators the spectrum is always a closed set. It is compact only for bounded operators.

We will use the functional calculus virtually everywhere throughout this paper. For example, it gives meaning to the operator e^{-itH} used in (2.4). We will look at the projection valued measures $\chi_M(A)$ more closely in chapter 7.

3.3. Some more functional analysis.

In this section we recall a few results from functional analysis and spectral theory and establish notations at the same time. In particular, we discuss the min-max principle and the Stone-Weierstraß theorem.

Let A be a selfadjoint (not necessarily bounded) operator on the (separable) Hilbert space $\mathcal{H}$ with domain $D(A)$. We denote the set of eigenvalues of A by $\varepsilon(A)$. Obviously, any eigenvalue of A belongs to the spectrum $\sigma(A)$. The *multiplicity* of an eigenvalue λ of A is the dimension of the eigenspace $\{\varphi \in D(A); A\varphi = \lambda\varphi\}$ associated to λ. If μ is the projection valued spectral measure of A, then the multiplicity of λ equals tr $\mu(\{\lambda\})$. An eigenvalue is called *simple* or *non degenerate* if its multiplicity is one, it is called *finitely degenerate* if its eigenspace is finite dimensional. An eigenvalue λ is called *isolated* if there is an $\varepsilon > 0$ such that $\sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) = \{\lambda\}$. Any isolated point in the spectrum is always an eigenvalue. The *discrete spectrum* $\sigma_{dis}(A)$ is the set of all isolated eigenvalues of finite multiplicity. The *essential spectrum* $\sigma_{ess}(A)$ is defined by $\sigma_{ess}(A) = \sigma(A) \setminus \sigma_{dis}(A)$.

The operator A is called *positive* if $\langle \phi, A\phi \rangle \geq 0$ for all ϕ in the domain $D(A)$, A is called *bounded below* if $\langle \phi, A\phi \rangle \geq -M\langle \phi, \phi \rangle$ for some M and all $\phi \in D(A)$. We define

$$\mu_0(A) \; = \; \inf \left\{ \langle \phi, A\phi \rangle \, ; \; \phi \in D(A), \|\phi\| = 1 \right\} \tag{3.41}$$

and for $k \geq 1$

$$\mu_k(A) \; = \; \sup_{\psi_1,\ldots,\psi_k \in \mathcal{H}} \inf \left\{ \langle \phi, A\phi \rangle \, ; \; \phi \in D(A), \|\phi\| = 1, \phi \perp \psi_1, \ldots, \psi_k \right\} \tag{3.42}$$

The operator A is bounded below iff $\mu_0(A) > -\infty$ and $\mu_0(A)$ is the infimum of the spectrum of A.

If A is bounded below and has purely discrete spectrum (i.e. $\sigma_{ess}(A) = \emptyset$), we can order the eigenvalues of A in increasing order and repeat them according to their multiplicity, namely

$$E_0(A) \leq E_1(A) \leq E_2(A) \leq \ldots \,. \tag{3.43}$$

If an eigenvalue E of A has multiplicity m it occurs in (3.43) exactly m times. The min-max principle relates the $E_k(A)$ with the $\mu_k(A)$.

THEOREM 3.1 (Min-max principle). *If the self adjoint operator A has purely discrete spectrum and is bounded below, then*

$$E_k(A) \;=\; \mu_k(A) \qquad \text{for all } k \geq 0 \;. \tag{3.44}$$

A proof of this important result can be found in [**115**]. The formulation there contains various refinements of our version. In particular [**115**] deals also with discrete spectrum below the infimum of the essential spectrum.

We state an application of Theorem 3.1. By $A \leq B$ we mean that the domain $D(B)$ is a subset of the domain $D(A)$ and $\langle \phi, A\phi \rangle \leq \langle \phi, B\phi \rangle$ for all $\phi \in D(B)$.

COROLLARY 3.2. *Let A and B are self adjoint operators which are bounded below and have purely discrete spectrum. If $A \leq B$ then $E_k(A) \leq E_k(B)$ for all k.*

The Corollary follows directly from Theorem 3.1.

We end this section with a short discussion of the Stone-Weierstraß Theorem in the context of spectral theory. The Stone-Weierstraß Theorem deals with subsets of the space $C_\infty(\mathbb{R})$, the set of all (complex valued) continuous functions on $\mathbb{R}$ which vanish at infinity..

A subset $\mathcal{D}$ is called an involutative subalgebra of $C_\infty(\mathbb{R})$, if it is a linear subspace and if for $f, g \in \mathcal{D}$ both the product $f \cdot g$ and the complex conjugate $\overline{f}$ belong to $\mathcal{D}$. We say that $\mathcal{D}$ *seperates points* if for $x, y \in \mathbb{R}$ there is a function $f \in \mathcal{D}$ such that $f(x) \neq f(y)$ and both $f(x)$ and $f(y)$ are non zero.

THEOREM 3.3 (Stone-Weierstraß). *If $\mathcal{D}$ is an involutative subalgebra of $C_\infty(\mathbb{R})$ which seperates points, then $\mathcal{D}$ is dense in $C_\infty(\mathbb{R})$ with respect to the topology of uniform convergence.*

A proof of this theorem is contained e.g. in [**117**]. Theorem 3.3 can be used to prove some assertion $\mathcal{P}(f)$ for the operators $f(A)$ for *all* $f \in C_\infty(\mathbb{R})$ if we know $\mathcal{P}(f)$ for *some* f. Suppose we know $\mathcal{P}(f)$ for all $f \in \mathcal{D}_0$. If we can show that $\mathcal{D}_0$ seperates points and that the set of all f satisfying $\mathcal{P}(f)$ is a closed involutative subalgebra of $C_\infty(\mathbb{R})$, then the Stone-Weierstraß theorem tells us that $\mathcal{P}(f)$ holds for all $f \in C_\infty(\mathbb{R})$.

Theorem 3.3 is especially useful in connection with resolvents. Suppose a property $\mathcal{P}(f)$ holds for all functions f in $\mathcal{R}$, the set of linear combinations of the functions $f_\zeta(x) = \frac{1}{x-\zeta}$ for all $\zeta \in \mathbb{C}\backslash\mathbb{R}$, so for resolvents of A and their linear combinations. The resolvent equations (or rather basic algebra of $\mathbb{R}$) tell us that $\mathcal{R}$ is actually an involutative algebra. So, if the property $\mathcal{P}(f)$ survives uniform limits, we can conclude that $\mathcal{P}(f)$ is valid for all $f \in C_\infty(\mathbb{R})$. The above procedure was dubbed the 'Stone-Weierstraß Gavotte' in [**30**]. More details can be found there.

3.4. Random potentials.

DEFINITION 3.4. *A* random variable *is a real valued measurable function on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

If X is a random variable we call the probability measure P_0 on $\mathbb{R}$ defined by

$$P_0(A) \; = \; \mathbb{P}\left(\{\omega \,|\, X(\omega) \in A\}\right) \qquad \textit{for any Borel set } A \qquad (3.45)$$

the distribution of X. If the distributions of the random variables X and Y agree we say that X and Y are identically distributed. We also say that X and Y have a common distribution in this case.

A family $\{X_i\}_{i \in I}$ of random variables is called independent if for any finite subset $\{i_1, \ldots, i_n\}$ of I

$$\mathbb{P}\left(\{\omega|\, X_{i_1}(\omega) \in [a_1, b_1], X_{i_2}(\omega) \in [a_2, b_2], \ldots\, X_{i_n}(\omega) \in [a_n, b_n]\}\right)$$
$$= \mathbb{P}\left(\{\omega|\, X_{i_1}(\omega) \in [a_1, b_1]\}\right) \cdot \ldots \cdot \mathbb{P}\left(\{\omega|\, X_{i_n}(\omega) \in [a_n, b_n]\}\right). \quad (3.46)$$

REMARK 3.5. If X_i are independent and identically distributed (iid) with common distribution P_0 then

$$\mathbb{P}\left(\{\omega|\, X_{i_1}(\omega) \in [a_1, b_1], X_{i_2}(\omega) \in [a_2, b_2], \ldots\, X_{i_n}(\omega) \in [a_n, b_n]\}\right)$$
$$= P_0([a_1, b_1]) \cdot P_0([a_2, b_2]) \cdot \ldots \cdot P_0([a_n, b_n]).$$

For the reader's convenience we state a very useful result of elementary probability theory which we will need a number of times in this text.

THEOREM 3.6 (Borel-Cantelli lemma). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of set in $\mathcal{F}$. Denote by A_∞ the set*

$$A_\infty \; = \; \{\omega \in \Omega \,|\, \omega \in A_n \text{ for infinitely many } n\} \qquad (3.47)$$

(1) *If $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(A_\infty) = 0$*

(2) *If the sets $\{A_n\}$ are independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then $\mathbb{P}(A_\infty) = 1$*

REMARK 3.7.

(1) We recall that a sequence $\{A_n\}$ of events (i.e. of sets from $\mathcal{F}$) is called *independent* if for any finite subsequence $\{A_{n_j}\}_{j=1,\ldots,M}$

$$\mathbb{P}\left(\bigcap_{j=1}^{M} A_{n_j}\right) \; = \; \prod_{j=1}^{M} \mathbb{P}\left(A_{n_j}\right) \qquad (3.48)$$

(2) The set A_∞ can be written as $A_\infty = \bigcap_N \bigcup_{n \geq N} A_n$.

For the proof of Theorem 3.6 see e.g. [12] or [95].

From now on we assume that the random variables $\{V_\omega(n)\}_{n \in \mathbb{Z}^d}$ are independent and identically distributed with common distribution P_0.

By supp P_0 we denote the *support* of the measure P_0, i.e.

$$\text{supp}\, P_0 \; = \; \{x \in \mathbb{R} \,|\, P_0\left((x - \varepsilon, x + \varepsilon)\right) > 0 \text{ for all } \varepsilon > 0\}. \qquad (3.49)$$

If supp P_0 is compact then the operator $H_\omega = H_0 + V_\omega$ is bounded. In fact, if supp $P_0 \subset [-M, M]$, with probability one

$$\sup_{j \in \mathbb{Z}^d} |V_\omega(j)| \le M \ .$$

Even if $\operatorname{supp} P_0$ is not compact the multiplication operator V_ω is selfadjoint on $D = \{\varphi \in \ell^2 | V_\omega \varphi \in \ell^2\}$. It is essentially selfadjoint on

$$\ell_0^2(\mathbb{Z}^d) = \{\varphi \in \ell^2(\mathbb{Z}^d) \mid \varphi(i) = 0 \text{ for all but finitely many points } i\} \ .$$

Since H_0 is bounded it is a fortiori Kato bounded with respect to V_ω. By the Kato-Rellich theorem it follows that $H_\omega = H_0 + V_\omega$ is essentially selfadjoint on $\ell_0^2(\mathbb{Z}^d)$ as well (see [**114**] for details).

In a first result about the Anderson model we are now going to determine its spectrum (as a set). In particular we will see that the spectrum $\sigma(H_\omega)$ is ($\mathbb{P}$-almost surely) a fixed non random set. First we prove a proposition which while easy is very useful in the following. Roughly speaking, this proposition tells us:

Whatever *can* happen, *will* happen, in fact infinitely often.

PROPOSITION 3.8. *There is a set Ω_0 of probability one such that the following is true: For any $\omega \in \Omega_0$, any finite set $\Lambda \subset \mathbb{Z}^d$, any sequence $\{q_i\}_{i \in \Lambda}$, $q_i \in \operatorname{supp} P_0$ and any $\varepsilon > 0$, there exists a sequence $\{j_n\}$ in $\mathbb{Z}^d$ with $||\, j_n\,||_\infty \to \infty$ such that*

$$\sup_{i \in \Lambda} |\, q_i - V_\omega(i + j_n)\, | < \varepsilon \ .$$

PROOF: Fix a finite set Λ, a sequence $\{q_i\}_{i \in \Lambda}$, $q_i \in \operatorname{supp} P_0$ and $\varepsilon > 0$. Then, by the definition of supp and the independence of the q_i we have for $A = \{\omega | \sup_{i \in \Lambda} |V_\omega(i) - q_i| < \varepsilon\}$

$$\mathbb{P}(A) > 0 \ .$$

Pick a sequence $\ell_n \in \mathbb{Z}^d$, such that the distance between any ℓ_n, ℓ_m $(n \ne m)$ is bigger than twice the diameter of Λ. Then, the events

$$A_n = A_n(\Lambda, \{q_i\}_{i \in \Lambda}, \varepsilon) = \{\omega | \sup_{i \in \Lambda} |V_\omega(i + \ell_n) - q_i| < \varepsilon\}$$

are independent and $\mathbb{P}(A_n) = \mathbb{P}(A) > 0$. Consequently, the Borel-Cantelli lemma (see Theorem 3.6) tells us that

$$\Omega_{\Lambda, \{q_i\}, \varepsilon} = \{\, \omega \mid \omega \in A_n \text{ for infinitely many } n\}$$

has probability one.

The set $\operatorname{supp} P_0$ contains a countable dense set R_0. Moreover, the system Ξ of all finite subsets of $\mathbb{Z}^d$ is countable. Thus the set

$$\Omega_0 := \bigcap_{\substack{\Lambda \,\in\, \Xi, \\ \{q_i\} \in R_0, n \in \mathbb{N}}} \Omega_{\Lambda, \{q_i\}, \frac{1}{n}}$$

has probability one. It is a countable intersection of sets of probability one. By its definition, Ω_0 satisfies the requirements of the assertion.

$\square$

We now turn to the announced theorem

THEOREM 3.9. *For $\mathbb{P}$-almost all ω we have $\sigma(H_\omega) = [0, 4d] + \operatorname{supp} P_0$.*

PROOF: The spectrum $\sigma(V)$ of the multiplication operator with $V(n)$ is given by the closure of the set $R(V) = \{V(n) | n \in \mathbb{Z}^d\}$. Hence $\sigma(V_\omega) = \operatorname{supp} P_0$ almost surely. Since $0 \le H_0 \le 4d$ we have

$$\begin{aligned}
\sigma(H_0 + V_\omega) &\subset \sigma(V_\omega) + [0, \|H_0\|] \\
&= \operatorname{supp} P_0 + [0, 4d] .
\end{aligned}$$

Let us prove the converse. We use the *Weyl criterion* (see [**117**] or [**141**]):

$$\lambda \in \sigma(H_\omega) \quad \Leftrightarrow \quad \exists \, \varphi_n \in D_0, \|\varphi_n\| = 1 : \quad \|(H_\omega - \lambda)\varphi_n\| \to 0 ,$$

where D_0 is any vector space such that H_ω is essentially selfadjoint on D_0. The sequence φ_n is called a *Weyl sequence*. In a sense, φ_n is an 'approximate eigenfunction'.

Let $\lambda \in [0, 4d] + \operatorname{supp} P_0$, say $\lambda = \lambda_0 + \lambda_1$, $\lambda_0 \in \sigma(H_0) = [0, 4d]$, $\lambda_1 \in \operatorname{supp} P_0$. Take a Weyl sequence φ_n for H_0 and λ_0, i. e. $\|(H_0 - \lambda_0)\varphi_n\| \to 0, \|\varphi_n\| = 1$. Since H_0 is essentially selfadjoint on $D_0 = \ell_0^2(\mathbb{Z}^d)$ (in fact H_0 is bounded), we may suppose $\varphi_n \in D_0$. Setting $\varphi^{(j)}(i) = \varphi(i - j)$, we easily see

$$H_0 \varphi^{(j)} = (H_0 \varphi)^{(j)} .$$

Due to Proposition 3.8 there is (with probability one) a sequence $\{j_n\}, \|j_n\|_\infty \to \infty$ such that

$$\sup_{i \in \operatorname{supp} \varphi_n} |V_\omega(i + j_n) - \lambda_1| < \frac{1}{n} . \tag{3.50}$$

Define $\psi_n = \varphi_n^{j_n}$. Then ψ_n is a Weyl sequence for H_ω and $\lambda = \lambda_0 + \lambda_1$. This proves the theorem. $\square$

The above result tells us in particular that the spectrum $\sigma(H_\omega)$ is (almost surely) non random. Moreover, an inspection of the proof shows that there is no discrete spectrum (almost surely), as the constructed Weyl sequence tends to zero weakly, in fact can be chosen to be orthonormal. Both results are valid in much bigger generality. They are due to ergodicity properties of the potential V_ω. We will discuss this topic in the following chapter.

Notes and Remarks

For further information see [**23**] and [**30**] or consult [**94**] and [**64, 65**].

4. Ergodicity properties

4.1. Ergodic stochastic processes.

Some of the basic questions about random Schrödinger can be formulated and answered most conveniently within the framework of 'ergodic operators'. This class of operators comprises many random operators, such as the Anderson model and its continuous analogs, the Poisson model, as well as random acoustic operators. Moreover, also operators with almost periodic potentials can be viewed as ergodic operators.

In these notes we only briefly touch the topic of ergodic operators. We just collect a few definitions and results we will need in the following chapters. We recommend the references cited in the notes at the end of this chapter for further reading.

Ergodic stochastic processes are a certain generalization of independent, identically distributed random variables. The assumption that the random variables X_i and X_j are independent for $|i - j| > 0$ is replaced by the requirement that X_i and X_j are 'almost independent' if $|i - j|$ is large (see the discussion below, especially (4.2), for a precise statement). The most important result about ergodic processes is the ergodic theorem (see Theorem 4.2 below), which says that the strong law of large numbers, one of the basic results about independent, identically distributed random variables, extends to ergodic processes.

At a number a places in these notes we will have to deal with ergodic processes. Certain important quantities connected with random operators are ergodic but not independent even if the potential V_ω *is* a sequence of independent random variables.

A family $\{X_i\}_{i \in \mathbb{Z}^d}$ of random variables is called a *stochastic process* (with index set $\mathbb{Z}^d$). This means that there is a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ ($\mathcal{F}$ a σ-algebra on Ω and $\mathbb{P}$ a probability measure on $(\Omega, \mathcal{F})$) such that the X_i are real valued, measurable functions on $(\Omega, \mathcal{F})$.

The quantities of interest are the probabilities of events that can be expressed through the random variables X_i, like

$$\{\omega | \lim_{N \to \infty} \frac{1}{|\Lambda_N|} \sum_{||i||_\infty \leq N} X_i(\omega) = 0\} \,.$$

The special way Ω is constructed is irrelevant. For example, one may take the set $\mathbb{R}^{\mathbb{Z}^d}$ as Ω. The corresponding σ-algebra $\mathcal{F}$ is generated by *cylinder sets* of the form

$$\{\omega \mid \omega_{i_1} \in A_1, \ldots, \omega_{i_n} \in A_n\} \tag{4.1}$$

where $A_1, \ldots, A_n$ are Borel subsets of $\mathbb{R}$. On Ω the random variables X_i can be realized by $X_i(\omega) = \omega_i$.

This choice of $(\Omega, \mathcal{F})$ is called the *canonical probability space* . For details in connection with random operators see e.g. [**58, 64**]. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we call a measurable mapping $T : \Omega \to \Omega$ a *measure preserving transformation* if $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. If $\{T_i\}_{i \in \mathbb{Z}^d}$ is a family of

measure preserving transformations we call a set $A \in \mathcal{F}$ *invariant* (under $\{T_i\}$) if $T_i^{-1} A = A$ for all $i \in \mathbb{Z}^d$.

A family $\{T_i\}$ of measure preserving transformations on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *ergodic* (with respect to the probability measure $\mathbb{P}$) if any invariant $A \in \mathcal{F}$ has probability zero or one. A stochastic process $\{X_i\}_{i \in \mathbb{Z}^d}$ is called *ergodic*, if there exists an ergodic family of measure preserving transformations $\{T_i\}_{i \in \mathbb{Z}^d}$ such that $X_i(T_j\,\omega) = X_{i-j}(\omega)$.

Our main example of an ergodic stochastic process is given by independent, identically distributed random variables $X_i(\omega) = V_\omega(i)$ (a random potential on $\mathbb{Z}^d$). Due to the independence of the random variables the probability measure $\mathbb{P}$ (on $\Omega = \mathbb{R}^{\mathbb{Z}^d}$) is just the infinite product measure of the probability measure P_0 on $\mathbb{R}$ given by $P_0(M) = \mathbb{P}(V_\omega(0) \in M)$. P_0 is the distribution of $V_\omega(0)$.

It is easy to see that the shift operators

$$(T_i\,\omega)_j = \omega_{j-i}$$

form a family of measure preserving transformations on $\mathbb{R}^{\mathbb{Z}^d}$ in this case.

It is not hard to see that the family of shift operators is ergodic with respect to the product measure $\mathbb{P}$. One way to prove this is to show that

$$\mathbb{P}(T_i^{-1}A \cap B) \to \mathbb{P}(A)\,\mathbb{P}(B) \tag{4.2}$$

as $\|\,i\,\|_\infty \to \infty$ for all $A,\ B \in \mathcal{F}$. This is obvious if both A and B are of the form (4.1). Moreover, the system of sets $A,\ B$ for which (4.2) holds is a σ-algebra, thus (4.2) is true for the σ-algebra generated by sets of the form (4.1), i.e. on $\mathcal{F}$.

Now let M be an invariant set. Then (4.2) (with $A = B = M$) gives

$$\mathbb{P}(M) = \mathbb{P}(M \cap M) = \mathbb{P}(T_i^{-1}M \cap M) \to \mathbb{P}(M)^2$$

proving that M has probability zero or one.

We will need two more results on ergodicity.

PROPOSITION 4.1. *Let $\{T_i\}_{i \in \mathbb{Z}^d}$ be an ergodic family of measure preserving transformations on a probability space $(\Omega, \mathcal{F}, P)$. If a random variable Y is invariant under $\{T_i\}$ (i.e. $Y(T_i\omega) = Y(\omega)$ for all $i \in \mathbb{Z}^d$) then Y is almost surely constant, i.e. there is a $c \in \mathbb{R}$, such that $\mathbb{P}(Y = c) = 1$.*

We may allow the values $\pm\infty$ for Y (and hence for c) in the above result. The proof is not difficult (see e.g. [**30**]).

The final result is the celebrated ergodic theorem by Birkhoff. It generalizes the strong law of large number to ergodic processes.

We denote by $\mathbb{E}(\cdot)$ the expectation with respect to the probability measure $\mathbb{P}$.

THEOREM 4.2. *If $\{X_i\}_{i \in \mathbb{Z}^d}$ is an ergodic process and $\mathbb{E}(|X_0|) < \infty$ then*

$$\lim_{L \to \infty} \frac{1}{(2L+1)^d} \sum_{i \in \Lambda_L} X_i \to \mathbb{E}(X_0)$$

for $\mathbb{P}$-almost all ω.

For a proof of this fundamental result see e.g. [**96**]. We remark that the ergodic theorem has important extensions in various directions (see [**91**]).

4.2. Ergodic operators.

Let $V_\omega(n)$, $n \in \mathbb{Z}^d$ be an ergodic process (for example, one may think of independent identically distributed $V_\omega(n)$).

Then there exist measure preserving transformations $\{T_i\}$ on Ω such that

(1) $V_\omega(n)$ satisfies
$$V_{T_i\omega}(n) = V_\omega(n - i) . \tag{4.3}$$

(2) Any measurable subset of Ω which is invariant under the $\{T_i\}$ has trivial probability (i.e. $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$) .

We define translation operators $\{U_i\}_{i \in \mathbb{Z}^d}$ on $\ell^2(\mathbb{Z}^d)$ by
$$(U_i\varphi)_m = \varphi_{m-i} \quad , \varphi \in \ell^2(\mathbb{Z}^d) . \tag{4.4}$$

It is clear that the operators U_i are unitary. Moreover, if we denote the multiplication operators with the function V by $\underline{V}$ then

$$\underline{V_{T_i\omega}} = U_i\underline{V_\omega}U_i^*. \tag{4.5}$$

The free Hamiltonian H_0 of the Anderson model (3.7) commutes with U_i, thus (4.5) implies

$$H_{T_i\omega} = U_iH_\omega U_i^* . \tag{4.6}$$

i.e. $H_{T_i\omega}$ and H_ω are unitarily equivalent.

Operators satisfying (4.6) (with ergodic T_i and unitary U_i) are called *ergodic operators* .

The following result is basic to the theory of ergodic operators.

THEOREM 4.3. *(Pastur) If H_ω is an ergodic family of selfadjoint operators, then there is a (closed, nonrandom) subset Σ of $\mathbb{R}$, such that*

$$\sigma(H_\omega) = \Sigma \quad \text{for } \mathbb{P}\text{-almost all } \omega.$$

Moreover, there are sets Σ_{ac}, Σ_{sc}, Σ_{pp} such that

$$\sigma_{ac}(H_\omega) = \Sigma_{ac}, \quad \sigma_{sc}(H_\omega) = \Sigma_{sc}, \quad \sigma_{pp}(H_\omega) = \Sigma_{pp}$$
$$\text{for } \mathbb{P}\text{-almost all } \omega.$$

REMARK 4.4.

(1) The theorem in its original form is due to Pastur [**111**]. It was extended in [**94**] and [**64**].
(2) We have been sloppy about the measurability properties of H_ω which have to be defined and checked carefully. They are satisfied in our case (i.e. for the Anderson model). For a precise formulation and proofs see [**64**].

(3) We denote by $\sigma_{ac}(H), \sigma_{sc}(H), \sigma_{pp}(H)$ the absolutely continuous (resp. singularly continuous, resp. pure point) spectrum of the operator H. For a definition and basic poperties we refer to Sections 7.2and 7.3.

PROOF *(Sketch)* : If H_ω is ergodic and f is a bounded (measurable) function then $f(H_\omega)$ is ergodic as well, i.e.

$$f(H_{T_i w}) = U_i f(H_\omega) U_i^* .$$

(see Lemma 4.5).

We have $(\lambda, \mu) \cap \sigma(H_\omega) \neq \emptyset$ if and only if $\chi_{(\lambda,\mu)}(H_\omega) \neq 0.$

This is equivalent to $Y_{\lambda,\mu}(\omega) := \operatorname{tr} \chi_{(\lambda,\mu)}(H_\omega) \neq 0.$

Since $\chi_{(\lambda,\mu)}(H_\omega)$ is ergodic, $Y_{\lambda,\mu}$ is an invariant random variable and consequently, by Proposition 4.1 $Y_{\lambda,\mu} = c_{\lambda,\mu}$ for all $\omega \in \Omega_{\lambda,\mu}$ with $P(\Omega_{\lambda,\mu}) = 1$.

Set

$$\Omega_0 = \bigcap_{\lambda,\mu \in \mathbb{Q}, \ \lambda \leq \mu} \Omega_{\lambda,\mu} .$$

Since Ω_0 is a *countable* intersection of sets of full measure, it follows that $P(\Omega_0) = 1$. Hence we can set

$$\Sigma = \{E \mid c_{\lambda,\mu} \neq 0 \text{ for all } \lambda < E < \mu, \quad \lambda, \mu \in \mathbb{Q}\} .$$

To prove the assertions on σ_{ac} we need that the projection onto $\mathcal{H}_{ac}$, the absolutely continuous subspace with respect to H_ω is measurable, the rest is as above. The same is true for σ_{sc} and σ_{pp} .

We omit the measurability proof and refer to [**64**] or [**23**]. $\qquad\square$

Above we used the following results

LEMMA 4.5. *Let A be a self adjoint operators and U a unitary operator, then for any bounded measurable function f we have*

$$f(U A U^*) \ = \ U\, f(A)\, U^* . \tag{4.7}$$

PROOF: For resolvents, i.e. for $f_z(\lambda) = \frac{1}{\lambda - z}$ with $z \in \mathbb{C} \setminus \mathbb{R}$ equation (4.7) can be checked directly. Linear combinations of the f_z are dense in $C_\infty(\mathbb{R})$, the continuous functions vanishing at infinity, by the Stone-Weierstraß theorem (see Section 3.3). Thus (4.7) is true for $f \in C_\infty(\mathbb{R})$.

If μ and ν are the projection valued measures for A and $B = U A U^*$ respectively, we have therefore for all $f \in C_\infty(\mathbb{R})$

$$
\begin{aligned}
\int f(\lambda)\, d\nu_{\varphi,\psi}(\lambda) \ &= \ \langle \varphi, f(B)\,\psi \rangle \\
&= \ \langle \varphi, U f(A)\, U^* \psi \rangle \\
&= \ \langle U^* \varphi, f(A)\, U^* \psi \rangle \\
&= \ \int f(\lambda)\, d\mu_{U^* \varphi, U^* \psi}(\lambda) \tag{4.8}
\end{aligned}
$$

holds for all . Thus the measures $\mu_{\varphi,\psi}$ and $\nu_{U^*\varphi,U^*\psi}$ agree. Therefore (4.8) holds for all bounded measurable f. □

Notes and Remarks

For further information see [**23**], [**58**], [**64**], [**65**], [**94**], [**111**] and [**112**]. An recent extensive review on ergodic operators can be found in [**55**].

5. The density of states

5.1. Definition and existence.

Here, as in the rest of the paper we consider the Anderson model, i.e. $H_\omega = H_0 + V_\omega$ on $\ell^2(\mathbb{Z}^d)$ with independent random variables $V_\omega(n)$ with a common distribution P_0.

In this section we define a quantity of fundamental importance for models in condensed matter physics: the density of states. The density of states measure $\nu([E_1, E_2])$ gives the 'number of states per unit volume' with energy between E_1 and E_2. Since the spectrum of our Hamiltonian H_ω is not discrete we can not simply count eigenvalues within the interval $[E_1, E_2]$ or, what is the same, take the dimension of the corresponding spectral projection. In fact, the dimension of any spectral projection of H_ω is either zero or infinite. Instead we restrict the spectral projection to the finite cube Λ_L (see 3.5) in $\mathbb{Z}^d$, take the dimension of its range and divide by $|\Lambda_L| = (2L + 1)^d$ the number of points in Λ_L. Finally, we send the parameter L to infinity. This procedure is sometimes called the thermodynamic limit.

For any bounded measurable function φ on the real line we define the quantity

$$\nu_L(\varphi) \;=\; \frac{1}{|\Lambda_L|} \, \operatorname{tr}\left(\chi_{\Lambda_L} \, \varphi(H_\omega) \, \chi_{\Lambda_L} \right) \;=\; \frac{1}{|\Lambda_L|} \, \operatorname{tr}\left(\varphi(H_\omega) \, \chi_{\Lambda_L} \right) . \qquad (5.1)$$

Here χ_Λ denotes the *characteristic function* of the set Λ, (i.e. $\chi_\Lambda(x) = 1$ for $x \in \Lambda$ and $= 0$ otherwise). The operators $\varphi(H_\omega)$ are defined via the spectral theorem (see Section 3.2). In equation (5.1) we used the cyclicity of the trace, (i.e.: $\operatorname{tr}(AB) = \operatorname{tr}(BA)$) and the fact that $\chi_\Lambda{}^2 = \chi_\Lambda$.

Since ν_L is a positive linear functional on the bounded continuous functions, by Riesz representation theorem, it comes from a measure which we also call ν_L, i.e.

$$\nu_L(\varphi) = \int_\mathbb{R} \varphi(\lambda) \, d\nu_L(\lambda). \qquad (5.2)$$

We will show in the following that the measures ν_L converge to a limit measure ν as $L \to \infty$ in the sense of vague convergence of measures for $\mathbb{P}$-almost all ω.

DEFINITION 5.1. *A series ν_n of Borel measures on $\mathbb{R}$ is said to* converge vaguely *to a Borel measure ν if*

$$\int \varphi(x) \, d\nu_n(x) \to \int \varphi(x) \, d\nu(x)$$

for all function $\varphi \in C_0(\mathbb{R})$, the set of continuous functions with compact support.

We start with a proposition which establishes the almost sure convergence of the integral of ν_L over a *given* function.

PROPOSITION 5.2. *If φ is a bounded measurable function, then for $\mathbb{P}$-almost all ω*

$$\lim_{L \to \infty} \frac{1}{|\Lambda_L|} \, \operatorname{tr}\left(\varphi(H_\omega) \, \chi_{\Lambda_L} \right) = \mathbb{E}\left(\, \langle \delta_0, \varphi(H_\omega) \delta_0 \rangle \, \right) \quad . \qquad (5.3)$$

REMARK 5.3. The right hand side of (5.3) defines a positive measure ν by

$$\int \varphi(\lambda)\, d\nu(\lambda) = \mathbb{E}(\langle \delta_0, \varphi(H_\omega)\delta_0\rangle) \ .$$

This measure satisfies $\nu(\mathbb{R}) = 1$, hence it is a probability measure (just insert $\varphi(\lambda) \equiv 1$).

DEFINITION 5.4. *The measure ν, defined by*

$$\nu(A) = \mathbb{E}\left(\langle \delta_0, \chi_A(H_\omega)\, \delta_0\rangle\right) \qquad \text{for } A \text{ a Borel set in } \mathbb{R} \tag{5.4}$$

is called the density of states measure .
The distribution function N of ν, defined by

$$N(E) = \nu\big((-\infty, E]\big) \tag{5.5}$$

is known as the integrated density of states .

PROOF *(Proposition)* :

$$\frac{1}{|\Lambda_L|}\ \mathrm{tr}\left(\varphi(H_\omega)\chi_{\Lambda_L}\right)$$

$$= \frac{1}{(2L+1)^d} \sum_{i\in\Lambda_L} \langle \delta_i, \varphi(H_\omega)\delta_i\rangle \tag{5.6}$$

The random variables $X_i = \langle \delta_i, \varphi(H_\omega)\delta_i\rangle$ form an ergodic stochastic process since the shift operators $\{T_i\}$ are ergodic and since

$$\begin{aligned}
X_i(T_j\omega) &= \langle \delta_i, \varphi(H_{T_j\omega})\, \delta_i\rangle \\
&= \langle \delta_i, U_j\, \varphi(H_\omega)\, U_j^*\, \delta_i\rangle \\
&= \langle U_j^*\, \delta_i, \varphi(H_\omega)\, U_j^*\, \delta_i\rangle \\
&= \langle \delta_{i-j}, \varphi(H_\omega)\, \delta_{i-j}\rangle \\
&= X_{i-j}(\omega) \ . \tag{5.7}
\end{aligned}$$

We used that $U_j^*\delta_i(n) = \delta_i(n+j) = \delta_{i-j}(n)$.
Since $|X_i| \leq ||\varphi||_\infty$, the X_i are integrable (with respect to $\mathbb{P}$). Thus we may apply the ergodic theorem (4.2) to obtain

$$\frac{1}{|\Lambda_L|}\, \mathrm{tr}(\varphi(H_\omega)\chi_{\Lambda_L}) = \frac{1}{(2L+1)^d} \sum_{i\in\Lambda_L} X_i \tag{5.8}$$

$$\longrightarrow \quad \mathbb{E}(X_0) = \mathbb{E}(\langle \delta_0, \varphi(H_\omega)\delta_0\rangle) \ . \tag{5.9}$$

$\square$

We have proven that (5.3) holds for *fixed* φ on a set of full probability. This set, let's call it Ω_φ, may (and will) depend on φ. We can conclude that (5.3) holds for *all* φ for $\omega \in \bigcap_\varphi \Omega_\varphi$. However, this is an *uncountable* intersection of sets of

probability one. We do *not* know whether this intersection has full measure, in fact we even don't know whether this set is measurable.

THEOREM 5.5. *The measures ν_L converge vaguely to the measure ν $\mathbb{P}$-almost surely, i.e. there is a set Ω_0 of probability one, such that*

$$\int \varphi(\lambda)\,d\nu_L(\lambda) \to \int \varphi(\lambda)\,d\nu(\lambda) \tag{5.10}$$

for all $\varphi \in C_0(\mathbb{R})$ and all $\omega \in \Omega_0$.

REMARK 5.6. The measure ν is *non random* by definition.

PROOF: Take a countable dense set D_0 in $C_0(\mathbb{R})$ in the uniform topology. With Ω_φ being the set of full measure for which (5.10) holds, we set

$$\Omega_0 = \bigcap_{\varphi \in D_0} \Omega_\varphi\,.$$

Since Ω_0 is a *countable* intersection of sets of full measure, Ω_0 has probability one. For $\omega \in \Omega_0$ the convergence (5.10) holds for all $\varphi \in D_0$.
By assumption on D_0, if $\varphi \in C_0(\mathbb{R})$ there is a sequence $\varphi_n \in D_0$ with $\varphi_n \to \varphi$ uniformly. It follows

$$|\int \varphi(\lambda)\,d\nu(\lambda) - \int \varphi(\lambda)\,d\nu_L(\lambda)|$$

$$\leq \quad |\int \varphi(\lambda)\,d\nu(\lambda) - \int \varphi_n(\lambda)\,d\nu(\lambda)|$$

$$+ \quad |\int \varphi_n(\lambda)\,d\nu(\lambda) - \int \varphi_n(\lambda)\,d\nu_L(\lambda)|$$

$$+ \quad |\int \varphi_n(\lambda)\,d\nu_L(\lambda) - \int \varphi(\lambda)\,d\nu_L(\lambda)|$$

$$\leq \quad ||\varphi - \varphi_n||_\infty \cdot \nu(\mathbb{R}) \; + \; ||\varphi - \varphi_n||_\infty \cdot \nu_L(\mathbb{R})$$

$$+ \quad |\int \varphi_n(\lambda)\,d\nu(\lambda) - \int \varphi_n(\lambda)\,d\nu_L(\lambda)|\,. \tag{5.11}$$

Since both $\nu(\mathbb{R})$ and $\nu_L(\mathbb{R})$ are bounded by 1 (in fact are equal to one) the first two terms can be made small by taking n large enough. We make the third term small by taking L large. $\square$

REMARKS 5.7.

   (1) As we remarked already in the above proof both ν_L and ν are probability measures. Consequently, the measures ν_L converge even weakly to ν, i. e. when integrated against a bounded continuous function (see e. g. [**12**]). Observe that the space of bounded continuous functions $C_b(\mathbb{R})$ does *not* contain a countable dense set, so the above *proof* does not work for C_b directly.

(2) In the continuous case the density of states measure is unbounded, even for the free Hamiltonian. So, in the continuous case, it does not make sense even to talk about weak convergence, we have to restrict ourselves to vague convergence in this case.

(3) Given a countable set D of bounded measurable functions we can find a set Ω_1 of probability one such that

$$\int \varphi(\lambda) d\nu_L(\lambda) \to \int \varphi(\lambda) d\nu(\lambda)$$

for all $\varphi \in D \cup C_b(\mathbb{R})$ and all $\omega \in \Omega_1$.

COROLLARY 5.8. *For $\mathbb{P}$-almost all ω the following is true:*
For all $E \in \mathbb{R}$

$$N(E) = \lim_{L \to \infty} \nu_L\big((-\infty, E]\big) . \tag{5.12}$$

REMARKS 5.9. It is an immediate consequence of Proposition 5.2 that for *fixed E* the convergence in (5.12) holds for almost all ω, with the set of exceptional ω being E-dependent. The statement of Corollary 5.8 is stronger: It claims the existence of an *E-independent* set of ω such that 5.12 is true for all E.

PROOF: We will prove (5.12) first for energies E where N is continuous.
Since N is monotone increasing the set of *discontinuity* points of N is at most countable (see Lemma 5.10 below). Consequently, there is a countable set S of continuity points of N which is dense in $\mathbb{R}$. By Proposition 5.2 there is a set of full $\mathbb{P}$-measure such that

$$\int \chi_{(-\infty, E]}(\lambda) \, d\nu_L(\lambda) \ \to \ N(E)$$

for all $E \in S$.
Take $\varepsilon > 0$. Suppose E is an arbitrary continuity point of N. Then, we find $E_+, E_- \in S$ with $E_- \leq E \leq E_+$ such that $N(E_+) - N(E_-) < \frac{\varepsilon}{2}$.
We estimate (N is monotone increasing)

$$N(E) - \int \chi_{(-\infty, E]}(\lambda) \, d\nu_L(\lambda) \tag{5.13}$$

$$\leq \ N(E_+) - \int \chi_{(-\infty, E_-]}(\lambda) \, d\nu_L(\lambda) \tag{5.14}$$

$$\leq \ N(E_+) - N(E_-) + \Big| N(E_-) - \int \chi_{(-\infty, E_-]}(\lambda) \, d\nu_L(\lambda) \Big| \tag{5.15}$$

$$\leq \ \varepsilon \tag{5.16}$$

for L large enough.

Analogously we get

$$N(E) - \int \chi_{(-\infty, E]}(\lambda)\, d\nu_L(\lambda) \tag{5.17}$$

$$\geq \quad N(E_-) - N(E_+) - \left| N(E_+) - \int \chi_{(-\infty, E_+]}(\lambda)\, d\nu_L(\lambda) \right| \tag{5.18}$$

$$\geq \quad -\varepsilon\, . \tag{5.19}$$

Hence

$$\left| N(E) - \int \chi_{(-\infty, E]}(\lambda)\, d\nu_L(\lambda) \right| \to 0$$

This proves (5.12) for continuity points. Since there are at most countably many points of discontinuity for N another application of Proposition 5.2 proves the result for all E. $\qquad\square$

Above we used the following Lemma.

LEMMA 5.10. *If the function $F : \mathbb{R} \to \mathbb{R}$ is monotone increasing then F has at most countably many points of discontinuity.*

PROOF: Since F is monotone both $F(t-) = \lim_{s \nearrow t} F(s)$ and $F(t+) = \lim_{s \searrow t} F(s)$ exist. If F is discontinuous at $t \in \mathbb{R}$ then $F(t+) - F(t-) > 0$. Set

$$D_n = \{ t \in \mathbb{R} \mid F(t+) - F(t-) > \frac{1}{n} \}$$

then the set D of discontinuity points of F is given by $\bigcup_{n \in \mathbb{N}} D_n$.

Let us assume that D is uncountable. Then also one of the D_n must be uncountable. Since F is monotone and defined on all of $\mathbb{R}$ it must be bounded on any bounded interval. Thus we conclude that $D_n \cap [-M, M]$ is finite for any M. It follows that $D_n = \bigcup_{M \in \mathbb{N}} \left(D_n \cap [-M, M] \right)$ is countable. This is a contradiction to the conclusion above. $\qquad\square$

REMARK 5.11.
The proof of Corollary 5.8 shows that we also have

$$\begin{aligned}
N(E-) &= \sup_{\varepsilon > 0} N(E - \varepsilon) \\
&= \int \chi_{(-\infty, E)}(\lambda)\, d\nu(\lambda) \\
&= \lim_{L \to \infty} \int \chi_{(-\infty, E)}(\lambda)\, d\nu_L(\lambda) \tag{5.20}
\end{aligned}$$

for *all* E and $\mathbb{P}$-almost all ω (with an E-independent set of ω).
Consequently, we also have $\nu(\{E\}) = \lim_{L \to \infty} \nu_L(\{E\})$.

PROPOSITION 5.12. *$supp(\nu) = \Sigma \quad (= \sigma(H_\omega))$.*

PROOF: If $\lambda \notin \Sigma$ then there is an $\epsilon > 0$ such that $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\omega) = 0$
$\mathbb{P}$-almost surely, hence

$$\nu\big((\lambda - \epsilon, \lambda + \epsilon)\big) = E\big(\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\omega)(0,0)\big) = 0 \,.$$

If $\lambda \in \Sigma$ then $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\omega) \neq 0$ $\mathbb{P}$-almost surely for any $\epsilon > 0$.
Since $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\omega)$ is a projection, it follows that for some $j \in \mathbb{Z}^d$

$$
\begin{aligned}
0 \;&\neq\; \mathbb{E}\big(\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\omega)(j,j)\big) \\
&=\; \mathbb{E}\big(\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(H_\omega)(0,0)\big) \\
&=\; \nu\big((\lambda - \epsilon, \lambda + \epsilon)\big) \,.
\end{aligned}
\tag{5.21}
$$

Here, we used that by Lemma 4.5

$$f(H_\omega)(j,j) = f(H_{T_j\,\omega})(0,0)$$

and the assumption that T_j is measure preserving. □

It is not hard to see that the integrated density of states $N(\lambda)$ is a continuous func-
tion, which is equivalent to the assertion that ν has no atoms, i.e. $\nu(\{\lambda\}) = 0$ for
all λ. We note, that an analogous result for the continuous case (i.e. Schrödinger
operators on $L^2(\mathbb{R}^d)$) is unknown in this generality.
We first state

LEMMA 5.13. *Let $\mathcal{V}_\lambda$ be the eigenspace of H_ω with respect to the eigenvalue λ*
then $\dim\big(\chi_{\Lambda_L}(\mathcal{V}_\lambda)\big) \leq C L^{d-1}$.

From this we deduce

THEOREM 5.14. *For any $\lambda \in \mathbb{R}$* $\nu(\{\lambda\}) = 0.$

PROOF *(of the Theorem assuming the Lemma)* :
By Proposition 5.2 and Theorem 5.5 we have

$$\nu(\{\lambda\}) = \lim_{L\to\infty} \frac{1}{(2L+1)^d} \,\mathrm{tr}\,\big(\chi_{\Lambda_L}\,\chi_{\{\lambda\}}(H_\omega)\big). \tag{5.22}$$

If f_i is an orthonormal basis of $\chi_{\Lambda_L}(\mathcal{V}_\lambda)$ and g_j an orthonormal basis of $\chi_{\Lambda_L}(\mathcal{V}_\lambda)^\perp$
we have, noting that $\chi_{\Lambda_L}(\mathcal{V}_\lambda)$ is finite dimensional,

$$
\begin{aligned}
&\mathrm{tr}\,\big(\chi_{\Lambda_L}\,\chi_{\{\lambda\}}(H_\omega)\big) \\
&=\; \sum_i \langle f_i, \chi_{\Lambda_L}\,\chi_{\{\lambda\}}(H_\omega)f_i\rangle + \sum_j \langle g_j, \chi_{\Lambda_L}\,\chi_{\{\lambda\}}(H_\omega)g_j\rangle \\
&=\; \sum_i \langle f_i, \chi_{\Lambda_L}\,\chi_{\{\lambda\}}(H_\omega)f_i\rangle \\
&\leq\; \dim \chi_{\Lambda_L}(\mathcal{V}_\lambda) \leq C L^{d-1}
\end{aligned}
\tag{5.23}
$$

hence (5.22) converges to zero. Thus $\nu\big(\{\lambda\}\big) = 0.$ □

PROOF *(Lemma)* :

We define $\widetilde{\Lambda}_L = \{ i \in \Lambda_L | \ (L-1) \leq ||\,i\,||_\infty \leq L \ \}$

$\widetilde{\Lambda}_L$ consists of the two outermost layers of Λ_L.

The values $u(n)$ of an eigenfunction u of H_ω with $H_\omega u = \lambda u$ can be computed from the eigenvalue equation for all $n \in \Lambda_L$ once we know its values on $\widetilde{\Lambda}_L$. So, the dimension of $\chi_{\Lambda_L}(\mathcal{V}_\lambda)$ is at most the number of points in $\widetilde{\Lambda}_L$. $\qquad\square$

5.2. Boundary conditions.

Boundary conditions are used to define differential operators on sets M with a boundary. A rigorous treatment of boundary conditions for differential operators is most conveniently based on a quadratic form approach (see [**115**]) and is out of the scope of this review. Roughly speaking boundary conditions restrict the domain of a differential operator D by requiring that functions in the domain of D have a certain behavior at the boundary of M. In particular, Dirichlet boundary conditions force the functions f in the domain to vanish at ∂M. Neumann boundary conditions require the normal derivative to vanish at the boundary. Let us denote by $-\Delta_M^D$ and $-\Delta_M^N$ the Laplacian on M with Dirichlet and Neumann boundary condition respectively.

The definition of boundary conditions for the discrete case are somewhat easier then in the continuous case. However, they are presumably less familiar to the reader and may look somewhat technical at a first glance. The reader might therefore skip the details for the first reading and concentrate of 'simple' boundary conditions defined below. Neumann and Dirichlet boundary conditions will be needed for this text only in chapter 6 in the proof of Lifshitz tails.

For our purpose the most important feature of Neumann and Dirichlet boundary conditions is the so called Dirichlet-Neumann bracketing . Suppose M_1 and M_2 are disjoint open sets in $\mathbb{R}^d$ and $M = (\overline{M_1 \cup M_2})^\circ$, ($^\circ$ denoting the interior) then

$$-\Delta_{M_1}^N \oplus -\Delta_{M_2}^N \leq -\Delta_M^N \leq -\Delta_M^D \leq -\Delta_{M_1}^D \oplus -\Delta_{M_2}^D \ . \qquad (5.24)$$

in the sense of quadratic forms. In particular the eigenvalues of the operators in (5.24) are increasing from left to right.

We recall that a bounded operator A on a Hilbert space $\mathcal{H}$ is called *positive* (or positive definite or $A \geq 0$) if

$$\langle \varphi, A\,\varphi \rangle \geq 0 \qquad \text{for all } \varphi \in \mathcal{H} \ . \qquad (5.25)$$

For unbounded A the validity of equation 5.25 is required for the (form-)domain of A only.

By $A \leq B$ for two operators A and B we mean $B - A \geq 0$.

For the lattice case we introduce boundary conditions which we call Dirichlet and Neumann conditions as well. Our choice is guided by the chain of inequalities(5.24).

The easiest version of boundary conditions for the lattice is given by the following procedure

DEFINITION 5.15. *The Laplacian with* simple boundary conditions *on* $\Lambda \subset \mathbb{Z}^d$ *is the operator on* $\ell^2(\Lambda)$ *defined by*

$$(H_0)_\Lambda(n, m) = \langle \delta_n, H_0 \, \delta_m \rangle \tag{5.26}$$

whenever both n and m belong to Λ. We also set $H_\Lambda = (H_0)_\Lambda + V$.
In particular, if Λ is finite, the operator H_Λ acts on a finite dimensional space, i.e. is a matrix.

We are going to use simple boundary conditions frequently in this work. At a first glance simple boundary conditions seem to be a reasonable analog of Dirichlet boundary conditions. However, they do not satisfy (5.24) as we will see later. Thus, we will have to search for other boundary conditions.
Let us define

$$\begin{aligned} \partial\Lambda \;\; = \;\; & \left\{ (n, m) \in \mathbb{Z}^d \times \mathbb{Z}^d \;\middle|\; \|n - m\|_1 = 1 \text{ and} \right. \\ & \left. \text{either } n \in \Lambda, m \notin \Lambda \text{ or } n \notin \Lambda, m \in \Lambda \right\} . \end{aligned} \tag{5.27}$$

The set $\partial\Lambda$ is the *boundary* of Λ. It consists of the *edges* connecting points in Λ with points outside Λ. We also define the *inner boundary* of Λ by

$$\partial^-\Lambda = \left\{ n \in \mathbb{Z}^d \;\middle|\; n \in \Lambda, \, \exists \, m \notin \Lambda \;\; (n, m) \in \partial\Lambda \right\} \tag{5.28}$$

and the *outer boundary* by

$$\partial^+\Lambda = \left\{ m \in \mathbb{Z}^d \;\middle|\; m \notin \Lambda, \, \exists \, n \in \Lambda \;\; (n, m) \in \partial\Lambda \right\} . \tag{5.29}$$

Hence $\partial^+\Lambda = \partial^-(\complement\Lambda)$ and the boundary $\partial\Lambda$ consists of edges between $\partial^-\Lambda$ and $\partial^+\Lambda$.
For any set Λ we define the boundary operator Γ_Λ by

$$\Gamma_\Lambda(n, m) = \begin{cases} -1 & \text{if } (n, m) \in \partial\Lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{5.30}$$

Thus for the Hamilitonian $H = H_0 + V$ we have the important relation

$$H = H_\Lambda \oplus H_{\complement\Lambda} + \Gamma_\Lambda . \tag{5.31}$$

In this equation we identified $\ell^2(\mathbb{Z}^d)$ with $\ell^2(\Lambda) \oplus \ell^2(\complement\Lambda)$.
More precisely

$$(H_\Lambda \oplus H_{\complement\Lambda})(n,m) = \begin{cases} H_\Lambda(n,m) & \text{if } n,m \in \Lambda, \\ H_{\complement\Lambda}(n,m) & \text{if } n,m \notin \Lambda, \\ 0 & \text{otherwise.} \end{cases} \qquad (5.32)$$

In other words $H_\Lambda \oplus H_{\complement\Lambda}$ is a block diagonal matrix and Γ_Λ is the part of H which connects these blocks.

It is easy to see, that Γ_Λ is neither negative nor positive definite. Consequently, the operator H_Λ will not satisfy any inequality of the type (5.24).

To obtain analogs to Dirichlet and Neumann boundary conditions we should substitute the operator Γ_Λ in (5.31) by a negative definite resp. positive definite operator and $H_\Lambda \oplus H_{\complement\Lambda}$ by an appropriate block diagonal matrix.

For the operator H_0 the diagonal term $H_0(i,i) = 2d$ gives the number of sites j to which i is connected (namely the $2d$ neighbors in $\mathbb{Z}^d$). This number is called the *coordination number* of the graph $\mathbb{Z}^d$. In the matrix H_Λ the edges to $\complement\Lambda$ are removed but the diagonal still contains the 'old' number of adjacent edges. Let us set $n_\Lambda(i) = |\{j \in \Lambda \mid \||j - i\||_1 = 1\}|$ to be the number of sites adjacent to i *in* Λ, the *coordination number* for the graph Λ. $n_\Lambda(i) = 2d$ as long as $i \in \Lambda \setminus \partial^- \Lambda$ but $n_\Lambda(i) < 2d$ at the boundary. We also define the *adjacency matrix* on Λ by

$$A_\Lambda(i,j) = \begin{cases} -1 & \text{if } i,j \in \Lambda, \ \||i - j\||_1 = 1 \\ 0 & \text{otherwise.} \end{cases} \qquad (5.33)$$

The operator $(H_0)_\Lambda$ on $\ell^2(\Lambda)$ is given by

$$(H_0)_\Lambda \quad = \quad 2d \ + \ A_\Lambda \qquad (5.34)$$

where $2d$ denotes a multiple of the identity.

DEFINITION 5.16. *The* Neumann Laplacian *on $\Lambda \subset \mathbb{Z}^d$ is the operator on $\ell^2(\Lambda)$ defined by*

$$(H_0)_\Lambda^N \quad = \quad n_\Lambda \ + \ A_\Lambda \, . \qquad (5.35)$$

Above n_Λ stands for the multiplication operator with the function $n_\Lambda(i)$ on $\ell^2(\Lambda)$.

REMARK 5.17.

(1) In $(H_0)_\Lambda$ the off diagonal term 'connecting' Λ to $\mathbb{Z}^d \setminus \Lambda$ are removed. However, through the diagonal term $2d$ the operator still 'remembers' there were $2d$ neighbors originally.

(2) The Neumann Laplacian H_0^N on Λ is also called the *graph Laplacian*. It is the canonical and intrinsic Laplacian with respect to the graph structure of Λ. It 'forgets' completely that the set Λ is imbedded in $\mathbb{Z}^d$.

(3) The quadratic form corresponding to $(H_0)_\Lambda^N$ is given by

$$\langle u, (H_0)_\Lambda^N v \rangle = \frac{1}{2} \sum_{\substack{n,m \in \Lambda \\ \||n-m\||_1 = 1}} \overline{(u(n) - u(m))}(v(n) - v(m)) \, .$$

DEFINITION 5.18. *The* Dirichlet Laplacian *on Λ is the operator on $\ell^2(\Lambda)$ defined by*

$$(H_0)_\Lambda^D = 2d + (2d - n_\Lambda) + A_\Lambda \ .$$

REMARK 5.19.

(1) The definition of the Dirichlet Laplacian may look a bit strange at the first glance. The main motivation for this definition is to preserve the properties (5.24) of the continuous analog.

(2) The Dirichlet Laplacian not only remembers that there were $2d$ neighboring sites before introducing boundary conditions, it even increases the diagonal entry by one for each adjacent edge which was removed. Very loosely speaking, one might say that the points at the boundary get an additional connection for every 'missing' link to points outside Λ.

It is not hard to see, that

$$(H_0)_\Lambda^N \leq (H_0)_\Lambda \leq (H_0)_\Lambda^D \tag{5.36}$$

and

$$
\begin{aligned}
H_0 &= (H_0)_\Lambda^N \oplus (H_0)_{\complement\Lambda}^N + \Gamma_\Lambda^N \tag{5.37} \\
&= (H_0)_\Lambda^D \oplus (H_0)_{\complement\Lambda}^D + \Gamma_\Lambda^D \tag{5.38}
\end{aligned}
$$

with

$$
\Gamma_\Lambda^N(i,j) = \begin{cases} 2d - n_\Lambda(i) & \text{if } i = j, i \in \Lambda, \\ 2d - n_{\complement\Lambda}(i) & \text{if } i = j, i \in \complement\Lambda, \\ -1 & \text{if } (i,j) \in \partial\Lambda, \\ 0 & \text{otherwise} \end{cases} \tag{5.39}
$$

and

$$
\Gamma_\Lambda^D(i,j) = \begin{cases} n_\Lambda(i) - 2d & \text{if } i = j, i \in \Lambda, \\ n_{\complement\Lambda}(i) - 2d & \text{if } i = j, i \in \complement\Lambda, \\ -1 & \text{if } (i,j) \in \partial\Lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{5.40}
$$

The operator Γ_Λ^N is positive definite as

$$\langle u, \Gamma_\Lambda^N v \rangle = \frac{1}{2} \sum_{(i,j) \in \partial\Lambda} \overline{\left(u(i) - u(j)\right)} \left(v(i) - v(j)\right)$$

is its quadratic form. In a similar way, we see that Γ_Λ^D is negative definite, since

$$\langle u, \Gamma_\Lambda^D v \rangle = -\frac{1}{2} \sum_{(i,j) \in \partial\Lambda} \overline{\left(u(i) + u(j)\right)} \left(v(i) + v(j)\right)$$

Hence we have in analogy to (5.24)

$$(H_0)_\Lambda^N \oplus (H_0)_{\complement\Lambda}^N \leq H_0 \leq (H_0)_\Lambda^D \oplus (H_0)_{\complement\Lambda}^D \ . \tag{5.41}$$

If $H = H_0 + V$ we define $H_\Lambda = (H_0)_\Lambda + V$, where in the latter expression V stands for the multiplication with the function V restricted to Λ. Similarly, $H_\Lambda^N = (H_0)_\Lambda^N + V$ and $H_\Lambda^D = (H_0)_\Lambda^D + V$.

These operators satisfy

$$H_\Lambda^N \oplus H_{\complement\Lambda}^N \leq H \leq H_\Lambda^D \oplus H_{\complement\Lambda}^D . \tag{5.42}$$

For $\Lambda_1 \subset \Lambda \subset \mathbb{Z}^d$ we have analogs of the 'splitting' formulae (5.31), (5.37) and (5.38). To formulate them it will be useful to define the 'relative' boundary $\partial_{\Lambda_2}\Lambda_1$ of $\Lambda_1 \subset \Lambda_2$ in Λ_2.

$$\partial_{\Lambda_2}\Lambda_1 = \partial\Lambda_1 \cap (\Lambda_2 \times \Lambda_2) = \partial\Lambda_1 \setminus \partial\Lambda_2 \tag{5.43}$$
$$= \left\{ (i,j) \,\big|\, ||\,i-j\,||_1 = 1 \text{ and } i \in \Lambda_1, j \in \Lambda_2 \setminus \Lambda_1 \text{ or } i \in \Lambda_2 \setminus \Lambda_1, j \in \Lambda_1 \right\}$$

The analogs of the splitting formulae are

$$H_{\Lambda_2} = H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} + \Gamma_{\Lambda_1}^{\Lambda_2} \tag{5.44}$$
$$H_{\Lambda_2}^N = H_{\Lambda_1}^N \oplus H_{\Lambda_2 \setminus \Lambda_1}^N + \Gamma_{\Lambda_1}^{\Lambda_2\,N} \tag{5.45}$$
$$H_{\Lambda_2}^D = H_{\Lambda_1}^D \oplus H_{\Lambda_2 \setminus \Lambda_1}^D + \Gamma_{\Lambda_1}^{\Lambda_2\,D} \tag{5.46}$$

with

$$\Gamma_{\Lambda_1}^{\Lambda_2}(i,j) = \begin{cases} 0 & \text{if } i = j \text{ and } i \in \Lambda_1 \\ 0 & \text{if } i = j \text{ and } i \in \Lambda_2 \setminus \Lambda_1 \\ -1 & \text{if } (i,j) \in \partial_{\Lambda_2}\Lambda_1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.47}$$

$$\Gamma_{\Lambda_1}^{\Lambda_2\,N}(i,j) = \begin{cases} n_{\Lambda_2}(i) - n_{\Lambda_1}(i) & \text{if } i = j \text{ and } i \in \Lambda_1 \\ n_{\Lambda_2}(i) - n_{\Lambda_2 \setminus \Lambda_1}(i) & \text{if } i = j \text{ and } i \in \Lambda_2 \setminus \Lambda_1 \\ -1 & \text{if } (i,j) \in \partial_{\Lambda_2}\Lambda_1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.48}$$

$$\Gamma_{\Lambda_1}^{\Lambda_2\,D}(i,j) = \begin{cases} n_{\Lambda_1}(i) - n_{\Lambda_2}(i) & \text{if } i = j \text{ and } i \in \Lambda_1 \\ n_{\Lambda_2 \setminus \Lambda_1}(i) - n_{\Lambda_2}(i) & \text{if } i = j \text{ and } i \in \Lambda_2 \setminus \Lambda_1 \\ -1 & \text{if } (i,j) \in \partial_{\Lambda_2}\Lambda_1 \\ 0 & \text{otherwise.} \end{cases} \tag{5.49}$$

In particular, for $\Lambda = \Lambda_1 \cup \Lambda_2$ with disjoint sets Λ_1 and Λ_2 we have

$$H_{\Lambda_1}^N \oplus H_{\Lambda_2}^N \leq H_\Lambda^N \leq H_\Lambda^D \leq H_{\Lambda_1}^D \oplus H_{\Lambda_2}^D. \tag{5.50}$$

since $\Gamma_{\Lambda_1}^{\Lambda\,N} \geq 0$ and $\Gamma_{\Lambda_1}^{\Lambda\,D} \leq 0$.

5.3. The geometric resolvent equation.

The equations (5.44), (5.45)and (5.46) allow us to prove the so called geometric resolvent equation. It expresses the resolvent of an operator on a larger set in terms of operators on smaller sets. This equality is a central tool of multiscale analysis. We do the calculations for simple boundary conditions (5.26) but the results are valid for Neumann and Dirichlet boundary conditions with the obvious changes. We start from equation (5.44) for $\Lambda_1 \subset \Lambda_2 \subset \mathbb{Z}^d$.

For $z \in \mathbb{C} \setminus \mathbb{R}$ this equation and the resolvent equation (3.18) imply

$$
\begin{aligned}
& (H_{\Lambda_2} - z)^{-1} \\
={} & (H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} - z)^{-1} - (H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} - z)^{-1} \Gamma_{\Lambda_1}^{\Lambda_2} (H_{\Lambda_2} - z)^{-1} \\
={} & (H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} - z)^{-1} - (H_{\Lambda_2} - z)^{-1} \Gamma_{\Lambda_1}^{\Lambda_2} (H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} - z)^{-1} \ .
\end{aligned}
$$
(5.51)

In fact, (5.51) holds for $z \notin \sigma(H_{\Lambda_1}) \cup \sigma(H_{\Lambda_2}) \cup \sigma(H_{\Lambda_2 \setminus \Lambda_1})$.

For $n \in \Lambda_1$, $m \in \Lambda_2 \setminus \Lambda_1$ we have

$$
H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1}(n, m) = 0
$$

hence

$$
(H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} - z)^{-1}(n, m) = 0 \ .
$$

Note that $(H_{\Lambda_1} \oplus H_{\Lambda_2 \setminus \Lambda_1} - z)^{-1} = (H_{\Lambda_2} - z)^{-1} \oplus (H_{\Lambda_2 \setminus \Lambda_1} - z)^{-1}$.

Thus (5.51) gives (for $n \in \Lambda_1$, $m \in \Lambda_2 \setminus \Lambda_1$)

$$
\begin{aligned}
& (H_{\Lambda_2} - z)^{-1}(n, m) \\
={} & - \sum_{k, k' \in \Lambda_2} (H_{\Lambda_1} \oplus H_{\Lambda_2} - z)^{-1}(n, k) \ \Gamma_{\Lambda_1}^{\Lambda_2}(k, k') \ (H_{\Lambda_2} - z)^{-1}(k', m) \\
={} & \sum_{\substack{(k, k') \in \partial \Lambda_1 \\ k \in \Lambda_1, \ k' \in \Lambda_2}} (H_{\Lambda_1} - z)^{-1}(n, k) \ (H_{\Lambda_2} - z)^{-1}(k', m) \ .
\end{aligned}
$$
(5.52)

We summarize in the following theorem

THEOREM 5.20 (Geometric resolvent equation).

If $\Lambda_1 \subset \Lambda_2$ *and* $n \in \Lambda_1$, $m \in \Lambda_2 \setminus \Lambda_1$ *and if* $z \notin \big(\sigma(H_{\Lambda_1}) \cup \sigma(H_{\Lambda_2}) \big)$, *then*

$$
\begin{aligned}
& (H_{\Lambda_2} - z)^{-1}(n, m) \\
={} & \sum_{\substack{(k, k') \in \partial \Lambda_1 \\ k \in \Lambda_1, \ k' \in \Lambda_2}} (H_{\Lambda_1} - z)^{-1}(n, k) \ (H_{\Lambda_2} - z)^{-1}(k', m) \ .
\end{aligned}
$$
(5.53)

Equation (5.53) is the *geometric resolvent equation*. It expresses the resolvent on a large set (Λ_2) in terms of the resolvent on a smaller set (Λ_1). Of course, the right hand side still contains the resolvent on the large set.

REMARK 5.21. Above we derived (5.53) only for $z \notin \sigma(H_{\Lambda_2 \setminus \Lambda_1})$. However, both sides of (5.53) exist and are analytic outside $\sigma(H_{\Lambda_1}) \cup \sigma(H_{\Lambda_2})$, so the formula is valid for all z outside $\sigma(H_{\Lambda_1}) \cup \sigma(H_{\Lambda_2})$

We introduce a short-hand notation for the matrix elements of resolvents

$$G_z^\Lambda(n, m) = (H_\Lambda - z)^{-1}(n, m). \tag{5.54}$$

The functions G_z^Λ are called *Green's functions* .

With this notation the geometric resolvent equation reads

$$G_z^{\Lambda_2}(n, m) = \sum_{\substack{(k, k') \in \partial\Lambda_1 \\ k \in \Lambda_1, \ k' \in \Lambda_2}} G_z^{\Lambda_1}(n, k) G_z^{\Lambda_2}(k', m) . \tag{5.55}$$

There are analogous equations to (5.53) for Dirichlet or Neumann boundary conditions which can be derived from (5.46) and (5.46) in the same way as above.

5.4. An alternative approach to the density of states.

In this section we present an alternative definition of the density of states measure. Perhaps, this is the more traditional one. We prove its equivalence to the definition given above.

In section 5.1 we defined the density of states measure by starting with a function φ of the Hamiltonian, taking its trace restricted to a cube Λ_L and normalizing this trace. In the second approach we first restrict the *Hamiltonian H_ω* to Λ_L with appropriate boundary conditions, apply the function φ to the restricted Hamiltonian and then take the normalized trace.

For any Λ let H_Λ^X be either H_Λ or H_Λ^N or H_Λ^D. We define the measures $\tilde{\nu}_L^X$ (i.e. $\tilde{\nu}_L$, $\tilde{\nu}_L^N$, $\tilde{\nu}_L^D$) by

$$\int \varphi(\lambda) \, d\tilde{\nu}_L^X(\lambda) = \frac{1}{|\Lambda_L|} \operatorname{tr} \varphi(H_{\Lambda_L}^X) . \tag{5.56}$$

Note that the operators $H_{\Lambda_L}^X$ act on the finite dimensional Hilbert space $\ell^2(\Lambda_L)$, so their spectra consist of eigenvalues $E_n(H_{\Lambda_L}^X)$ which we enumerate in increasing order

$$E_0(H_{\Lambda_L}^X) \leq E_1(H_{\Lambda_L}^X) \leq \ldots \quad .$$

In this enumeration we repeat each eigenvalue according to its multiplicity (see also (3.43)).

With this notation (5.56) reads

$$\int \varphi(\lambda) \, d\tilde{\nu}_L^X(\lambda) = \frac{1}{|\Lambda_L|} \sum_n \varphi(E_n(H_{\Lambda_L}^X)) .$$

The measure $\tilde{\nu}_L^X$ is concentrated on the eigenvalues of $H_{\Lambda_L}^X$. If E is an eigenvalue of $H_{\Lambda_L}^X$ then $\tilde{\nu}_L^X(\{E\})$ is equal to the dimension of the eigenspace corresponding to E.

We define the eigenvalue counting function by

$$N(H_\Lambda^X, E) = \left| \{ n \mid E_n(H_\Lambda^X) < E \} \right| \qquad (5.57)$$

(where $|M|$ is the number of elements of M). Then $\frac{1}{|\Lambda_L|} N(H_{\Lambda_L}^X, E)$ is the distribution function of the measure $\tilde{\nu}_L^X$, i. e.

$$\frac{1}{|\Lambda_L|} N(H_{\Lambda_L}^X, E) = \int \chi_{(-\infty, E)}(\lambda) \, d\tilde{\nu}_L^X(\lambda) \, . \qquad (5.58)$$

THEOREM 5.22. *The measures $\tilde{\nu}_L$, $\tilde{\nu}_L^D$ and $\tilde{\nu}_L^N$ converge $\mathbb{P}$-almost surely vaguely to the density of states measure ν.*

PROOF: We give the proof for $\tilde{\nu}_L$. An easy modification gives the result for $\tilde{\nu}_L^D$ and $\tilde{\nu}_L^N$ as well. To prove that $\tilde{\nu}_L$ converges vaguely to ν it suffices to prove

$$\int \varphi(\lambda) \, d\tilde{\nu}_L(\lambda) \to \int \varphi(\lambda) \, d\nu(\lambda)$$

for all φ of the form

$$\varphi(x) = r_z(x) = \frac{1}{x - z} \quad \text{for } z \in \mathbb{C} \setminus \mathbb{R}$$

because linear combination of these functions are dense in $\mathbb{C}_\infty(\mathbb{R})$ by the Stone-Weierstraß Theorem (see Section 3.3). ($C_\infty(\mathbb{R})$ are the continuous functions vanishing at infinity.)
We have

$$\int r_z(\lambda) \, d\tilde{\nu}_L(\lambda) = \frac{1}{(2L+1)^d} \, \text{tr} \left((H_{\Lambda_L} - z)^{-1} \right) = \frac{1}{(2L+1)^d} \sum_{n \in \Lambda_L} (H_{\Lambda_L} - z)^{-1}(n, n)$$

and

$$\int r_z(\lambda) \, d\nu_L(\lambda) = \frac{1}{(2L+1)^d} \, \text{tr} \left(\chi_{\Lambda_L} (H - z)^{-1} \right) = \frac{1}{(2L+1)^d} \sum_{n \in \Lambda_L} (H - z)^{-1}(n, n)$$

We use the resolvent equation in the form (5.51) for $n \in \Lambda_L$:

$$\left| \sum_{n \in \Lambda_L} (H_{\Lambda_L} - z)^{-1}(n,n) - (H - z)^{-1}(n,n) \right|$$

$$= \left| \sum_{\substack{n \in \Lambda_L \\ (k,k') \in \partial \Lambda_L \\ k \in \Lambda_L, \ k' \in \complement \Lambda_L}} (H_{\Lambda_L} - z)^{-1}(n,k) \ (H - z)^{-1}(k',n) \right|$$

$$\leq \sum_{\substack{(k,k') \in \partial \Lambda_L \\ k \in \Lambda_L, \ k' \in \complement \Lambda_L}} \left(\sum_n |(H_{\Lambda_L} - z)^{-1}(n,k)|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_n |(H - z)^{-1}(k',n)|^2 \right)^{\frac{1}{2}}$$

$$= \sum_{\substack{(k,k') \in \partial \Lambda_L \\ k \in \Lambda_L, \ k' \in \complement \Lambda_L}} ||(H_{\Lambda_L} - z)^{-1} \delta_k|| \cdot ||(H - z)^{-1} \delta_{k'}||$$

$$\leq c \, L^{d-1} \, ||(H_{\Lambda_L} - z)^{-1}|| \cdot ||(H - z)^{-1}||$$

$$\leq \frac{c}{(\operatorname{Im} z)^2} \, L^{d-1} \, . \tag{5.59}$$

Hence

$$\left| \int r_z(\lambda) \, d\tilde{\nu}_L(\lambda) - \int r_z(\lambda) \, d\nu_L(\lambda) \right| \leq \frac{c'}{(\operatorname{Im} z)^2} \cdot \frac{1}{L} \ \to \ 0 \qquad \text{as } L \to \infty \, .$$

$\square$

5.5. The Wegner estimate.

We continue with the celebrated 'Wegner estimate'. This result due to Wegner [140] shows not only the regularity of the density of states, it is also a key ingredient to prove Anderson localization. We set $N_\Lambda(E) := N(H_\Lambda, E)$.

THEOREM 5.23. *(Wegner estimate) Suppose the measure P_0 has a bounded density g, (i.e. $P_0(A) = \int_A g(\lambda)d\lambda$, $||g||_\infty < \infty$) then*

$$\mathbb{E} \left(N_\Lambda(E + \varepsilon) - N_\Lambda(E - \varepsilon) \right) \leq C \, || \, g \, ||_\infty \, |\Lambda| \, \varepsilon \, . \tag{5.60}$$

Before we prove this estimate we note two important consequences.

COROLLARY 5.24. *Under the assumption of Theorem 5.23 the integrated density of states is absolutely continuous with a bounded density $n(E)$.*

Thus $N(E) = \int_{-\infty}^E n(\lambda) \, d\lambda$. We call $n(\lambda)$ the *density of states*. Sometimes, we also call N the density of states, which, we admit, is an abuse of language.

COROLLARY 5.25. *Under the assumptions of Theorem 5.23 we have for any E and Λ*

$$\mathbb{P} \left(\operatorname{dist}(E, \sigma(H_\Lambda)) < \varepsilon \right) \leq C \, || \, g \, ||_\infty \, \varepsilon \, |\Lambda| \, . \tag{5.61}$$

PROOF *(Corollary 5.25)* :   By the Chebycheff inequality we get

$$\mathbb{P}\left(\text{dist}(E, \sigma(H_\Lambda)) < \varepsilon\right)$$
$$= \mathbb{P}\left(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon) \geq 1\right)$$
$$\leq \mathbb{E}\left(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon)\right)$$
$$\leq C \, \|\, g \,\|_\infty \, \varepsilon \, |\Lambda| \qquad \text{by Theorem (5.23).} \tag{5.62}$$

$\square$

PROOF *(Corollary 5.24)* :   By Theorem 5.23 we have

$$N(E+\varepsilon) - N(E-\varepsilon) = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \, \mathbb{E}\left(N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon)\right)$$

$$\leq C \, \|\, g \,\|_\infty \, \varepsilon \, . \tag{5.63}$$

$\square$

We turn to the proof of the theorem.

PROOF *(Wegner estimate)* :   Let ϱ be a non decreasing C^∞-function with $\varrho(\lambda) = 1$ for $\lambda \geq \varepsilon$, $\varrho(\lambda) = 0$ for $\lambda \leq -\varepsilon$ and consequently $0 \leq \varrho(\lambda) \leq 1$. Then

$$\begin{aligned} 0 \;\leq\;\; & \chi_{(-\infty,E+\varepsilon)}(\lambda) - \chi_{(-\infty,E-\varepsilon)}(\lambda) \\ \leq\;\; & \varrho(\lambda - E + 2\varepsilon) - \varrho(\lambda - E - 2\varepsilon) \end{aligned}$$

hence

$$\begin{aligned} 0 \;\leq\;\; & \chi_{(-\infty,E+\varepsilon)}(H_\Lambda) - \chi_{(-\infty,E-\varepsilon)}(H_\Lambda) \\ \leq\;\; & \varrho(H_\Lambda - E + 2\varepsilon) - \varrho(H_\Lambda - E - 2\varepsilon). \end{aligned}$$

Consequently,

$$\begin{aligned} & N_\Lambda(E+\varepsilon) - N_\Lambda(E-\varepsilon) \\ =\;\; & \text{tr}\left(\chi_{(-\infty,E+\varepsilon)}(H_\Lambda) - \chi_{(-\infty,E-\varepsilon)}(H_\Lambda)\right) \\ \leq\;\; & \text{tr}\, \varrho(H_\Lambda - E + 2\varepsilon) - \text{tr}\, \varrho(H_\Lambda - E - 2\varepsilon) \, . \tag{5.64} \end{aligned}$$

To compute the expectation of (5.64) we look upon the operators H_Λ (and their eigenvalues $E_n(H_\Lambda)$) as functions of the values $V_\Lambda = \{V_i\}_{i\in\Lambda}$ of the potential inside Λ. More precisely, we view the mapping

$$V_\Lambda \to H_\Lambda = H_\Lambda(V_\Lambda)$$

as a matrix-valued function on $\mathbb{R}^{|\Lambda|}$. This function is differentiable and

$$\left(\frac{\partial H_\Lambda}{\partial V_i}\right)_{\ell m} = \delta_{\ell m}\delta_{\ell i} \, . \tag{5.65}$$

The function

$$(E, V_\Lambda) \to \text{tr}\, \varrho\big(H_\Lambda(V_\Lambda) - E\big) \, .$$

is differentiable as well. Furthermore, since

$$H_\Lambda(V_\Lambda) - E = H_\Lambda(V_\Lambda - E).$$

(5.66)

it follows

$$\text{tr }\varrho\big(H_\Lambda(V_\Lambda) - E\big) = F\big(\{V_i - E\}_{i\in\Lambda}\big)$$

(5.67)

and consequently

$$\frac{\partial}{\partial E}\left(\text{tr }\varrho\big(H_\Lambda(V_\Lambda) - E\big)\right) = -\sum_{i\in\Lambda}\frac{\partial}{\partial V_i}\left(\text{tr }\varrho\big(H_\Lambda(V_\Lambda) - E\big)\right)$$

(5.68)

Therefore, with (5.64)

$$
\begin{aligned}
N_\Lambda(E + \varepsilon) - N_\Lambda(E - \varepsilon) &\le \text{ tr }\varrho\left(H_\Lambda - E + 2\varepsilon\right) - \text{tr }\varrho\left(H_\Lambda - E - 2\varepsilon\right)\\
&= -\big(\text{tr }\varrho\left(H_\Lambda - (E + 2\varepsilon)\right) - \text{tr }\varrho\left(H_\Lambda - (E - 2\varepsilon)\right)\big)\\
&= -\int_{E-2\varepsilon}^{E+2\varepsilon}\frac{\partial}{\partial\eta}\left(\varrho\big(H_\Lambda(V_\Lambda) - \eta\big)\right)d\eta\\
&= \int_{E-2\varepsilon}^{E+2\varepsilon}\sum_{j\in\Lambda}\frac{\partial}{\partial V_j}\ \text{tr }\varrho\big(H_\Lambda(V_\Lambda - \eta)\big)\ d\eta.
\end{aligned}
$$

(5.69)

Therefore

$$
\begin{aligned}
&\mathbb{E}\left(N_\Lambda(E + \varepsilon) - N_\Lambda(E - \varepsilon)\right)\\
&\le \mathbb{E}\left(\sum_{j\in\Lambda}\int_{E-2\varepsilon}^{E+2\varepsilon}\frac{\partial}{\partial V_j}\ \text{tr}\left(\varrho\left(H_\Lambda(V_\Lambda) - \eta\right)\right)d\eta\right)\\
&= \sum_{j\in\Lambda}\int_{E-2\varepsilon}^{E+2\varepsilon}\mathbb{E}\left(\frac{\partial}{\partial V_j}\ \text{tr}\left(\varrho\left(H_\Lambda(V_\Lambda) - \eta\right)\right)\right)d\eta.
\end{aligned}
$$

(5.70)

Since the random variables $V_\omega(i)$ are independent and have the common distribution $dP_0(V_i) = g(V_i)\,dV_i$, the expectation $\mathbb{E}$ is just integration with respect to the product of these distributions. Moreover, since supp P_0 is compact the integral over the variable V_i can be restricted to $[-M, +M]$ for some M large enough.

Hence

$$\mathbb{E}\left(\frac{\partial}{\partial V_j} \operatorname{tr}\left(\varrho\left(H_\Lambda(V_\Lambda) - \eta\right)\right)\right)$$

$$= \int_{-M}^{+M} \cdots \int_{-M}^{+M} \frac{\partial}{\partial V_j} \operatorname{tr}\left(\varrho(H_\Lambda(V_\Lambda) - \eta)\right) \prod_{i\in\Lambda} g(V_i) \prod_{i\in\Lambda} dV_i$$

$$= \int_{-M}^{+M} \cdots \int_{-M}^{+M} \left(\int_{-M}^{+M} \frac{\partial}{\partial V_j} \operatorname{tr}\left(\varrho(H_\Lambda(V_\Lambda) - \eta)\right) g(V_j)\, dV_j\right) \prod_{\substack{i\in\Lambda \\ i\neq j}} g(V_i)\, dV_i \;.$$

$$(5.71)$$

Since $\operatorname{tr}\left(\varrho(H_\Lambda(V_\Lambda) - \eta)\right)$ is non decreasing in V_j we can estimate

$$\int_{-M}^{+M} \frac{\partial}{\partial V_j} \operatorname{tr}\left(\varrho(H_\Lambda(V_\Lambda) - \eta)\right) g(V_j)\; dV_j$$

$$\leq \|g\|_\infty \left(\operatorname{tr}\left(\varrho(H_\Lambda(V_\Lambda, V_j = M) - \eta)\right) - \operatorname{tr}\left(\varrho(H_\Lambda(V_\Lambda, V_j = -M) - \eta)\right)\right)$$

$$(5.72)$$

where $H_\Lambda(V_\Lambda, V_j = a) = H_{0\Lambda} + \tilde{V}$ is the Anderson Hamiltonian on Λ with potential

$$\tilde{V}_i = \begin{cases} V_i & \text{for} \quad i \neq j \\ a & \text{for} \quad i = j. \end{cases} \tag{5.73}$$

To estimate the right hand side of inequality (5.72) we will use the following Lemma:

LEMMA 5.26. *Let A be a selfadjoint operator bounded below with purely discrete spectrum and eigenvalues $E_0 \leq E_1 \leq \ldots$ repeated according to multiplicity. If B is a symmetric positive rank one operator then $\tilde{A} = A + B$ has eigenvalue $\tilde{E}_n$ with $E_n \leq \tilde{E}_n \leq E_{n+1}$.*

Given the Lemma we continue the proof of the theorem.
We set $A = H_\Lambda(V_\Lambda, V_j = -M)$ and $\tilde{A} = H_\Lambda(V_\Lambda, V_j = +M)$. Obviously their difference is a (positive) rank one operator

$$\operatorname{tr} \varrho(\tilde{A} - \eta) - \operatorname{tr} \varrho(A - \eta)$$

$$= \sum_n \left(\varrho(\tilde{E}_n - \eta) - \varrho(E_n - \eta)\right)$$

$$\leq \sum_n \left(\varrho(E_{n+1} - \eta) - \varrho(E_n - \eta)\right)$$

$$\leq \sup_{\lambda,\mu} \varrho(\lambda) - \varrho(\mu)$$

$$= 1\,. \tag{5.74}$$

Thus from (5.72) we have

$$\int_{-M}^{+M} \frac{\partial}{\partial V_j} \, \text{tr} \left(\varrho(H_\Lambda(V_\Lambda) - \eta) \right) \, g(V_j) \, dV_j \, \leq \, || \, g ||_\infty \, .$$

Since $\int_{-M}^{+M} g(v)dv = 1$ we conclude from (5.71) and (5.72) that

$$\mathbb{E} \left(\frac{\partial}{\partial V_j} \, \text{tr} \left(\varrho \left(H_\Lambda(V_\Lambda) - \eta \right) \right) \right) \, \leq \, || \, g ||_\infty.$$

So, (5.70) implies

$$\mathbb{E} \left(N_\Lambda(E + \varepsilon) - N_\Lambda(E - \varepsilon) \right) \, \leq \, 4 \, || \, g ||_\infty \, | \, \Lambda | \, \varepsilon \, . \tag{5.75}$$

$\square$

PROOF *(Lemma)* : Since B is a positive symmetric rank one operator it is of the form $B = c \, |h\rangle\langle h|$ with $c \geq 0$, i.e. $B \, \varphi = c \, \langle h, \varphi \rangle \, h$ for some h.
By the min-max principle (Theorem 3.1)

$$\begin{aligned}
\tilde{E}_n & = \sup_{\psi_1, \ldots, \psi_{n-1}} \inf_{\substack{\varphi \perp \psi_1, \ldots, \psi_{n-1} \\ ||\varphi||=1}} \langle \varphi, A \, \varphi \rangle + c \, |\langle \varphi, h \rangle|^2 \\
& \leq \sup_{\psi_1, \ldots, \psi_{n-1}} \inf_{\substack{\varphi \perp \psi_1, \ldots, \psi_{n-1} \\ \varphi \perp h \quad ||\varphi||=1}} \langle \varphi, A \, \varphi \rangle + c \, |\langle \varphi, h \rangle|^2 \\
& \leq \sup_{\psi_1, \ldots, \psi_{n-1}} \inf_{\substack{\varphi \perp \psi_1, \ldots, \psi_{n-1},h \\ ||\varphi||=1}} \langle \varphi, A \, \varphi \rangle \\
& \leq \sup_{\psi_1, \ldots, \psi_{n-1},\psi_n} \inf_{\substack{\varphi \perp \psi_1, \ldots, \psi_{n-1},\psi_n \\ ||\varphi||=1}} \langle \varphi, A \, \varphi \rangle \\
& = E_{n+1} \, . \tag{5.76}
\end{aligned}$$

$\square$

By the Wegner estimate we know that any given energy E is *not* an eigenvalue of $(H_\omega)_{\Lambda_L}$ for almost all ω. On the other hand it is clear that for any given ω there are (as a rule $|\Lambda_L|$) eigenvalues of $(H_\omega)_{\Lambda_L}$.

This simple fact illustrates that we are not allowed to interchange 'any given E' and 'for $\mathbb{P}-$almost all ω' in assertions like the one above. What goes wrong is that we are trying to take an uncountable union of sets of measure zero. This union may have any measure, if it is measurable at all.

In the following we demonstrate a way to overcome these difficulties (in a sense). This idea is extremely useful when we want to prove pure point spectrum.

THEOREM 5.27. *If Λ_1, Λ_2 are disjoint finite subsets of $\mathbb{Z}^d$, then*

$$\mathbb{P} \left(\text{ There is an } E \in \mathbb{R} \text{ such that } \text{dist}(E, \sigma(H_{\Lambda_1})) < \varepsilon \text{ and } \text{dist}(\sigma(E, H_{\Lambda_2}))) < \varepsilon \right)$$

$$\leq 2 \, C \, || \, g \, ||_\infty \, \varepsilon \, | \, \Lambda_1 || \, \Lambda_2 | \, .$$

We start the proof with the following lemma.

LEMMA 5.28. *If* Λ_1, Λ_2 *are disjoint finite subsets of* $\mathbb{Z}^d$ *then*

$$\mathbb{P}\left(\,\mathrm{dist}(\sigma(H_{\Lambda_1}), \sigma(H_{\Lambda_2})) < \varepsilon\,\right) \leq C \,\|\,g\,\|_\infty\, \varepsilon\, |\Lambda_1||\Lambda_2|\,.$$

PROOF *(Lemma)* : Since $\Lambda_1 \cap \Lambda_2 = \emptyset$ the random potentials in Λ_1 and Λ_2 are independent of each other and so are the eigenvalues $E_0^{(1)} \leq E_1^{(1)} \leq ...$ of H_{Λ_1} and the eigenvalues $E_0^{(2)} \leq E_1^{(2)} \leq ...$ of H_{Λ_2}.
We denote the probability (resp. the expectation) with respect to the random variables in Λ by $\mathbb{P}_\Lambda$ (resp. $\mathbb{E}_\Lambda$). Since the random variables $\{V_\omega(n)\}_{n \in \Lambda_1}$ and $\{V_\omega(n)\}_{n \in \Lambda_2}$ are independent for $\Lambda_1 \cap \Lambda_2 = \emptyset$ we have that for such sets $\mathbb{P}_{\Lambda_1 \cup \Lambda_2}$ is the product measure $\mathbb{P}_{\Lambda_1} \otimes \mathbb{P}_{\Lambda_2}$.
We compute

$$\begin{aligned}
\mathbb{P}\left(\,\mathrm{dist}(\sigma(\Lambda_1), \sigma(\Lambda_2)) < \varepsilon\,\right) &= \mathbb{P}\left(\,\min_i\ \mathrm{dist}(E_i^{(1)}, \sigma(H_{\Lambda_2})) < \varepsilon\,\right) \\
&\leq \sum_{i=1}^{|\Lambda_1|} \mathbb{P}\left(\,\mathrm{dist}(E_i^{(1)}, \sigma(H_{\Lambda_2})) < \varepsilon\,\right) \\
&\leq \sum_{i=1}^{|\Lambda_1|}\ \mathbb{P}_{\Lambda_1} \otimes \mathbb{P}_{\Lambda_2}\left(\,\mathrm{dist}(E_i^{(1)}, \sigma(H_{\Lambda_2})) < \varepsilon\,\right) \\
&\leq \sum_{i=1}^{|\Lambda_1|} \mathbb{E}_{\Lambda_1}\left(\,\mathbb{P}_{\Lambda_2}\left(\,\mathrm{dist}(E_i^{(1)}, \sigma(H_{\Lambda_2})) < \varepsilon\,\right)\right).
\end{aligned}$$
$$(5.77)$$

From Theorem 5.23 we know that

$$\mathbb{P}_{\Lambda_2}\left(\,\mathrm{dist}(E, \sigma(H_{\Lambda_2})) < \varepsilon\,\right)\ \leq\ C \,\|\,g\,\|_\infty\, \varepsilon\, |\Lambda_2|\,.$$

Hence, we obtain

$$(5.77) \leq C \,\|\,g\,\|_\infty\, \varepsilon\, |\Lambda_2||\Lambda_1|\,.$$

$\square$

The proof of the theorem is now easy.

PROOF *(Theorem)* :

$$\begin{aligned}
\mathbb{P}\left(\text{ There is an } E \in \mathbb{R} \text{ such that } \mathrm{dist}(\sigma(H_{\Lambda_1}), E) < \varepsilon \text{ and } \mathrm{dist}(\sigma(H_{\Lambda_2}, E))) < \varepsilon\,\right) \\
\leq \mathbb{P}\left(\,\mathrm{dist}(\sigma(H_{\Lambda_1}), \sigma(H_{\Lambda_2})) < 2\,\varepsilon\right) \\
\leq 2\,C \,\|\,g\,\|_\infty\, \varepsilon\, |\Lambda_1||\Lambda_2|
\end{aligned}$$

by the lemma.

$\square$

Notes and Remarks

General references for the density of states are [**109**], [**63**], [**10**] and [**36**], [**58**] [**121**] and [**138**]. A thorough discussion of the geometric resolvent equation in the context of perturbation theory can be found in [**40**], [**47**] and [**127**].

In the context of the discrete Laplacian Dirichlet and Neumann boundary conditions were introduced and investigated in [**121**]. See also [**68**].

For discrete ergodic operators the integrated density of states N is log-Hölder continuous, see [**29**]. Our proof of the continuity of N is tailored after [**36**].

For results concerning the Wegner estimates see [**140**], [**59**], [**130**] [**27**], [**26**],and [**138**], as well as references given there.

6. Lifshitz tails

6.1. Statement of the Result.

Already in the 1960s, the physicist I. Lifshitz observed that the low energy behavior of the density of states changes drastically if one introduces disorder in a system. More precisely, Lifshitz found that

$$N(E) \sim C\, (E - E_0)^{\frac{d}{2}} \qquad E \searrow E_0 \tag{6.1}$$

for the ordered case (i.e. periodic potential), E_0 being the infimum of the spectrum, and

$$N(E) \sim C_1 e^{-C\,(E-E_0)^{-\frac{d}{2}}} \qquad E \searrow E_0 \tag{6.2}$$

for the disordered case. The behavior (6.2) of N is now called *Lifshitz behavior* or *Lifshitz tails*. We will prove (a weak form of) Lifshitz tails for the Anderson model. This result is an interesting and important result on its own. It is also used as an input for the proof of Anderson localization.

If P_0 is the common distribution of the independent random variables $V_\omega(i)$, we denote by a_0 the infimum of the support supp P_0 of P_0. From Theorem 3.9 we have $E_0 = \inf \sigma(H_\omega) = a_0$ $\mathbb{P}$-almost surely. We assume that P_0 is not trivial, i.e. is not concentrated in a single point. Moreover, we suppose that

$$P_0([a_0, a_0 + \epsilon]) \geq C\epsilon^\kappa, \quad \text{for some } C, \kappa > 0 . \tag{6.3}$$

Under these assumptions we prove:

THEOREM 6.1 (Lifshitz-tails).

$$\lim_{E \searrow E_0} \frac{\ln |\ln N(E)|}{\ln(E - E_0)} = -\frac{d}{2} . \tag{6.4}$$

REMARK 6.2. (6.4) is a weak form of (6.2). The asymptotic formula (6.2) suggests that we should expect *at least*

$$\lim_{E \searrow E_0} \frac{\ln N(E)}{(E - E_0)^{-\frac{d}{2}}} = -C . \tag{6.5}$$

Lifshitz tails can be proven in the strong form (6.5) for the Poisson random potential (see [38] and [108]). In general, however, there can be a logarithmic correction to (6.5) (see [102]) so that we can only expect the weak form (6.4). This form of the asymptotics is called the 'doublelogarithmic' asymptotics.

To prove the theorem, we show an upper and a lower bound.

6.2. Upper bound.

For the upper bound we will need Temple's inequality , which we state and prove for the reader's convenience.

LEMMA 6.3 (Temple's inequality). *Let A be a self-adjoint operator and $E_0 = \inf \sigma(A)$ be an isolated non degenerate eigenvalue. We set $E_1 = \inf \big(\sigma(A)\backslash\{E_0\}\big)$. If $\psi \in D(A)$ with $||\psi|| = 1$ satisfies*

$$\langle \psi, A\psi \rangle < E_1 \ ,$$

then

$$E_0 \ \geq \ \langle \psi, A\psi \rangle - \frac{\langle \psi, A^2\psi \rangle - \langle \psi, A\psi \rangle^2}{E_1 - \langle \psi, A\psi \rangle} \ .$$

PROOF: By assumption we have

$$(A - E_1)(A - E_0) \geq 0.$$

Hence, for any ψ with norm 1

$$\langle \psi, A^2\psi \rangle - E_1\langle \psi, A\psi \rangle - E_0\langle \psi, A\psi \rangle + E_1 E_0 \geq 0 \ .$$

This implies

$$E_1 E_0 - E_0\langle \psi, A\psi \rangle \geq E_1\langle \psi, A\psi \rangle - \langle \psi, A\psi \rangle^2 - (\langle \psi, A^2\psi \rangle - \langle \psi, A\psi \rangle^2) \ .$$

Since $E_1 - \langle \psi, A\psi \rangle > 0$, we obtain

$$E_0 \geq \langle \psi, A\psi \rangle - \frac{\langle \psi, A^2\psi \rangle - \langle \psi, A\psi \rangle^2}{E_1 - \langle \psi, A\psi \rangle} \ .$$

$\square$

We proceed with the upper bound.

PROOF *(upper bound)* :
By adding a constant to the potential we may assume that $a_0 = \inf \operatorname{supp}(P_0) = 0$, so that $V_\omega(n) \geq 0$. By (5.50) we have that

$$\begin{aligned} N(E) \ &\leq \ \frac{1}{|\Lambda_L|} \, \mathbb{E}\big(N(H_{\Lambda_L}^N, E)\big) \\ &\leq \ \mathbb{P}\big(E_0(H_{\Lambda_L}^N) < E\big) \end{aligned} \qquad (6.6)$$

for any L, since $N(H_{\Lambda_L}^N, E) \leq |\Lambda_L|$.
At the end of the proof, we will choose an optimal L.
To estimate the right hand side in (6.6) from above we need an estimate of $E_0(H_{\Lambda_L}^N)$ from below which will be provided by Temple's inequality. As a test function ψ for Temple's inequality we use the ground state of $(H_0)_{\Lambda_L}^N$, namely

$$\psi_0(n) = \frac{1}{|\Lambda_L|^{\frac{1}{2}}} \qquad \text{for all } n \in \Lambda_L \ .$$

In fact $(H_0)^N_{\Lambda_L} \psi_0 = 0$. We have

$$
\begin{aligned}
\langle \psi_0, H^N_{\Lambda_L} \psi_0 \rangle &= \langle \psi_0, V_\omega \psi_0 \rangle \\
&= \frac{1}{(2L+1)^d} \sum_{i \in \Lambda_L} V_\omega(i) \,.
\end{aligned}
\tag{6.7}
$$

Observe that this is an arithmetic mean of independent, identically distributed random variables. Hence, (6.7) converges to $\mathbb{E}(V_\omega(0)) > 0$ almost surely.
To apply Temple's inequality, we would need

$$
\frac{1}{(2L+1)^d} \sum_{i \in \Lambda_L} V_\omega(i) < E_1(H^N_{\Lambda_L})
$$

which is certainly wrong for large L since $E_1(H^N_{\Lambda_L}) \to 0$. We estimate

$$
E_1(H^N_{\Lambda_L}) \geq E_1((H_0)^N_{\Lambda_L}) \geq cL^{-2} \,.
$$

The latter inequality can be obtained by direct calculation. Now we define

$$
V^{(L)}_\omega(i) = \min\{V_\omega(i), \, \frac{c}{3}L^{-2}\} \,.
$$

For fixed L, the random variables $V^{(L)}_\omega$ are still independent and identically distributed, but their distribution depends on L. Moreover, if

$$
H^{(L)} = (H_0)^N_{\Lambda_L} + V^{(L)}_\omega
$$

then $E_0(H^N_{\Lambda_L}) \geq E_0(H^{(L)})$ by the min-max principle (Theorem 3.1).
We get

$$
\langle \psi_0, H^{(L)} \psi_0 \rangle = \frac{1}{(2L+1)^d} \sum_{i \in \Lambda_L} V^{(L)}_\omega(i) \leq \frac{c}{3}L^{-2}
\tag{6.8}
$$

by definition of $V^{(L)}_\omega$, consequently

$$
\langle \psi_0, H^{(L)} \psi_0 \rangle \leq \frac{c}{3}L^{-2} < E_1((H_0)^N_{\Lambda_L}) \leq E_1(H^{(L)}) \,.
$$

Thus, we may use Temples inequality with ψ_0 and $H^{(L)}$:

$$
\begin{aligned}
E_0(H_{\Lambda_L}^N) \;&\geq\; E_0(H^{(L)}) \\[2mm]
&\geq\; \langle \psi_0, H^{(L)}\psi_0\rangle - \frac{\langle \psi_0, (H^{(L)})^2\psi_0\rangle}{cL^{-2} - \langle \psi_0, H^{(L)}\psi_0\rangle} \\[2mm]
&\geq\; \frac{1}{(2L+1)^d}\sum_{i\in\Lambda_L} V_\omega^{(L)}(i) - \frac{\frac{1}{(2L+1)^d}\sum_{i\in\Lambda_L}(V_\omega(i))^2}{(c-\frac{c}{3})L^{-2}} \\[2mm]
&\geq\; \frac{1}{(2L+1)^d}\sum_{i\in\Lambda_L} V_\omega^{(L)}(i) - \frac{1}{(2L+1)^d}\sum_{i\in\Lambda_L} V_\omega^{(L)}\left(\frac{\frac{c}{3}L^{-2}}{\frac{2}{3}cL^{-2}}\right) \\[2mm]
&\geq\; \frac{1}{2}\frac{1}{(2L+1)^d}\sum_{i\in\Lambda_L} V_\omega^{(L)}(i)\,. && (6.9)
\end{aligned}
$$

Collecting the estimates above, we arrive at

$$
N(E) \leq \mathbb{P}\left(E_0(H_{\Lambda_L}^N) < E\right) \leq \mathbb{P}\left(\frac{1}{(2L+1)^d}\sum_{i\in\Lambda_L} V_\omega^{(L)} < \frac{E}{2}\right). \qquad (6.10)
$$

Now we choose L. We try to make the right hand side of (6.10) as small as possible. Since $V_\omega^{(L)} \leq \frac{c}{3}L^{-2}$, the probability in (6.10) will be one if L is too big. So we certain want to choose L in such a way that $\frac{c}{3}L^{-2} > \frac{E}{2}$. Thus, a reasonable choice seems to be

$$
L := \lfloor \beta E^{-\frac{1}{2}}\rfloor
$$

with some β small enough and $\lfloor x\rfloor$ the largest integer not exceeding x.
We single out an estimate of the probability in (6.10)

LEMMA 6.4. *For $L = \lfloor \beta E^{-\frac{1}{2}}\rfloor$ with β small and L large enough*

$$
\mathbb{P}\left(\frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_L} V_\omega^{(L)}(i) < \frac{E}{2}\right) \leq e^{-\gamma|\Lambda_L|}
$$

with some $\gamma > 0$.

Given the lemma, we proceed

$$
\begin{aligned}
N(E) \;&\leq\; \mathbb{P}\left(\frac{1}{|\Lambda_L|}\sum_{i\in\Lambda_L} V_\omega^{(L)}(i) < \frac{E}{2}\right) \\[2mm]
&\leq\; e^{-\gamma|\Lambda_L|} \\[2mm]
&=\; e^{-\gamma(2\lfloor \beta E^{-\frac{1}{2}}\rfloor^d + 1)} \\[2mm]
&\leq\; e^{-\gamma' E^{-\frac{d}{2}}}\,. && (6.11)
\end{aligned}
$$

This estimate is the desired upper bound on $N(E)$. $\qquad\square$

To finish the proof of the upper bound, it remains to prove Lemma 6.4. This lemma is a typical large deviation estimate: By our choice of L we have $\mathbb{E}(V_\omega^{(L)}) > \frac{E}{2}$ if β is small enough; thus, we estimate the probability that an arithmetic mean of independent random variables deviates from its expectation value. What makes the problem somewhat nonstandard is the fact that the random variables $V^{(L)}$ depend on the parameter L, which is also implicit in E.

PROOF *(Lemma)* :

$$\mathbb{P}\left(\frac{1}{|\Lambda_L|}\sum V_\omega^L(i) < \frac{E}{2}\right)$$

$$\leq \quad \mathbb{P}\left(\frac{1}{|\Lambda_L|}\sum V_\omega^L(i) < \frac{\beta^2}{2}L^{-2}\right)$$

$$\leq \quad \mathbb{P}\left(\#\{\,i \mid V_\omega^L(i) < \frac{c}{3}L^{-2}\} \geq (1 - \frac{3\beta^2}{c})|\Lambda_L|\right) . \qquad (6.12)$$

Indeed, if less than $(1 - \frac{3\beta^2}{c})|\Lambda_L|$ of the $V(i)$ are below $\frac{c}{3}L^{-2}$ than more than $\frac{3\beta^2}{c}|\Lambda_L|$ of them are at least $\frac{c}{3}L^{-2}$ (in fact equal to). In this case

$$\frac{1}{|\Lambda_L|}\sum V_\omega^L(i) \quad \geq \quad \frac{1}{|\Lambda_L|}\frac{3\beta^2}{c}|\Lambda_L|\frac{c}{3}L^{-2}$$

$$= \quad \frac{\beta^2}{2}L^{-2} .$$

Since $P(V(i) > 0) > 0$ there is a $\gamma > 0$ such that $q := P(V(i) < \gamma) < 1$.
We set $\xi_i = \begin{cases} 1 & \text{if } V_i < \gamma, \\ 0 & \text{otherwise.} \end{cases}$
The random variables ξ_i are independent and identically distributed, $\mathbb{E}(\xi_i) = q$.
Let us set $r = 1 - \frac{3\beta^2}{c}$. By taking β small we can ensure that $q < r < 1$.
Then, for L sufficient large

$$(6.12) \quad \leq \quad \mathbb{P}\left(\#\{i \mid V_\omega^L(i) < \frac{c}{3}L^{-2}\} \geq r|\Lambda_L|\right)$$

$$\leq \quad \mathbb{P}\left(\#\{i \mid V_\omega^L(i) < \gamma\} \geq r|\Lambda_L|\right)$$

$$\leq \quad \mathbb{P}\left(\frac{1}{|\Lambda_L|}\sum \xi_i \geq r\right) . \qquad (6.13)$$

Through our somewhat lengthy estimate above we finally arrived at the standard large deviations problem (6.13). To estimate the probability in (6.13) we use the inequality

$$\mathbb{P}\left(X > a\right) \leq e^{-ta}\,\mathbb{E}(e^{tX}) \qquad \text{for } t \geq 0 .$$

Indeed

$$\begin{aligned}
\mathbb{P}(\,X > a\,) &= \int \chi_{\{X>a\}}(\omega)\, d\,\mathbb{P}(\omega) \\
&\leq \int e^{-ta} e^{tX} \chi_{\{X>a\}}(\omega)\, d\,\mathbb{P}(\omega) \\
&\leq \int e^{-ta} e^{tX}\, d\,\mathbb{P}\,.
\end{aligned}$$

We obtain

$$\begin{aligned}
(6.13) \quad &\leq \; e^{-|\Lambda_L|\,tr} \cdot \mathbb{E}\Big(\prod_{i \in \Lambda_L} e^{t\xi_i} \Big) \\
&= \; e^{-|\Lambda_L|(rt - \ln \mathbb{E}(e^{t\xi_0}))}\,.
\end{aligned}$$

Set $f(t) = rt - \ln \mathbb{E}(e^{t\xi_0})$. If we can choose t such that $f(t) > 0$, the result is proven. To see that this is possible, we compute

$$f'(t) = r - \frac{\mathbb{E}(\xi_0 e^{t\xi_0})}{\mathbb{E}(e^{t\xi_0})}$$

So $f'(0) = r - q\ > 0$.
Since $f(0) = 0$, there is a $t > 0$ with $f(t) > 0$.

$\square$

Thus, we have shown

$$\overline{\lim}_{E \searrow E_0} \frac{\ln |\ln N(E)|}{\ln(E - E_0)} \leq -\frac{d}{2}\,. \tag{6.14}$$

6.3. Lower bound.

We proceed with the lower bound. By (5.50) we estimate

$$\begin{aligned}
N(E) &\geq \frac{1}{|\Lambda_L|}\, \mathbb{E}\left(N(H^D_{\Lambda_L}, E) \right) \\
&\geq \frac{1}{|\Lambda_L|}\, \mathbb{P}\left(E_0(H^D_{\Lambda_L}) < E \right). \tag{6.15}
\end{aligned}$$

As in the upper bound, the above estimate holds for any L.
To proceed, we have to estimate $E_0(H^D_{\Lambda_L})$ from above.
This is easily done via the min-max principle (Theorem 3.1):

$$\begin{aligned}
E_0(H^D_{\Lambda_L}) &\leq \; \langle \psi, H^D_{\Lambda_L} \psi \rangle \\
&= \; \langle \psi, (H_0)^D_{\Lambda_L} \psi \rangle + \sum_{i \in \Lambda_L} V_\omega(i)\, |\psi(i)|^2 \tag{6.16}
\end{aligned}$$

for any ψ with $||\psi|| = 1$. Now we try to find ψ which minimizes the right hand side of (6.16). First we deal with the term

$$\langle \psi, (H_0)_{\Lambda_L}^D \psi \rangle . \tag{6.17}$$

Since $(H_0)_{\Lambda_L}^D$ adds a positive term to $(H_0)_{\Lambda_L}^N$ at the boundary, it seems desirable to choose $\psi(n) = 0$ for $|n| = L$. On the other hand, to keep (6.17) small we don't want ψ to change too abruptly.

So, we choose

$$\psi_1(n) = L - ||\,n\,||_\infty \qquad , n \in \Lambda_L$$

and

$$\psi(n) = \frac{1}{||\psi_1||} \psi_1(n) .$$

We have

$$\sum_{n \in \Lambda_L} |\psi_1(n)|^2 \geq \sum_{n \in \Lambda_{\frac{L}{2}}} |\psi_1(n)|^2 \geq |\Lambda_{\frac{L}{2}}| \, (\frac{L}{2})^2 \geq c \, L^{d+2}$$

and

$$
\begin{aligned}
\langle \psi_1, (H_0)_{\Lambda_L}^D \psi_1 \rangle \quad &\leq \quad \sum_{\substack{(n,n') \in \Lambda_L \\ ||\,n - n'\,||_1 = 1}} \big|\,\psi_1(n') - \psi_1(n)\,\big|^2 \\
&\leq \quad \Big|\,\{\,(n, n') \in \Lambda_L \times \Lambda_L \,;\, ||\,n - n'\,||_1 = 1\}\,\Big| \\
&\leq \quad c_1 \, L^d .
\end{aligned}
$$

Above, we used that $|\psi_1(n) - \psi_1(n')| \leq 1$ if $||\,n - n'\,||_1 = 1$.

Collecting these estimates, we obtain

$$
\begin{aligned}
E_0\Big((H_0)_{\Lambda_L}^D\Big) \quad &\leq \quad \frac{\langle \psi_1, (H_0)_{\Lambda_L}^D \psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} \\
&\leq \quad c_0 \, L^{-2} .
\end{aligned} \tag{6.18}
$$

The bounds of (6.15), (6.16) and (6.18) give

$$
\begin{aligned}
N(E) \quad &\geq \quad \frac{1}{|\Lambda_L|} \, \mathbb{P} \left(\sum_{i \in \Lambda_L} V_\omega(i) \, |\psi(i)|^2 < E - c_0 L^{-2} \right) \\
&\geq \quad \frac{1}{|\Lambda_L|} \, \mathbb{P} \left(\frac{c_2}{|\Lambda_L|} \sum_{i \in \Lambda_{L/2}} V_\omega(i) < E - c_0 L^{-2} \right) . \tag{6.19}
\end{aligned}
$$

In the last estimate, we used that for $\| i \|_\infty \leq L/2$

$$
\begin{aligned}
\psi(i) &= \frac{1}{\|\psi_1\|}\left(L - \|i\|_\infty\right) \\
&\geq \bar{c}\,\frac{L - \|i\|_\infty}{L^{\frac{d+2}{2}}} \\
&\geq \bar{c}\,\frac{L/2}{L^{\frac{d+2}{2}}} \\
&= \widetilde{c}\,L^{-d/2}\ .
\end{aligned}
$$

The probability in (6.19) is again a large deviation probability. As above, the $L-$independence of the right hand side is nonstandard. We estimate (6.19) in a somewhat crude way by

$$
\begin{aligned}
(6.19) \quad &\geq \frac{1}{|\Lambda_L|}\,\mathbb{P}\left(\text{For all } i \in \Lambda_{L/2}\,,\quad V_\omega(i) < \frac{1}{c_2}(E - c_0 L^{-2})\right) \\
&= \frac{1}{|\Lambda_L|}\,\mathbb{P}\left(V_\omega(0) < \frac{1}{c_2}(E - c_0\,L^{-2})\right)^{|\Lambda_{L/2}|}\ . \tag{6.20}
\end{aligned}
$$

If we take L so large that $c_0 L^{-2} < \frac{E}{2}$ (i.e. $L \sim E^{-1/2}$ as for the upper bound), we obtain

$$
(6.20) \geq \frac{1}{|\Lambda_L|}\,P_0\Big([0, E/2)\Big)^{c_3 L^d}\ .
$$

Using assumption (6.3), we finally get

$$
\begin{aligned}
N(E) &\geq c_4 L^{-d} E^{c_3 \kappa L^d} \\
&= c_4 L^{-d} e^{(\ln E)c_3 \kappa L^d}\ .
\end{aligned}
$$

We remind the reader that κ is the exponent occurring in (6.3).
So

$$
N(E) \geq c_4' E^{d/2} e^{c_3'(\ln E)E^{-d/2}}\ .
$$

This gives the lower bound

$$
\underline{\lim}_{E \searrow E_0} \frac{\ln|\ln N(E)|}{\ln(E - E_0)} \geq -\frac{d}{2}\ .
$$

Notes and Remarks

There are various approaches to Lifshitz tails by now. The first is through the Donsker-Varadhan theory of large deviations, see [38], [108] and [110]. Related results and further references can be found in [15]. For an alternative approach, see [132].

The results contained in these lecture and variants can be found for example in [66], [69], [58], [121] and [129]. See also [72].

For an approach using periodic approximation see [81], [83], [84], [85], and references therein. In [102] the probabilistic and the spectral point of view were combined.

There are also results on other band edges than the bottom of the spectrum, so called internal Lifshitz tails, [103], [122], [86], [87] and [107].

Magnetic fields change the Lifshitz behavior drastically, see [20], [43], [139].

For a recent survey about the density of states, see [67].

7. The spectrum and its physical interpretation

7.1. Generalized Eigenfunctions and the spectrum.

In this section we explore the connection between generalized eigenfunctions of (discrete) Hamiltonians H and their spectra. A function f on $\mathbb{Z}^d$ is called *polynomially bounded* if

$$|f(n)| \leq C \left(1 + ||\, n\,||_\infty\right)^k \tag{7.1}$$

for some constants $k, C > 0$. We say that λ is a *generalized eigenvalue* if there is a polynomially bounded solution ψ of the finite difference equation

$$H\psi = \lambda\psi \,. \tag{7.2}$$

ψ is called a *generalized eigenfunction*. Note that we do not require $\psi \in \ell^2(\mathbb{Z}^d)$!
We denote the set of generalized eigenvalues of H by $\varepsilon_g(H)$.
We say that the sets $A, B \in \mathcal{B}(\mathbb{R})$ *agree up to a set of spectral measure zero* if $\chi_{A\setminus B}(H) = \chi_{B\setminus A}(H) = 0$ where $\chi_I(H)$ is the projection valued spectral measure associated with H (see Section 3.2).
The goal of this section is to prove the following theorem.

THEOREM 7.1. *The spectrum of a (discrete) Hamilitonian H agrees up to a set of spectral measure zero with the set $\varepsilon_g(H)$ of all generalized eigenvalues.*

As a corollary to the proof of Theorem 7.1 we obtain the following result

COROLLARY 7.2. *Any generalized eigenvalue λ of H belongs to the spectrum $\sigma(H)$, moreover*

$$\sigma(H) \;=\; \overline{\varepsilon_g(H)} \tag{7.3}$$

REMARK 7.3. The proof shows that in Theorem 7.1 as well as in Corollary 7.2 the set $\varepsilon_g(H)$ can be replaced by the set of those generalized eigenvalues with a corresponding generalized eigenfunction satisfying

$$|\,\psi(n)\,| \leq C \left(1 + ||\, n\,||_\infty\right)^{\frac{d}{2}+\varepsilon} \tag{7.4}$$

for some $\varepsilon > 0$.

The proof of Theorem 7.1 and Corollary 7.2 we present now is quite close to [123], but the arguments simplify considerably in the discrete (ℓ^2-) case we consider here. For $\Delta \subset \mathbb{R}$ a Borel set, let $\mu(\Delta) = \chi_\Delta(H)$ be the projection valued measure associated with the self adjoint operator H (see Section 3.2). Thus,

$$\langle \varphi, H\psi \rangle = \int \lambda \, d\mu_{\varphi,\psi}(\lambda)$$

with

$$\mu_{\varphi,\psi}(\Delta) = \langle \varphi, \mu(\Delta)\psi \rangle \,.$$

In the case of $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, we set

$$\mu_{n,m}(\Delta) = \langle \delta_n, \mu(\Delta)\delta_m \rangle \ .$$

If $\{\alpha_n\}_{n\in\mathbb{Z}^d}$ is a sequence of real numbers with $\alpha_n > 0$, $\sum \alpha_n = 1$, we define

$$\rho(\Delta) = \sum_{n\in\mathbb{Z}^d} \alpha_n \, \mu_{n,n}(\Delta) \ . \tag{7.5}$$

ρ is a finite positive Borel measure of total mass $\rho(\mathbb{R}) = 1$. We call ρ a *spectral measure* (sometimes *real valued spectral measure* to distinguish it from μ, the *projection valued spectral measure*). It is easy to see that

$$\rho(\Delta) = 0 \qquad \text{if and only if} \qquad \mu(\Delta) = 0 \tag{7.6}$$

Thus, A and B agree up to a set of spectral measure zero if $\rho(A \setminus B) = 0$ and $\rho(B \setminus A) = 0$. Moreover, the support of ρ is the spectrum of H. Although the spectral measure is not unique (many choices for the α_n), its measure class and its support are uniquely defined by (7.5).

We are ready to prove one half of Theorem 7.1, namely

PROPOSITION 7.4. *Let ρ be a spectral measure for $H = H_0 + V$. Then, for ρ-almost all λ there exists a polynomially bounded solution of the difference equation*

$$H\psi = \lambda\psi \ .$$

PROOF: By the Cauchy-Schwarz inequality, we have

$$|\mu_{n,m}(\Delta)| \leq \mu_{n,n}(\Delta)^{\frac{1}{2}} \, \mu_{m,m}(\Delta)^{\frac{1}{2}}$$

Consequently, the $\mu_{n,m}$ are absolutely continuous with respect to ρ, i.e.

$$\rho(\Delta) = 0 \quad \Longrightarrow \quad \mu_{n,m}(\Delta) = 0 \ . \tag{7.7}$$

Hence, the Radon-Nikodym theorem tells us that there exist measurable functions $F_{n,m}$ (densities) such that

$$\mu_{n,m}(\Delta) = \int_\Delta F_{n,m}(\lambda) \, d\rho(\lambda) \ . \tag{7.8}$$

The functions $F_{n,m}$ are defined up to sets of ρ-measure zero and, since $\mu_{n,n} \geq 0$ the functions $F_{n,n}$ are non negative ρ-almost surely (ρ-a.s.). Moreover

$$\begin{aligned}
\rho(\Delta) &= \sum \alpha_n \, \mu_{n,n} \\
&= \sum \alpha_n \int_\Delta F_{n,n}(\lambda) \, d\rho(\lambda) \\
&= \int_\Delta \sum \alpha_n F_{n,n}(\lambda) \, d\rho(\lambda) \ .
\end{aligned} \tag{7.9}$$

Hence, $\sum \alpha_n F_{n,n}(\lambda) = 1$ (ρ-a.s.). In particular

$$F_{n,n}(\lambda) \leq \frac{1}{\alpha_n} \; . \tag{7.10}$$

It follows

$$
\begin{aligned}
\left| \int_{\Delta} F_{n,m}(\lambda) \, d\rho(\lambda) \right| \;&=\; |\mu_{n,m}(\Delta)| \\[1mm]
&\leq\; \mu_{n,n}(\Delta)^{\frac{1}{2}} \, \mu_{m,m}(\Delta)^{\frac{1}{2}} \\[1mm]
&=\; \left(\int_{\Delta} F_{n,n}(\lambda) \, d\rho(\lambda) \right)^{\frac{1}{2}} \left(\int_{\Delta} F_{m,m}(\lambda) \, d\rho(\lambda) \right)^{\frac{1}{2}} \\[1mm]
&\leq\; \alpha_n^{-\frac{1}{2}} \alpha_m^{-\frac{1}{2}} \rho(\Delta) \; .
\end{aligned} \tag{7.11}
$$

Thus

$$|F_{n,m}| \leq \alpha_n^{-\frac{1}{2}} \alpha_m^{-\frac{1}{2}} \; . \tag{7.12}$$

Equation (7.8) implies that for any bounded measurable function f

$$\langle \delta_n, f(H)\delta_m \rangle = \int f(\lambda) F_{n,m}(\lambda) \, d\rho(\lambda) \; .$$

In particular, for $f(\lambda) = \lambda \, g(\lambda)$ (g of compact support)

$$
\begin{aligned}
&\int \lambda \, g(\lambda) \, F_{n,m}(\lambda) \, d\rho(\lambda) \\
&= \langle \delta_n, H \, g(H) \, \delta_m \rangle \\
&= \langle H \, \delta_n, g(H) \, \delta_m \rangle \\
&= \sum_{|e|=1} \big(-\langle \delta_{n+e}, g(H) \, \delta_m \rangle \big) \; + \; \big(V(n) + 2d \big) \langle \delta_n, g(H) \, \delta_m \rangle \\
&= \sum_{|e|=1} \left(-\int g(\lambda) \, F_{n+e,m}(\lambda) \, d\rho(\lambda) \right) \; + \; \int g(\lambda) \, (V(n) + 2d) F_{n,m}(\lambda) \, d\rho(\lambda) \\
&= \int g(\lambda) \, H^{(n)} F_{n,m}(\lambda) \, d\rho(\lambda)
\end{aligned} \tag{7.13}
$$

where $H^{(n)} F_{n,m}(\lambda)$ is the operator H applied to the function $n \mapsto F_{n,m}(\lambda)$. Thus,

$$\int g(\lambda) \, \lambda \, F_{n,m}(\lambda) \, d\rho(\lambda) = \int g(\lambda) \, H^{(n)} F_{n,m}(\lambda) \, d\rho(\lambda)$$

for any bounded measurable function g with compact support.
It follows that for ρ-almost all λ and for any fixed $m \in \mathbb{Z}^d$, the function $\psi(n) = F_{n,m}(\lambda)$ is a solution of $H\psi = \lambda\psi$. By (7.12) the function ψ satisfies

$$|\psi(n)| \; \leq \; C_0 \, \alpha_n^{-\frac{1}{2}} \; . \tag{7.14}$$

So far, the sequence α_n has only to fulfill $\alpha_n > 0$ and $\sum \alpha_n = 1$. Now, we choose $\alpha_n = c \, (1 + || \, n ||_\infty)^{-\beta}$ for an arbitrary $\beta > d$, hence

$$| \, \psi(n)| \leq C \, (1 + || \, n||_\infty)^{\frac{d}{2} + \varepsilon}$$

for an $\varepsilon > 0$.

This proves the proposition as well as the estimate (7.4). $\qquad\square$

We turn to the proof of the opposite direction of Theorem 7.1. As usual, we equip $\mathbb{Z}^d$ with the norm $|| \, n ||_\infty = \max_{i=1,...,d} |n_i|$. So $\Lambda_L = \{|n| \leq L\}$ is a cube of side length $2L + 1$.

For a subset S of $\mathbb{Z}^d$ we denote by $||\psi||_S$ the l^2- norm of ψ over the set S.

We begin with a lemma:

LEMMA 7.5. *If ψ is polynomially bounded ($\neq 0$) and l is a positive integer, then there is a sequence $L_n \to \infty$ such that*

$$\frac{||\psi||_{\Lambda_{L_n + l}}}{||\psi||_{\Lambda_{L_n}}} \to 1 \; .$$

PROOF: Suppose the assertion of the Lemma is wrong. Then there exists $a > 1$ and L_0 such that for all $L \geq L_0$

$$||\psi||_{\Lambda_{L+l}} \geq a ||\psi||_{\Lambda_L} \; .$$

So

$$||\psi||_{\Lambda_{L_0 + lk}} \geq a^k ||\psi||_{\Lambda_{L_0}} \tag{7.15}$$

but by the polynomial boundedness of ψ we have

$$||\psi||_{\Lambda_{L_0 + lk}} \leq C_1 \, (L_0 + l \, k)^M \leq C \, k^M \tag{7.16}$$

for some $C, M > 0$ which contradicts (7.15). $\qquad\square$

We are now in position to prove the second half of Theorem 7.1.

PROPOSITION 7.6. *If the difference equation $H\psi = \lambda\psi$ admits a polynomially bounded solution ψ, then λ belongs to the spectrum $\sigma(H)$ of H.*

PROOF: We set $\psi_L(n) = \begin{cases} \psi(n) & \text{for } |n| \leq L, \\ 0 & \text{otherwise .} \end{cases}$

Set $\varphi_L = \frac{1}{||\psi_L||} \psi_L$. Then ψ_L is 'almost' a solution of $H\psi = \lambda\psi$, more precisely

$$(H - \lambda)\psi_L(n) = 0$$

as long as $n \notin S_L := \{m \mid L - 1 \leq |m| \leq L + 1\}$ and

$$\begin{aligned}
\|(H-\lambda)\,\psi_L\|^2 &\leq \|\psi\|^2_{S_L} \\
&= \sum_{m\in S_L} |\psi(m)|^2 \\
&= \|\psi\|^2_{\Lambda_{L+1}} - \|\psi\|^2_{\Lambda_{L-2}} \; .
\end{aligned} \tag{7.17}$$

By Lemma (7.5) there is a sequence $L_n \to \infty$ such that

$$\frac{\|\psi\|^2_{\Lambda_{Ln+1}}}{\|\psi\|^2_{\Lambda_{Ln-2}}} \to 1 \; ,$$

so

$$\|(H-\lambda)\varphi_{L_n}\|^2 \leq \frac{\|\psi\|^2_{\Lambda_{Ln+1}} - \|\psi\|^2_{\Lambda_{Ln-2}}}{\|\psi\|_{\Lambda_{Ln-2}}} \to 0 \; .$$

Thus, φ_{L_n} is a Weyl sequence for H and λ and $\lambda \in \sigma(H)$. $\qquad\square$

PROOF *(Corollary)*:   We have already seen in Proposition 7.6 that

$$\varepsilon_g(H) \subset \sigma(H) \; .$$

Since $\sigma(H)$ is closed, it follows

$$\overline{\varepsilon_g(H)} \subset \sigma(H) \; .$$

By the Theorem 7.1, we know that

$$\rho\left(\complement\,\varepsilon_g(H)\right) = 0$$

hence

$$\complement\,\overline{\varepsilon_g(H)} \cap \sigma(H) = \complement\,\overline{\varepsilon_g(H)} \cap \operatorname{supp}\rho = \emptyset \; .$$

So,

$$\sigma(H) \subset \overline{\varepsilon_g(H)} \; .$$

$\qquad\square$

7.2. The measure theoretical decomposition of the spectrum.

The spectrum gives the physically possible energies of the system described by the Hamiltonian H. Hence, if $E \notin \sigma(H)$, no (pure) state of the system can have energy E. It turns out that the fine structure of the spectrum gives important information on the dynamical behavior of the system, more precisely on the long time behavior of the state $\psi(t) = e^{-itH}\psi_0$.

To investigate this fine structure we have to give a little background in measure theory. By the term *bounded Borel measure* (or bounded measure, for short) we

mean in what follows a complex-valued σ-additive function ν on the Borel sets $\mathcal{B}(\mathbb{R})$ such that the total variation

$$\|\nu\| = \sup \{ \sum_i^N |\nu(A_i)|;\ A_i \in \mathcal{B}(\mathbb{R}) \quad \text{pairwise disjoint} \}$$

is finite. By a *positive Borel measure* we mean a non-negative σ-additive function m on the Borel sets such that $m(A)$ is finite for any bounded Borel set A.

A bounded Borel measure ν on $\mathbb{R}$ is called a *pure point measure* if ν is concentrated on a countable set, i.e. if there is a countable set $A \in \mathcal{B}(\mathbb{R})$ such that $\nu(\mathbb{R}\backslash A) = 0$. The points $x_i \in \mathbb{R}$ with $\nu(\{x_i\}) \neq 0$ are called the atoms of ν. A pure point measure ν can be written as $\nu = \sum \alpha_i \delta_{x_i}$, where δ_{x_i} is the Dirac measure at the point x_i and $\alpha_i = \nu(\{x_i\})$.

A measure ν is called *continuous* if ν has no atoms, i.e. $\nu(\{x\}) = 0$ for all $x \in \mathbb{R}$.

A bounded measure ν is called *absolutely continuous* with respect to a positive measure m (in short $\nu \ll m$) if there is a measurable function $\varphi \in L^1(\nu)$ such that $\nu = \varphi(m)$, i.e. $\nu(A) = \int_A \varphi(x) dm(x)$.

The Theorem of Radon-Nikodym asserts that ν is absolutely continuous with respect to m if (and only if) for any Borel set A, $m(A) = 0$ implies $\nu(A) = 0$.

By saying ν is *absolutely continuous* we always mean ν is absolutely continuous with respect to Lebesgue measure L.

A measure is called *singular continuous* if it is continuous and it lives on a set N of Lebesgue measure zero, i.e. $\nu(\{x\}) = 0$ for all $x \in \mathbb{R}$, $\nu(\mathbb{R}\backslash N) = 0$ and $L(N) = 0$.

The Lebesgue-decomposition theorem tells us that any bounded Borel measure ν on $\mathbb{R}$ admits a unique decomposition

$$\nu = \nu_{pp} + \nu_{sc} + \nu_{ac}$$

where ν_{pp} is a pure point measure, ν_{sc} a singular continuous measure and ν_{ac} is absolutely continuous (with respect to Lebesgue measure). We call ν_{pp} the pure point part of ν etc.

Let H be a self adjoint operator on a Hilbert space $\mathcal{H}$ with domain $D(H)$ and μ be the corresponding *projection valued spectral measure* (see Section 3.2). So, for any Borel set $A \subset \mathbb{R}$, $\mu(A)$ is a projection operator,

$$\langle \varphi, \mu(A)\psi \rangle = \mu_{\varphi,\psi}(A)$$

is a complex valued measure and

$$\langle \varphi, H\psi \rangle = \int \lambda \, d\mu_{\varphi,\psi}(\lambda) \, .$$

We also set $\mu_\varphi = \mu_{\varphi,\varphi}$ which is a positive measure. Note that

$$
\begin{aligned}
|\mu_{\psi,\varphi}(A)| \quad &= \quad |\langle \psi, \mu(A)\varphi \rangle| \\
&= \quad |\langle \mu(A)\psi, \mu(A)\varphi \rangle| \\
&\leq \quad (\langle \mu(A)\psi, \mu(A)\psi \rangle)^{\frac{1}{2}} \ (\langle \mu(A)\varphi, \mu(A)\varphi \rangle)^{\frac{1}{2}} \\
&= \quad \mu_{\psi}(A)^{\frac{1}{2}} \, \mu_{\varphi}(A)^{\frac{1}{2}} \ .
\end{aligned}
\tag{7.18}
$$

We define $\mathcal{H}_{pp} = \{\varphi \in \mathcal{H} \mid \mu_{\varphi} \text{ is pure point}\}$ and analogously $\mathcal{H}_{sc}$ and $\mathcal{H}_{ac}$. These sets are closed subspaces of $\mathcal{H}$ which are mutually orthogonal and

$$
\mathcal{H} = \mathcal{H}_{pp} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{ac} \ .
$$

The operator H maps each of these spaces into itself (see e.g. [**117**]). We set $H_{pp} = H \mid_{\mathcal{H}_{pp} \cap D(H)}, \quad H_{sc} = H \mid_{\mathcal{H}_{sc} \cap D(H)} , \quad H_{ac} = H \mid_{\mathcal{H}_{ac} \cap D(H)}$. We define the *pure point spectrum* $\sigma_{pp}(H)$ of H to be the spectrum $\sigma(H_{pp})$ of H_{pp}, analogously the *singular continuous spectrum* $\sigma_{sc}(H)$ of H to be $\sigma(H_{sc})$ and the *absolutely continuous spectrum* $\sigma_{ac}(H)$ to be $\sigma(H_{ac})$. It is clear that

$$
\sigma(H) \ = \ \sigma_{pp}(H) \cup \sigma_{sc}(H) \cup \sigma_{ac}(H)
$$

but this decomposition of the spectrum is not a *disjoint* union in general.

This measure theoretic decomposition of the spectrum is defined in a rather abstract way and we should ask: Is there any *physical* meaning of the decomposition? The answer is *YES* and will be given in the next section.

7.3. Physical meaning of the spectral decomposition.

The measure theoretic decomposition of the Hilbert space and the spectrum may look more like a mathematical subtleness than like a physically relevant classification. In fact, in physics one is primarily interested in long time behavior of wave packets. For example, one distinguishes bound states and scattering states. It turns out, there is an intimate connection between the classification of states by their long time behavior and the measure theoretic decomposition of the spectrum. We explore this connection in the present section.

The circle of results we present here was dubbed 'RAGE-theorem' in [**30**] after the pioneering works by Ruelle [**120**], Amrein, Georgescu [**8**] and Enss [**42**] on this topic.

If E is an eigenvalue of H (in the ℓ^2-sense) and φ a corresponding eigenfunction, then the spectral measure μ has an atom at E, and μ_{φ} is a pure point measure concentrated at the point E. Thus, all eigenfunctions and the closed subspace generated by them belong to the pure point subspace $\mathcal{H}_{pp}$. The converse is also true, i.e. the space $\mathcal{H}_{pp}$ is exactly the closure of the linear span of all eigenvectors. It follows that the set $\varepsilon(H)$ of all eigenvalues of H is always contained in the pure point spectrum $\sigma_{pp}(H)$ and that $\varepsilon(H)$ is dense in $\sigma_{pp}(H)$. The set $\varepsilon(H)$ is countable (as our Hilbert space is always assumed to be separable), it may have accumulation points, in fact $\varepsilon(H)$ may be dense in a whole interval.

Let us look at the time evolution of a function $\psi \in \mathcal{H}_{pp}$. To start with, suppose ψ is an eigenfunction of H with eigenvalue E. Then

$$e^{-itH}\psi \;=\; e^{-itE}\psi$$

so that $|e^{-itH}\psi(x)|^2$ is independent of the time t. We may say that the particle, if starting in an eigenstate, stays where it is for all t. It is easy to see that for general ψ in $\mathcal{H}_{pp}$ the function $e^{-itH}\psi(x)$ is almost periodic in t. A particle in a state ψ in $\mathcal{H}_{pp}$ will stay inside a compact set with high probability for arbitrary long time, in the following sense:

THEOREM 7.7. *Let H be a self adjoint operator on $\ell^2(\mathbb{Z}^d)$, take $\psi \in \mathcal{H}_{pp}$ and let Λ_L denote a cube in $\mathbb{Z}^d$ centered at the origin with side length $2L+1$.*
Then

$$\lim_{L\to\infty}\ \sup_{t\ge 0}\ \left(\sum_{x\in\Lambda_L} |\,e^{-itH}\psi(x)\,|^2 \right) \;=\; \|\psi\|^2 \qquad (7.19)$$

and

$$\lim_{L\to\infty}\ \sup_{t\ge 0}\ \left(\sum_{x\notin\Lambda_L} |\,e^{-itH}\psi(x)\,|^2 \right) \;=\; 0\,. \qquad (7.20)$$

REMARK 7.8. Equations (7.19), (7.20) can be summarized in the following way: Given any error bound $\varepsilon > 0$ there is a cube Λ_L such that for arbitrary time t we find the particle inside Λ_L with probability $1 - \varepsilon$. In other words, the particle will not escape to infinity. Thus a state $\psi \in \mathcal{H}_{pp}$ can be called a *bound state* .

PROOF: Since e^{-itH} is unitary, we have for all t

$$\begin{aligned} \|\psi\|^2 \;&=\; \|\,e^{-itH}\psi\,\|^2 \\ &=\; \sum_{x\in\Lambda} |\,e^{-itH}\psi(x)\,|^2 \;+\; \sum_{x\notin\Lambda} |\,e^{-itH}\psi(x)\,|^2\,. \qquad (7.21) \end{aligned}$$

Consequently, (7.19) follows from (7.20).
Above we saw that (7.20) is valid for eigenfunctions ψ. To prove it for other vectors in $\mathcal{H}_{pp}$, we introduce the following notation: By P_L we denote the projection onto $\complement\Lambda_L$. Then equation (7.20) claims that

$$\| \, P_L\, e^{-itH}\psi\, \| \;\to\; 0$$

uniformly in t as $L \to \infty$. If ψ is a (finite) linear combination of eigenfunctions, say $\psi = \sum_{m=1}^{M} \alpha_k\psi_k,\ H\psi_k = E_k\psi_k$, then

$$\| P_L \, e^{-itH} \psi \| \quad = \quad \| \sum_{m=1}^{M} \alpha_k \, P_L \, e^{-itH} \psi_k \|$$

$$\leq \quad \sum_{m=1}^{M} |\alpha_k| \, \| P_L \, e^{-itH} \psi_k \| = \sum_{m=1}^{M} |\alpha_k| \, \| P_L \, e^{-itE_k} \psi_k \|$$

$$= \quad \sum_{m=1}^{M} |\alpha_k| \, \| P_L \, \psi_k \| \, . \tag{7.22}$$

By taking L large enough, each term in the sum above can be made smaller then $\left(\sum_{m=1}^{M} |\alpha_k| \right)^{-1} \varepsilon$.

If now ψ is an arbitrary element of $\mathcal{H}_{pp}$, there is a linear combination of eigenfunctions $\psi^{(M)} = \sum_{m=1}^{M} \alpha_k \psi_k$ such that $\| \psi - \psi^{(M)} \| < \varepsilon$. We conclude

$$\| P_L \, e^{-itH} \psi \| \quad \leq \quad \| P_L \, e^{-itH} \psi^{(M)} \| + \| P_L \, e^{-itH} (\psi - \psi^{(M)}) \|$$

$$\leq \quad \| P_L \, e^{-itH} \psi^{(M)} \| + \| \psi - \psi^{(M)} \| \, . \tag{7.23}$$

By taking M large enough the second term of the right hand side can be made arbitrarily small. By choosing L large, we can finally make the first term small as well. $\qquad\square$

We turn to the interpretation of the continuous spectrum. Let us start with a vector $\psi \in \mathcal{H}_{ac}$. Then, by definition the spectral measure μ_ψ is absolutely continuous. From estimate (7.18) we learn that $\mu_{\varphi,\psi}$ is absolutely continuous for any $\varphi \in \mathcal{H}$ as well. It follows that the measure $\mu_{\varphi,\psi}$ has a density h with respect to Lebesgue measure, in fact $h \in L^1$. Hence, for any $\phi \in \mathcal{H}$ and $\psi \in \mathcal{H}_{ac}$

$$\langle \varphi, e^{-itH} \psi \rangle \quad = \quad \int e^{-it\lambda} \, d\mu_{\varphi,\psi}(\lambda)$$

$$= \quad \int e^{-it\lambda} \, h(\lambda) \, d\lambda \, . \tag{7.24}$$

The latter expression is the Fourier transform of the (L^1-)function h. Thus, by the Riemann-Lebesgue-Lemma (see e.g. [**117**]), it converges to 0 as t goes to infinity. We warn the reader that the decay of the Fourier transform of a measure does *not* imply that the measure is absolutely continuous. There are examples of singular continuous measures whose Fourier transforms decay.

If the underlying Hilbert space is $\ell^2(\mathbb{Z}^d)$ we may choose $\varphi = \delta_x$ for any $x \in \mathbb{Z}^d$, then $\langle \varphi, e^{-itH} \psi \rangle = e^{-itH} \psi(x)$, thus we have immediately

THEOREM 7.9. *Let H be a self adjoint operator on $\ell^2(\mathbb{Z}^d)$, take $\psi \in \mathcal{H}_{ac}$ and let Λ denote a finite subset of $\mathbb{Z}^d$. Then*

$$\lim_{t \to \infty} \left(\sum_{x \in \Lambda} |\, e^{-itH} \psi(x)\,|^2 \right) \; = \; 0 \tag{7.25}$$

or equivalently

$$\lim_{t \to \infty} \left(\sum_{x \notin \Lambda} |\, e^{-itH} \psi(x)\,|^2 \right) \; = \; \|\,\psi\,\|^2 \,. \tag{7.26}$$

REMARK 7.10. As $\psi \in \mathcal{H}_{pp}$ may be interpreted as a particle staying (essentially) in a finite region for all time, a particle $\psi \in \mathcal{H}_{ac}$ runs out to infinity as time evolves (and came out from infinity as t goes to $-\infty$). So, in contrast to the bound states ($\psi \in \mathcal{H}_{pp}$), we might call the states in $\mathcal{H}_{ac}$ *scattering states* . Observe, however, that this term is used in scattering theory in a more restrictive sense.

In the light of these results, states in the pure point subspace are interpreted as bound states with low mobility. Consequently, electrons in such a state should not contribute to the electrical conductivity of the system. In contrast, states in the absolutely continuous subspace are highly mobile. They are the carrier of transport phenomena like conductivity.

A (relatively) simple example of a quantum mechanical system is a Hydrogen atom. After removal of the center of mass motion it consists of one particle moving under the influence of a Coulomb potential $V(x) = -\frac{Z}{|x|}$. The spectrum of the corresponding Schrödinger operator consists of infinitely many eigenvalues $\left(-\frac{C}{n^2}\right)$ which accumulate at 0 and the interval $[0, \infty)$ representing the absolutely continuous spectrum. The eigenfunction corresponding to the negative eigenvalues represent electrons in bound states, the orbitals. The states of the a.c.-spectrum correspond to electrons coming from infinity being scattered at the nucleus and going off to infinity again.

The Hydrogen atom is typical for the classical picture of a quantum system: Above an energy threshold there is purely absolutely continuous spectrum due to scattering states, below the threshold there is a finite or countable set of eigenvalues accumulating at most at the threshold. For the harmonic oscillator there is a purely discrete spectrum, for periodic potentials the spectrum consists of bands with purely absolutely continuous spectrum. Until a few decades ago almost all physicists believed that all quantum systems belonged to one of the above spectral types.

We have seen above that there may be pure point spectrum which is dense in a whole interval and we will see this is in fact typically the case for random operators. So far, we have not discussed the long time behavior for states in the singular continuous spectrum. Singularly continuous spectrum seems to be particularly exotic and unnatural. In fact, one might tend to believe it is only a mathematical sophistication which never occurs in physics. This point of view is proved to be wrong. In fact, singularly continuous spectrum is typical for systems with aperiodic long range order, such as quasicrystals.

The definition of singularly continuous measures is a quite indirect one. Indeed, we have not defined them by what they are but rather by what they are not. In other

words: Singular continuous measures are those that remain if we remove pure point and absolutely continuous measures. There is a characterization of *continuous* measures (i.e. those without atoms) by their Fourier transform which goes back to Wiener.

THEOREM 7.11 (Wiener). *Let μ be a bounded Borel measure on $\mathbb{R}$ and denote its Fourier transform by $\hat{\mu}(t) = \int e^{-it\lambda}\,d\mu(\lambda)$. Then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\,dt = \sum_{x\in\mathbb{R}} |\mu(\{x\})|^2\ .$$

COROLLARY 7.12. *μ is a continuous measure if and only if*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\,dt\ =\ 0\ .$$

PROOF *(Theorem)* :

$$\frac{1}{T}\int_0^T |\hat{\mu}(t)|^2 dt$$
$$= \frac{1}{T}\int_0^T \left(\int_{\mathbb{R}} e^{-it\lambda}\,d\mu(\lambda)\int_{\mathbb{R}} e^{it\varrho}\,d\bar{\mu}(\varrho)\right)\,dt$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(\frac{1}{T}\int_0^T e^{it(\varrho-\lambda)}dt\right)\,d\bar{\mu}(\varrho)\,d\mu(\lambda)\ . \tag{7.27}$$

Here $\bar{\mu}$ denotes the complex conjugate of the measure μ. The functions

$$f_T(\varrho,\lambda) = \frac{1}{T}\int_0^T e^{it(\varrho-\lambda)}dt$$

are bounded by one. Moreover for $\varrho \neq \lambda$

$$f_T(\varrho,\lambda) = \frac{1}{i(\varrho-\lambda)T}(e^{iT(\varrho-\lambda)} - 1) \to\ 0 \quad \text{as} \quad T\to\infty$$

and

$$f_T(\varrho,\varrho) = 1\ .$$

Thus, $f_T(\varrho,\lambda) \to \chi_D(\varrho,\lambda)$ with $D = \{(x,y) \mid x = y\}$. By Lebesgue's dominated convergence theorem, it follows that

$$\frac{1}{T} \int_0^T |\hat{\mu}(t)|^2 dt$$

$$\rightarrow \int \int \chi_D(\varrho, \lambda) \, d\bar{\mu}(\varrho) d\mu(\lambda)$$

$$= \int \bar{\mu}(\{\lambda\}) \, d\mu(\lambda) = \sum |\mu(\{\lambda\})|^2 . \qquad (7.28)$$

$\square$

This enables us to prove an analog of Theorem 7.7 and Theorem 7.9 for continuous measures. It says that states in the singularly continuous subspace represent particles which go off to infinity (at least) in the time average.

THEOREM 7.13. *Let H be a self adjoint operator on $\ell^2(\mathbb{Z}^d)$, take $\psi \in \mathcal{H}_c$ and let Λ be a finite subset of $\mathbb{Z}^d$.*
Then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\sum_{x \notin \Lambda} |e^{-itH}\psi(x)|^2 \right) dt = \| \psi \|^2 \qquad (7.29)$$

or equivalently

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\sum_{x \in \Lambda} |e^{-itH}\psi(x)|^2 \right) dt = 0 . \qquad (7.30)$$

PROOF: The equivalence of (7.29) and (7.30) follows from (7.21).
We prove (7.30). Let ψ be in $\mathcal{H}_c$. From estimate (7.18), we learn that for any $x \in \mathbb{Z}^d$ the measure $\mu_{\delta_x, \psi}$ is continuous. We have

$$\frac{1}{T} \int_0^T \sum_{x \in \Lambda} |e^{-itH}\psi(x)|^2 \, dt = \sum_{x \in \Lambda} \frac{1}{T} \int_0^T |\hat{\mu}_{\delta_x, \psi}|^2 \, dt .$$

The latter term converges to 0 by Theorem (7.11). $\square$

We close this section with a result which allows us to express the projections onto the pure point subspace and the absolutely continuous subspace as dynamical quantities.

THEOREM 7.14. *Let H be a self adjoint operator on $\ell^2(\mathbb{Z}^d)$, let P_c and P_{pp} be the orthogonal projection onto $\mathcal{H}_c$ and $\mathcal{H}_{pp}$ respectively, and let Λ_L denote a cube in $\mathbb{Z}^d$ centered at the origin with side length $2L + 1$. Then, for any $\psi \in \ell^2(\mathbb{Z}^d)$*

$$\|P_c \psi\|^2 = \lim_{L \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left(\sum_{x \notin \Lambda_L} |e^{-itH}\psi(x)|^2 \right) dt \qquad (7.31)$$

and

$$\|P_{pp}\,\psi\|^2 \;=\; \lim_{L\to\infty}\;\lim_{T\to\infty}\;\frac{1}{T}\int_0^T\left(\sum_{x\in\Lambda_L}|\,e^{-itH}\psi(x)\,|^2\right)\,dt\,. \qquad (7.32)$$

PROOF:　As in (7.21) we have

$$\begin{aligned}
\|\,P_c\,\psi\,\|^2 \;=\;& \frac{1}{T}\int_0^T\Big(\sum_{x\notin\Lambda_L}|\,e^{-itH}\psi(x)\,|^2\Big)\,dt \\
&-\frac{1}{T}\int_0^T\Big(\sum_{x\notin\Lambda_L}|\,e^{-itH}P_{pp}\psi(x)\,|^2\Big)\,dt \\
&+\frac{1}{T}\int_0^T\Big(\sum_{x\in\Lambda_L}|\,e^{-itH}P_c\psi(x)\,|^2\Big)\,dt\,. \qquad (7.33)
\end{aligned}$$

By Theorem 7.7 and Theorem 7.13 the second and the third term in (7.33) tend to zero as T and (then) L go to infinity. This proves (7.31). Assertion (7.32) is proved in a similar way. □

Notes and Remarks

Most of the material in this chapter is based on [30], [123] and the lecture notes [134] by Gerald Teschl. Teschl's excellent notes are only available on the internet. For further reading we recommend [8], [13], [42], [113], [120] and [141].

8. Anderson localization

8.1. What physicists know.

Since the ground breaking work of P. Anderson in the late fifties, physicists like Mott, Lifshitz, Thouless and many others have developed a fairly good knowledge about the measure theoretic nature of the spectrum of random Schrödinger operators, i.e. about the dynamical properties of wave packets.

By Theorem 3.9 (see also Theorem 4.3) we know that the (almost surely non random) spectrum Σ of H_ω is given by $\text{supp}(P_0) + [0, 4d]$ where P_0 is the probability distribution of $V_\omega(0)$. Thus if $\text{supp}(P_0)$ consists of finitely many points or intervals the spectrum Σ has a band structure in the sense that it is a union of (closed) intervals.

In the following we report on the picture physicists developed about the measure theoretic structure of the spectrum of H_ω. This picture is supported by convincing physical arguments and is generally accepted among theoretical physicists. Only a part of it can be shown with mathematical rigor up to now. We will discuss this issue in the subsequent sections.

There is a qualitative difference between one dimensional disordered systems ($d = 1$) and higher dimensional ones ($d \geq 3$). For one dimensional (disordered) systems one expects that the whole spectrum is pure point. Thus, there is a complete system of eigenfunctions. The corresponding (countably many) eigenvalues form a dense set in Σ ($= \cup[a_i, b_i]$). The eigenfunctions decay exponentially at infinity. This phenomenon is called *Anderson localization* or *exponential localization*. In the light of our discussion in section 7, we conclude that Anderson localization corresponds to low mobility of the electrons in our system. Thus, one dimensional disordered systems ('thin wires with impurities') should have low or even vanishing conductivity.

In arbitrary dimension, an ordered quantum mechanical system should have purely absolutely continuous spectrum. This is known for periodic potentials in any dimension. Thus, in one dimension, an *arbitrarily small disorder* will change the total spectrum from absolutely continuous to pure point and hence a conductor to an insulator. Anderson localization in the one dimensional case can be proved with mathematical rigor for a huge class of disordered systems. We will not discuss the one dimensional case in detail in this paper.

In dimension $d \geq 3$ the physics of disordered systems is much richer (and consequently more complicated). As long as the randomness is not too strong Anderson localization occurs only near the band edges of the spectrum. Thus near any band edge a there is an interval $[a, a+\delta]$ (resp. $[a-\delta, a]$) of pure point spectrum and the corresponding eigenfunctions are 'exponentially localized' in the sense that they decay exponentially fast at infinity.

Well inside the bands, the spectrum is expected to be absolutely continuous at small disorder ($d \geq 3$). Since the corresponding (generalized) eigenfunctions are certainly not square integrable, one speaks of *extended states* or *Anderson delocalization* in this regime. If the randomness of the system increases the pure point

spectrum will expand and the absolutely continuous part of the spectrum will shrink correspondingly. So, according to physical intuition, there is a phase transition from an insulating phase to a conducting phase. A transition point between these phases is called a *mobility edge*.

At a certain degree of randomness, the a.c. spectrum should be 'eaten up' by the pure point spectrum. The physical implications of the above picture are that we expect an energy region for which the corresponding states do not contribute to the conductance of the system (pure point spectrum) and an energy region corresponding to states with good mobility which constitute the conductivity of the system (a.c. spectrum).

In the above discussion we have deliberately avoided the case of space dimension $d = 2$. The situation in two dimensions was under debate in the theoretical physics community until a few years ago. At present, the general believe seems to be that we have complete Anderson localization for $d = 2$ similar to the case $d = 1$. However, the pure point spectrum is expected to be less stable for $d = 2$, for example a magnetic field might be able to destroy it.

8.2. What mathematicians prove.

For more than 25 years, mathematicians have been working on random Schrödinger operators. Despite of this, the mathematically rigorous knowledge about these operators is far from being complete.

As mentioned above, the results on the one dimensional case are fairly satisfactory. One can prove Anderson localization for all energies for a huge class of one dimensional random quantum mechanical systems.

For quite a number of models in $d \geq 2$ we also have proofs of Anderson localization, even in the sense of dynamical localization (see Section 8.4), at low energies or high disorder. There are also results about localization at spectral edges (other than the bottom of the spectrum).

The model which is best understood in the continuous case is the alloy-type model with potential (2.5)

$$V_\omega(x) = \sum q_i(\omega) f(x - i) . \tag{8.1}$$

The q_i are assumed to be independent with common distribution P_0. Until very recently, all known localization proof (for $d \geq 2$) required some kind of regularity of the probability measure P_0, for example the existence of a bounded density with respect to Lebesgue measure. In any case, these assumptions *exclude* the case when P_0 is concentrated in finitely many points. From a physical point of view such measures with a finite support are pretty natural. They model a random alloy with finitely many constituents. A few years ago, Bourgain and Kenig [19] proved localization for the Bernoulli alloy type model, i.e. a potential as in (8.1) with P_0 concentrated on $\{0, 1\}$.

Their proof works in the continuous case, but it does *not* for the (discrete) Anderson model. In the continuous case Bourgain and Kenig strongly use that eigenfunctions

of a Schrödinger operator on $\mathbb{R}^d$ can not decay faster than a certain exponential bound. This is a strong quantitative version of the unique continuation theorem which says that a solution of the Schrödinger equation which is zero on an open set vanishes everywhere.

Such a unique continuation theorem is wrong on the lattice, so a fortiori the lower bound on eigenfunctions is not valid on $\mathbb{Z}^d$. This is the main reason why the proof by Bourgain-Kenig does not extend to the discrete case.

Using ideas from Bourgain-Kenig [19], Germinet, Hislop and Klein [48] proved Anderson localization for the Poisson model (2.8). Until their paper nothing was known about Anderson localization for the Poisson model in dimension $d \geq 2$. (For $d = 1$ see [131]).

It is certainly fair to say that by now mathematicians know quite a bit about Anderson *localization*, i.e. about the insulating phase.

The contrary is true for Anderson *delocalization*. There is *no* proof of existence of absolutely continuous spectrum for *any* of the models we have discussed so far. In particular it is not known whether there *is* a conducting phase or a mobility edge at all.

Existence of absolutely continuous spectrum is known, however, for the so called Bethe lattice (or Cayley tree). This is a graph ("lattice") without loops (hence a tree) with a fixed number of edges at every site. One considers the graph Laplacian on the Bethe lattice, which is analogously defined to the Laplacian on the graph $\mathbb{Z}^d$ (see [75], [76], [77]) and an independent identically distributed potential on the sites of the graph.

There are also 'toy'-models similar to the Anderson model but with non identically distributed $V_\omega(i)$ which are more and more diluted (or 'weak') as $||i||_\infty$ becomes large. For these models, the mobility can be determined. (see [61], [60] and [54]).

8.3. Localization results.

We state the localization result we are going to prove in the next chapters. For convenience, we repeat our assumptions. They are stronger than necessary but allow for an easier, we hope more transparent, proof.

Assumptions:

(1) H_0 is the finite difference Laplacian on $\ell^2(\mathbb{Z}^d)$.
(2) $V_\omega(i)$, $i \in \mathbb{Z}^d$ are independent random variables with a common distribution P_0.
(3) P_0 has a bounded density g, i.e. $\mathbb{P}(V_\omega(i) \in A) = P_0(A) = \int_A g(\lambda)d\lambda$ and $||g||_\infty < \infty$.
(4) supp P_0 is compact.

DEFINITION 8.1. *We say that the random operator H_ω exhibits* spectral localization *in an energy interval I (with $I \cap \sigma(H_\omega) \neq \emptyset$) if for $\mathbb{P}$-almost all ω*

$$\sigma_c(H_\omega) \cap I = \emptyset. \tag{8.2}$$

We will show spectral localization for low energies and for strong disorder. To measure the degree of disorder of $P_0 = g\, d\lambda$, we introduce the 'disorder parameter' $\delta(g) := ||g||_\infty^{-1}$. If $\delta(g)$ is large, i.e. $||g||_\infty$ is small, then the probability density g (recall $\int g = 1$) is rather extended. So one may, in deed, say that $\delta(g)$ large is an indicator for large disorder. (If $\delta(g)$ is small then g might be concentrated near a small number of points. This, however, is not a convincing indicator of small disorder.) Let us denote by E_0 the bottom of the (almost surely constant) spectrum of $H_\omega = H_0 + V_\omega$.

In the following chapters we will prove:

THEOREM 8.2. *There exists $E_1 > E_0 = \inf(\sigma(H_\omega))$ such that the spectrum of H_ω exhibits spectral decomposition in the interval $I = [E_0, E_1]$.*
In particular, the spectrum inside I is pure point almost surely and the corresponding eigenfunctions decay exponentially.

THEOREM 8.3. *For any interval $I \neq \emptyset$, there is a δ_0 such that for any $\delta(g) \geq \delta_0$ the operator of H_ω exhibits spectral localization in I.*
The spectrum inside I is pure point almost surely and the corresponding eigenfunctions decay exponentially.

8.4. Further Results.

As we discussed in the previous chapter, physicists are not primarily interested in spectral properties of random Hamiltonians but rather in dynamical properties, i.e. in the longtime behavior of e^{-itH_ω}. Consequently Anderson localization should have dynamical consequences, as we might expect from the considerations in section 7.3.

It seems reasonable to expect that the following property holds in the localization regime.

DEFINITION 8.4. *We say that the random operator H_ω exhibits* dynamical localization *in an energy interval I (with $I \cap \sigma(H_\omega) \neq \emptyset$) if for all φ in the Hilbert space and all $p \geq 0$*

$$\sup_{t \in \mathbb{R}} ||\, |X|^p\, e^{-itH_\omega}\, \chi_I(H_\omega)\, \varphi\, || \; < \; \infty \tag{8.3}$$

for $\mathbb{P}$-almost all ω.

Above, $\chi_I(H_\omega)$ denotes the spectral projection for H_ω onto the interval I (see Section 3.2) and $|X|$ is the multiplication operator defined by $|X|\, \psi(n) = ||\, n\, ||_\infty \, \psi(n)$. Intuitively, dynamical localization tells us that the particle is concentrated near the origin uniformly for all times. We will not prove dynamical localization here. We refer to the references given in the notes and in particular to the review [78].

We turn to the question of the relation between spectral and dynamical localization.

THEOREM 8.5. *Dynamical localization implies spectral localization.*

PROOF: From Theorem 7.14 we know

$$\|P_c\,\psi\|^2 \;=\; \lim_{L\to\infty}\,\lim_{T\to\infty}\,\frac{1}{T}\int_0^T\left(\sum_{j\notin\Lambda_L}|\,e^{-itH}\psi(j)\,|^2\right)\,dt\;. \qquad (8.4)$$

For $\psi = P_I(H_\omega)\,\varphi$, we have

$$\sum_{j\notin\Lambda_L}|\,e^{-itH}\psi(j)\,|^2 \;=\; \sum_{\|j\|_\infty>L}\frac{1}{\|j\|_\infty^{2p}}\left|\,|X|^p\,e^{-itH}\psi(j)\,\right|^2$$

$$\leq\;\big|\big|\,|X|^p\,e^{-itH_\omega}\psi\,\big|\big|\sum_{\|j\|_\infty>L}\frac{1}{\|j\|_\infty^{2p}} \qquad (8.5)$$

By (8.3) we have that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\big|\big|\,|X|^p\,e^{-itH_\omega}\psi\,\big|\big|\,dt\;\leq\;C\;<\;\infty\;. \qquad (8.6)$$

Thus, for p large enough,

$$\|P_c\,\psi\|^2\;\leq\;C\,\lim_{L\to\infty}\sum_{\|j\|_\infty>L}\frac{1}{\|j\|_\infty^{2p}}\;=\;0\;. \qquad (8.7)$$

Hence, there is only pure point spectrum inside the interval I. $\square$

It turns out that the converse is not true, in general. There are examples of operators with pure point spectrum without dynamical localization [**35**].

Notes and Remarks

For an overview on the physics of Anderson localization / delocalization we refer to the papers [**9**], [**98**] and [**135**], [**136**]. For the mathematical aspects we refer to [**23**], [**112**] and [**128**].

In this lecture notes we have to omit many important results about the one dimensional case. We just mention a few of the most important papers about one dimensional localization here: [**52**], [**104**], [**111**], [**88**], as well as [**22**], [**21**], [**31**].

In the multidimensional case there exist two quite different approaches to localization. The first (in chronological order) is the multiscale analysis based on the fundamental paper [**47**]. This is the method we are going to present in the following chapters. For further references see the literature cited there.

The second method, the method of fractional moments, is also called the Aizenman-Molchanov method after the basic paper [**4**]. At least for the lattice case, this method is in many ways easier than the multiscale analysis. Moreover, it gives a number of additional results. On the other hand its adaptation to the continuous case is rather involved. We refer to [**1**], [**53**], [**3**], [**5**], [**2**] for further developments. We will not discuss this method here due to the lack of space and time.

It was realized by Martinelli and Scoppolla [**101**] that the result of multiscale analysis implies absence of a.c. spectrum. The first proofs of spectral localization were given independently in [**46**] (see also [**40**]), [**37**] and [**126**] . The latter papers develop the method of spectral averaging which goes partly back to [**89**].

The result, that dynamical localization implies spectral localization, was proved in [**94**] (see also [**30**]). These authors also prove dynamical localization for the one dimensional case. An example with spectral localization which fails to exhibit dynamical localization was given in [**35**].

The first proof of dynamical localization for the multidimensional Anderson model was given by Aizenman [**1**]. This paper uses and extends the Aizenman-Molchanov method of fractional moments to prove localization. De Bièvre and Germinet [**14**] proved dynamical localization both for the Anderson model and for alloy-type models (in the continuum). Their proof is based on multiscale ideas. Damanik and Stollmann [**32**] proved that the multiscale analysis actually implies dynamical localization. They proved a version of dynamical localization (strong dynamical localization) which is stronger than ours.

There are various even stronger versions of dynamical localization, we just mention strong Hilbert-Schmidt dynamical localization which was proven by Germinet and Klein [**49**]. We refer to the survey [**78**] by Abel Klein for this kind of questions.

In theoretical physics, the theory of conductivity goes much beyond a characterization of the spectral type of the Hamiltonian. One of the main topics is the linear response theory and the Kubo-formula. This approach is investigated from a mathematical point of view in [**3**], [**17**], [**79**] (see also [**74**]).

For delocalization on the Bethe lattice see: [**75**], [**76**], [**77**]. See also [**7**], [**6**] and [**45**] for new proofs and further developments.

Delocalization for potentials with randomness decaying at infinity was investigated in [**92**], [**93**], [**60**], [**18**], [**118**]. A localization / delocalization transition was proved for such potentials in [**61**], [**54**].

Dynamical delocalization was shown for a random dimer model in [**56**] and for a random Landau Hamiltonian in [**51**]. Dynamical delocalization means that dynamical localization is violated in some sense. It does not imply delocalization in the sense of a.c. spectrum. Moreover, in the above cited papers dynamical delocalization is only shown at special energies of Lebesgue measure zero.

9. The Green's function and the spectrum

9.1. Generalized eigenfunctions and the decay of the Green's function.

Here we start to prove Anderson localization via the multiscale method. The proof will require the whole rest of this text. For the reader who might get lost while trying to understand the proof, we provided a roadmap through these chapters in chapter 12.

We begin our discuss of multiscale analysis. This method is used to show exponential decay of Green's functions. In this section we investigate some of the consequences of that estimate on the spectral properties of H_ω. The multiscale estimates are discussed in the next chapter.

Let us start by defining what we mean by exponential decay of Green's functions . We recall some of the notations introduced in previous chapters. $\Lambda_L(n)$ is the cube of side length $(2L + 1)$ centered at $n \in \mathbb{Z}^d$ (see (3.5)), and Λ_L denotes a cube around the origin. $|| m ||_\infty = \sup_{i=1,\dots,d} |m_i|$.

The inner boundary $\partial^- \Lambda_L(n)$ of $\Lambda_L(n)$ consists of the outermost layer of lattice points in $\Lambda_L(n)$, namely (see 5.28)

$$
\begin{aligned}
\partial^- \Lambda_L(n) &= \{ m \in \mathbb{Z}^d \mid m \in \Lambda_L(n), \ \exists\, m' \notin \Lambda_L(n) \ (m, m') \in \partial \Lambda_L(n) \} \\
&= \{ m \in \mathbb{Z}^d \mid \ || m - n ||_\infty = L \} \,.
\end{aligned}
\tag{9.1}
$$

Similarly, the outer boundary of $\Lambda_L(n)$ is defined by

$$
\begin{aligned}
\partial^+ \Lambda_L(n) &= \{ m \in \mathbb{Z}^d \mid m \notin \Lambda_L(n), \ \exists\, m' \in \Lambda_L(n) \ (m, m') \in \partial \Lambda_L(n) \} \\
&= \{ m \in \mathbb{Z}^d \mid \ || m - n ||_\infty = L + 1 \} \,.
\end{aligned}
\tag{9.2}
$$

For $A \subset \mathbb{Z}^d$ we denote the number of lattice points inside A by $|A|$. So, $|\Lambda_L| = (2L+1)^d$ and $|\partial^- \Lambda_L| = 2d\,(2L)^{d-1}$. By $A_m \nearrow \mathbb{Z}^d$ we mean: $A_m \subset A_{m+1} \subset \mathbb{Z}^d$ and $\bigcup A_m = \mathbb{Z}^d$.

The Green's function $G_E^\Lambda(n, m)$ is the kernel of the resolvent of H_Λ given by

$$
G_E^\Lambda(n, m) = (H_\Lambda - E)^{-1}(n, m) = \langle \delta_n, (H_\Lambda - E)^{-1} \delta_m \rangle \,.
\tag{9.3}
$$

DEFINITION 9.1.

(1) We will say that the Green's functions $G_E^{\Lambda_L(n_0)}(n, m)$ for energy E and potential V *decays exponentially* on $\Lambda_L(n_0)$ with rate γ $(\gamma > 0)$ if E is not an eigenvalue for $H_{\Lambda_L(n_0)} = (H_0 + V)_{\Lambda_L(n_0)}$ and

$$
|G_E^{\Lambda_L(n_0)}(n, m)| = |(H_{\Lambda_L} - E)^{-1}(n, m)| \le e^{-\gamma L}
\tag{9.4}
$$

for all $n \in \Lambda_{L^{1/2}}(n_0)$ and all $m \in \partial^- \Lambda_L(n_0)$.

(2) If the Green's function $G_E^{\Lambda_L(n_0)}$ decays exponentially with rate $\gamma > 0$ we call the cube $\Lambda_L(n_0)$ (γ, E)-*good* for V.

(3) We call an energy E γ-*good* (for V_ω) if the there is a sequence of cubes $\Lambda_{\ell_m} \nearrow \mathbb{Z}^d$ such that all Λ_{ℓ_m} are (γ, E)-good. (Note, that γ is independent of Λ_{ℓ_m}!)

Note that, by definition, $E \notin \sigma(H_{\Lambda_L(n_0)})$ if $\Lambda_L(n_0)$ is (γ, E)-good.

The behavior of the Green's function has important consequences for the behavior of (generalized) eigenfunctions. Suppose that the function ψ is a solution of the difference equation

$$H\psi = E\psi . \tag{9.5}$$

Then (see equations (5.30) and (5.31))

$$0 = (H - E)\psi = (H_\Lambda \oplus H_{\complement\Lambda} + \Gamma_\Lambda - E)\psi , \tag{9.6}$$

hence

$$((H_\Lambda \oplus H_{\complement\Lambda}) - E)\psi = -\Gamma_\Lambda\psi . \tag{9.7}$$

So, for any $n_0 \in \Lambda$ we have

$$(H_\Lambda - E)\psi(n_0) = (-\Gamma_\Lambda\psi)(n_0) . \tag{9.8}$$

Suppose that E is not an eigenvalue of H_Λ, then $(n_0 \in \Lambda)$

$$\psi(n_0) = -[(H_\Lambda - E)^{-1}\Gamma_\Lambda\psi](n_0) . \tag{9.9}$$

So

$$\psi(n_0) = - \sum_{\substack{(k,m)\in\partial\Lambda \\ k\in\partial^-\Lambda, m\in\partial^+\Lambda}} G_E^\Lambda(n_0, k)\psi(m) . \tag{9.10}$$

This enables us to prove a crucial observation.

THEOREM 9.2. *If E is γ-good for V then E is not a generalized eigenvalue of $H = H_0 + V$.*

PROOF: Suppose ψ is a polynomially bounded eigenfunction of H with (generalized) eigenvalue E, hence

$$H\psi = E\psi \quad \text{and} \quad |\psi(m)| \leq c\,|m|^r \qquad \text{for } m \neq 0 .$$

Take any $n \in \mathbb{Z}^d$, then $n \in \Lambda_{L_k^{1/2}}(n_0)$ for k large enough. Thus by (9.10)

$$|\psi(n)| \quad \leq \quad \left| \sum_{\substack{(m',m) \in \partial \Lambda_{L_k} \\ m' \in \Lambda_{L_k}}} G_E^{\Lambda_{L_k}}(n,m')\psi(m) \right|$$

$$\leq \quad c_1 \, L_k^{d-1} \, e^{-\gamma L_k} \sup_{m \in \partial^+ \Lambda_{L_k}} |\psi(m)| \tag{9.11}$$

$$\leq \quad c_2 \, L_k^{d-1+r} \, e^{-\gamma L_k} \tag{9.12}$$

$$\to \quad 0 \qquad \text{as} \quad k \to \infty \,. \tag{9.13}$$

Hence $\psi \equiv 0$. Consequently, there are no non zero polynomially bounded eigensolutions. $\qquad \square$

There are two immediate yet remarkable consequences of Theorem (9.2).

COROLLARY 9.3. *If every $E \in [E_1, E_2]$ is γ-good for V then $\sigma(H_0 + V) \cap (E_1, E_2) = \emptyset$.*

COROLLARY 9.4. *If Lebesgue-almost all $E \in [E_1, E_2]$ are γ-good for V then*

$$\sigma_{ac}(H_0 + V) \cap (E_1, E_2) = \emptyset \,.$$

PROOF: The assumption of Corollary 9.3 implies by Theorem 9.2 that there are no generalized eigenvalues in (E_1, E_2). By Theorem 7.1 (or Proposition 7.4) it follows that there is no spectrum there.

If there is any absolutely continuous spectrum in (E_1, E_2) the spectral measure restricted to that interval must have an absolutely continuous component. Hence, by Theorem 7.1, there must be a set of generalized eigenvalues of *positive* Lebesgue measure. However, this is not possible by the assumption of Corollary 9.4 and Theorem 9.2. $\qquad \square$

9.2. From multiscale analysis to absence of a.c. spectrum.

The results of the previous section indicate a close relation between the existence of (γ, E)-good cubes and the spectrum of the (discrete) Schrödinger operator. The following theorem gives first hints to a probabilistic analysis of this connection.

THEOREM 9.5. *If there is a sequence $R_k \to \infty$ of integers such that for every k, every $E \in I = [E_1, E_2]$ and a constant $\gamma > 0$*

$$\mathbb{P} \left(\, \Lambda_{R_k} \text{ is not } (\gamma, E)\text{-good} \, \right) \quad \to \quad 0 \tag{9.14}$$

then with probability one

$$\sigma_{ac}(H_\omega) \cap (E_1, E_2) = \varnothing \,. \tag{9.15}$$

PROOF: Set $p_k = \mathbb{P}\left(\Lambda_{R_k} \text{ is not } (\gamma, E)\text{-good}\right)$. By passing to a subsequence, if necessary, we may assume that the R_k are increasing and that $\sum p_k < \infty$. Consequently, from the Borel-Cantelli-Lemma (see Theorem 3.6)) we learn that with probability one, there is a k_0 such that all Λ_{R_k} are $(\gamma, E)-$good for $k \geq k_0$. Hence for $\mathbb{P}-$almost every ω any given $E \in [E_1, E_2]$ is $\gamma-$good.
We set

$$
\begin{aligned}
\mathcal{N} &= \left\{ (E, \omega) \in [E_1, E_2] \times \Omega \;\middle|\; E \text{ is not } \gamma-\text{good for } V_\omega \right\} & (9.16) \\
\mathcal{N}_E &= \left\{ \omega \in \Omega \;\middle|\; E \text{ is not } \gamma-\text{good for } V_\omega \right\} & (9.17) \\
\mathcal{N}_\omega &= \left\{ E \in [E_1, E_2] \;\middle|\; E \text{ is not } \gamma-\text{good for } V_\omega \right\}. & (9.18)
\end{aligned}
$$

Above we proved $\mathbb{P}(\mathcal{N}_E) = 0$ for any $E \in [E_1, E_2]$.
Denoting the Lebesgue measure on $\mathbb{R}$ by λ we have by Fubini's theorem

$$
\begin{aligned}
\lambda \otimes \mathbb{P}(\mathcal{N}) &= \int_{E_1}^{E_2} \mathbb{P}(\mathcal{N}_E) \, d\lambda(E) \\
&= \int \lambda(\mathcal{N}_\omega) \, d\mathbb{P}(\omega).
\end{aligned}
$$

Since $\mathbb{P}(\mathcal{N}_E) = 0$ for all $E \in [E_1, E_2]$ we conclude that

$$
0 = \int_{E_1}^{E_2} \mathbb{P}(\mathcal{N}_E) \, d\lambda(E) = \int \lambda(\mathcal{N}_\omega) \, d\mathbb{P}(\omega).
$$

Thus, for almost all ω we have $\lambda(\mathcal{N}_\omega) = 0$. Consequently, by Corollary 9.4 there is no absolutely continuous spectrum in (E_1, E_2) for these ω. $\square$

One might be tempted to think the assumption that all $E \in [E_1, E_2]$ are $\gamma-$good $\mathbb{P}$-almost surely would imply that there are *no* generalized eigenvalues in $[E_1, E_2]$. This would exclude *any* spectrum inside (E_1, E_2), not only absolutely continuous one. This reasoning is *wrong*. The problem with the argument is the following: Under this assumption, we know that for any *given* energy E, there are no generalized eigenvalues with probability one, i.e. the set $\mathcal{N}_E$ is a set of probability zero. Thus, for $\omega \in \Omega_0 := \bigcup_{E \in [E_1, E_2]} \mathcal{N}_E$ there are no generalized eigenvalues in the interval $[E_1, E_2]$. However, the set Ω_0 is an *uncountable* union of sets of measure zero, therefore, we cannot conclude that it has zero measure.

Theorem 9.5 immediately triggers two kind of questions: First, is (9.14) true under certain assumptions, and how can we prove it? This is exactly what the multiscale analysis does. We will discuss this result in the following section 9.3 and prove it in chapter 10.

The other question raised by the theorem is whether or not 'good' cubes might help to prove even *pure point* spectrum, not only the absence of absolutely continuous spectrum.

It turns out that the condition (9.14) alone is not sufficient to prove pure point spectrum. There are examples of operators with (almost periodic) potential V satisfying condition (9.14) inside their spectrum, having no (ℓ^2-)eigenvalues at all (see e.g. [30]). So, these operators have purely singular continuous spectrum in the region

where (9.14) holds. This effect is due to some kind of 'long range' order of almost periodic potentials.

In the situation of the Anderson model, we have the independence of the random variables $V_\omega(i)$. This assumption, we may hope, prevents the potential from 'conspiring' against pure point spectrum through long range correlations. However, the above example of an almost periodic potential makes clear that some extra work is required to go beyond the absence of a.c. spectrum and prove pure point spectrum. This question will be addressed in section 9.5 after some preparation in section 9.4.

9.3. The results of multiscale analysis .

We define a length scale L_k inductively. The initial length L_0 will be defined later depending on the specific parameters (disorder, energy region, etc.) of the problem considered. The length L_{k+1} is defined by $L_k{}^\alpha$ for an α with $1 < \alpha < 2$ to be further specified later. The constant α will only depend on some general parameters like the dimension d. The condition $\alpha > 1$ ensures that $L_k \to \infty$, while $\alpha < 2$ makes the estimates to come easier. Finally, we will have to choose α close to one. Observe, that the length scale L_k is growing very fast, in fact superexponentially.

A main result of multiscale analysis will be the following probabilistic estimate, which holds for certain intervals $[E_1, E_2]$.

RESULT 9.6 (multiscale analysis - weak form). *For some $\alpha > 1$, $p > 2d$ and a $\gamma > 0$ and for all $E \in I = [E_1, E_2]$*

$$\mathbb{P}\left(\Lambda_{L_k} \text{ is not } (\gamma, E)-\text{good for } V_\omega \right) \ \leq \ \frac{1}{L_k^p} \ . \tag{9.19}$$

REMARKS 9.7.

  (1) We will prove this result in the next two chapters.
  (2) To prove Result 9.6, we need to assume that the probability distribution P_0 of the random variables $V_\omega(i)$ has a bounded density. This ensures that we can apply Wegner's estimate (Theorem 5.23) which is a key tool in our proof. Recently, Bourgain and Kenig [**19**] were able to do the multiscale analysis for some V_ω without a density for P_0.
  (3) We will proof Result 9.6 for $I = [E_1, E_2]$ when I is close to the bottom of the spectrum or for given I if the disorder is sufficiently strong.
  (4) As the proof shows we have to take $\alpha < \frac{2p}{2d+p}$ which is bigger than 1 since $p > 2d$.

The proof of Result 9.6 and its variants (see below) will take two chapters. We prove the result by induction, i. e. we prove (9.19) for the initial scale L_0 and then prove the induction step, namely: If (9.19) holds for a certain k, it holds for $k + 1$ as well.

The initial scale estimate will be done in chapter 11. It is only here where we need assumptions about the energy interval I (e.g. I is close to the bottom of the spectrum or to an other band edge) or about the strength of the disorder. Thus, the specific parameters of the model enter only here.

In contrast to this, the induction step can be done under quite general conditions for all energies and any degree of disorder. This step will be presented in chapter 10.

The multiscale estimate Result 9.6 obviously implies the absence of absolutely continuous spectrum inside I via Theorem 9.5. The estimate (9.19) per se does not imply pure point spectrum (see the discussion at the end of the previous section). However, for the *Anderson model* one can use Result 9.6 to deduce pure point spectrum, provided P_0 has a bounded density. This can be done using a technique known as *spectral averaging*. The basic idea goes back to Kotani [**89**] and was further developed and applied to the Anderson model by various authors (see e.g. [**37, 90, 126, 27**]). The paper [**126**] triggered also the development of the theory of rank one perturbations [**124**]. We will not discuss this method here and refer to the papers cited.

Instead, we will present another proof of pure point spectrum which goes back to [**46**] and [**40**]. It consists in a version of the estimate (9.19) which is 'uniform' in energy E. Taken literally a uniform version of (9.19) would be

$$\mathbb{P}\left(\text{ There is an } E \in I \text{ such that } \Lambda_{L_k} \text{ is not } (\gamma, E)-\text{good for } V_\omega\right) \leq L_k^{-p} \ . \quad (9.20)$$

However, it is easy to see by inspecting the proof of Theorem 9.5 that (9.20) implies that any $E \in I$ is γ-good, thus there is *no* spectrum inside I by Corollary 9.3 above. In other words: condition (9.20) is 'too strong' to imply pure point spectrum.

A way out of this dilemma is indicated by the 'uniform' version of Wegner's estimate (Theorem 5.27). There, uniformity in energy is required only for *pairs* of disjoint cubes. This leads us to a uniform version of Result 9.6 for pairs of cubes.

RESULT 9.8 (multiscale analysis - strong form). *For some $p > 2d$, an α with $1 < \alpha < \frac{2p}{p+2d}$ and a $\gamma > 0$ we have: For any disjoint cubes $\Lambda_1 = \Lambda_{L_k}(n)$ and $\Lambda_2 = \Lambda_{L_k}(m)$*

$$\mathbb{P}\left(\text{ For some } E \in I \text{ both } \Lambda_1 \text{ and } \Lambda_2 \text{ are not } (\gamma, E)-\text{good}\right) \leq L_k^{-2p} \ . \quad (9.21)$$

The proof of this result is an induction procedure analogous to the one discussed above. In fact, the initial step will be the same as for Result 9.6, see Chapter 11.

In the induction step we assume the validity of estimate (9.21) for k and deduce the assertion for $k + 1$ from this assumption. The general idea of this step is quite close to the induction step for the weaker version (9.19), but it is technically more involved. Therefore, we present the proof of the weak version first and then discuss the necessary changes for the strong ('uniform') version.

9.4. An iteration procedure.

One of the crucial ingredients of multiscale analysis is the observation that the estimate (9.11) in the proof of Theorem 9.2 can be iterated.

A first version of this procedure is the contents of the following result.

We say that a subset $A \subset \mathbb{Z}^d$ is *well inside* a set Λ ($A \Subset \Lambda$) if $A \subset \Lambda$ and $A \cap \partial^-\Lambda = \emptyset$. For any set $\Lambda \subset \mathbb{Z}^d$ we define the *collection of $L-$cubes inside Λ*

by

$$C_L(\Lambda) = \{\Lambda_L(n) \mid \Lambda_L(n) \Subset \Lambda\} . \tag{9.22}$$

We also set $\partial_L^- \Lambda = \{m \in \Lambda \mid \mathrm{dist}(m, \partial^- \Lambda) \leq L\}$ (where $\mathrm{dist}(m, A) = \inf_{k \in A} \|m - k\|_\infty$).

THEOREM 9.9. *Suppose that each cube in $\mathcal{C}_M(A)$, $A \subset \mathbb{Z}^d$ finite, is (γ, E)-good and M is large enough. If ψ is a solution of $H\psi = E\psi$ in A and $n_0 \in A$ with*

$$\mathrm{dist}(n_0, \partial^- A) \geq k(M+1) , \tag{9.23}$$

then

$$|\psi(n_0)| \leq e^{-\gamma' kM} \sup_{m \in \partial_M^- \Lambda} |\psi(m)| \tag{9.24}$$

for some $\gamma' > 0$.

REMARK 9.10. Let us set

$$r = 2d \, (2M+1)^{d-1} \, e^{-\gamma M} \tag{9.25}$$

and

$$\gamma' = \gamma - \frac{1}{M} \ln \left(2d \, (2M+1)^{d-1} \right) \tag{9.26}$$

such that

$$r = e^{-\gamma' M} . \tag{9.27}$$

Then the phrase 'M large enough' in the theorem means that $r < 1$ and the theorem holds with γ' as in (9.26). Note that $\gamma' < \gamma$, but the 'error' term $\gamma - \gamma' = \frac{1}{M}\left((d - 1)\ln(2M+1) + \ln 2d\right)$ decreases in M and goes to zero if M tends to infinity.

The theorem may look a bit clumsy at first sight. Nevertheless, it contains some of the main ideas of multiscale analysis. The estimate (9.24) says that any solution ψ decays exponentially in regions which are filled with good cubes. In other words: The tunneling probability of a quantum particle through such a region is exponentially small. This will finally lead to the induction step in multiscale analysis.

To illustrate Theorem 9.9, we state the following Corollary which is essentially a reformulation of the theorem. The Corollary follows immediately from the Theorem.

COROLLARY 9.11. *Suppose each cube in $\mathcal{C}_M(A)$, $A \subset \mathbb{Z}^d$ finite, is (γ, E)-good and $M \geq C$ is large enough. Take $n_0 \in A$ with $d(n_0) = \mathrm{dist}(n_0, \partial^- A)$ so large that $\frac{d(n_0)}{M} \geq D$. If ψ is a solution of $H\psi = E\psi$ in A, then*

$$|\psi(n_0)| \leq e^{-\gamma'' d(n_0)} \sup_{m \in \partial_M^- \Lambda} |\psi(m)| \tag{9.28}$$

with

$$\gamma'' \;=\; \gamma - \frac{1}{M} \, \ln \left(2d \, (2M+1)^{d-1} \right) \left(1 - \frac{1}{C} - \frac{1}{D} \right) . \tag{9.29}$$

Observe, that the error term $\frac{1}{M} \ln \left(2d \, (2M+1)^{d-1} \right) \left(1 - \frac{1}{C} - \frac{1}{D} \right)$ is small if both M and the ratio of $d(n_0)$ and M are big.

PROOF *(Theorem)*: Since $n_0 \in A$ and $\mathrm{dist}(n_0, \partial^- A) \geq (M+1)$ we have $\Lambda_M(n_0) \in \mathcal{C}_M(A)$.
Thus by (9.10), we have

$$
\begin{aligned}
|\psi(n_0)| \;&\leq\; \sum_{\substack{(q,q') \in \partial \Lambda_M(n_0) \\ q \in \Lambda_M(n_0)}} |G_E^{\Lambda_M(n_0)}(n_0, q)| |\psi(q')| \\
&\leq\; |\partial \Lambda_M(n_0)| \; e^{-\gamma M} \sup_{q' \in \partial^+ \Lambda_M(n_0)} |\psi(q')| \\
&\leq\; 2d \, (2M+1)^{d-1} \, e^{-\gamma M} \, |\psi(n_1)| \\
&=\; r \, |\psi(n_1)| \tag{9.30}
\end{aligned}
$$

for some $n_1 \in \partial^+ \Lambda_M(n_0)$.
If $n_1 \in \partial_M^- A$, this is estimate (9.24) for $k = 1$. Note that $\mathrm{dist}(n_1, \partial_M^- A) \geq \mathrm{dist}(n_0, \partial_M^- A) - (M+1)$.
So $n_1 \in \partial_M^- A$ can only happen if $k = 1$.
If $n_1 \notin \partial_M^- A$, we have $\Lambda_M(n_1) \in \mathcal{C}_M(A)$ and we can iterate the estimate (9.30) to obtain

$$|\psi(n_1)| \leq r |\psi(n_2)| \tag{9.31}$$

with some $n_2 \in \partial^+ \Lambda_M(n_1)$, so

$$|\psi(n_0)| \leq r^2 |\psi(n_2)| \, . \tag{9.32}$$

(Note, that for $n_1 \in \partial_M^- A$ the iteration might get us out of A!)
For n_2 we have

$$
\begin{aligned}
\mathrm{dist}(n_2, \partial_M^- A) \;&\geq\; \mathrm{dist}(n_1, \partial_M^- A) - (M+1) \\
&\geq\; \mathrm{dist}(n_0, \partial_M^- A) - 2(M+1) \, .
\end{aligned}
$$

So $n_2 \in \partial_M^- \Lambda$ can happen only if $k \leq 2$.
If $n_2 \notin \partial_M^- \Lambda$, then we may iterate (9.30) again. We obtain

$$|\psi(n_0)| \leq r |\psi(n_1)| \leq r^2 |\psi(n_2)| \leq r^3 |\psi(n_3)| \leq \ldots \leq r^\ell |\psi(n_\ell)| \, .$$

This iteration process works fine as long as the new point $n_\ell \notin \partial_M^- A$. Consequently, by the assumption on n_0, we can iterate at least k times.
Thus, we obtain

$$|\psi(n_0)| \leq r^{k'} \sup_{q \in \partial_M^- \Lambda} |\psi(q)| \tag{9.33}$$

with some $k' \geq k$.

We conclude

$$|\psi(n_0)| \leq e^{-\gamma' kM} \sup_{q \in \partial_M^- \Lambda} |\psi(q)| \, . \tag{9.34}$$

$\square$

REMARK 9.12. For $\psi(n_0) \neq 0$, the above iteration procedure must finally reach ∂_L^-. Otherwise, we have

$$|\psi(n_0)| \leq r^\ell \sup_{q \in \Lambda} |\psi(q)|$$

for *any* $\ell \in \mathbb{N}$ which implies $\psi(n_0) = 0$. For $\psi(n_0) = 0$ the theorem is trivially fulfilled.

9.5. From multiscale analysis to pure point spectrum.

In this section we prove that the strong version (Result 9.8) of the multiscale estimate implies pure point spectrum inside the interval where the estimate holds.

THEOREM 9.13. *If Result 9.8 holds for an interval $I = [E_1, E_2]$, then with probability one*

$$\sigma_c(H_\omega) \cap (E_1, E_2) = \emptyset \, .$$

The spectrum of H_ω inside (E_1, E_2) consists of pure point spectrum, the corresponding eigenfunctions decay exponentially at infinity.

REMARK 9.14. The theorem includes the case $(E_1, E_2) \cap \sigma(H_\omega) = \emptyset$ but we will choose E_1, E_2 such that there is some spectrum inside (E_1, E_2) when we apply the theorem.

PROOF:

Step 1

We begin with a little geometry. As before we choose a sequence L_k by setting $L_k = L_{k-1}^\alpha$ with an $\alpha > 1$ and L_0 to be determined later. We consider the cubes $\Lambda_{L_k} = \Lambda_{L_k}(0)$ and annuli A_k which cover the region between the boundaries of Λ_{L_k} and $\Lambda_{L_{k+1}}$, more precisely

$$A_k = \Lambda_{6L_{k+1}} \setminus \Lambda_{3L_k} \, . \tag{9.35}$$

So, $n \in A_k$ if $||\,n\,||_\infty \leq 6L_{k+1}$ and $||\,n\,||_\infty > 3L_k$. It is clear that

$$A_k \cap A_{k+1} \neq \emptyset \tag{9.36}$$

and

$$\bigcup A_k = \mathbb{Z}^d \setminus \Lambda_{3L_0} . \qquad (9.37)$$

We will need also an enlarged version A_k^+ of the A_k namely

$$A_k^+ = \Lambda_{8L_{k+1}} \setminus \Lambda_{2L_k} . \qquad (9.38)$$

Obviously, $A_k \subset A_k^+$ and any $n \in A_k$ has a certain 'security' distance from ∂A_k^+, in fact we have:

LEMMA 9.15. *For each $n \in A_k$*

$$\operatorname{dist}(n, \partial A_k^+) \geq \frac{1}{3} || n ||_\infty .$$

PROOF *(Lemma)*:   If $|| n ||_\infty \geq 3L_k$ we have

$$
\begin{aligned}
\operatorname{dist}(n, \partial \Lambda_{2L_k}) &= || n ||_\infty - 2L_k \\
&\geq || n ||_\infty - \frac{2}{3} || n ||_\infty \\
&= \frac{1}{3} || n ||_\infty .
\end{aligned}
$$

If $|| n ||_\infty \leq 6L_{k+1}$

$$
\begin{aligned}
\operatorname{dist}(n, \partial \Lambda_{8L_{k+1}}) &= 8L_{k+1} - || n ||_\infty \\
&\geq \frac{8}{6} || n ||_\infty - || n ||_\infty \\
&= \frac{1}{3} || n ||_\infty .
\end{aligned}
$$

If $n \in A_k$ we have $3L_k \leq || n ||_\infty \leq 6L_{k+1}$, so

$$
\begin{aligned}
\operatorname{dist}(n, \partial A_k^+) &= \min\{\operatorname{dist}(n, \partial \Lambda_{6L_{k+1}}), \operatorname{dist}(n, \partial \Lambda_{3L_k})\} \\
&\geq \frac{1}{3} || n ||_\infty .
\end{aligned}
$$

$\square$

Step 2

Now, we investigate the probability that Λ_{L_k} is *not* (E, γ)-good and, at the same time, one of the L_k-cubes in A_k is also *not* (E, γ)-good.
Let us abbreviate

$$\mathcal{C}_k^+ = \mathcal{C}_{L_k}(A_k^+) = \{\Lambda_{L_k}(m) | \Lambda_{L_k}(m) \Subset A_k^+\} .$$

For a given k, define p_k to be the probability of the event

$$B_k = \left\{\omega \,\middle|\, \text{For some } E \in [E_1, E_2] , \ \Lambda_{L_k} \text{ and at least one cube in } \mathcal{C}_k^+ \text{ are } not \ (E, \gamma)\text{-good} \right\} .$$

We will prove

LEMMA 9.16. *If Result 9.8 holds for $I = [E_1, E_2]$, then there is a constant C such that for all k*

$$p_k \leq \frac{C}{L_k^{2p-\alpha d}} \; . \tag{9.39}$$

REMARK 9.17. The constants α and p are given in Result 9.8.

PROOF *(Lemma)*: If $\Lambda_{L_k}(m)$ is a fixed cube in $\mathcal{C}_k^+$ then

$$\mathbb{P}\left(\text{For some } E \in [E_1, E_2] \text{ both } \Lambda_{L_k}(m) \text{ and } \Lambda_{L_k} \text{ are } not \ (E, \gamma)\text{-good }\right)$$
$$\leq \frac{1}{L_k^{2p}} \; . \tag{9.40}$$

Hence

$$\mathbb{P}(B_k) \leq |\mathcal{C}_k^+| \frac{1}{L_k^{2p}} \quad \leq \quad C \, L_{k+1}^d \, \frac{1}{L_k^{2p}}$$
$$\leq \; C \, (L_k^\alpha)^d \, \frac{1}{L_k^{2p}} \; \leq \; \frac{C}{L_k^{2p-\alpha d}} \; . \tag{9.41}$$

$\square$

Since $\alpha < \frac{2p}{d}$ (by Result 9.8) we have $2p - \alpha d > 0$. Thus

$$\sum_k \mathbb{P}(B_k) \; < \; \infty \; . \tag{9.42}$$

Hence, by the Borel-Cantelli-Lemma (Theorem 3.6), we have

$$\mathbb{P}\left(\{\omega \,|\, \omega \in B_k \text{ for infinitely many } k \}\right) \; = \; 0 \; . \tag{9.43}$$

Thus we have shown

PROPOSITION 9.18. *If Result 9.8 holds for $I = [E_1, E_2]$, then for $\mathbb{P}$-almost all ω, there is a $k_0 = k_0(\omega)$ such that for all $k \geq k_0$:*
For any $E \in [E_1, E_2]$ either Λ_{L_k} is (E, γ)-good or all cubes $\Lambda_{L_k}(m)$ in $\mathcal{C}_{L_k}^+$ are (E, γ)-good.

Step 3

In this final step, we take ω such that the assertion of Proposition 9.18 is true.
Suppose now that $E \in [E_1, E_2]$ is a generalized eigenvalue. It follows from Theorem 9.2 that there is no sequence L_k' (with $L_k' \to \infty$) such that all $\Lambda_{L_k'}$ are (E, γ)-good. Hence by Proposition 9.18, we conclude that for *all $k > k_1$, all* cubes in $\mathcal{C}_k^+$ are (E, γ)-good.
Let ψ be a generalized eigenfunction corresponding to the generalized eigenvalue E. Take any $n \in \mathbb{Z}^d$ with $||n||_\infty$ large enough. Then there is a k, $k \geq k_1$, so

that $n \in A_k$ (hence $3L_k \leq ||n||_\infty < 6L_{k+1}$). It follows from Lemma 9.15 that $\text{dist}(n, \partial A_k^+) \geq \frac{1}{3}||n||_\infty$. Thus we may apply theorem 9.9 to conclude

$$|\psi(n)| \leq e^{-\gamma''||n||_\infty} \sup_{m \in A_k^+} |\psi(m)|. \qquad (9.44)$$

Since ψ is polynomially bounded by assumption, we have for $m \in A_k^+$ and for some r

$$\begin{aligned} |\psi(m)| &\leq C_0 \, (8L_{k+1})^r \\ &\leq C_1 \, L_k^{\alpha r} \\ &\leq C_2 \, ||n||_\infty^{\alpha r}. \end{aligned}$$

Thus

$$|\psi(n)| \leq e^{-\tilde{\gamma}||n||_\infty}. \qquad (9.45)$$

We have therefore shown that any generalized eigenfunction of H_ω with eigenvalues in $[E_1, E_2]$ decays exponentially fast. A fortiori, any generalized eigenfunction is ℓ^2, so the corresponding generalized eigenvalue is a bona fide eigenvalue. Thus, the spectrum in (E_1, E_2) is pure point.

$\square$

REMARK 9.19. Observe that eigenfunctions ψ_1, ψ_2 to different eigenvalues are orthogonal to each other. Since the Hilbert space $\ell^2(\mathbb{Z}^d)$ is separable, there are only countably many $E \in [E_1, E_2]$ with exponentially decaying eigensolutions.

Notes and Remarks

multiscale analysis is based on the ground breaking paper by Fröhlich and Spencer [47]. That the MSA result implies absence of a.c. spectrum was realized by Martinelli and Scoppolla [101]. An alternative appoach to exclude a.c. spectrum can be found in [125].

The first proofs of Anderson localization were given independently in [46], [37], [126]. The latter papers develop the method of spectral averaging which goes partly back to [89].

The method to prove Anderson localization we present above is due to [40] which is related to [46]. Germinet and Klein [50] investigate the relation between Localization and multiscale analysis in great detail. They characterize a certain version of localization in terms of the multiscale estimate.

For the literature on the continuous case, i.e. for Schrödinger operators on $L^2(\mathbb{R}^d)$, we refer to the Notes at the end of the next chapter.

10. Multiscale analysis

10.1. Strategy.

We turn to the proof of the multiscale analysis result.

Multiscale analysis (MSA) is an *induction* procedure which starts with a certain length scale L_0 and then proves the validity of the multiscale estimate (9.6 and 9.8) for $L_{k+1} = L_k^\alpha$ assuming the estimate holds for L_k. The value of α will be fixed later. To get an increasing sequence L_k we obviously need $\alpha > 1$. We will also choose $\alpha < 2$ for reasons that will become clear later. In fact, later we will have to choose α close to one.

In this chapter we will present the induction step (from L_k to L_{k+1}) deferring the initial step (for L_0) to the next chapter. The induction step can be done for all energies E and for arbitrary degree of disorder (provided there is *some* disorder, of course). Thus, it is the *initial step* which distinguishes between energy regions with pure point spectrum and those energies where we might have (absolutely) continuous spectrum. As explained in chapter 8, we expect certain energy regions with absolutely continuous spectrum, but are not (yet) able to prove it.

The proof of the induction step consists of an analytical and a probabilistic part. We start with analytic estimates.

For the rest of this chapter, we set for brevity $l = L_k$ and $L = L_{k+1}$, so we do the induction step from l to $L = l^\alpha$. By taking L_0 sufficiently large we can always assume that l and, a fortiori, L is big enough, i. e. bigger than a certain constant. Since $\alpha > 1$ we have $L \gg l$. Below, we will need that both l and L are integers. To ensure this we should actually choose L to be the smallest integer bigger or equal to l^α. We will neglect this point, it would complicate the notation. However, the reasoning of the proof remains the same.

The analytic estimate is a puzzle with different types of cubes. There are (small) cubes $\Lambda_l(r)$ of size $l = L_k$ and (big) cubes $\Lambda_L(m)$ of size $L = L_{k+1} = l^\alpha$. The goal is to prove that that the Green's function $(H_{\Lambda_L} - E)^{-1}(m, n)$ decays exponentially.

By induction hypothesis the probability that a small cube (of size l) is (γ, E)-good is very high. Thus, we expect that most of the small cubes $\Lambda_l(n)$ inside Λ_L are (γ, E)−good. Let us suppose for the moment, that actually *all* cubes of size l inside Λ_L are (γ, E)-good. Then, using the geometric resolvent identity (5.53) and iterating it just as we did in the proof of Theorem 9.9 will give us an estimate for the Green's function $G_E^{\Lambda_L}$ of the form

$$|G_E^{\Lambda_L}(n, m)| \leq e^{-\tilde{\gamma} k l} |G_E^{\Lambda_L}(n_k, m)|. \tag{10.1}$$

This estimate results from applying the geometric resolvent equation k times. This step can be iterated as long as the point n_k is not too close to the boundary of Λ_L (so that the cube of size l around n_k belongs to Λ_L) and the cube $\Lambda_l(n_k)$ is a (γ, E)-good cube. If all cubes of size l inside Λ_L are good, we expect that we can iterate roughly $\frac{L}{l}$ times before we reach the boundary and conclude

$$|G_E^{\Lambda_L}(n,m)| \;\leq\; e^{-\tilde\gamma L}\,|G_E^{\Lambda_L}(n',m)|. \tag{10.2}$$

We may *hope* that we can obtain an estimate of the type (10.2) even if not all l-cubes in Λ_L are good but, at least, an overwhelming majority of them is.

Once we have (10.2) we need a *rough* a priori bound on $G_E^{\Lambda_L}(n',m)$ to obtain the desired exponential estimate for $G_E^{\Lambda_L}(n,m)$, i. e. we need to know that Λ_L is not an *extremely* bad cube. We say that a cube is extremely bad, if it is *resonant* in the sense of the following definition.

DEFINITION 10.1. *We call a cube* $\Lambda_L(n)$ E-*resonant if* $\mathrm{dist}(E,\sigma(H_{\Lambda_L(n)})) < e^{-\sqrt{L}}$.

From Wegner's estimate (Theorem 5.23) we immediately learn that it is very unlikely (at least for large L) that a cube is E-resonant, in fact

PROPOSITION 10.2. *If the (single-site) measure P_0 has a bounded density, then*

$$\mathbb{P}(\Lambda_L(n) \text{ is } E\text{-resonant}) \;\leq\; C\,(2L+1)^d\, e^{-\sqrt{L}}\,. \tag{10.3}$$

If $\Lambda_L(n)$ is not E-resonant, we know that the Green's function $G_E^{\Lambda_L(n)}$ exists, because E is not in the spectrum. We even have a rough estimate on the Green's function which tells us that Λ_L is not 'extremely bad'.

PROPOSITION 10.3. *If the cube $\Lambda_L(n)$ is not E-resonant, then for all* $m, m' \in \Lambda_L(n)$

$$|G_E^{\Lambda_L(n)}(m,m')| \leq e^{\sqrt{L}}\,. \tag{10.4}$$

PROOF: If Λ_L is not E-resonant then

$$
\begin{aligned}
|G_E^{\Lambda_L}(m,m')| \;&=\; |(H_{\Lambda_L} - E)^{-1}(m,m')| \\
&\leq\; \|\,(H_{\Lambda_L} - E)^{-1}\| \\
&\leq\; \frac{1}{\mathrm{dist}(E,\sigma(H_{\Lambda_L}))} \\
&\leq\; e^{\sqrt{L}}\,.
\end{aligned}
\tag{10.5}
$$

$\square$

Thus, if the cube Λ_L is not resonant and if we have (10.2), we get an estimate of the form

$$
\begin{aligned}
|G_E^{\Lambda_L}(n,m)| \;&\leq\; e^{-\tilde\gamma L}\, e^{\sqrt{L}} \tag{10.6}\\
&\leq\; e^{-\gamma' L}\,. \tag{10.7}
\end{aligned}
$$

What we finally shall prove in (the analytical part of) the induction step is:

If an overwhelming majority of the cubes $\Lambda_l(m)$ in Λ_L is $(\gamma, E)-$good and Λ_L itself is not E-resonant, then Λ_L is $(\gamma', E)-$good.

Note that the exponential rates differ. In fact, $\gamma' < \gamma$. That is to say, we can not avoid to *decrease* the decay rate in each and every induction step. As a result we get a sequence of rates $\gamma_0, \gamma_1, \ldots$ (for induction step 0, 1, ...) . Of course, if $\gamma_n \to 0$ (or becomes negative) the whole result is pretty useless. So, we have to prove that $\gamma_n \searrow \gamma_\infty > 0$.

Once we have an analytic estimate of the above type, the induction step will be completed by a *probabilistic* estimate. We have to prove that with high probability most cubes $\Lambda_l(j)$ inside of Λ_L are (γ, E)-good and Λ_L is not E-resonant. This probability has to be bigger than $1 - L^{-p}$. To prove that most cubes $\Lambda_l(j)$ are good, we use the induction hypothesis. That Λ_L is not resonant with high probability follows from the Wegner estimate Theorem 5.23.

We have deliberately used the vague terms 'most cubes' and 'an overwhelming majority'. What they exactly mean is yet to be defined.

10.2. Analytic estimate - first try .

We start with a first attempt to do the analytic part of the induction step. This first try assumes that *all* cubes of size l inside Λ_L are $(\gamma, E)-$good. We recall that $\mathcal{C}_l(\Lambda_L) = \{\Lambda_l(m) \,|\, \Lambda_l(m) \Subset \Lambda_L\}$.

The main idea of the approach is already contained in the proof of Theorem 9.9.

PROPOSITION 10.4. *Suppose all cubes in $\mathcal{C}_l(\Lambda_L)$ are $(\gamma, E)-$good. Then for any $\bar{\gamma} < \gamma$ there is an l_0 such that for $l \geq l_0$*

$$|G_E^{\Lambda_L}(m,n)| = |(H_{\Lambda_L} - E)^{-1}(m,n)| \leq \frac{1}{\operatorname{dist}(E, \sigma(H_{\Lambda_L}))} \, e^{-\bar{\gamma} L} \qquad (10.8)$$

for any $m \in \Lambda_{L^{1/2}}$ and any $n \in \partial^- \Lambda_L$.

PROOF: Take $m \in \Lambda_{L^{1/2}}$. Since $\operatorname{dist}(m, \partial^- \Lambda_L) \geq l + 1$ if l_0 and hence l is large enough, we have $\Lambda_l(m) \in \mathcal{C}_l(\Lambda_L)$ and we may apply the geometric resolvent equation (5.53). Thus, we have

$$|G_E^{\Lambda_L}(m,n)| \leq \sum_{\substack{(q,q') \in \partial \Lambda_l(m) \\ q \in \Lambda_l(m)}} |G_E^{\Lambda_l(m)}(m,q)| \, |G_E^{\Lambda_L}(q',n)| \qquad (10.9)$$

$$\leq 2d \, (2l+1)^{d-1} \, e^{-\gamma l} \, |G_E^{\Lambda_L}(n_1,n)| \qquad (10.10)$$

$$\leq e^{-\tilde{\gamma} l} \, |G_E^{\Lambda_L}(n_1,n)| \qquad (10.11)$$

with

$$\tilde{\gamma} = \gamma - \frac{(d-1)\ln(2l+1)}{l} - \frac{\ln 2d}{l} \qquad (10.12)$$

for some $n_1 \in \partial^+ \Lambda_l(m)$.

If $\text{dist}(n_1, \partial^- \Lambda_L) \geq l + 1$, we may repeat this estimate with $\Lambda_l(m)$ replaced by $\Lambda_l(n_1)$ and obtain

$$|G_E^{\Lambda_L}(m, n)| \leq e^{-\tilde{\gamma} 2l} |G_E^{\Lambda_L}(n_2, n)|$$

with $n_2 \in \partial^+ \Lambda_l(n_1)$.

Note that $\text{dist}(n_1, \partial^- \Lambda_L) \geq L - \sqrt{L} - (l + 1)$, since $n_1 \in \partial^+ \Lambda_l(m)$.

So, the second estimation step is certainly possible if $L - \sqrt{L} - (l + 1) \geq l + 1$.

If this is so, we may try to iterate (10.9) a second time. This is possible if

$$L - \sqrt{L} - 2(l + 1) \geq l + 1$$

and the result is

$$|G_E^{\Lambda_L}(m, n)| \leq e^{-\tilde{\gamma} 3l} |G_E^{\Lambda_L}(n_3, n)| .$$

We may apply this procedure k times as long as $L - \sqrt{L} - k(l + 1) \geq l + 1$, i. e. for

$$k \leq \frac{L}{l + 1} - \frac{\sqrt{L}}{l + 1} - 1 . \tag{10.13}$$

The largest integer k_0 satisfying (10.13) is at least

$$k_0 \geq \frac{L}{l + 1} - \frac{\sqrt{L}}{l + 1} - 2 . \tag{10.14}$$

Consequently, we obtain

$$
\begin{aligned}
|G_E^{\Lambda_L}(m, n)| &\leq e^{-\tilde{\gamma} k_0 l} |G_E^{\Lambda_L}(n_{k_0}, n)| \\
&\leq \|(H_{\Lambda_L} - E)^{-1}\| \; e^{-\tilde{\gamma} k_0 l} \\
&= \frac{1}{\text{dist}(E, \sigma(H_\Lambda))} e^{-\tilde{\gamma} k_0 l} .
\end{aligned}
\tag{10.15}
$$

As long as $\tilde{\gamma} > 0$, we have

$$
\begin{aligned}
e^{-\tilde{\gamma} k_0 l} &\leq e^{-\tilde{\gamma}(L \frac{l}{l+1} - \sqrt{L} \frac{l}{l+1} - 2l)} \\
&= e^{-\tilde{\gamma}(\frac{l}{l+1} - \frac{1}{\sqrt{L}} \frac{l}{l+1} - 2\frac{l}{L})L} \\
&\leq e^{-\tilde{\gamma}(1 - \frac{1}{l+1} - \frac{1}{\sqrt{L}} \frac{l}{l+1} - 2\frac{l}{L})L} \\
&\leq e^{-\tilde{\gamma}(1 - \frac{1}{l} - \frac{1}{l^{\alpha/2}} - 2\frac{1}{l^{\alpha-1}})L} .
\end{aligned}
\tag{10.16}
$$

So, estimate (10.8) holds if

$$\left(\gamma - \frac{(d - 1) \ln(2l + 1)}{l} - \frac{2d}{l} \right) \left(1 - \frac{1}{l} - \frac{1}{l^{\alpha/2}} - 2\frac{1}{l^{\alpha-1}} \right) \geq \bar{\gamma} . \tag{10.17}$$

By taking l large enough we can assure that (10.17) holds. $\qquad \square$

If we assume that Λ_L is not E-resonant (see Definition 10.1), we can further estimate expression (10.8).

THEOREM 10.5. *If the cube Λ_L is not E-resonant and if all the cubes in $\mathcal{C}_l(\Lambda_L)$ are $(\gamma, E)-good$ and $\gamma' < \gamma$, then*

$$\Lambda_L \text{ is } (\gamma', E)-\text{good}$$

if l is large enough.

PROOF: By (10.8) and the assumption that Λ_L is not resonant (see 10.5) we obtain

$$
\begin{aligned}
|G_E^{\Lambda_L}(m, n)| &\leq e^{-\bar{\gamma} L} e^{L^{1/2}} \\
&\leq e^{-\gamma' L} \qquad\qquad (10.18)
\end{aligned}
$$

with $\gamma' = \bar{\gamma} - \frac{1}{l^{\alpha/2}}$. $\qquad\qquad\square$

COROLLARY 10.6. *If the cube Λ_L is not E-resonant and if all the cubes in $\mathcal{C}_l(\Lambda_L)$ are $(\gamma, E)-good$, then Λ_L is $(\gamma', E)-good$ with*

$$\gamma' \geq \gamma \left(1 - \frac{4}{l^{\alpha-1}}\right) - \left(\frac{3d \ln(2l+1)}{l} + \frac{1}{l^{\alpha/2}}\right). \qquad (10.19)$$

Moreover, for $l \geq C_0$, with C_0 depending only on α and the dimension d, we have

$$\gamma' \geq \gamma \left(1 - \frac{4}{l^{\alpha-1}}\right) - \frac{2}{l^{\alpha/2}}. \qquad (10.20)$$

PROOF: Estimate (10.19) follows from (10.17), (10.18) and the observation that

$$\alpha - 1 < \alpha/2 < 1 \qquad\qquad (10.21)$$

since $1 < \alpha < 2$.

Moreover, there is a constant $C_0 = C_0(\alpha, d)$ such that for $l \geq C_0$ we have $\frac{3d \ln(2l+1)}{l} \leq \frac{1}{l^{\alpha/2}}$ which implies (10.20). $\qquad\qquad\square$

An obvious problem with the above result is the fact that we have to decrease the rate γ of the exponential decay in each induction step. Suppose we start with a rate γ_0 for length scale L_0. Let us assume $L_0 \geq C_0$, the constant appearing before (10.20). We call $\gamma_k \leq \gamma_0$ the decay rate we obtain from Theorem 10.5 and Corollary 10.6 in the kth step, i. e. for $L_k = (L_{k-1})^\alpha$.
We get the lower bound

$$
\begin{aligned}
\gamma_{k+1} &\geq \gamma_k - \gamma_k \frac{4}{L_k^{\alpha-1}} - \frac{2}{L_k^{\alpha/2}} \qquad\qquad (10.22) \\
&\geq \gamma_k - \gamma_0 \frac{4}{L_k^{\alpha-1}} - \frac{2}{L_k^{\alpha/2}}. \qquad\qquad (10.23)
\end{aligned}
$$

Thus

$$\gamma_\infty = \liminf \gamma_k \geq \gamma_0 - \gamma_0 \sum_{k=0}^{\infty} \frac{4}{L_k^{\alpha-1}} - \sum_{k=0}^{\infty} \frac{2}{L_k^{\alpha/2}}. \qquad (10.24)$$

To estimate the right hand side of (10.24), we use the following lemma.

LEMMA 10.7. *For $\beta > 0$ and L_0 large enough we have*

$$\sum_{k=0}^{\infty} \frac{1}{L_k{}^\beta} \leq \frac{2}{L_0{}^\beta} . \tag{10.25}$$

REMARK 10.8. In the lemma L_0 large means: $L_0{}^{\beta(\alpha-1)} \geq 2$.

PROOF:

$$r_k \quad := \quad \frac{1}{L_k^\beta} \leq \frac{1}{\left(L_0{}^{\alpha^k}\right)^\beta} \leq \frac{1}{\left(L_0^\beta\right)^{\alpha^k}}$$

$$\leq \quad \frac{1}{\left(L_0^\beta\right)^{1+k(\alpha-1)}} \leq \frac{1}{L_0^\beta} \left(\frac{1}{L_0^{\beta(\alpha-1)}}\right)^k . \tag{10.26}$$

Above, we used $\alpha^k \geq 1 + k(\alpha - 1)$.
From these estimates we obtain for $L_0{}^{\beta(\alpha-1)} \geq 2$

$$r := \sum_{k=0}^{\infty} r_k \leq \frac{1}{1 - L_0{}^{-\beta(\alpha-1)}} \frac{1}{L_0^\beta} \leq \frac{2}{L_0^\beta} . \tag{10.27}$$

$\square$

From this lemma we learn that the 'final' decay rate γ_∞ is positive if L_0 and γ_0 are not too small, more precisely:

PROPOSITION 10.9. *If L_0 is big enough and*

$$\gamma_0 \geq \frac{16}{L_0{}^{\alpha/2}} \tag{10.28}$$

then

$$\gamma_\infty = \inf \gamma_k \geq \frac{1}{2} \gamma_0 . \tag{10.29}$$

REMARK 10.10. L_0 big enough means

$$L_0{}^{\alpha-1} \geq 32 \qquad \text{and} \qquad L_0{}^{(\alpha-1)^2} \geq 2 . \tag{10.30}$$

PROOF: Since $\alpha < 2$, we know $\frac{\alpha}{2} \geq (\alpha - 1)$. So, if $L_0{}^{(\alpha-1)^2} \geq 2$, by Lemma 10.7 we have

$$\sum_{k=0}^{\infty} \frac{1}{L_k{}^{\alpha/2}} \leq \frac{2}{L_0{}^{\alpha/2}} \tag{10.31}$$

and

$$\sum_{k=0}^{\infty} \frac{1}{L_k{}^{\alpha-1}} \leq \frac{2}{L_0{}^{\alpha-1}} . \tag{10.32}$$

Thus, (10.28) and (10.30) inserted in (10.24) give

$$\gamma_\infty \geq \gamma_0 - \frac{8}{L_0{}^{\alpha-1}}\gamma_0 - \frac{4}{L_0{}^{\alpha/2}} \tag{10.33}$$

$$\geq \frac{3}{4}\gamma_0 - \frac{1}{4}\gamma_0 = \frac{1}{2}\gamma_0 . \tag{10.34}$$

$$\square$$

Let us pause to summarize what we have done so far.

THEOREM 10.11. *Define the length scale* $L_{k+1} = L_k{}^\alpha$ *with* $1 < \alpha < 2$ *and a suitable* L_0*, which is not too small.*
If for a certain k

    (1) *all the cubes in* $\mathcal{C}_{L_k}(\Lambda_{L_{k+1}})$ *are* (γ_k, E)*- good and*
    (2) *the cube* $\Lambda_{L_{k+1}}$ *is not* E*-resonant*

then the cube $\Lambda_{L_{k+1}}$ *is* (γ_{k+1}, E)*- good with a rate* γ_{k+1} *satisfying*

$$\gamma_{k+1} \geq \gamma_k - \gamma_k \frac{4}{L_k{}^{\alpha-1}} - \frac{2}{L_k{}^{\alpha/2}} . \tag{10.35}$$

Moreover, we have some control on the sequence γ_k.

COROLLARY 10.12. *If the initial rate* γ_0 *satisfies* $\gamma_0 \geq \frac{16}{L_0{}^{\alpha/2}}$ *and* L_0 *is large enough, then the* γ_k *(as in (10.35)) satisfy* $\gamma_k \geq \frac{\gamma_0}{2}$ *for all* k.

Thus, we have done a first version of the analytic part of the MSA-proof. So far for the good news about Theorem 10.11.

We are left with the probabilistic estimates, namely:
Prove that if Λ_{L_k} is good with high probability then the hypothesis' (1) and (2) in Theorem 10.11 above are true with high probability. More precisely, we *would like* to prove:
If

$$\mathbb{P}(\Lambda_l \text{ is not } (\gamma, E)-\text{good}) \leq \frac{1}{l^p}$$

then

$$\mathbb{P}(\Lambda_L \text{ is not } (\gamma, E)-\text{good}) \leq \frac{1}{L^p} \tag{10.36}$$

with $L = l^\alpha$.

Here comes the bad news: There is *no chance* for such an estimate.
In fact, Theorem 10.11 allows us to estimate

    $\mathbb{P}(\Lambda_L \text{ is not } (\gamma, E)-\text{good})$

$\leq \mathbb{P}(\Lambda_L \text{ is not } E\text{-resonant}) + \mathbb{P}(\text{at least one cube in } \mathcal{C}_l(\Lambda_L) \text{ is not } (\gamma, E)-\text{good}) .$
$$\tag{10.37}$$

The first term in (10.37) can be estimated by the Wegner estimate (5.23). However the second term is certainly *bigger* than $\mathbb{P}(\Lambda_L(0)$ is not $(\gamma, E)-$good). The only estimate we have for this is $\frac{1}{l^p}$. So the best we can possibly hope for is an estimate like

$$\mathbb{P}(\Lambda_L \text{ is not } (\gamma, E)-\text{good}) \leq \frac{1}{l^p} = \frac{1}{L^{p/\alpha}} . \tag{10.38}$$

This is much worse than estimate (10.36).

What goes wrong here is that the probability that *all* small cubes are good is too small. Consequently, we have to accept at least one or even a few cubes in $\mathcal{C}_l(\Lambda_L)$ which are not $(\gamma, E)-$good. Dealing with bad cubes in $\mathcal{C}_l(\Lambda_L)$ requires a refined version of the above analytic reasoning.

10.3. Analytic estimate - second try.

Now, we try to do the induction step allowing a few bad cubes in $\mathcal{C}_l(\Lambda_L)$. We start with just *one* bad cube. More precisely, we suppose now that $\mathcal{C}_l(\Lambda_L)$ does not contain two disjoint cubes which are not $(\gamma, E)-$good.

If two cubes overlap, events connected with these cubes are *not* independent, so probability estimates are hard in this case. That is why we insist above on non overlapping sets.

The above assumption implies that there is an $m_0 \in \Lambda_L$ such that all the cubes $\Lambda_l(m) \in \mathcal{C}_l(\Lambda_L)$ with $\|m - m_0\|_\infty > 2l$ are $(\gamma, E)-$good. Consequently, there are no bad cubes with centers outside $\Lambda_{2l}(m_0)$. The cube $\Lambda_{2l}(m_0)$ is the 'dangerous' region which requires special care.

As in the proof of Proposition 10.4, we use and iterate the geometric resolvent equation to estimate

$$|G_E^{\Lambda_L}(m, n)| \leq e^{-\tilde{\gamma} lr} |G_E^{\Lambda_L}(n_r, n)| \tag{10.39}$$

as long as possible. With a bad cube inside Λ_L, this procedure can stop not only when n_r is near the boundary of Λ_L but also if n_r reaches the problematic region around m_0 where cubes $\Lambda_l(m)$ might be bad.

Let us concentrate for a moment how we can handle sites n_r inside the dangerous region $\Lambda_{2l}(m_0)$. So, suppose that $u := n_r \in \Lambda_{2l}(m_0)$. Hence we can*not* be sure the cube $\Lambda_l(u)$ is good. We can still try to apply the geometric resolvent equation and obtain

$$|G_E^{\Lambda_L}(u, n)| \leq \sum_{\substack{(q,q')\in\partial\Lambda_l(u) \\ q \in \Lambda_l(u)}} |G_E^{\Lambda_l(u)}(u, q)| \, |G_E^{\Lambda_L}(q', n)| . \tag{10.40}$$

If we assume *nothing* about the cube $\Lambda_l(u)$, there is no chance to estimate $G_E^{\Lambda_l(u)}(u, q)$. In fact, this Green's function may be arbitrarily large or even non existing. It seems reasonable to suppose that the 'trouble making' region, the cube $\Lambda_{2l}(m_0)$, is 'not completely bad' in the sense, that $\Lambda_{2l}(m_0)$ is not E-resonant. This allows us to estimate

$$|G_E^{\Lambda_L}(u,n)| \quad \le \sum_{\substack{(q,q')\in\partial\Lambda_{2l}(m_0) \\ q\in\Lambda_{2l}(m_0)}} |G_E^{\Lambda_{2l}(m_0)}(u,q)|\,|G_E^{\Lambda_L}(q',n)|$$

$$\le \quad 2d\,(4l+1)^{d-1}\,e^{\sqrt{2l}}\,|G_E^{\Lambda_L}(u',n)| \tag{10.41}$$

for a $u' \in \Lambda_L\backslash\Lambda_{2l}(m_0)$.

Observe, that the cube $\Lambda_l(u')$ is $(\gamma_l, E)-$good by induction hypothesis since $u' \notin \Lambda_{2l}(m_0)$. Therefore, the *next* iteration of the geometric resolvent estimate will give us an *exponentially decreasing* term

$$|G_E^{\Lambda_L}(u,n)| \le \; 2d\,(4l+1)^{d-1}\,e^{\sqrt{2l}}\,|G_E^{\Lambda_L}(u',n)|$$

$$\le \quad (2d)^2\,(4l+1)^{d-1}\,(2l+1)^{d-1}\,e^{\sqrt{2l}}\,e^{-\gamma_l\,l}\,|G_E^{\Lambda_L}(n_{r+1},n)|\,. \tag{10.42}$$

In the double step (10.41) and (10.42), we pick up a factor

$$\rho := (2d)^2\,(4l+1)^{d-1}\,(2l+1)^{d-1}\,e^{\sqrt{2l}}\,e^{-\gamma_l\,l}\,. \tag{10.43}$$

The second step (10.42) compensates the first one (10.41) if $\rho \le 1$. This is the case if

$$\gamma_l \ge \frac{\sqrt{2}}{\sqrt{l}} + \frac{2\,\ln(2d)\,+2\,(d-1)\,\ln(4l+1)}{l} \tag{10.44}$$

which is fulfilled for

$$\gamma_l \ge \frac{2}{\sqrt{l}} \tag{10.45}$$

if l is bigger than a constant depending only on the dimension.

In the proof of Theorem 10.5, we could choose (see (10.22))

$$\gamma_{k+1} \ge \gamma_k - \gamma_k\,\frac{4}{L_k^{\alpha-1}} - \frac{2}{L_k^{\alpha/2}}\,. \tag{10.46}$$

An induction argument using (10.46) shows

LEMMA 10.13. *If $L_0 \ge M$, a constant depending only on α and d, and if (10.46) holds, then $\gamma_0 \ge \frac{2}{L_0^{1/2}}$ implies*

$$\gamma_k \;\ge\; \frac{2}{L_k^{1/2}} \qquad \text{for all } k\,. \tag{10.47}$$

This Lemma ensures that we can iterate the induction step in the multiscale analysis even if we hit the dangerous region $\Lambda_{2l}(m_0)$. In fact, once we start with $\gamma_0 \ge \frac{2}{L_0^{1/2}}$, we can be sure that all the the rates satisfy the condition $\gamma_k \ge \frac{2}{L_k^{1/2}}$.

PROOF: By taking L_0 large enough we can ensure that:

$$\frac{4}{L_k^{\alpha-1}} \;\le\; \frac{1}{2} \quad \text{and} \quad \frac{4}{L_k^{\alpha/2}} \;\le\; \frac{1}{L_k^{1/2}} \qquad \text{for all } k\,. \tag{10.48}$$

So, if $\gamma_k \geq \frac{2}{L_k^{1/2}}$, then

$$\gamma_{k+1} \geq \gamma_k \left(1 - \frac{4}{L_k^{\alpha-1}}\right) - \frac{2}{L_k^{\alpha/2}}$$

$$\geq \frac{1}{2}\gamma_k - \frac{2}{L_k^{\alpha/2}}$$

$$\geq \frac{1}{L_k^{1/2}} - \frac{2}{L_k^{\alpha/2}}$$

$$\geq \frac{2}{L_k^{\alpha/2}}$$

$$\geq \frac{2}{L_{k+1}^{1/2}}$$

Thus, the Lemma follows by induction. $\qquad\square$

Knowing how to deal with the cubes inside $\Lambda_{2l}(m_0)$, we now sketch our strategy. We use the geometric resolvent equation to estimate the resolvent on the big cube of size L in terms of the resolvent of small cubes of size l. As long as the first argument n_r of the Green's function (for Λ_L) belongs to a good cube, we use an exponential bound as in (10.11). If n_r belongs to the 'bad' region which may contain cubes that are not good, then we do the double step estimate (10.41) and (10.42). This procedure can be repeated until we get close to the boundary of Λ_L. The number of times we do the exponential bound in this procedure is at least of the order L/l. In fact, analogously to (10.13) the number k_0 of 'good' steps is at least

$$k_0 \geq \frac{L}{l+1} - \frac{\sqrt{L}}{l+1} - C_1 \ . \tag{10.49}$$

Consequently, the estimates of the previous section can be redone if we allow 'one' bad cube with the following changes

- We need $L_0 \geq C_2$ with a constant C_2 (possibly) bigger than the previous one.
- We have to take $\gamma_0 \geq 2\,L_0^{-1/2}$
- The procedure requires that all cubes of size $2l$ inside Λ_L are non resonant. While we need this only for the cube $\Lambda_{2l}(m_0)$ around the 'bad' cube, we do not know, where the bad cube is, so we require non resonance for all cubes of the appropriate size.

Thus, we have shown the following improvement of Theorem 10.11.

THEOREM 10.14. *Suppose L_0 is large enough and $L_{k+1} = L_k{}^\alpha$ with $1 < \alpha < 2$. If for a certain k ($l := L_k$ and $L := L_{k+1}$)*

(1) *there do not exist two disjoint cubes in $\mathcal{C}_l(\Lambda_L)$ which are not (γ_k, E)-good with a rate $\gamma_k \geq \frac{2}{l^{1/2}}$,*

(2) *no cube $\Lambda_{2l}(m)$ in Λ_L is E-resonant and*

(3) *the cube Λ_L is not E-resonant,*

then the cube Λ_L is (γ_{k+1}, E)- good with a rate γ_{k+1} satisfying $\gamma_{k+1} \geq \frac{2}{L^{1/2}}$.
Moreover we can choose the rate γ_{k+1} such that

$$\gamma_{k+1} \geq \gamma_k - \gamma_k \frac{C}{{L_k}^{\alpha-1}} - \frac{C}{{L_k}^{\alpha/2}} . \qquad (10.50)$$

As above, we can estimate the decay rates as follows.

COROLLARY 10.15. *If the initial rate γ_0 satisfies $\gamma_0 \geq \frac{C}{{L_0}^{1/2}}$ and L_0 is large enough, then the γ_k in Theorem 10.14 satisfy $\gamma_k \geq \frac{\gamma_0}{2}$ for all k.*

This result allows us to prove the multiscale estimate in its weak form (9.6) as we will show in the next section 10.4 where we do the corresponding probabilistic estimates.

The above analytic results (especially the counterpart of Theorem 10.14) can be shown for the strong version (Result 9.8) as well with not too much difficulties. Unfortunately, the probabilistic estimate breaks down for the strong form, as we will discuss below. To make the probabilistic part of the argument work for the strong case, we have to allow more than just one bad l-cube inside the L-cubes. In Section 10.6, we show how to deal with this problem.

10.4. Probabilistic estimates - weak form.

We turn to the probablistic estimates of the induction step in multiscale analysis. Here, we will prove the multiscale result in its weak form (Result 9.6).

In the whole section we assume that the probability distribution P_0 of the independent, identically distributed random variables $V_\omega(i)$ has a *bounded density*, i. e.

$$P_0(A) \quad := \quad \mathbb{P}(V_\omega(i) \in A) = \int_A g(\lambda)d\lambda,$$
$$\text{with} \quad ||g||_\infty = \sup_\lambda |g(\lambda)| < \infty). \qquad (10.51)$$

This condition is assumed throughout this section even when not explicitly stated. The main result is

THEOREM 10.16. *Assume that the probability distribution P_0 has a bounded density. Suppose L_0 is large enough, $\gamma \geq \frac{1}{{L_0}^{1/2}}$, $p > 2d$ and $1 < \alpha < \frac{2p}{p+2d}$. If*

$$\mathbb{P}(\Lambda_{L_0} \text{ is not } (2\gamma, E)-\text{good}) \leq \frac{1}{{L_0}^p}, \qquad (10.52)$$

then for all k

$$\mathbb{P}(\Lambda_{L_k} \text{ is not } (\gamma, E)-\text{good}) \leq \frac{1}{{L_k}^p} . \qquad (10.53)$$

REMARK 10.17.

- Note that $p > 2d$ ensures that we can choose $\alpha > 1$.

- We need the assumption (10.51) on P_0 (only) in order to have the Wegner estimate (Theorem 5.23).

This theorem reduces the multiscale analysis to the initial scale estimate (10.52) which we discuss in chapter 11. As we remarked above, Theorem 10.16 is proved by induction. Thus, under the assumptions of Theorem 10.16 and with the rates γ_k as in Theorem 10.14, we have to prove the following theorem.

THEOREM 10.18. *If*

$$\mathbb{P}(\Lambda_{L_k} \text{ is not } (\gamma_k, E)-\text{good}) \leq \frac{1}{L_k^p} \,, \tag{10.54}$$

then

$$\mathbb{P}(\Lambda_{L_{k+1}} \text{ is not } (\gamma_{k+1}, E)-\text{good}) \leq \frac{1}{L_{k+1}^p} \,. \tag{10.55}$$

PROOF: As usual, we set $l = L_k$, $L = L_{k+1}$ and $\gamma = \gamma_k$, $\gamma' = \gamma_{k+1}$. To prove Theorem 10.18, we use Theorem 10.14 to estimate

$$\begin{aligned}
\mathbb{P}(\; \Lambda_L \text{ is not } (\gamma', E)-\text{good} \;) & \\
\leq \quad & \mathbb{P}(\; \Lambda_L \text{ is } E-\text{resonant} \;) & (10.56)\\
& + \mathbb{P}(\; \text{One of the cubes } \Lambda_{2l}(m) \subset \Lambda_L \text{ is } E-\text{resonant} \;) & (10.57)\\
& + \mathbb{P}(\; \text{There are two disjoint cubes in } \mathcal{C}_l(\Lambda_L) & \\
& \quad \text{which are not } (\gamma, E)-\text{good} \;) \,. & (10.58)
\end{aligned}$$

Both (10.56) and (10.57) can be bounded using the Wegner estimate (Theorem 5.23)

$$\begin{aligned}
\mathbb{P}(\; \Lambda_L \text{ is } E-\text{resonant} \;) \quad & \leq \quad (2L+1)^d e^{-\sqrt{L}} \\
& \leq \quad \frac{1}{3}\frac{1}{L^p} & (10.59)
\end{aligned}$$

provided L is large enough, and

$$\begin{aligned}
\mathbb{P}(\; \text{One of the cubes } \Lambda_{2l}(m) \subset \Lambda_L \text{ is } E-\text{resonant} \;) & & (10.60)\\
\leq \quad (2L+1)^d \; \mathbb{P}(\; \text{The cube } \Lambda_{2l}(0) \text{ is } E-\text{resonant} \;) & & (10.61)\\
\leq \quad (2L+1)^d \, (4l+1)^d \; e^{-\sqrt{2l}} & & \\
\leq \quad (2L+1)^d \, (4L^{1/\alpha}+1)^d \; e^{-\sqrt{2}\,L^{\frac{1}{2\alpha}}} & & \\
\leq \quad \frac{1}{3}\frac{1}{L^p} & & (10.62)
\end{aligned}$$

if L is large enough.

Using the induction hypothesis (10.54), we can estimate the term (10.58) by

$$\sum_{\substack{i,j\in\Lambda_L \\ \Lambda_l(i)\cap\Lambda_l(j)=\emptyset}} \mathbb{P}\left(\Lambda_l(i) \text{ and } \Lambda_l(j) \text{ are both not } (\gamma, E)\text{-good}\right)$$

$$\leq \sum_{i,j\in\Lambda_L} \mathbb{P}\left(\Lambda_l(i) \text{ is not } (\gamma, E)\text{-good}\right) \mathbb{P}\left(\Lambda_l(j) \text{ is not } (\gamma, E)\text{-good}\right)$$

$$\leq (2L+1)^{2d} \frac{1}{l^{2p}}$$

$$\leq \frac{C}{L^{\frac{2p}{\alpha}-2d}}$$

$$\leq \frac{1}{3}\frac{1}{L^p} \tag{10.63}$$

provided L is large.

We used above that $\alpha < \frac{2p}{p+2d}$ implies $\frac{2p}{\alpha} - 2d > p$.

Summing up, we get

$$\mathbb{P}\left(\Lambda_L \text{ is not } (\gamma', E)-\text{good}\right) \leq \frac{1}{L^p}.$$

$\square$

10.5. Towards the strong form of the multiscale analyis.

When we try to prove the 'uniform' Result 9.8, i.e. the strong form of the multiscale estimate, we may proceed in the same manner as above for awhile. Let us suppose we consider two disjoint cubes $\Lambda_1 = \Lambda_L(n)$ and $\Lambda_2 = \Lambda_L(m)$. We want to prove

$$\mathbb{P}\left(\text{For some } E \in I \text{ both}\Lambda_1 \text{ and } \Lambda_2 \text{ are not } (\gamma', E)-\text{good}\right) \leq L^{-2p}. \tag{10.64}$$

We set

$$\begin{aligned}
A_1(E) &= \{\Lambda_1 \text{ is not } (\gamma', E)-\text{good}\} \\
R_1(E) &= \{\Lambda_1 \text{ or a cube in } \mathcal{C}_{2l}(\Lambda_1) \text{ is not } E\text{-resonant}\} \tag{10.65} \\
B_1(E) &= \{\mathcal{C}_l(\Lambda_1) \text{ contains two disjoint cubes which are not } (\gamma, E)-\text{good}\}.
\end{aligned}$$

We define $A_2(E), R_2(E), B_2(E)$ analogously for the cube Λ_2.

The event we are interested in (see 10.64) can be expressed through $A_1(E), A_2(E)$, namely

$$\{\exists_{E\in I} \text{ such that } \Lambda_1 \text{ and } \Lambda_2 \text{ are not } (\gamma', E)-\text{good}\} \tag{10.66}$$

$$= \bigcup_{E\in I} \left(A_1(E) \cap A_2(E)\right). \tag{10.67}$$

Theorem 10.14 implies that

$$\mathbb{P}\left(\bigcup_{E\in I}\left(A_1(E)\cap A_2(E)\right)\right)$$

$$\leq \quad \mathbb{P}\left(\bigcup_{E\in I}\left(R_1(E)\cap R_2(E)\right)\right) \qquad (10.68)$$

$$+ \quad \mathbb{P}\left(\bigcup_{E\in I}\left(B_1(E)\cap B_2(E)\right)\right) \qquad (10.69)$$

$$+ \quad \mathbb{P}\left(\bigcup_{E\in I}\left(R_1(E)\cap B_2(E)\right)\right) \qquad (10.70)$$

$$+ \quad \mathbb{P}\left(\bigcup_{E\in I}\left(B_1(E)\cap R_2(E)\right)\right). \qquad (10.71)$$

The term (10.68) can be estimated using the 'uniform' Wegner estimate (Theorem 5.27) and (10.69) will be handled using the induction hypothesis. It turns out that the critical terms are the mixed ones (10.70) and (10.71).

The only effective way we know to estimate (10.71) is

$$\mathbb{P}\left(\bigcup_{E\in I}\left(B_1(E)\cap R_2(E)\right)\right) \quad \leq \quad \mathbb{P}\left(\bigcup_{E\in I} B_1(E)\right)$$

$$\leq \quad L^{2d}\,\frac{1}{l^{2p}}$$

$$\leq \quad \frac{1}{L^{2p/\alpha-2d}} \qquad (10.72)$$

where we used the induction hypothesis and the fact that there are at most L^{2d} disjoint cubes of side length l in Λ_1.

Observe that the term $\bigcup_{E\in I} R_2(E)$ which we neglected above does not have small probability as long as there is spectrum inside I.

Since we need $\alpha > 1$, the exponent in (10.72) is certainly *less* than $2p$. Consequently there is no way to do the induction step the way we tried above. The induction step would require that (10.72) is less than $\frac{1}{L^{2p}}$.

Observe that the situation is completely analogous to the one in Section 10.2 (see (10.38)). There we needed to allow more (namely one) bad cubes. The same idea remedies the present situation: We have to accept 'three' bad cubes, as will be explained in the next section.

10.6. Estimates - third try .

In a third round, we accept 'three' bad cubes. More precisely: We assume that the cube Λ_L does not contain four disjoint cubes of side length l which are not (γ, E)-good. Then, there are (at most) three cubes, $\Lambda_{2l}(m_1), \Lambda_{2l}(m_2), \Lambda_{2l}(m_3) \subset \Lambda_L$, such that there are no bad cubes outside $M = \bigcup_{\nu=1}^{3} \Lambda_{2l}(m_\nu)$.

As in Section 10.3, we use the geometric resolvent equation and an exponential bound on the Green's function as long as we do not enter one of the $\Lambda_{2l}(m_\nu)$. Once we enter such a set, we would like to use the geometric resolvent equation in

connection with a Wegner-type bound for $\Lambda_{2l}(m_\nu)$ as in the following expression for $u \in \Lambda_{2l}(m_\nu)$:

$$| G_E^{\Lambda_L}(u,n)| \;\; \leq \;\; \sum_{\substack{(q,q') \in \partial \Lambda_{2l}(m_\nu) \\ q \in \Lambda_{2l}(m_\nu)}} | G_E^{\Lambda_{2l}(m_\nu)}(u,q)| \, | G_E^{\Lambda_L}(q',n)| \; . \quad (10.73)$$

If we assume that $\Lambda_{2l}(m_\nu)$ is not E-resonant we can estimate the first term on the right hand side of (10.73) by $e^{\sqrt{2l}}$. If the site q' is the center of a good cube, we may estimate the second term $G_E^{\Lambda_L}(q',n)$ by applying the geometric resolvent equation for the cube $\Lambda_l(q')$ and using the exponential bound for this cube. However, it is not guaranteed that $\Lambda_l(q')$ is (γ, E)-good. q' could belong to one of the other 'dangerous' cubes $\Lambda_{2l}(m_{\nu'})$. The problem here is that two (or all three) of these cubes could touch or intersect.

To get rid of this problem, we redefine the 'dangerous' regions where we use the Wegner bound instead of the exponential bound. We say that two subsets A and B of $\mathbb{Z}^d$ *touch* if $A \cap B \neq \emptyset$ or if there are points $x \in A$ and $y \in B$ such that $||x - y||_\infty = 1$.

As before we use the geometric resolvent equation iteratively to estimate the Green's function $G_E^{\Lambda_L}$. We define sets M_1, M_2, M_3 - the dangerous regions - where we use the Wegner estimate, i. e. we will assume that the M_i are not E-resonant. We construct the M_i in such a way that for all sites x outside the M_i, the cube $\Lambda_l(x)$ is (γ, E)-good. Moreover, any two of the M_i do not touch.

If the cubes $\Lambda_{2l}(m_\nu)$ do not touch each other, we set $M_\nu = \Lambda_{2l}(m_\nu)$.

If two of the $\Lambda_{2l}(m_\nu)$ touch, say $\Lambda_{2l}(m_1)$ and $\Lambda_{2l}(m_2)$, we set $M' = \Lambda_{6l+1}(m_1)$. Then $\Lambda_{2l}(m_1) \cup \Lambda_{2l}(m_2) \subset M'$. Indeed, if $\Lambda_{2l}(m_1)$ and $\Lambda_{2l}(m_2)$ touch, there are points $x \in \Lambda_{2l}(m_1)$ and $y \in \Lambda_{2l}(m_2)$ with $||x - y||_\infty \leq 1$. If $z \in \Lambda_{2l}(m_2)$ we have

$$\begin{aligned} ||z - m_1||_\infty \;&\leq ||z - m_2||_\infty + ||m_2 - y||_\infty + ||y - x||_\infty + ||x - m_1||_\infty \\ &\leq 6\,l + 1 \; . \end{aligned} \quad (10.74)$$

If M' and $\Lambda_{2l}(m_3)$ do not touch we set $M_1 = M'$ and $M_2 = \Lambda_{2l}(m_3)$ (The set M_3 is not needed, we may formally set $M_3 = \emptyset$.). If M' and $\Lambda_{2l}(M_3)$ do touch then $M', \Lambda_{2l}(m_3) \subset \Lambda_{10l+2}(m_1)$ which is shown by a calculation analogous to (10.74). In this case, we set $M_1 = \Lambda_{10l+2}(m_1)$ and $M_2 = M_3 = \emptyset$.

We have shown

LEMMA 10.19. *If there are not four disjoint cubes in $\mathcal{C}_l(\Lambda_L)$ which are not (γ, E)-good, then either*

- *There are three cubes $M_1, M_2, M_3 \in \mathcal{C}_{2l}(\Lambda_L)$ which do not touch and such that any cube in $\mathcal{C}_l(\Lambda_L)$ with center outside the M_i is (γ, E)-good, or*

- *There is a cube $M_1 \in \mathcal{C}_{6l+1}(\Lambda_L)$ and a cube $M_2 \in \mathcal{C}_{2l}(\Lambda_L)$ which do not touch such that any cube in $\mathcal{C}_l(\Lambda_L)$ with center outside the M_i is (γ, E)-good,*

  or

- *There is a cube $M_1 \in \mathcal{C}_{10l+2}(\Lambda_L)$ such that any cube in $\mathcal{C}_l(\Lambda_L)$ with center outside M_1 is (γ, E)-good.*

We are now in a position to prove the analytic part of the induction step of multi-scale analysis in the final form.

THEOREM 10.20. *Suppose L_0 is large enough and $L_{k+1} = L_k{}^\alpha$ with $1 < \alpha < 2$. If for a certain k ($l := L_k$ and $L := L_{k+1}$)*

    (1) *there do not exist four disjoint cubes in $\mathcal{C}_l(\Lambda_L)$ which are not (γ_k, E)-good with a rate $\gamma_k \geq \frac{12}{l^{1/2}}$,*

    (2) *no cube in*

$$\mathcal{C}_{2l}(\Lambda_L) \ \cup \ \mathcal{C}_{6l+1}(\Lambda_L) \ \cup \ \mathcal{C}_{10l+2}(\Lambda_L) \tag{10.75}$$

    is E-resonant and

    (3) *the cube Λ_L is not E-resonant,*

then the cube Λ_L is (γ_{k+1}, E)- good with a rate γ_{k+1} satisfying $\gamma_{k+1} \geq \frac{12}{L^{1/2}}$. Moreover we can choose the rate γ_{k+1} such that

$$\gamma_{k+1} \ \geq \ \gamma_k \ - \gamma_k \, \frac{C}{L_k{}^{\alpha-1}} \ - \ \frac{C}{L_k{}^{\alpha/2}} \ . \tag{10.76}$$

As above, we can estimate the decay rates as follows.

COROLLARY 10.21. *If L_0 is large enough and the initial rate γ_0 satisfies $\gamma_0 \geq \frac{12}{L_0{}^{1/2}}$ then the γ_k in Theorem 10.20 satisfy $\gamma_k \geq \frac{\gamma_0}{2}$ for all k.*

PROOF: We set $\gamma = \gamma_k$ and $\gamma' = \gamma_{k+1}$. From Lemma 10.19 we know that there are three cubes M_1, M_2, M_3 of side length $2l$, $6l + 1$ or $10l + 2$ (or 0 if $M_i = \emptyset$) such that the M_i contain all cubes in $\mathcal{C}_l(\Lambda_L)$ which are not (γ, E)-good.
Starting with $m \in \Lambda_{\sqrt{L}}$ and $n \in \partial^- \Lambda_L$, we use the geometric resolvent equation repeatedly.
If m does not belong to one of the 'dangerous' cubes M_i we know $\Lambda_l(m)$ is (γ, E)-good, so we estimate

$$|\,G_E^{\Lambda_L}(m,n)| \ \leq \sum_{\substack{(q,q')\in\partial\Lambda_l(m) \\ q\in\Lambda_l(m)}} |\,G_E^{\Lambda_l(m)}(m,q)|\,|\,G_E^{\Lambda_L}(q',n)| \tag{10.77}$$

$$\leq \ 2d\,(2l+1)^{d-1}\,e^{-\gamma l}\,|\,G_E^{\Lambda_L}(n_1,n)| \tag{10.78}$$

$$\leq \ e^{-\tilde{\gamma}l}\,|\,G_E^{\Lambda_L}(n_1,n)| \ . \tag{10.79}$$

We call such a step an exponential bound. We do this repeatedly, as long as the new point $n_1, n_2, \ldots$ neither belongs to one of the M_i nor is close to the boundary of Λ_L.

If n_j belongs to one of the M_i, say to M_1, we use a Wegner-type bound

$$
\begin{aligned}
|G_E^{\Lambda_L}(n_j, n)| &\leq \sum_{\substack{(q,q') \in \partial M_1 \\ q \in M_1}} |G_E^{M_1}(n_j, q)| \, |G_E^{\Lambda_L}(q', n)| \\
&\leq 2d \, (20l + 5)^{d-1} \, e^{\sqrt{10l+2}} \, |G_E^{\Lambda_L}(n_j', n)| \qquad (10.80)
\end{aligned}
$$

for a certain $n_j' \in \partial^+ M_1$. Since the M_i do not touch, we can be sure that $\Lambda_l(n_j')$ is (γ, E)-good. Consequently, we can always (as long as n_j' is not near the boundary of Λ_L) do an exponential bound after a Wegner-type bound and obtain

$$
\begin{aligned}
&|G_E^{\Lambda_L}(n_j, n)| \\
&\leq 2d \, (20l + 5)^{d-1} \, e^{\sqrt{10l+2}} \, |G_E^{\Lambda_L}(n_j', n)| \\
&\leq (2d)^2 \, (20l + 5)^{d-1} \, (2l + 1)^{d-1} \, e^{\sqrt{10l+2}} \, e^{-\gamma l} \, |G_E^{\Lambda_L}(n_{j+1}, n)| . \qquad (10.81)
\end{aligned}
$$

If l is larger than a certain constant and $\gamma \geq \frac{12}{l^{1/2}}$, we have

$$
\rho = (2d)^2 \, (20l + 5)^{d-1} \, (2l + 1)^{d-1} \, e^{\sqrt{10l+2}} \, e^{-\gamma l} \leq 1 \qquad (10.82)
$$

thus

$$
(10.81) \leq |G_E^{\Lambda_L}(n_{j+1}, n)| . \qquad (10.83)
$$

Whenever the point n_j does not belong to one of the 'dangerous' regions M_i, we know that $\Lambda_l(n_j)$ is (γ, E)-good. Hence, we obtain an exponential bound of the Green's function and gain an exponential factor $e^{-\gamma l}$. This step can be done roughly $\frac{L}{l}$ times. Hence, we get the desired bound. The details are as in the previous sections. $\qquad \square$

Now, we do the probabilistic estimate.

THEOREM 10.22. *Assume that the probability distribution P_0 has a bounded density. Suppose L_0 is large enough, $\gamma \geq \frac{12}{L_0^{1/2}}$, $p > 2d$ and $1 < \alpha < \frac{2p}{p+2d}$. If for any disjoint cubes $\Lambda_{L_0}(n)$ and $\Lambda_{L_0}(m)$*

$$
\begin{aligned}
\mathbb{P} \big(\quad &\text{For some } E \in I \text{ both } \Lambda_{L_0}(n) \text{ and } \Lambda_{L_0}(m) \\
&\text{are not } (2\gamma, E)\text{-good} \big) \quad \leq \quad \frac{1}{L_0^{2p}} \qquad (10.84)
\end{aligned}
$$

then for all k and all disjoint cubes $\Lambda_{L_k}(n)$ and $\Lambda_{L_k}(m)$

$$
\begin{aligned}
\mathbb{P} \big(\quad &\text{For some } E \in I \text{ both } \Lambda_{L_k}(n) \text{ and } \Lambda_{L_k}(m) \\
&\text{are not } (\gamma, E)\text{-good} \big) \quad \leq \quad \frac{1}{L_k^{2p}} . \qquad (10.85)
\end{aligned}
$$

PROOF: The prove works by induction. So, we suppose, we know (10.85) already for k. We try to prove it for $k+1$.

As usual, we set $l = L_k$, $L = L_{k+1}$, $\gamma = \gamma_k$, and $\gamma' = \gamma_{k+1}$.

We also abbreviate $\Lambda_1 = \Lambda_{L_{k+1}}(n)$ and $\Lambda_2 = \Lambda_{L_{k+1}}(m)$.

Similar to (10.65) we define ($i = 1, 2$)

$$
\begin{aligned}
A_i(E) &= \{ \Lambda_i \text{ is not } (\gamma', E)-\text{good} \} \\
Q_i(E) &= \{ \Lambda_i \text{ or a cube in } \mathcal{C}_{2l}(\Lambda_i) \cup \mathcal{C}_{6l+1}(\Lambda_i) \cup \mathcal{C}_{10l+2}(\Lambda_i) \text{ is not } E\text{-resonant} \} \\
D_i(E) &= \{ \mathcal{C}_l(\Lambda_i) \text{ contains four disjoint cubes which are not } (\gamma, E)-\text{good} \} .
\end{aligned}
$$

Let us denote by $\mathcal{S}(E)$ the set of all cubes of side length l which are *not* (γ, E)-good. Like in Section 10.5, we estimate

$$
\mathbb{P} \left(\exists_{E \in I} \text{ such that } \Lambda_1 \text{ and } \Lambda_2 \text{ are not } (\gamma', E)-\text{good} \right)
$$

$$
= \mathbb{P} \left(\bigcup_{E \in I} \left(A_1(E) \cap A_2(E) \right) \right)
$$

$$
\leq \mathbb{P} \left(\bigcup_{E \in I} \left(Q_1(E) \cap Q_2(E) \right) \right) + \mathbb{P} \left(\bigcup_{E \in I} \left(D_1(E) \cap D_2(E) \right) \right)
$$

$$
+ \mathbb{P} \left(\bigcup_{E \in I} \left(Q_1(E) \cap D_2(E) \right) \right) + \mathbb{P} \left(\bigcup_{E \in I} \left(D_1(E) \cap Q_2(E) \right) \right)
$$

$$
\leq \mathbb{P} \left(\bigcup_{E \in I} \left(Q_1(E) \cap Q_2(E) \right) \right) + \mathbb{P} \left(\bigcup_{E \in I} D_1(E) \right) \mathbb{P} \left(\bigcup_{E \in I} D_2(E) \right)
$$

$$
+ \mathbb{P} \left(\bigcup_{E \in I} D_1(E) \right) + \mathbb{P} \left(\bigcup_{E \in I} D_2(E) \right)
$$

$$
\leq \mathbb{P} \left(\bigcup_{E \in I} \left(Q_1(E) \cap Q_2(E) \right) \right) + 3 \, \mathbb{P} \left(\bigcup_{E \in I} D_1(E) \right) .
$$

Let us first estimate the latter term:

$$
\mathbb{P} \left(\bigcup_{E \in I} D_1(E) \right)
$$

$$
= \mathbb{P} \left(\underset{E \in I}{\exists} \ \underset{\substack{C_1, C_2, C_3, C_4 \in \mathcal{C}_l(\Lambda_L) \\ \text{pairwise disjoint}}}{\exists} C_1, C_2, C_3, C_4 \in \mathcal{S}(E) \right)
$$

$$
\leq \sum_{\substack{C_i \in \mathcal{C}_l(\Lambda_L) \\ \text{pairwise disjoint}}} \mathbb{P} \left(\underset{E \in I}{\exists} \ C_1 \in \mathcal{S}(E), C_2 \in \mathcal{S}(E), C_3 \in \mathcal{S}(E) \text{ and } C_4 \in \mathcal{S}(E) \right)
$$

$$\leq \sum_{\substack{C_i \in \mathcal{C}_l(\Lambda_L) \\ \text{pairwise disjoint}}} \mathbb{P}\left(\left(\underset{E \in I}{\exists} \quad C_1 \in \mathcal{S}(E) \text{ and } C_2 \in \mathcal{S}(E) \right) \text{ and} \right.$$

$$\left. \left(\underset{E \in I}{\exists} \quad C_3 \in \mathcal{S}(E) \text{ and } C_4 \in \mathcal{S}(E) \right) \quad \right)$$

$$\leq \sum_{\substack{C_i \in \mathcal{C}_l(\Lambda_L) \\ \text{pairwise disjoint}}} \mathbb{P}\left(\underset{E \in I}{\exists} \quad C_1, C_2 \in \mathcal{S}(E) \right) \mathbb{P}\left(\underset{E \in I}{\exists} \quad C_3, C_4 \in \mathcal{S}(E) \right)$$

$$\leq \quad C\, L^{4d} \left(\frac{1}{l^{2p}} \right)^2 \quad \leq \quad \frac{C}{L^{4p/\alpha - 4d}} \quad \leq \quad \frac{1}{4}\, \frac{1}{L^{2p}} \ .$$

In the last step, we used that $p > 2d$ and $1 < \alpha < \frac{2p}{p+2d}$.

We turn to the estimate of

$$\mathbb{P}\left(\bigcup_{E \in I} \left(Q_1(E) \cap Q_2(E) \right) \right)$$

By setting $\mathcal{Q}_i = \mathcal{C}_{2l}(\Lambda_i) \cup \mathcal{C}_{6l+1}(\Lambda_i) \cup \mathcal{C}_{10l+2}(\Lambda_i) \cup \{\Lambda_i\}$, we get

$$\mathbb{P}\left(\bigcup_{E \in I} \left(Q_1(E) \cap Q_2(E) \right) \right) \leq \sum_{c_1 \in \mathcal{Q}_1, c_2 \in \mathcal{Q}_2} \mathbb{P}\left(\underset{E \in I}{\exists}\ c_1 \text{ and } c_2 \text{ are } E\text{-resonant} \right) .$$
$$(10.86)$$

Each term in the sum in (10.86) can be estimated using Theorem 5.27 by a term of the form $C\, L^k\, e^{L^{\frac{1}{2}}}$ and the sum does not have more than $C\, L^d$ terms, thus the sum can certainly be bounded by $\frac{1}{4}\, \frac{1}{L^{2p}}$.

This finishes the proof. $\qquad \square$

Notes and Remarks

The celebrated paper by Fröhlich and Spencer [**47**] laid the foundation for multi-scale analysis. This technique was further developed and substantially simplified in the paper by Dreifus and Klein [**40**]. Germinet and Klein [**49**] developed a 'Bootstrap multiscale analysis' which uses the output of a multiscale estimate as the input of a new multiscale procedure. These authors obtain the best available estimates of this kind. In fact, in [**50**] they prove that their result characterizes the regime of 'strong localization'.

Multiscale analysis can be transferred to the continuous case as well, see e.g. [**100, 24, 11, 59, 80, 44, 70, 71, 137**].

11. The initial scale estimate

11.1. Large disorder.

In this final chapter, we will prove an initial scale estimate for two cases, namely for energies near the bottom of the spectrum with arbitrary disorder and for arbitrary energies at large disorder.

We prove the initial scale estimate first for the case of high disorder. As usual we have to assume that the random variables $V_\omega(n)$ are independent and identically distributed with a bounded density $g(\lambda)$. We may say that *the disorder is high* if the norm $\| g \|_\infty$ is small. In fact, small $\| g \|_\infty$ reflects a wide spreading of the random variables.

THEOREM 11.1. *Suppose the distribution P_0 has a bounded density g.*
Then for any L_0 and any $\gamma > 0$, there is a $\rho > 0$ such that:
If $\| g \|_\infty < \rho$ and $\Lambda_1 = \Lambda_{L_0}(n)$, $\Lambda_2 = \Lambda_{L_0}(m)$ are disjoint, then

$$\mathbb{P} \left(\, \exists_E \, \Lambda_1 \text{ and } \Lambda_2 \text{ are both not } (\gamma, E)\text{-good} \, \right) \leq \frac{1}{L_0^{2p}} \, . \qquad (11.1)$$

PROOF: Since $| G_E^{\Lambda_i}(m, n)| \leq \| (H_{\Lambda_i} - E)^{-1} \|$ we have

$$\mathbb{P} \left(\, \exists_E \quad \Lambda_1 \text{ and } \Lambda_2 \text{ are both not } (\gamma, E)\text{-good} \, \right)$$
$$\leq \quad \mathbb{P} \left(\, \exists_E \quad \| (H_{\Lambda_1} - E)^{-1} \| > e^{-\gamma \, L_0} \text{ and } \| (H_{\Lambda_2} - E)^{-1} \| > e^{-\gamma \, L_0} \, \right)$$
$$\leq \quad \mathbb{P} \left(\, \exists_E \quad \text{dist}(E, \sigma(H_{\Lambda_1}) \leq e^{\gamma \, L_0} \text{ and dist}(E, \sigma(H_{\Lambda_2}) \leq e^{\gamma \, L_0} \, \right)$$
$$\leq \quad 2 \, C \, \| g \|_\infty \, e^{\gamma \, L_0} \, (2 \, L_0 \, + \, 1)^{2d}$$

where we used the 'uniform' Wegner estimate, Theorem 5.27, in the final estimate. By choosing ρ and, hence, $\| g \|_\infty$ very small we obtain the desired estimate. $\qquad \square$

11.2. The Combes-Thomas estimate.

To prove the initial scale estimate for small energies, the following bound is crucial.

THEOREM 11.2 (Combes-Thomas estimate). *If $H = H_0 + V$ is a discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ and $\text{dist}(E, \sigma(H)) = \delta \leq 1$, then for any $n, m \in \mathbb{Z}^d$*

$$\left| (H - E)^{-1}(n, m) \right| \quad \leq \quad \frac{2}{\delta} \quad e^{- \frac{\delta}{12 \, d} \| n - m \|_1} \, . \qquad (11.2)$$

REMARK 11.3. Theorem 11.2 can be improved in various directions, see for example the discussion of the Combes-Thomas estimate in [**128**]. In particular, the condition $\delta \leq 1$ which we need for technical reasons is rather unnatural. Our proof can easily be extended to $\delta \leq C$ for any $C < \infty$ but then the exponent $\frac{\delta}{12 \, d}$ in the right hand side of (11.2) has to be adjusted depending on the value of C.

PROOF: For fixed $\mu > 0$ to be specified later and fixed n_0, we define the multiplication operator $F = F_{n_0}$ on $\ell^2(\mathbb{Z}^d)$ by

$$F\,u(n)\ =\ F_{n_0}\,u(n)\ =\ e^{\mu\,||\,n_0-n||_1}\,u(n)\,. \tag{11.3}$$

Then for any operator A we have

$$\left(F_{n_0}^{-1}\,A\,F_{n_0}\right)(n,m)\ =\ e^{-\mu\,||\,n_0-n||_1}\,A(n,m)\,e^{\mu\,||\,n_0-m||_1}\,. \tag{11.4}$$

Hence, with $F = F_n$

$$\begin{aligned}
\left|\,(H-E)^{-1}(n,m)\,\right| &= e^{-\mu\,||\,n-m||_1}\,\left|\,F^{-1}\,(H-E)^{-1}\,F\,(n,m)\right| \\
&= e^{-\mu\,||\,n-m||_1}\,\left|\,(F^{-1}\,H\,F - E)^{-1}\,(n,m)\right| \\
&\le e^{-\mu\,||\,n-m||_1}\,\left\|\,(F^{-1}\,H\,F - E)^{-1}\right\|\,. \tag{11.5}
\end{aligned}$$

To compute the norm of the operator $(F^{-1}\,H\,F - E)^{-1}$, we use the resolvent equation to conclude

$$\begin{aligned}
&(F^{-1}\,H\,F - E)^{-1} \\
&= (H-E)^{-1} - (F^{-1}\,H\,F - E)^{-1}\big(F^{-1}HF - H\big)(H-E)^{-1}\,.
\end{aligned}$$

This implies

$$(F^{-1}\,H\,F - E)^{-1}(1 + (F^{-1}HF - H)(H-E)^{-1}) = (H-E)^{-1}\,.$$

If $\|(F^{-1}HF-H)(H-E)^{-1}\| \le 1$, we may invert $(1+(F^{-1}HF-H)(H-E)^{-1})$ and obtain

$$\begin{aligned}
&(F^{-1}HF - E)^{-1} \\
&=\ (H-E)^{-1}\Big(1 + (F^{-1}HF - H)(H-E)^{-1}\Big)^{-1}\,. \tag{11.6}
\end{aligned}$$

We compute the norm of the operator $F^{-1}HF - H$. If an operator A on $\ell^2(\mathbb{Z}^d)$ has matrix elements $A(u,v)$, then A is bounded if

$$a_1\ =\ \sup_{u\in\mathbb{Z}^d}\sum_{v\in\mathbb{Z}^d}\,\big|A(u,v)\big|\ <\ \infty \qquad \text{and}$$

$$a_2\ =\ \sup_{v\in\mathbb{Z}^d}\sum_{u\in\mathbb{Z}^d}\,\big|A(u,v)\big|\ <\ \infty\,.$$

Moreover, we have

$$\|\,A\,\|\ \le\ a_1^{\,1/2}\,a_2^{\,1/2} \tag{11.7}$$

(see e.g. [141]). We estimate using (11.4)

$$\sum_{v\in\mathbb{Z}^d}\,|\,\big(F_n^{-1}HF_n - H\big)(u,v)\,|\ \le\ \sum_{v:||\,v-u||_1=1}\,||\,e^{-\mu||\,n-u||_1}\,e^{\mu||\,n-v||_1} - 1||$$

$$\le\ 2\,d\ \mu\,e^{\mu}\,. \tag{11.8}$$

The last inequality results from an elementary calculation:

For $\|u - v\|_1 \le 1$ we have

$$\big|\,\|n - u\|_1 - \|n - v\|_1\,\big| \;\le\; \|u - v\|_1 \;\le\; 1\,.$$

Moreover, for $|a| \le 1$ and $\mu > 0$

$$\big|\,e^{\mu a} - 1\,\big| \;\le\; \big|\,e^{\mu} - 1\,\big| \;=\; e^{\mu} - 1$$
$$\le\; \int_0^1 \mu\,e^{\mu t} \;\le\; \mu\,e^{\mu}$$

which proves (11.8).

Estimate (11.8) and an analogous estimate with the role of u and v interchanged imply using (11.7)

$$\| F^{-1}HF - H \| \;\le\; 2\,d\,\mu\,e^{\mu}\,. \tag{11.9}$$

Now we *choose* $\mu = \frac{\delta}{12\,d}$. As $\mathrm{dist}(E, \sigma(H)) = \delta \le 1$ we conclude

$$\|(F^{-1}HF - H)(H - E)^{-1}\| \;\le\; \|(F^{-1}HF - H)\|\,\|(H - E)^{-1}\|$$
$$\le\; 2\,d\,\mu\,e^{\mu}\,\frac{1}{\delta}$$
$$=\; 2\,d\,\frac{\delta}{12\,d}\,e^{\frac{\delta}{12\,d}}\,\frac{1}{\delta}$$
$$\le\; \frac{1}{2} \tag{11.10}$$

Above we used $e^{\frac{\delta}{12\,d}} \le e \le 3$ since $\delta \le 1$.

It follows that the operator $1 + (F^{-1}HF - H)(H - E)^{-1}$ is indeed invertible and, using the Neumann series, we conclude that

$$\left\| \left(1 + (F^{-1}HF - H)(H - E)^{-1}\right)^{-1} \right\| \;\le\; 2 \tag{11.11}$$

Thus, by (11.6) we have

$$\left\|(F^{-1}HF - E)^{-1}\right\| \;=\; \left\|(H - E)^{-1}\left(1 + (F^{-1}HF - H)\,(H - E)^{-1}\right)^{-1}\right\|$$
$$\le\; \frac{2}{\delta} \tag{11.12}$$

and (11.5) gives

$$\big|(H - E)^{-1}(n, m)\big| \;\le\; e^{-\mu\,\|n - m\|_1}\,\|(F^{-1}\,H\,F - E)^{-1}\|$$
$$\le\; \frac{2}{\delta}\,e^{-\frac{\delta}{12\,d}\,\|n - m\|_1}. \tag{11.13}$$

$\square$

11.3. Energies near the bottom of the spectrum.

For energies near the bottom of the spectrum, we prove the following estimate.

THEOREM 11.4. *Suppose the distribution P_0 has a bounded support. Denote by E_0 the infimum of the spectrum of H_ω. Then for arbitrary large L_0, any C and p there is an energy $E_1 > E_0$ such that*

$$\mathbb{P} \left(\Lambda_{L_0} \text{ is not } (\frac{C}{L_0^{1/2}}, E)\text{-regular for some } E \leq E_1 \right) \leq \frac{1}{L_0^p} . \qquad (11.14)$$

REMARK 11.5. By the results of Chapter 10 the above result implies pure point spectrum for energies near the bottom of the spectrum.

PROOF: If $E_0(H_{\Lambda_L}) \geq 2\gamma$ then Theorem 11.2 implies that Λ_L is (γ, E)-regular for any $E \leq \gamma$. Indeed, for such an E

$$\text{dist}\left(E, \sigma(H_{\Lambda_L}) \right) \geq E_0(H_{\Lambda_L}) - \gamma \geq \gamma . \qquad (11.15)$$

From our study of Lifshitz tails (Chapter 6), we have already a lower bound on some $E_0(H_{\Lambda_L}^N) \leq E_0(H_{\Lambda_L})$, namely:
By (6.10) and Lemma 6.4 there exist ℓ_0 and β such that

$$\mathbb{P} \left(E_0(H_{\Lambda_{\ell_0}}^N) < \frac{1}{\beta \ell_0^2} \right) \leq e^{-c \ell_0^d} . \qquad (11.16)$$

This estimate tells us that for $E \leq \gamma = \frac{1}{2 \beta \ell_0^2}$, the cube Λ_{ℓ_0} is (γ, E)-good with very high probability.
This sounds like it is exactly what we need for the initial scale estimate. Unfortunately, it is *not quite* what makes the machine work.
The multiscale scheme requires for the initial step the assumption (see Theorem 10.22)

$$\gamma \geq \frac{C}{L_0^{1/2}} \qquad (11.17)$$

but the γ we obtain from (11.16) is much smaller than the rate required by (11.17). On the other hand, the right hand side of (11.16) is much better than what we need (exponential versus polynomial bound). So, we may hope we can 'trade probability for rate'. This is exactly what we do now.
We build a big cube Λ_{L_0} by piling up disjoint copies of the cube Λ_{ℓ_0}, more precisely

$$\Lambda_{L_0} = \bigcup_{j \in R} \Lambda_{\ell_0}(j) . \qquad (11.18)$$

Indeed, for any odd integer r we may take $L_0 = r \ell_0 + \frac{r-1}{2}$. The set R in (11.18) contains r^d points.
By (5.50) we have

$$H_{\Lambda_{L_0}}^N \geq \bigoplus_{j \in R} H_{\Lambda_{\ell_0}}^N(j) , \qquad (11.19)$$

hence

$$E_0\left(H^N_{\Lambda_{L_0}}\right) \;\geq\; \inf_{j\in R}\; E_0\left(H^N_{\Lambda_{\ell_0}}(j)\right). \tag{11.20}$$

It follows that

$$\mathbb{P}\left(\; E_0\left(H^N_{\Lambda_{L_0}}\right) \;\leq\; 2\,\gamma\; \right)$$

$$\leq\; \mathbb{P}\left(\; \inf_{j\in R} E_0\left(H^N_{\Lambda_{\ell_0}}(j)\right) \;\leq\; 2\,\gamma\; \right)$$

$$\leq\; \mathbb{P}\left(\; E_0\left(H^N_{\Lambda_{\ell_0}}(j)\right) \;\leq\; 2\,\gamma \;\text{ for some } j \in R \right)$$

$$\leq\; r^d\;\; \mathbb{P}\left(\; E_0\left(H^N_{\Lambda_{\ell_0}}\right) \;\leq\; 2\,\gamma\; \right). \tag{11.21}$$

If we choose $\gamma = \frac{1}{2\,\beta\,\ell_0{}^2}$, we may use (11.16) to estimate (11.21) and obtain

$$\mathbb{P}\left(\; E_0\left(H^N_{\Lambda_{L_0}}\right) \;\leq\; 2\,\gamma\; \right) \;\leq\; r^d\; e^{-\,c\,\ell_0{}^d}. \tag{11.22}$$

Now, we choose r and hence L_0 in such a way that $\gamma > \frac{C}{L_0{}^{1/2}}$. This leads to setting $r \sim \ell_0{}^3$, thus $L_0 \sim \ell_0{}^4$. With this choice, (11.22) gives

$$\mathbb{P}\left(\; E_0\left(H^N_{\Lambda_{L_0}}\right) \;\leq\; 2\,\gamma\; \right) \;\leq\; C_1\, L_0{}^d\; e^{-\,c'\,L_0{}^{d/4}}. \tag{11.23}$$

Since the right hand side of (11.23) is smaller than $\frac{1}{L_0{}^p}$, this proves the initial scale estimate.

$$\square$$

Notes and Remarks

Already the paper [47] contained the proof for high disorder localization we gave above. The idea to use Lifshitz tails to prove localization for small energies goes back to [100] and was further developed in [70] (see also [73]), but an intimate connection between Lifshitz tails and Anderson localization was clear to physicists for a long time (see [98]).

The Combes-Thomas inequality was proved in [28]. It was improved in [11], see also [128]. We took the proof above from [1].

12. Appendix: Lost in Multiscalization – A guide through the jungle

This is a short guide to the proof of Anderson localization via multiscale analysis given in this text.

The core of the localization proof is formed by the estimates stated in Section 9.3 as Result 9.6 and 9.8. The first estimate (9.6) says that for a given energy E, exponential decay of the Green's function is very likely on large cubes. Cubes with exponentially decaying Green's functions will be called 'good' cubes. In Section 9.2 we prove that the estimate in Result 9.6 implies the absence of absolutely continuous spectrum.

The strong version (Result 9.8) of the multiscale estimate considers a whole energy interval I and two disjoint cubes. The result tells us that with high probability for all energies in I at least one of the cubes has an exponentially decaying resolvent. This result is a strong version of the former result as it is *uniform* in the energy. The price to be paid is the consideration of a second cube. A single cube cannot be good for all energies in I if there is spectrum at all in I (see 9.2). We show in Section 9.5 that the strong form of the multiscale estimate implies pure point spectrum inside I. This is done using the exponential decay of eigenfunctions which we deduce from the key Theorem 9.9. The connection between spectrum and (generalized) eigenfunctions is discussed in Chapter 7.

The proofs of the multiscale estimates (Results 9.6 and 9.8) are contained in the Chapters 10 and 11. We prove the estimates inductively for cubes of side length $L_k, k = 0, 1, 2 \ldots$. The length scale is such that $L_{k+1} = L_k^\alpha$ for an $\alpha > 1$.

The induction step from L_k to L_{k+1} is done in Chapter 10. In a first attempt (Section 10.2) to do this for the weaker form we prove that if all the small cubes (of size L_k) inside a big cube (of size L_{k+1}) are good, then the big cube itself is good if we have a rough a priori estimate for the big cube. This a priori bound is provided by the 'Wegner estimate', a key ingredient to our proof. We prove the Wegner estimate in Section 5.5. Unfortunately, the probability that *all* small cubes inside the big one are good is rather small. So, this 'first try' is not appropriate to prove that the big cube is good with high enough probability.

In the 'second try' we allow *one* bad small cube inside the big cube. (For the precise formulation see Section 10.3). To prove that this still implies that the big cube is good requires more work. We need again that the big cube and also the 'bad' small cube allow an a priori bound of the Wegner type. The advantage of allowing one bad cube is that this event has a much higher probability. In this way, we prove the induction step for the weak form of the multiscale analysis.

The strong form of the multiscale analysis is then treated in Section 10.6. Here we have to allow even a few bad cubes among the small ones. This makes the proof yet a bit more complicated.

So far we have done the induction step. Of course, we still have to prove the estimate for the initial length L_0. This is done in Chapter 11. We prove that the initial estimate is satisfied if either the disorder is large or the energy is close to the bottom of the spectrum. An important tool in this chapter is the Combes-Thomas inequality. We prove this result in section 11.2.

The strategy of proof outlined above is certainly not the fastest one to prove local-ization via multiscale analysis. However, we believe that for a first reading, it is easier to learn the subject this way than in a streamlined turbo version.

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