Research Article

W. Kirsch and H. Najar

Lifshitz tails for a percolation model in the continuum

Abstract: In this work we study Lifshitz tails for Laplacians in a percolation model on $\mathbb{R}^d$. At any lattice point i in $\mathbb{R}^d$ we remove a set $S+i$ with a certain probability p. We consider the Laplacian on the remaining subset of $\mathbb{R}^d$ with either Dirichlet or Neumann boundary conditions. We prove that the integrated density of states exhibits Lifshitz behavior at the bottom of the spectrum when we consider Dirichlet boundary conditions, while when we consider Neumann boundary conditions , it exhibits a van Hove behavior.

Keywords: Spectral theory, random operators, integrated density of states, Lifshitz tails, percolation, random graphs.

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W. Kirsch: FernUniversität Hagen, Germany. e-mail: werner.kirsch@fernuni-hagen.de

H. Najar:
Département de Mathématiques Fac. Sci. Monastir. Avenue de l'Environnement - Monastir - 5000 Tunisie, e-mail: hatemnajar@ipeim.rnu.tn

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1 Introduction

In this paper we study the integrated density of states in the context of a quantum percolation in the continuum. The Hamiltonian we consider is the Laplacian on a random subset D_ω of $\mathbb{R}^d$. The set D_ω is constructed as follows:
At any lattice point $i \in \mathbb{Z}^d$ we remove from $\mathbb{R}^d$ with probability p and independently from the other lattice points a set around i, more precisely we remove the set $S+i = \{x+i \mid x \in S\}$ where S is a compact subset of $\mathbb{R}^d$. D_ω is the what remains of $\mathbb{R}^d$ after removing these copies of S. Let us denote by H_ω the Dirichlet-Laplacian and by $\widetilde{H}_\omega$ the Neumann-Laplacian on the set D_ω respectively.

We will be interested in the integrated density of states N_D resp. N_N of these operators and in particular their behavior near the bottom of the spectrum. It turns out that under suitable conditions on S and/or p the density of states shows Lifshitz behavior in the Dirichlet case, but doesn't in the Neumann case.

The integrated density of states (IDS) measures the number of energy levels per unit volume, below a given energy, more precisely: Let $P_{(-\infty,E]}$ be the spectral projection of a random Schrödinger operator H_ω, Λ_L be a cube in $\mathbb{R}^d$ of side length L around the origin and χ_A the characteristic function of the set

$A \subset \mathbb{R}^d$. We consider

$$N(E) = \lim_{L\to\infty} \frac{1}{\mid \Lambda_L \mid} tr(\chi_{\Lambda_L} P(-\infty, E]). \tag{1.1}$$

Under quite general assumption, this limit exists and is non random. This is in particular true for our model operators H_ω and $\widetilde{H}_\omega$. The quantity N called the *integrated density of states* of H_ω. See (11) and references given there for an overview on the IDS.

The question we are interested in here concerns the behavior of N at the bottom of the spectrum of H_ω. In 1964, Lifshitz (14) argued that, for a Schrödinger operator of the form $H_\omega = -\Delta + V_\omega$, there exists $c_1, c_2 > 0$ such that $N(E)$ satisfies the asymptotic:

$$N(E) \simeq c_1 \exp(-c_2(E - E_0)^{-\frac{d}{2}}), \quad E \searrow E_0. \tag{1.2}$$

Here E_0 is the bottom of the spectrum of H_ω. The behavior (1.2) is known as *Lifshitz tails*. In the last thirty years, there has been vast literature, both physical and mathematical, concerning Lifshitz tails and related phenomena. We do not try to give an exhaustive account of this literature. The paper (11) gives a survey of such results and basic references on this subject. Below, we give results on the IDS behavior in the context of our percolation operators H_ω and $\widetilde{H}_\omega$.

A quantum percolation Hamiltonian was studied already by de Gennes et al in (3; 4), where the Hamiltonian of binary solid solution was considered. It is proved that the spectrum of these percolation Hamiltonians is pure point if the fraction p is less than the critical value p_c. We recall that p_c is the value of the well-known critical probability of the percolation theory: If $p < p_c$, no infinite active cluster exists almost surely, and for $p > p_c$ there exists almost surely one infinite cluster. Theses facts are given in (5) and were mainly obtained by Hammersley in the late fifties. We notice that uniqueness of the infinite cluster, was proved only thirty years later by Aizenman, Kesten and Newmann, see (5).

If the concentration of active sites is above the critical value, one speaks of the *percolation regime*. For this regime it is argued (3) that the spectrum contains a continuous part. In (12), it is proved that in the non-percolation case, $p \in]0, p_c[$, the spectrum of the Laplacian is $\mathbb{P}$-almost surely only a dense pure-point spectrum with infinitely degenerate eigenvalues.

Bond-percolation graphs are random subgraphs of the d-dimensional integer lattice generated by a standard bond-percolation. The associated graph Laplacians, subject to Dirichlet or Neumann conditions at these cluster boundary, represent bounded, self-adjoint, ergodic random operators with an off-diagonal disorder. They have almost surely a non-random spectrum.

In (2) the authors considered the site dilution model on the hyper-cubic lattice $\mathbb{Z}^d$, for $d \geq 2$. They investigated the density of states for the tight-binding Hamiltonian projected onto an infinite cluster. It is shown that, almost surely, the IDS is discontinuous on a set of energies which is dense in the band. This is proved by constructing states supported on finite regions of the infinite cluster.

In the same context, in (19), Veselić studied Hamiltonians (a finite hopping range operators) corresponding to site percolation on the lattice $\mathbb{Z}^d$ and graphs with an amenable group action and characterize the set of energies which are almost surely eigenvalues with infinitely supported eigenfunctions. It is proved that this set of energies is a dense subset of algebraic integers and this set of energies corresponds to the discontinuity point of the IDS.

Spectral theory of random graphs, however, is still a widely open field. The recent contributions (1; 6; 15) take a probabilistic point of view to derive heat-kernel estimates for Laplacians on supercritical Bernoulli bond-percolation graphs in the d-dimensional hyper-cubic lattice. On the other hand, spectral

theory methods are used by Kirsch and Müller in (12) to study spectral properties of the Laplacian on bond-percolation graphs. Indeed they investigate the IDS of Laplacian on subcritical bond-percolation graphs. Depending on the boundary condition that is chosen at cluster borders, two different types of Lifshitz asymptotics at spectral edges were proved, precisely at the lower spectral edge for bond probabilities $p < p_c$, the IDS, $\widetilde{N}(E)$ of the Neumann Laplacian satisfies

$$\lim_{E\to 0^+} \frac{\log|\log\big(\widetilde{N}(E)-\widetilde{N}(0)\big)|}{\log E} = -\frac{1}{2}. \tag{1.3}$$

Here one notices that the Lifshitz exponent $\frac{1}{2}$ in (1.3) is independent of the spatial dimension d. (see equation 1.2, for the usual dependence). This is due to the fact that asymptotically, $\widetilde{N}$ is dominated by the smallest eigenvalues which are caused by very long linear clusters. In contrast, for $p < p_c$, it is proved that the integrated density of states $N(E)$ of the Dirichlet Laplacian satisfies

$$\lim_{E\to 0^+} \frac{\log|\log\big(N(E)\big)|}{\log E} = -\frac{d}{2}. \tag{1.4}$$

In (1.4), the Lifshitz exponent is in the classical form i.e is $\frac{d}{2}$, this is explained by the fact that this is, the dominating small Dirichlet eigenvalues arise from large fully connected cube-or sphere-like clusters. We notice that due to some symmetries the Lifshitz tails at the upper spectral edge are related to the ones at the lower spectral edge whereas at the upper spectral edge, the behavior is reversed.
For the dual case to (12), Müller and Stollmann in (16) pursue the investigation of (12) and studied spectral asymptotics of the Laplacian on supercritical ($p < p_c$) bond-percolation graphs. They studied the influence and the contribution of the existence of the infinite cluster. The situation is different. Indeed, in the present situation it is proved that N_N exhibits van Hove asymptotics. Precisely

$$\lim_{E\to 0^+} \frac{\log\big(\widetilde{N}(E)-\widetilde{N}(0)\big)}{\log E} = \frac{d}{2}. \tag{1.5}$$

We notice that (1.5) is due to the existence of the d-dimensional infinite grid. In contrast to the Neumann case, for the Dirichlet Laplacian (1.4) is still true for $p \geq p_c$ situation.

Lifshitz tails for Neumann Laplacian on Erdös-Rényi random graphs at the lower spectral edge $E = 0$, are considered in (7).

2 Model and results

2.1 The Quantum Percolation Hamiltonian

We start by describing our percolation Hamiltonian. Let $S_0 \subset \mathbb{R}^d$ be a bounded open set and denote by S the closure of S_0. Furthermore, let $\{\omega_\gamma\}_{\gamma\in\mathbb{Z}^d}$ be a sequence of independent random variables with $\mathbb{P}(\omega_\gamma = 1) = p$ and $\mathbb{P}(\omega_\gamma = 0) = 1 - p$ and set $\Xi_\omega = \{\gamma \mid \omega_\gamma = 1\}$. Then we define the random sets

$$\Gamma_\omega = \bigcup_{i\in\Xi_\omega} (S + i) \qquad \text{and} \qquad D_\omega = \mathbb{R}^d \setminus \Gamma_\omega. \tag{2.6}$$

Finally we denote by H_ω and $\widetilde{H}_\omega$ the Laplacian on D_ω with Dirichlet resp. Neumann boundary conditions at ∂D_ω.

We will always follow the convention to decorate quantities related to *Neumann boundary conditions* at ∂D_ω by a tilde, for example the corresponding operator is denoted by $\widetilde{H}_\omega$, its integrated density of states by $\widetilde{N}(E)$ and so on. The quantities for *Dirichlet boundary conditions* will not be decorated, i. e. will be denoted by H_ω, $N(E)$ etc.

For the operators H_ω and $\widetilde{H}_\omega$ we will also have to consider the restrictions to $\Lambda_L = [-\frac{L}{2}, \frac{L}{2}]^d$. We denote the restriction of H_ω to Λ_L with Dirichlet and Neumann conditions at $\partial\Lambda_L$ by $H_L^D(\omega)$ and $H_L^N(\omega)$ respectively. Similarly, $\widetilde{H}_L^D(\omega)$ and $\widetilde{H}_L^D(\omega)$ denote the restriction of $\widetilde{H}_\omega$ to Λ_L with Dirichlet resp. Neumann boundary conditions at the boundary of the cube Λ_L. So, for example, the operator $\widetilde{H}_L^D$ is defined on $L^2(\Lambda_L \cap D_\omega)$ and has Neumann boundary conditions on $\partial D_\omega \cap \Lambda_L$ and Dirichlet boundary conditions on $\partial\Lambda_L \setminus D_\omega$. Finally we will take the liberty to supress the argument ω when the dependence of a quantity on ω is clear from the context.

In an informal way we may write $H_\omega = -\Delta + V_\omega$, with the random 'potential':

$$V_\omega = \sum_{\gamma\in\mathbb{Z}^d} \omega_\gamma\, f(x-\gamma) \tag{2.7}$$

$$\text{with} \qquad f(x) = \begin{cases} \infty, & \text{for } x \in \mathcal{S} \\ 0 & \text{elsewhere} \end{cases} \tag{2.8}$$

If the set S is contained in $]-\frac{1}{2}, \frac{1}{2}[^d$ then the set D_ω contains a unique unbounded cluster independent of the value of p. In fact, D_ω always contains the set

$$D_1 = \mathbb{R}^d \setminus \bigcup_{i\in\mathbb{Z}^d} (S+i). \tag{2.9}$$

Following our general convention, we denote by H_1 the Laplacian on D_1 with Dirichlet boundary conditions and by $\widetilde{H}_1$ the same operator with Neumann boundary conditions.

Whenever D_1 contains an unbounded component then D_ω will as well. On the other hand if D_1 contains no unbounded cluster then D_ω may or may not contain an unbounded cluster depending on the value of p and the shape of the set S. If D_ω contains only bounded components then both H_ω and $\widetilde{H}_\omega$ have pure point spectra. If not stated otherwise we always assume from now on that $S \subset]-\frac{1}{2}, \frac{1}{2}[^d$.

The families H_ω and $\widetilde{H}_\omega$ are ergodic families of self-adjoint operators, more precisely:
Let $\mathcal{U}_i$ be the unitary translation operator on $L^2(\mathbb{R}^d)$ given by

$$\mathcal{U}_i\psi(x) = \psi(x-i), \quad \forall\psi \in L^2(\mathbb{R}^d) \text{ and } \ x \in \mathbb{R}^d.$$

As the probability measure $\mathbb{P}$ is ergodic with respect to the group of translation $(\mathcal{T}_i)_{i\in\mathbb{Z}^d}$, acting as $\mathcal{T}_i(\omega) = (\omega_{\gamma+i})_{\gamma\in\mathbb{Z}^d}$, we get

$$\mathcal{T}_i^{-1} H_\omega \mathcal{T}_i = H_{(\mathcal{T}_i\omega)}, \ \ \forall i \in \mathbb{Z}^d, \ \omega \in \Omega. \tag{2.10}$$

We may therefor apply the methods from (8; 10; 17) to conclude that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc} closed and non-random sets of $\mathbb{R}$ such that Σ is the spectrum of H_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) denote the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of H_ω, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability one.

There is a little subtlety connected with the question of measurability in our case. Since the Hilbert spaces the operators H_ω and $\widetilde{H}_\omega$ act on *depend* on ω we need a notion of measurability appropriate for our situation. We get around this problem by noticing that the kernel of the operators e^{-tH_ω} and $e^{-t\widetilde{H}_\omega}$ can be expressed via the Feynman-Kac formula (see for example (9) and references given there). We extend the operators e^{-tH_ω} and $e^{-t\widetilde{H}_\omega}$ to $L^2(\mathbb{R}^d)$ by extending it by the zero operator on $L^2(\mathbb{R}^d \setminus D_\omega)$. The corresponding operators are easily seen to be measurable in the sense of (10), as the kernels are explicitly measurable. Moreover these operators form ergodic families. Consequently their spectra and the above defined parts of the spectra are non random. Thus, the same assertion is true for H_ω and $\widetilde{H}_\omega$ as their spectra can be computed from the spectra of e^{-tH_ω} and $e^{-t\widetilde{H}_\omega}$.

The following Lemma gives the precise location of the spectrum.

Lemma 2.1. *Suppose that* $S \subset]-\frac{1}{2},\frac{1}{2}[^d$.

1. *For* $0 \leq p < 1$, *the spectrum* Σ, *of* H_ω *is* $[0,+\infty[$ *with probability one.*
2. *For* $0 \leq p \leq 1$, *the spectrum* $\widetilde{\Sigma}$, *of* $\widetilde{H}_\omega$ *is* $[0,+\infty[$ *with probability one.*

Remark 2.2. *Unless* S *is empty the infimum of the spectrum of* H_1 *will be strictly positive, so in part 1 of the above lemma we had to exclude the case* $p = 1$. *In contrast to this the spectrum of* $\widetilde{H}_1$ *always contains* 0. *In fact the constant function on* D_1 *(or on* D_ω *in general) is a generalized eigenfunction for* $E = 0$.

Proof: First let us notice that for any $\omega \in \Omega$, we have

$$H_\omega \geq 0 \quad \text{and} \quad \widetilde{H}_\omega \geq 0 \tag{2.11}$$

so

$$\Sigma \subset [0,\infty[\quad \text{and} \quad \widetilde{\Sigma} \subset [0,\infty[\tag{2.12}$$

To complete the proof we have to show the opposite inclusion, i.e

$$[0,+\infty[\subset \Sigma \text{ for } \mathbb{P}-\text{almost every} \omega \in \Omega. \tag{2.13}$$

We do this for H_ω, the proof for $\widetilde{H}_\omega$ is essentially the same.
For this, let $\overline{\Omega}$, be the following events

$$\begin{aligned}\overline{\Omega} = \Big\{\omega \in \Omega \mid & \text{ For any } n \in \mathbb{N}, \text{there exists a} \\ & x_n \big(= x_n(\omega)\big) \in \mathbb{Z}^d \text{ such that } \big(x_n + \Lambda_{n+1}\big) \cap D_\omega = \emptyset\Big\}.\end{aligned} \tag{2.14}$$

Let $E \in [0,+\infty[= \Sigma(-\Delta)$ be arbitrarily fixed. Using Weyl criterion, we know that there exists a Weyl sequence $(\varphi_{E,n})_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$, for $-\Delta$. Thus $\|\varphi_{E,n}\| = 1$, for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|(\Delta + E\cdot\mathbb{I})\varphi_{E,n}\| = 0 \tag{2.15}$$

Notice that for any $i \in \mathbb{Z}^d$, $(\mathcal{U}_i\varphi_{E,n})_{n\in\mathbb{N}}$ is also a Weyl sequence. Without loss of generality, we assume that the sequence $(\varphi_{E,n})_{n\in\mathbb{N}}$ is compactly supported. So for any $\omega \in \overline{\Omega}$, there exists a Weyl sequences $(\varphi^\omega_{E,n})_{n\in\mathbb{N}}$ for $(-\Delta)$ such that

$$\operatorname{supp} \varphi^\omega_{E,n} \subset x_n + \Lambda_n \tag{2.16}$$

with x_n as in (2.14). By definition of $\overline{\Omega}$ we have

$$\operatorname{supp} \varphi^{\omega}_{E,n} \cap D_\omega = \emptyset. \tag{2.17}$$

Consequently for any $n \in \mathbb{N}$ and $\omega \in \widetilde{\Omega}$, $\varphi^{\omega}_{E,n}$ is in the domain of the operator H_ω and we get

$$\|(H_\omega - E\mathbb{I})\varphi^{\omega}_{E,n})\| = \|(\Delta + E \cdot \mathbb{I})\varphi^{\omega}_{E,n}\|. \tag{2.18}$$

Hence, $(\varphi^{\omega}_{E,n})_{n\in\mathbb{N}}$ is also a Weyl sequence for H_ω. So we get (2.13) for any $\omega \in \overline{\Omega}$.

It remains to check that $\mathbb{P}(\overline{\Omega}) = 1$. Define

$$\begin{aligned} \overline{\Omega}_n = \Big\{ \omega \in \Omega \mid & \text{ there exists a sequence } y_k \in \mathbb{Z}^d \\ & \text{such that for all } k \quad \big(y_k + \Lambda_{n+1}\big) \cap D_\omega = \emptyset \Big\}. \end{aligned} \tag{2.19}$$

By the Borel-Cantelli lemma we know that $\mathbb{P}\big(\overline{\Omega}_n\big) = 1$. Since $\overline{\Omega} \supset \bigcap \overline{\Omega}_n$ we conclude that $\mathbb{P}(\overline{\Omega}) = 1$. □

2.2 The main results

We investigate the integrated densities of states $N_D(E)$ and $N_N(E)$ for energies E near 0, the bottom of the almost sure spectrum of both H^D_ω and H^N_ω. The first result says that in the Dirichlet case our Percolation Hamiltonian has a *Lifshitz singularity* there, as one might guess from (2.7) and (2.8):

Theorem 2.3. *Assume $S \subset]-\frac{1}{2},\frac{1}{2}[^d$ and $p \in]0,1[$, then*

$$\lim_{E\to 0^+} \frac{\log|\log N_D(E)|}{\log E} = -\frac{d}{2}. \tag{2.20}$$

At a first glance one might be tempted to expect the same behavior for N_N. However, this is not the case. It turns out, that 0 is a stable boundary for the Neumann operator in the sense of (17) and we have a *van Hove singularity* in that case:

Theorem 2.4. *Assume $S \subset]-\frac{1}{2},\frac{1}{2}[^d$ and $p \in [0,1]$, then there is a constant $C > 0$ such that for small $E > 0$*

$$N_N(E) \geq C\, E^{d/2}. \tag{2.21}$$

If the set S has a 'hole' in the sense that the complement $\complement S$ of S contains a bounded connected components M then the constant function on $i + M$ is an eigenstate to energy $E = 0$ whenever $\omega_i = 1$. Thus 0 is an eigenvalue of $\widetilde{H}^X_{\Lambda_L}$ for $X = N, D$ whose multiplicity is proportional to $\operatorname{vol}(\Lambda_L)$ for typical ω. Hence, the integrated density of states will be discontinuous at 0 whenever $\complement S$ is not connected and $E = 0$ is an eigenvalue of $\widetilde{H}_\omega$ with infinite multiplicity.

On the other hand, if S has no 'holes', i. e. if $\complement S$ is connected, then 0 is not an eigenvalue of $\widetilde{H}_\omega$ almost surely.

3 Proofs

3.1 Preliminaries

We start by recalling the following result and giving some properties of the IDS.

Proposition 3.1. *Let $\varphi \in C_0^\infty(\mathbb{R}^d)$, then*

$$\lim_{L\to\infty} \frac{1}{vol(\Lambda_L)} tr(\varphi(H_\omega)\chi_{\Lambda_L}) = \mathbb{E}\Big(tr(\chi_{\Lambda_1}\varphi(H_\omega)\chi_{\Lambda_1})\Big), \tag{3.22}$$

for $\mathbb{P}$-almost all ω. Here $\mathbb{E}$, is the expectation with respect to the probability measure $\mathbb{P}$.

Proof: First we write $\Lambda_L = \sum_{i\in\Lambda_L\cap\mathbb{Z}^d} \Lambda_1(i)$, Here $\Lambda_1(i)$, is the cube of center i end side length 1. We set $\zeta_i = tr(\varphi(H_\omega)\chi_{\Lambda_1(i)})$. So ζ_i is an ergodic sequence (with respect to $\mathbb{Z}^d$) of random variables. So

$$\frac{1}{\mathrm{vol}(\Lambda_L)} tr(\varphi(H_\omega)\chi_{\Lambda_L}) = \frac{1}{\mathrm{vol}(\Lambda_L)} \sum_{i\in\Lambda_L\cup\mathbb{Z}^d} \zeta_i. \tag{3.23}$$

By the Birkhoff's ergodic theorem, the sum in (3.23) converges to its expectation value. This ends the proof of (3.22). □

Now, we notice that both sides of (3.22), are positive linear functionals on the bounded, continuous functions. So, they define positives measures respectively μ_L and μ. i.e

$$\int_{\mathbb{R}} \varphi(\lambda) d\mu_L(\lambda) = \frac{1}{\mathrm{vol}(\Lambda_L)} tr(\varphi(H_\omega)\chi_{\Lambda_L})$$

and

$$\int_{\mathbb{R}} \varphi(\lambda) d\mu(\lambda) = \mathbb{E}\Big(tr(\chi_{\Lambda_1}\varphi(H_\omega)\chi_{\Lambda_1})\Big).$$

For those two measures we have the following result proved in (19),

Theorem 3.2. *For almost all $\omega \in \Omega$ and for all $\varphi \in C_0^\infty(\mathbb{R})$ we have*

$$\lim_{k\to\infty} \langle \varphi, d\mu_L\rangle = \langle \varphi, d\mu\rangle.$$

Remark 3.3. *We call the non-random probability measure μ the density of states measure. It verifies the following fundamental properties*

$$N(E) = \mu((-\infty, E]),$$

$$\Sigma(H_\omega) = supp(\mu).$$

As mentioned above we denote the operator H_ω restricted to $L^2(\Lambda_L)$ with Dirichlet boundary conditions at $\partial\Lambda_L$ by $H^D_{\Lambda_L} = H^D_{\Lambda_L}(\omega)$, and the corresponding operator with Neumann boundary conditions at $\partial\Lambda_L$ by $H^N_{\Lambda_L} = H^N_{\Lambda_L}(\omega)$. We use an analogous notation for $\widetilde{H}_\omega$.

Since these operators have compact resolvents their spectra are purely discrete. We order their eigenvalues in increasing order with repetition of eigenvalues according to multiplicity. The eigenvalues of $H^X_{\Lambda_L}$ with $X = D$ or $X = N$ are denoted by

$$E^X_1(\Lambda_L) \le E^X_2(\Lambda_L) \le \cdots \le E^X_n(\Lambda_L) \le \cdots \tag{3.24}$$

and those of $\widetilde{H}^X_{\Lambda_L}$ by

$$\widetilde{E}^X_1(\Lambda_L) \leq \widetilde{E}^X_2(\Lambda_L) \leq \cdots \leq \widetilde{E}^X_n(\Lambda_L) \leq \cdots \tag{3.25}$$

If A is any operator with discrete spectrum, bounded below and $E \in \mathbb{R}$ define $N(A, E)$ to be the number of eigenvalues of A less than or equal to E, of course counted with their multiplicities.

To prove Theorem 2.3, we prove a lower and an upper bounds on $N(E)$. The upper and lower bounds are proven separately and based on the following result (see (9) or (17)).

$$\frac{1}{\mid \Lambda_L \mid}\mathbb{E}\{N(H^D_{\Lambda_L}(\omega), E)\} \leq N(E) \leq \frac{1}{\mid \Lambda_L \mid}\mathbb{E}\{N(H^N_{\Lambda_L}(\omega), E)\}. \tag{3.26}$$

and

$$\frac{1}{\mid \Lambda_L \mid}\mathbb{E}\{N(\widetilde{H}^D_{\Lambda_L}(\omega), E)\} \leq \widetilde{N}(E) \leq \frac{1}{\mid \Lambda_L \mid}\mathbb{E}\{N(\widetilde{H}^N_{\Lambda_L}(\omega), E)\}. \tag{3.27}$$

Inequalities in (3.26) and (3.27) are based on the method of Neumann-Dirichlet bracketing (see (18) and (9)). Indeed

$$H^N_{\Lambda_1}(\omega) \oplus H^N_{\Lambda_2} \leq H^N_{\Lambda_1 \cup \Lambda_2}(\omega) \tag{3.28}$$

and

$$H^D_{\Lambda_1 \cup \Lambda_2}(\omega) \leq H^D_{\Lambda_1}(\omega) \oplus H^D_{\Lambda_2}(\omega), \tag{3.29}$$

hold on $L^2(\Lambda_1 \cup \Lambda_2)$, for all bounded cubes $\Lambda_1, \Lambda_2 \subset \mathbb{R}^d$ whenever the interior of $\Lambda_1 \cap \Lambda_2$ is empty ((18)).

3.2 The Dirichlet case:

3.2.1 The upper bound

The upper bound is proved by a comparison procedure. Indeed, let,

$$\widetilde{V}_\omega = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma \chi_{\mathcal{S}}(x - \gamma),$$

and

$$\widetilde{H}_\omega = -\Delta + \widetilde{V}_\omega$$

For any $\Lambda_L \subset \mathbb{R}^d$, we set

$$\mathcal{Q}^{\Lambda_L}_\omega(\varphi, \psi) = \langle \varphi, H_0 \psi \rangle, \quad \varphi, \psi \in H^1_0(\Lambda_L \backslash \Gamma_\omega) = \mathcal{D}_{\Lambda_L}, \tag{3.30}$$

and

$$\widetilde{\mathcal{Q}}^{\Lambda_L}_\omega(\varphi, \psi) = \langle \varphi, \widetilde{H}_\omega \psi \rangle, \quad \varphi, \psi \in H^1_0(\Lambda_L) = \widetilde{\mathcal{D}}_{\Lambda_L}. \tag{3.31}$$

From (18), we recall the following result,

Lemma 3.4. *For any $L, k \in \mathbb{N}^*$, we have*

$$\sup_{\varphi_1, \cdots, \varphi_{k-1} \in \widetilde{\mathcal{D}}_{\Lambda_L}} \inf_{\psi \in [\varphi_1, \cdots, \varphi_{k-1}]^\perp \cap \widetilde{\mathcal{D}}_{\Lambda_L}, \|\psi\|=1} \widetilde{\mathcal{Q}}^{\Lambda_L}_\omega(\psi, \psi) \leq$$

$$\sup_{\psi_1, \cdots, \psi_{k-1} \in \mathcal{D}_{\Lambda_L}} \inf_{\varphi \in [\psi_1, \cdots, \psi_{k-1}]^\perp \cap \mathcal{D}_{\Lambda_L}, \|\varphi\|=1} \mathcal{Q}^{\Lambda_L}_\omega(\varphi, \varphi). \tag{3.32}$$

From Lemma 3.4, one deduces that for any $n \in \mathbb{N}^*$, we have

$$E_n(\widetilde{H}_\omega(\Lambda_L)) \leq E_n(H_\omega(\Lambda_L)). \tag{3.33}$$

Thus, we get that for any $E \in \mathbb{R}$,

$$N(H_{\Lambda_L}(\omega), E) \leq N(\widetilde{H}_{\Lambda_L}(\omega), E) \tag{3.34}$$

We notice that for $\widetilde{H}_\omega$ it is already known that it exhibits Lifshitz tails, by the result of Kirsch and Simon (13). This ends the proof of the upper bound. □

3.2.2 The lower bound

We recall that for any $\Lambda_L \subset \mathbb{R}^d$, we have

$$H_\omega \leq H^D_{\Lambda_L}(\omega) \leq H^D_{1,\Lambda_L}. \tag{3.35}$$

So by the min-max argument we get that

$$E^D_1(\Lambda_L) \leq E^D_1(H_{1,\Lambda_L}). \tag{3.36}$$

Using equation (3.26) one gets,

$$\begin{aligned} N(E) &\geq \frac{1}{L^d} \cdot \mathbb{P}\{E^D_1(\Lambda_L) \leq E\} \\ &\geq \frac{1}{L^d} \cdot \mathbb{P}\{E^D_1(\Lambda_L) \leq E \text{ and } \forall \gamma \in \Lambda_L \cap \mathbb{Z}^d,\ \omega_\gamma = 0\} \\ &= \frac{1}{L^d} \cdot \mathbb{P}\{E^D_1(H_{0,\Lambda_L}) \leq E \text{ and } \forall \gamma \in \Lambda_L \cap \mathbb{Z}^d,\ \omega_\gamma = 0\} \\ &\geq \mathbb{P}\{\omega_0 = 0\}^{|\Lambda_L|} = (1-p)^{L^d}. \end{aligned} \tag{3.37}$$

By this, we deal with the estimate of the volume of Λ_L i.e the order of L. As H_{0,Λ_L} is the free Laplacian restricted to Λ_L, it is known that $E^D_1(-\Delta_{\Lambda_L}) \simeq \frac{1}{L^2}$. So to be less than E, L should be $c \cdot E^{-\frac{d}{2}}$. This ends the proof of the lower bound.

3.3 The Neumann case

We estimate:

$$\begin{aligned} N_N(E) &\geq \frac{1}{L^d}\, \mathbb{E}\Big(N\big(\widetilde{H}^D_{\Lambda_L}(\omega), E\big)\Big) \\ &\geq \frac{1}{L^d}\, \mathbb{P}\Big(E_1\big(\widetilde{H}^D_{\Lambda_L}(\omega)\big) \leq E\Big) \end{aligned} \tag{3.38}$$

for arbitrary L. By the min-max-principle we have for any $\psi \in \mathcal{Q}\big(\widetilde{H}^D_{\Lambda_L}(\omega)\big)$, $\psi \neq 0$, we have

$$E_1\big(\widetilde{H}^D_{\Lambda_L}(\omega)\big) \ \leq\ \frac{\langle \nabla\psi, \nabla\psi\rangle_{D^L_\omega}}{\langle \psi, \psi\rangle_{D^L_\omega}}$$

where we used $\langle \cdot, \cdot \rangle_{D^L_\omega}$ to denote the scalar product in the space $L^2(D^L_\omega) = L^2(D_\omega \cap \Lambda_L)$. We conclude

$$N_N(E) \ \geq\ \frac{1}{L^d}\, \mathbb{P}\Big(\langle \nabla\psi, \nabla\psi\rangle_{D^L_\omega} \ \leq\ E\, \langle \psi, \psi\rangle_{D^L_\omega}\Big) \tag{3.39}$$

Now we construct a test function $\psi\,(=\phi_L)$ as follows: Let ϕ be a smooth function on $\mathbb{R}^d$ with $\operatorname{supp}\phi \subset]-\frac{1}{2},\frac{1}{2}[^d$, $0 \le \phi(x) \le 1$ and $\phi(x) = 1$ for $x \in [-\frac{1}{4},\frac{1}{4}]^d$. We set:

$$\phi_L(x) \;=\; \frac{1}{L^{d/2}}\,\phi\big(\frac{x}{L}\big) \tag{3.40}$$

It follows that ϕ_L (or rather its restriction to D_ω^L) belongs to $\mathcal{Q}(\widetilde{H}^D_{\Lambda_L}(\omega))$ for all ω, so we may take ϕ_L as a test function in (3.39).

We have

$$\begin{aligned}\langle\nabla\phi_L,\nabla\phi_L\rangle_{D_\omega^L} &\le \frac{1}{L^d}\int\limits_{\Lambda_L}\frac{1}{L^2}\,\big|(\nabla\phi)\big(\frac{x}{L}\big)\big|^2\,dx\\ &\le C_1\,\frac{1}{L^2}\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\langle\phi_L,\phi_L\rangle_{D_\omega^L} &\ge \frac{1}{L^d}\int\limits_{D_1^L}\big|\phi(\frac{x}{L}\big|^2\,dx\\ &\ge \frac{1}{L^d}\,\operatorname{vol}(D_1\cap\Lambda_{L/4})\\ &\ge C_2\end{aligned} \tag{3.42}$$

where $C_2 > 0$ is a constant which only depends on the volume $\operatorname{vol}(M)$ of the set M.

Thus we have proved that $N_N(E) \ge \frac{1}{L^d}$ as longs as $C_1\frac{1}{L^2} \le C_2E$. This can be guaranteed by choosing $L = C'\,E^{-1/2}$ with a suitable constant C'. Thus we proved:

$$N_N(E) \;\ge\; C\,E^{d/2} \tag{3.43}$$

for some constant $C > 0$.

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