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# Completions of Jordan Division Algebras of Degree Three over Fields with a discrete Valuation<sup>1</sup>

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**1.** Let  $K$  be a field and  $v : K \rightarrow \mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$  a discrete valuation. Writing  $\hat{K}$  for the completion of  $K$  relative to  $v$  and  $v : \hat{K} \rightarrow \mathbb{Z}_\infty$  also for the unique extension of  $v$  to a discrete valuation of  $\hat{K}$ , we will be concerned here with a question that has been raised by Richard Weiss in an Email of December 29, 2005, to the author and may be phrased as follows.

Given a Jordan division algebra  $J$  of degree 3 and dimension at least 4 over  $K$  whose generic trace is not identically zero, let us assume that for any cubic subfield  $E/K$  of any isotope of  $J$ , the base change  $(E \otimes \hat{K})/\hat{K}$  ( $\otimes = \otimes_K$ ) continues to be a field. Then the question is whether  $J \otimes \hat{K}$  is a Jordan division algebra over  $\hat{K}$ . In the sequel, we wish to answer this question affirmatively.

**2.** We begin with a few preliminaries of a more general nature. To this end, let  $k$  be any field and  $\mathcal{M} = (W, N, \sharp, 1)$  a finite-dimensional cubic norm structure over  $k$ . We write  $J = J(\mathcal{M})$  for its associated Jordan algebra and  $T : J \times J \rightarrow k$  for the bilinear trace of  $\mathcal{M}$ . Free use will be made of the McCrimmon formalism [3].

**Proposition 1.** *If  $J$  is a division algebra, then the following statements are equivalent.*

- (i)  $T$  is degenerate.
- (ii)  $T = 0$ .
- (iii)  $k$  has characteristic 3 and  $J/k$  is a purely inseparable field extension of exponent at most 1.

*Proof.* (i)  $\implies$  (ii). For  $x \in J$  we have  $T(x, x^\sharp) = 3N(x)$  [3, (13)], and since  $J$  is a division algebra,  $k$  has characteristic 3. But then  $\{0\} \neq \text{rad } T \subseteq J$ , always being an outer ideal [2, 16.12], is in fact an ideal, so (ii) holds.

(ii)  $\implies$  (iii). This direction is already implicit in the proof of [8, Theorem 3.1]. We claim that  $N : J \rightarrow k^+$  is a homomorphism of Jordan rings, i.e., of Jordan algebras over  $\mathbb{Z}$ . Indeed, since  $T = 0$ ,  $N$  is additive, and the relations  $N(1) = 1$ ,  $N(U_x y) = N(x)^2 N(y) = U_{N(x)} N(y)$  prove the assertion. Furthermore,  $3 = T(1, 1) = 0$  in  $k$ , so  $k$  has characteristic 3. Since  $J$  is a division algebra, the homomorphism  $N$  must be injective, showing that  $J$ , being isomorphic to a (linear) Jordan division subring of  $k^+$ , must be a field. The minimum equation of any  $x \in J$  reduces to  $x^3 = N(x)1$ , so that field is purely inseparable of exponent at most 1 over  $k$ .

(iii)  $\implies$  (i). Obvious. □

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We say that  $J$  is *separable* if every field extension of  $J$  has nil radical equal to zero, this being automatic if  $T$  is non-degenerate [7, Theorem 6].  $J$  is called *absolutely simple* if all its base field extensions are simple.

**Proposition 2.** *If  $J$  is simple and separable of dimension at least 4, then  $J$  is absolutely simple.*

*Proof.* Let  $k_s$  be the separable closure of  $k$ . We must show that  $J_s = J \otimes k_s$  is simple. If not, [9, Theorem 1] combined with [2, 10.17] allows an identification  $J_s = k_s \oplus J'$  as a direct sum of ideals, where  $J'$  is the Jordan algebra of a non-degenerate quadratic form with base point having dimension at least 3. Thus  $J'$  is simple, and  $c := 1_{k_s} \oplus 0$  is the only absolutely primitive idempotent of  $J_s$  whose Peirce components (in the labelling of Loos [2, § 5]) satisfy the relation  $(J_s)_1(c) = \{0\}$ . We conclude that  $c$  remains fixed under the action of the absolute Galois group of  $k$  and hence belongs to  $J$ . Since we still have  $J_1(c) = \{0\}$ , the Peirce rules imply  $J = kc \oplus J_0(c)$  as a direct sum of ideals, contradicting the simplicity of  $J$ .  $\square$

**3.** After these preparations, we can now turn to the question raised in **1.** and answer it as follows.

**Proposition 3.** *Notations being as in **1.**, let  $J$  be a finite-dimensional Jordan division algebra of degree 3 and dimension at least 4 over  $K$  whose generic trace is not identically zero. If for any cubic subfield  $E/K$  of any isotope of  $J$  the base change  $(E \otimes \hat{K})/\hat{K}$  is a field, then  $J \otimes \hat{K}$  is a Jordan division algebra over  $\hat{K}$ .*

*Proof.* As in **2.**, we have  $J = J(\mathcal{M})$  for some cubic norm structure  $\mathcal{M} = (W, N, \sharp, 1)$  over  $K$ . The map  $w : J \rightarrow \mathbb{Q}_\infty := \mathbb{Q} \cup \{\infty\}$  defined by

$$(1) \quad w(x) := \frac{1}{3}v(N(x))$$

for  $x \in J$  satisfies  $w(x) = \infty \Leftrightarrow x = 0$  and  $w(U_x y) = 2w(x) + w(y)$  for all  $x, y \in J$ . Hence it will be a discrete valuation in the sense of [5, §§ 2,3] once we have shown the non-archimedean triangle inequality  $w(x + y) \geq \min(w(x), w(y))$ . To do so, we may assume  $y = 1_J$  by passing to an appropriate isotope and then that  $x$  belongs to some cubic subfield  $E/K$  of  $J$ . But since  $\hat{E} := E \otimes \hat{K}$  is a field by hypothesis, (1) defines the unique valuation of  $\hat{E}$  extending  $v$  on  $\hat{K}$ . In particular, the non-archimedean inequality holds in  $\hat{E}$ , hence a fortiori in  $E$ , and we are done.

Summing up, we have shown that  $w$  is a discrete valuation of  $J$  extending  $v$ . On the other hand, by Propositions 1,2 combined,  $J$  is absolutely simple, allowing us to argue as in [6, **2.**, p.126]. Accordingly, write  $\hat{J}$  for the completion of  $J$  relative to  $w$ , which is a division algebra [4, Satz 3.3.10], and observe that, thanks to the universal property of completions, the composite map  $K \rightarrow J \rightarrow \hat{J}$  extends to a homomorphism  $\hat{K} \rightarrow \hat{J}$ , giving  $\hat{J}$  the structure of a  $\hat{K}$ -algebra. Hence the embedding  $J \rightarrow \hat{J}$  induces a  $\hat{K}$ -homomorphism  $J \otimes \hat{K} \rightarrow \hat{J}$ , which, by absolute simplicity, is injective. Thus  $J \otimes \hat{K}$ , being a finite-dimensional  $\hat{K}$ -subalgebra of a Jordan division algebra, must be a Jordan division algebra itself.  $\square$

**4.** By Proposition 1, the only case excluded from Proposition 3 is the one of a purely inseparable field extension  $F/K$  of exponent at most 1 over a field of characteristic 3. One can avoid this exclusion by dealing with Henselizations instead of completions throughout, see [10] for the relevant facts about Henselizations. In particular, the Henselization,  $K^H$ , of  $K$  relative to  $v$  is separably algebraic, so  $F \otimes K^H$  must be a field [1, Theorem 8.46]. Hence the Henselian analogue of Proposition

3 holds for  $F$ . Unfortunately, it is not clear whether the same can be said about the algebras actually allowed in Proposition 3, the difficulty being that Henselizations (in analogy to completions) of Jordan division rings don't seem to make sense at the moment, see [4, § 4] for further discussions.

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