Composition algebras over commutative rings[1]

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Conference on linear algebraic groups and cohomology
Emory University, Atlanta, GA, USA, May 16-20, 2007
Friday, May 18, 2007, 11:25-12:10.

0. Introduction.

First of all, I would like to thank the organizers, most notably my good fried Skip Garibaldi, for the invitation and for the opportunity to give a talk at this wonderful conference.

My purpose in this lecture will be to give a survey of what is presently known about composition algebras over arbitrary commutative rings. While composition algebras over fields, thanks to their intimate connection with quadratic forms, are reasonably well understood, precious little is known about the subject in full generality. In fact, one could argue with some justification that the only things really new in this lecture are related in one way or another to the three open problems that I will discuss at the end. But rather than beginning at the end, I prefer to begin with the

1. Definition and elementary properties.

Even when it comes to the definition of composition algebras over fields, the literature offers a rather confusing picture, with many different concepts floating around under the same name. In this lecture, I will follow a suggestion of Ottmar Loos to define the concept of a composition algebra as follows.

Given an arbitrary commutative ring k, let us first fix some notation. I will always write k-**alg** for the category of commutative k-algebras (with 1). For a k-module M and a $R \in k$-**alg**, I abbreviate $M_R = M \otimes R$ as R-modules.

1.1. Definition. By a *composition algebra* over k, I mean a non-associative k-algebra C (so C is a k-module together with a bilinear multiplication, subject to no further restrictions) satisfying the following conditions.

- C has a unit.
- C is faithful and finitely generated projective as a k-module.
- C has a *norm*, i.e., there exists a quadratic form $n : C \to k$ such that

    (i) *n permits composition*: $n(xy) = n(x)n(y)$.

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(ii) n is *separable*: Writing

$$n(x, y) := n(x + y) - n(x) - n(y)$$

for the bilinearization of n, then for all *fields* $K \in k$-**alg**, the extended quadratic form $n = n_K : C_K \to K$ is non-degenerate, so for all $x \in C_K$,

$$n(x) = n(x, y) = 0 \ (y \in C_K) \Longrightarrow x = 0.$$

Standard arguments as given in Knus [2, II §7], for example, then lead to what I call the

1.2. Basic facts. Composition algebras of rank r over k

- are invariant under base change,
- have a unique norm $n = n_C$, so we are allowed to call C *non-singular* if the bilinearization of n_C is non-singular in the sense that it determines a linear isomorphism from the k-module C onto its dual C^*,
- are *quadratic*: $x^2 - t_C(x)x + n_C(x)1 = 0$, where $t_C := n_C(1, -)$ is the *trace* of C,
- exist only in ranks $1, 2, 4, 8$,
- are non-singular unless $C \cong k$ and $2 \notin k^\times$, in which case they are not,
- are *alternative*: The *associator*

$$C \times C \times C \longrightarrow C, \quad (x, y, z) \longmapsto [x, y, z] := (xy)z - x(yz)$$

  is alternating,
- are associative iff $r \leq 4$,
- are commutative associative iff $r \leq 2$,
- have a *canonical involution*

$$\iota_C : C \longrightarrow C, \quad x \longmapsto \overline{x} := t_C(x)1 - x.$$

Before proceeding, let's give some

1.3. Examples of composition algebras. Let C be a composition algebra of rank r over k. For

- $r = 1$, $C \cong k$, $n_C(\alpha) = \alpha^2$,
- $r = 2$, C is a quadratic étale algebra, $n_C(x) = \det L_x$,
- $r = 4$, C is a *quaternion algebra*, i.e., an Azumaya algebra of degree 2, $n_C = \mathrm{Nrd}_C$,
- $r = 8$, C by definition is an *octonion algebra*.

2. Construction methods.

Actually, Example 1.3 for $r = 8$ doesn't give us more than the definition of an octonion algebra. We are therefore in desperate need of construction methods. We begin with the

2.1. Cayley-Dickson construction. (Petersson [5], Pumplün [7]) The
Input consists of

- a non-singular associative composition algebra B over k,
- a right B-module P that is *locally free of rank* 1, so locally looks just like B as a right B-module,
- a non-singular hermitian form $h : P \times P \to B$ that is *diagonal* in the sense that $h(x, x) \in k = k1_B$ for all $x \in P$.

Then the
Output is

- a composition algebra $C = \mathrm{Cay}(B, P, h)$ that lives on the k-module $B \oplus P$ under the multiplication
$$(u \oplus x)(v \oplus y) := \big(uv + h(y, x)\big) \oplus (x\overline{v} + yu).$$

We then have:

- B embeds into C as a subalgebra through the first factor,
- $\mathrm{rk}(B) = r \Longrightarrow \mathrm{rk}(C) = 2r$,
- $n_C(u \oplus x) = n_B(u) - h(x, x)$.

Conversely, one can prove

2.2. Theorem. *Let C be a composition algebra of rank $2r$ over k and $B \subseteq C$ a non-singular composition subalgebra of rank r. Then there exist P, h as above such that the inclusion $B \hookrightarrow C$ extends to an isomorphism* $\mathrm{Cay}(B, P, h) \xrightarrow{\sim} C$. □

2.3. Examples. If B is a quaternion algebra over k, the Cayley-Dickson construction produces an octonion algebra $C = \mathrm{Cay}(B, P, h)$; conversely, every octonion algebra over k containing a quaternion subalgebra arises in this way. While this extra condition is always fulfilled if we are working over a field, it doesn't hold in general. In fact, there are famous examples due to Knus-Parimala-Sridharan [3] of octonion algebras C over the polynomial ring $k = K[X_1, \ldots, X_n]$, K a field of characteristic not 2, $n \geq 2$, having the trace zero elements

$$C_0 := \{x \in C \,|\, t_C(x) = 0\}$$

as an indecomposable quadratic subspace of rank 7.

The second construction method we would like to discuss is the

2.4. Zorn construction. (Thakur [8] for $\frac{1}{2} \in k$, Petersson [6]) Here the
Input consists of

- a non-singular commutative associative composition algebra D over k,
- a ternary hermitian space (V, h) over D with trivial discriminant,
- a *trivialization* of $\bigwedge^3(V, h)$, i.e., an isometry

$$\Delta : \bigwedge^3(V, h) \xrightarrow{\sim} (D, \langle 1 \rangle),$$

giving rise to the induced hermitian vector product

$$V \times V \longrightarrow V, \quad (x, y) \longmapsto x \times_{h,\Delta} y,$$

via

$$h(x \times_{h,\Delta} y, z) = \Delta(x \wedge y \wedge z).$$

Then the
Output is

- a composition algebra $C = \mathrm{Zor}(D, V, h, \Delta)$ that lives on the k-module $D \oplus V$ under the multiplication

$$(a \oplus x)(b \oplus y) := \big(ab - h(x, y)\big) \oplus (xb + y\bar{a}).$$

We have

- D embeds into C as a subalgebra through the first factor,
- $\mathrm{rk}(D) = r \Longrightarrow \mathrm{rk}(C) = 4r$,
- $n_C(a \oplus x) = n_D(a) + h(x, x)$.

Conversely, one can prove

2.5. Theorem. *Let C be a composition algebra of rank $4r$ over k and $D \subseteq C$ a non-singular composition subalgebra of rank r. Then there exist V, h, Δ as above such that the inclusion $D \hookrightarrow C$ extends to an isomorphism* $\mathrm{Zor}(D, V, h, \Delta) \xrightarrow{\sim} C$. ☐

2.6. Remark. Again the Zorn construction doesn't apply to the examples of Knus-Parimala-Sridharan. However, it

- yields all quaternion algebras for $r = 1$ if 2 is a unit and, at the other extreme,
- it does apply if 2 is sufficiently far away from being a unit in the following sense.

2.7. Proposition. *Let C be a composition algebra of rank > 1 over k and suppose* $2 \in \mathrm{Jac}(k)$, *the Jacobson radical of k. Then C contains a quadratic étale subalgebra.* ☐

2.8. The case D split quadratic étale. (Petersson [5]) If $D = k \oplus k$, the Zorn construction boils down to the following. The

Input consists of

- a finitely generated projective k-module M of rank 3 with trivial determinant,
- a *trivialization* of $\bigwedge^3(M)$, i.e., an isomorphism

  $$\theta : \bigwedge^3(M) \xrightarrow{\sim} k,$$

  giving rise to vector products

  $$\times_\theta : M \times M \longrightarrow M^*, \quad \times_\theta : M^* \times M^* \longrightarrow M$$

  defined by

  $$\langle w, u \times_\theta v \rangle = \theta(u \wedge v \wedge w), \quad \langle u^* \times v^*, w^* \rangle = \theta^{*-1}(u^* \wedge v^* \wedge w^*),$$

Then the

Output is

- an octonion algebra $C = \mathrm{Zor}(M, \theta)$ living on the k-module

  $$\mathrm{Zor}(M, \theta) = \begin{pmatrix} k & M^* \\ M & k \end{pmatrix}$$

  under the multiplication

  $$\begin{pmatrix} \alpha_1 & v^* \\ v & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & w^* \\ w & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 + \langle w, v^* \rangle & \alpha_1 w^* + \beta_2 v^* + v \times_\theta w \\ \beta_1 v + \alpha_2 w - v^* \times_\theta w^* & \langle v, w^* \rangle + \alpha_2\beta_2 \end{pmatrix}.$$

The octonion algebra $\mathrm{Zor}(M, \theta)$ will be called a *twisted Zorn vector matrix algebra.* If $M = k^3$ is free and $\theta = \det$ is the the ordinary determinant, we speak of the Zorn vector matrix algebra over k.

3. Open problems.

As I said earlier, there are three problems that I would like to discuss here. Of course, I do so in the fond hope that you might be able to give me some clues how to solve them. Anyway, here they are. I begin with the

3.1. Norm equivalence problem. Are composition algebras classified by their norms, i.e., if C, C' are composition algebras over k, does it follow that

$$C \cong C' \Longleftrightarrow n_C \cong n_{C'} \mathbf{?}$$

Here are a few comments.

- For k a field, the answer is yes, by Witt cancellation of quadratic forms (Jacobson [1], van der Blij-Springer [9]).
- For $\mathrm{rk}(C) \leq 2$, the answer is easily seen to be yes.
- For $\mathrm{rk}(C) = 4$, the answer is yes (Knus [2]).
- For $\mathrm{rk}(C) = 8$, **???**

Closely related to the norm equivalence problem is the isotopy problem. To let you appreciate this problem, we take a detour through

3.2. The notion of isotopy. (McCrimmon [4]) Let C be a composition algebra over k.

- $a \in C$ is defined to be *invertible* iff $n_C(a) \in k^\times$. In this case, $a^{-1} := n_C(a)^{-1}\overline{a}$ is an honest-to-goodness inverse: $aa^{-1} = a^{-1}a = 1$. We put
$$C^\times := \{a \in C \mid a \text{ is invertible}\}.$$
- Let $a, b \in C^\times$ and $C^{(a,b)}$ be the k-algebra that lives on the k-module C under the multiplication
$$x \cdot_{a,b} y := (xa)(by).$$
Then one checks easily that
- $C^{(a,b)}$ is a composition algebra with unit $1^{(a,b)} = (ab)^{-1}$ and norm $n_{C^{(a,b)}} = \langle n_C(ab)\rangle \,.\, n_C$.

We can now phrase

3.3. The isotopy problem. Does isotopy of composition algebras reduce to isomorphism, i.e., given a composition algebra C over k and $a, b \in C^\times$, does it follow that
$$C^{(a,b)} \cong C\,\textbf{?}$$

Again I have a few comments.

- For $\mathrm{rk}(C) \leq 4$, the answer is yes since C is associative and
$$L_{ab} : C^{(a,b)} \xrightarrow{\sim} C$$
is easily seen to be an isomorphism.
- For $\mathrm{rk}(C) = 8$ **???**
- A positive answer to the norm equivalence problem yields a positive answer to the isotopy problem since
$$L_{ab} : n_{C^{(a,b)}} \xrightarrow{\sim} n_C$$
is an isometry, so a positive solution to the norm equivalence problem would imply $C^{(a,b)} \cong C$. In particular:
- The answer is yes if k is a field.
- We may always assume $b = a^{-1}$ since one checks that
$$L_{ab} : C^{(a,b)} \xrightarrow{\sim} C^{(a^2b,b^{-1}a^{-2})}$$
is an isomorphism. Setting
$$C^a := C^{(a^{-1},a)},$$
we have
- $(C^a)^b = C^{ab}$,
- $C^b \cong C^{ba^3} \cong C^{a^2ba} \cong C^{a^6b}$, in particular
- $C^a \cong C$ if a is a third power, so 3 is indeed a bad prime for the group G_2,
- $C^a \cong C$ if $k[a]^\perp \cap C^\times \neq \emptyset$,
- $C^a \cong C$ if a has trace zero: $t_C(a) = 0$.

3.4. The isotopy problem and the Cayley-Dickson construction. Let

- B be a non-singular associative composition algebra over k,
- P a locally free right B-module of rank 1,
- $h : P \times P \to B$ a non-singular hermitian form,
- $C = \mathrm{Cay}(B, P, h)$,
- $a \in B^{\times}$

Then

- $C^a = \mathrm{Cay}(B, P^a, h^a)$, where
- $P^a = P$ as k-modules with the twisted B-action

$$P^a \times B \longrightarrow P^a, \quad (x, u) \longmapsto x(aua^{-1}),$$

- $h^a : P^a \times P^a \to B$, $h^a(x, y) = a^{-1}h(x, y)a$,
- $C^a \cong C$ **???**

3.5. The isotopy problem and the Zorn construction. Let

- D be a non-singular commutative associative composition algebra over k,
- (V, h) a ternary hermitian space over D with trivial discriminant,
- $\Delta : \bigwedge^3(V, h) \xrightarrow{\sim} (D, \langle 1 \rangle)$ a trivialization of $\bigwedge^3(V, h)$,
- $C = \mathrm{Zor}(D, V, h, \Delta)$,
- $a \in D^{\times}$.

Then

- $s := a\overline{a}^{-1} \in D$ has norm 1, so $s\Delta$ is another trivialization of $\bigwedge^3(V, h)$ and
- $C^a = \mathrm{Zor}(D, V, h, s\Delta)$,
- $C^a \cong C$ **???**

Since this is a conference on algebraic groups and cohomology, I would like to draw a connection between

3.6. The isotopy problem and invariants of F_4. If k is a field, the 3-invariant mod 2 of a group of type F_4 may be described as follows: The group is completely determined by an Albert algebra A over k, which in turn can be co-ordinatized by 3-by-3 hermitian matrices having entries in some octonion algebra C over k and the Arason invariant of n_C, the corresponding 3-fold Pfister form, is the group invariant we are looking for. The whole point of this is, of course, that A determines C uniquely.

It doesn't seem at all clear whether this uniqueness continues to hold if k is a ring. In fact, the best one can possibly hope for is uniqueness up to isotopy since it is easy to see that if C co-ordinatizes A in the manner described above, so does every isotope. But then it could very well be that passing to isotopes of C is just about the only degree of freedom one is allowed here, in which case at least the norm of C would be an invariant. So passing to the corresponding Arason invariant (assuming it exists) would yield a group invariant over rings that generalizes the 3-invariant mod 2 over fields.

But I have promised you three open problems. Here is the third one.

3.7. The co-ordinatization problem. For $i = 1, 2$, let M_i be a finitely generated projective k-module of rank 3 and $\theta_i : \bigwedge^3(M_i) \xrightarrow{\sim} k$ a trivialization of $\bigwedge^3(M)$. Find conditions in terms of (M_1, θ_1) and (M_2, θ_2) that are necessary and sufficient for the octonion algebras $\mathrm{Zor}(M_1, \theta_1)$ and $\mathrm{Zor}(M_2, \theta_2)$ to be isomorphic.

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