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RESEARCH ARTICLE

Albert algebras over $\mathbb{Z}$ and other rings

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Abstract
Albert algebras, a specific kind of Jordan algebra, are naturally distinguished objects among commutative nonassociative algebras and also arise naturally in the context of simple affine group schemes of type F_4, E_6, or E_7. We study these objects over an arbitrary base ring R, with particular attention to the case $R = \mathbb{Z}$. We prove in this generality results previously in the literature in the special case where R is a field of characteristic different from 2 and 3.

Contents

1. Introduction

Albert algebras, which are a specific kind of Jordan algebra, are naturally distinguished objects among commutative nonassociative algebras and also arise naturally in the context of simple affine group schemes of type F_4, E_6, or E_7. We study these objects over an arbitrary base ring R, with particular attention to the case $R = \mathbb{Z}$. We prove in this generality results previously in the literature in the special case where R is a field of characteristic different from 2 and 3.

Why Albert algebras?

In the setting of semisimple algebraic groups over a field, a standard technique for computing with elements of a group — especially an anisotropic group — is to interpret the group in terms of automorphisms of some algebraic structure, such as viewing an adjoint group of type B_n as the special orthogonal group of a quadratic form of dimension $2n+1$, or an adjoint group of inner type A_n as the automorphism group of an Azumaya algebra of rank $(n+1)^2$. This approach can be seen in many references, from [Weil], through [KnMRT] and [Conrad]. In this vein, Albert algebras appear as a natural tool for computations related to F_4, E_6, and E_7 groups, as we do below.

In the setting of nonassociative algebras, Albert algebras also arise naturally. Among commutative not-necessarily-associative algebras under additional mild hypotheses (the field has characteristic $\neq 2, 3, 5$ and the algebra is metrized), every algebra satisfying a polynomial identity of degree ≤ 4 is a Jordan algebra (see [ChG, Proposition A.8]). Jordan algebras have an analogue of the Wedderburn-Artin theory for associative algebras [J68, p. 201, Corollary 2], and one finds that all the simple Jordan algebras are closely related to associative algebras (more precisely, "are special") except for one kind, the Albert algebras (see, for example [J68, p. 210, Theorem 11] or [McCZ]).

Our contribution

In the setting of nonassociative algebras, we prove a classification of Albert algebras over $\mathbb{Z}$ (Theorem 14.3), which was viewed as an open question in the context of nonassociative algebra; here, we see that it is equivalent to the classification of groups of type F_4, which was known (see [Conrad], which leverages [Gr] and [ElkiesGr]). We also prove new results about ideals in Albert algebras (Theorem 8.2), about isotopy of Albert algebras over semilocal rings (Theorem 13.3), and about the number of generators of an Albert algebra (Proposition 12.1). We have not seen Lemma 15.1 in the literature, even in the case of a base field of characteristic different from 2 and 3.

In the setting of affine group schemes, the language of Albert algebras provides a way to give concrete descriptions of the affine group schemes over $\mathbb{Z}$ (see Section 18). In that language, a clever computation in [ElkiesGr] appears as an example of a general mechanism known as isotopy (see Definition 14.1). To facilitate these applications, we present the definition of Albert algebras in a streamlined way (see Definition 7.1). Note that they are defined as a type of what was formerly called a "quadratic" Jordan algebra — because instead of a bilinear multiplication, one has a quadratic map, the U-operator — and that the definition makes sense whether or not 2 is invertible in the base ring. Applying the definition here allows one to replace, in some proofs, "global" computations over $\mathbb{Z}$ as one finds in [Conrad] with "local" computations over an algebraically closed field that exist in several places in the literature (see, for example, the proof of Lemma 9.1 and Section 18).

A different definition

The definition of Albert algebra over a ring R given here (Definition 7.2) is in the context of para-quadratic algebras as recalled at the beginning of Section 5 — such an algebra is an R-module M with a distinguished element 1_M and a quadratic map $U\colon M \to \mathrm{End}_R(M)$ such that $U_{1_M} = \mathrm{Id}_M$. There are no other axioms to check. We then define a specific para-quadratic algebra, $\mathrm{Her}_3(\mathrm{Zor}(R))$, in Definition 6.7, and define an Albert R-algebra J to be a para-quadratic R-algebra such that $J \otimes S \cong \mathrm{Her}_3(\mathrm{Zor}(R)) \otimes S$ for some faithfully flat R-algebra S.

A different approach is taken by references such as [Pe19, Section 6.1] or [Als21]. They define an Albert R-algebra to be a cubic Jordan R-algebra (Definition 6.2) J whose underlying R-module is projective of rank 27 and $J \otimes F$ is a simple algebra for every homomorphism from R to a field F. This definition involves axioms that in principle need to be verified over all R-algebras. The two definitions give the same objects. Theorem 17 in [Pe19] states that an Albert algebra in the sense of that paper is an Albert algebra in the sense of this paper by proving the existence of the required faithfully flat R-algebra; a detailed proof has not been published but closely follows an argument for octonion algebras from [LoPR]. For the converse, an Albert algebra in the sense of this paper has a projective underlying module (because $\mathrm{Her}_3(\mathrm{Zor}(R))$ does), is a cubic Jordan algebra (Proposition 10.1), and satisfies the simplicity condition (Corollary 8.5).

2. Notation

Rings, by definition, have a 1. We put $\mathbb{Z}$-**alg** for the category of commutative rings, where $\mathbb{Z}$ is an initial object. For any $R \in \mathbb{Z}$-**alg**, we put R-**alg** for the category of pairs (S, f) with $S \in \mathbb{Z}$-**alg** and $f: R \to S$, that is, the coslice category $R \downarrow \mathbb{Z}$-**alg**. Below, R will typically denote an element of $\mathbb{Z}$-**alg**. (The interested reader is invited to mentally replace R by a base scheme $\mathbf{X}$, R-**alg** with the category of schemes over $\mathbf{X}$, finitely generated projective R-modules with vector bundles over $\mathbf{X}$, etc., thereby translating results below into a language closer to that in [CalF].) An R-algebra S is said to be *fppf* if it is faithfully flat and finitely presented.

We write $\mathrm{Mat}_n(R)$ for the ring of n-by-n matrices with entries from R, Id for the identity matrix, and $\langle \alpha_1, \ldots, \alpha_n \rangle \in \mathrm{Mat}_n(R)$ for the diagonal matrix whose (i,i)-entry is α_i. The transpose of a matrix x is denoted $x^\top$. We write $\mathrm{GL}_n(R)$ for the group of invertible elements in $\mathrm{Mat}_n(R)$.

Suppose now that $\mathbf{G}$ is a finitely presented group scheme over R. For each fppf $S \in R$-**alg**, we write $H^1(S/R, \mathbf{G})$ for Čech cohomology of the sheaf of groups $\mathbf{G}$ relative to $R \to S$ (see, for example [Gir, Section III.3.6] or [Wa, Chapter 17]). The set $H^1(S/R, \mathbf{G})$ does not depend on the choice of structure homomorphism $R \to S$, and more is true: Every morphism $S \to T$ in k-**alg** gives a morphism $H^1(S/R, \mathbf{G}) \to H^1(T/R, \mathbf{G})$ that is injective and does not depend on the choice of arrow $S \to T$ [Gir, Remark III.3.6.5]. The subcategory of fppf elements of R-**alg** has a small skeleton, so the colimit

$$H^1(R, \mathbf{G}) := \varinjlim_{\text{fppf } S \in R\text{-}\mathbf{alg}} H^1(S/R, \mathbf{G})$$

is a set. We call it the nonabelian fppf cohomology of $\mathbf{G}$. In case $\mathbf{G}$ is smooth, it agrees with étale H^1. If additionally R is a field, then it agrees with the nonabelian Galois cohomology defined in, for example, [Serre].

Unimodular elements

Let M be an R-module. An element $m \in M$ is said to be *unimodular* if Rm is a free R-module of rank 1 and a direct summand of M, equivalently, if there is some $\lambda \in M^*$ (the dual of M) such that $\lambda(m) = 1$. When M is finitely generated projective, this is equivalent to: $m \otimes 1$ is not zero in $M \otimes F$ for every field $F \in R$-**alg** (see, for example [Lo, 0.3]). If $m \in M$ is unimodular, then so is $m \otimes 1 \in M \otimes S$ for every $S \in R$-**alg**. In the opposite direction, if M is finitely generated projective, S is a Zariski cover of R (i.e., $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is surjective), and $m \otimes 1$ is unimodular in $M \otimes S$, it follows that m is unimodular as an element of M.

3. Background on polynomial laws

We may identify an R-module M with a functor $\mathbf{W}(M)$ from R-**alg** to the category of sets defined via $S \mapsto M \otimes S$. For R-modules M, N, a *polynomial law* (in the sense of [Roby] or [BouA2, Section IV.5, Exercise 9]) $f: \mathbf{W}(M) \to \mathbf{W}(N)$ is a morphism of functors, that is, a collection of set maps $f_S: M \otimes S \to N \otimes S$ varying functorially with S. We put $\mathscr{P}_R(M, N)$ for the collection of polynomial laws $\mathbf{W}(M) \to \mathbf{W}(N)$, omitting the subscript R when it is understood. Note that $\mathscr{P}_R(M, N)$ is an R-module.

Lemma 3.1. *Let M be a finitely generated projective R-module, and suppose $f \in \mathscr{P}(M, N)$ is such that $f_R(0) = 0$. If $m \in M$ has $f_R(m)$ unimodular in N, then m is unimodular.*

In the case $N = R$, the condition that $f_R(m)$ is unimodular means that $f_R(m) \in R^\times$.

Proof. Replacing f with λf, where $\lambda \in N^*$ is such that $\lambda(f_R(m)) = 1$, we may assume $N = R$ and $f_R(m) = 1$. If m is not unimodular, then there is a field $F \in R$-**alg**, such that $m \otimes 1 = 0$ in $M \otimes F$, and $f_F(m \otimes 1) = 0$, whence $f_R(m)$ belongs to the kernel of $R \to F$ a contradiction. □

A polynomial law is *homogeneous of degree* $d \geq 0$ if $f_S(sx) = s^d f_S(x)$ for every $S \in R$-**alg**, $s \in S$, and $x \in M \otimes S$ (see [Roby, p. 226]). We put $\mathscr{P}^d_R(M, N)$ for the submodule of $\mathscr{P}_R(M, N)$ of polynomial laws that are homogeneous of degree d. The polynomial laws that are homogeneous of degree 0 are constants, and those of degree 1 are linear transformations, that is, the natural maps

$$N \to \mathscr{P}^0_R(M, N) \quad \text{and} \quad \mathrm{Hom}_R(M, N) \to \mathscr{P}^1_R(M, N)$$

are isomorphisms (see [Roby, pp. 230, 231]). For degree 2, $\mathscr{P}^2_R(M, N)$ is canonically identified with the maps $f\colon M \to N$ that are quadratic in the sense that $f(rm) = r^2 f(m)$ and the map $M \times M \to N$ defined by $f(m_1, m_2) := f(m_1 + m_2) - f(m_1) - f(m_2)$ is bilinear [Roby, p. 236, Proposition II.1]. A *form of degree d* on M is a polynomial law $\mathbf{W}(M) \to \mathbf{W}(R)$ that is homogeneous of degree d. The forms of degree 2 are commonly known as quadratic forms on M.

Directional derivatives

For $f \in \mathscr{P}(M, N)$, $v \in M$, t an indeterminate, and $n \geq 0$, we define a polynomial law $\nabla^n_v f$ as follows. For $S \in R$-**alg** and $x \in M \otimes S$, $f_{S[t]}(x + v \otimes t)$ is an element of $N \otimes S[t]$, and we define $\nabla^n_v f_S(x) \in N \otimes S$ to be the coefficient of t^n. This defines a polynomial law called the *n-th directional derivative $\nabla^n_v f$ of f in the direction v*. One finds that $\nabla^0_v f = f$ regardless of v. We abbreviate $\nabla_v f := \nabla^1_v f$; it is linear in v.

If f is homogeneous of degree d and $0 \leq n \leq d$, then $\nabla^n_v f(x)$ is homogeneous of degree $d - n$ in x and degree n in v. The symmetry implicit in the definition of the directional derivative gives $\nabla^n_v f(x) = \nabla^{d-n}_x f(v)$ for $x \in M$.

Lemma 3.2. *Suppose M, N are R-modules and A is a unital associative R-algebra and $g \in \mathscr{P}(M, A)$ is a polynomial law such that there is an element $m \in M$ such that $g_R(m) \in A$ is invertible. If $f \in \mathscr{P}^d(M, N)$ satisfies*

$$g_S(x) \in (A \otimes S)^\times \Rightarrow f_S(x) = 0$$

*for all $S \in R$-**alg** and $x \in M \otimes S$, then $f = 0$.*

Proof. Since the hypotheses are stable under base change, it suffices to show that $f(v) = 0$ for all $v \in M$. Replacing g by $L \circ g \in \mathscr{P}(M, A)$, where $L \in \mathrm{End}_R(A)$ is multiplication in A on the left by the inverse of $g_R(m)$, we may assume $g_R(m) = 1_A$. Set $S := R[\varepsilon]/(\varepsilon^{d+1})$. For $v \in M$, the element

$$g_S(m + \varepsilon v) = 1_A + \sum_{n=1}^{d} \varepsilon^n \nabla^n_v g_R(m)$$

is invertible in A_S, so by hypothesis,

$$0 = f_S(m + \varepsilon v) = \sum_{n=0}^{d} \varepsilon^n \nabla^n_v f_R(m).$$

Focusing on the coefficient of ε^d in that equation gives

$$0 = \nabla_v^d f_R(m) = \nabla_m^0 f_R(v) = f_R(v),$$

as required. □

While Lemma 3.2 has some similarity with the principle of extension of algebraic identities as in [BouA2, Section IV.2.3], that result imposes some hypothesis on R.

The module of polynomial laws

In the following, we write $\mathsf{S}^n M$ for the n-th symmetric power of M, that is, the R-module $\otimes^n M$ modulo the submodule generated by elements $x - \sigma(x)$ for $x \in \otimes^n M$ and σ a permutation of the n factors.

Lemma 3.3. *Let M and N be finitely generated projective R-modules. Then for each $d \geq 0$:*

1. *$\mathscr{P}^d(M, N)$ is a finitely generated projective R-module.*
2. *If $T \in R$-**alg** is flat, the natural map $\mathscr{P}_R^d(M, N) \otimes T \to \mathscr{P}_T^d(M \otimes T, N \otimes T)$ is an isomorphism.*
3. *The natural map $\mathsf{S}^d(M^*) \otimes N \to \mathscr{P}^d(M, N)$ is an isomorphism.*
4. *The natural map $\mathscr{P}^d(M, R) \otimes N \to \mathscr{P}^d(M, N)$ is an isomorphism.*

Proof. To establish notation, we write $R = \prod_{i=0}^n R_i$ for some n such that $M = \prod_i M_i$ and $N = \prod_i N_i$ with each M_i, N_i an R_i-module of finite constant rank.

Next write $\Gamma_d(M)$ for the module of degree d divided powers on M as defined in [BouA2, Section IV.5, Exercise 2]. We claim that it is finitely generated projective, and therefore, by [Stacks, Tag 00NX], finitely presented. If M is free, then $\Gamma_d(M)$ is free. If M is projective of constant rank, then there exists $S \in R$-**alg** faithfully flat such that $M \otimes S$ is free. Because Γ_d commutes with base change [BouA2, Section IV.5, Exercise 7], $\Gamma_d(M) \otimes S \cong \Gamma_d(M \otimes S)$ is free, and we again find that $\Gamma_d(M)$ is finitely generated projective [Stacks, Tags 03C4, 05A9]. In the general case, $\Gamma_d(M) = \prod_i \Gamma_d(M_i)$, and the claim is verified.

To verify (2), we note that $\mathscr{P}_R^d(M, N)$ is naturally isomorphic to $\mathrm{Hom}_R(\Gamma_d(M), N)$ by [Roby, Theorem IV.1]. Then $\mathscr{P}_R^d(M,N) \otimes T \cong \mathrm{Hom}_R(\Gamma_d(M), N) \otimes T$, which in turn is $\mathrm{Hom}_T(\Gamma_d(M) \otimes T, N \otimes T)$ because T is flat and $\Gamma_d(M)$ is finitely presented [BouCA, Section I.2.10, Proposition 11]. Since Γ_d commutes with base change, we have verified (2).

(3): If M and N are free modules, then the map is an isomorphism by [Roby, p. 232]. If M and N have constant rank, then there is a faithfully flat $T \in R$-**alg** such that $M \otimes T$ and $N \otimes T$ are free. Since (3) holds over T by the free case, (2) and faithfully flat descent give that (3) holds. In the general case, $\mathscr{P}^d(M, N) = \prod \mathscr{P}^d(M_i, N_i)$ and $\mathsf{S}^d(M^*) \otimes N = \prod (\mathsf{S}^d(M_i^*) \otimes N_i)$ and the claim follows by the constant rank case.

(4) follows trivially from (3). For (1), note that M^* is finitely generated projective, so are $\mathsf{S}^d(M^*)$ and the tensor product $\mathsf{S}^d(M^*) \otimes N$. Applying (3) gives the claim. □

One can create new polynomial laws from old by twisting by a line bundle, that is, by a rank 1 projective module.

Lemma 3.4. *Let M and N be finitely generated projective R-modules. Then for every $d \geq 0$ and every line bundle L, we have:*

1. *There is a natural isomorphism $\mathscr{P}^d(M, N) \otimes (L^*)^{\otimes d} \to \mathscr{P}^d(M \otimes L, N)$.*
2. *There is a natural isomorphism $\mathscr{P}^d(M, N) \cong \mathscr{P}^d(M \otimes L, N \otimes L^{\otimes d})$.*

Proof. For (1), since L^* is a line bundle, the natural map $(L^*)^{\otimes d} \to \mathsf{S}^d(L^*)$ is an isomorphism because it is so after faithfully flat base change. Since $\mathsf{S}^d(M^*) \otimes \mathsf{S}^d(L^*)$ is naturally identified with $\mathsf{S}^d((M \otimes L)^*)$, combining Lemma 3.3(3),(4) then gives (1).

For (2), there are isomorphisms $\mathscr{P}^d(M \otimes L, N \otimes L^{\otimes d}) \xrightarrow{\sim} \mathscr{P}^d(M, N) \otimes (L^*)^{\otimes d} \otimes L^{\otimes d}$ by (1) and Lemma 3.3(4). Since $L^{\otimes d} \otimes (L^*)^{\otimes d} \cong R$, the claim follows. □

Example 3.5. (References: [Stacks, Tag 03PK], [CalF, Section 2.4.3], [Kn, Section III.3]) Suppose L is a line bundle and there is an isomorphism $h\colon L^{\otimes d} \to R$ for some $d \geq 1$. We call such a pair $[L, h]$ a *d-trivialized line bundle*. (In the case $d = 2$, they are sometimes called *discriminant modules*.) Applying h to identify $N \otimes L^{\otimes d} \xrightarrow{\sim} N$ in Lemma 3.4(2) gives a construction that takes $f \in \mathscr{P}^d(M, N)$ and gives an element of $\mathscr{P}^d(M \otimes L, N)$, which we denote by $[L, h] \cdot (M, f)$.

For example, for each $\alpha \in R^\times$, define $\langle \alpha \rangle$ to be $[L, h]$ as in the preceding paragraph, where $L = R$ and h is defined by $h(\ell_1 \otimes \cdots \otimes \ell_d) = \alpha \prod \ell_i$. Clearly, $\langle \alpha\beta^d \rangle \cong \langle \alpha \rangle$ for all $\alpha, \beta \in R^\times$. Applying the construction in the previous paragraph, we find $\langle \alpha \rangle \cdot (M, f) \cong (M, \alpha f)$.

Every $[L, h]$ with $L = R$ is necessarily isomorphic to $\langle \alpha \rangle$ for some $\alpha \in R^\times$. In particular, if $\mathrm{Pic}(R)$ has no d-torsion elements other than zero — for example, if R is a semilocal ring or a UFD [Stacks, Tags 0BCH, 02M9] — then each $[L, h]$ is isomorphic to $\langle \alpha \rangle$ for some α. The group scheme μ_d of d-th roots of unity is the automorphism group of each $[L, h]$, where μ_d acts by multiplication on L. The group $H^1(R, \mu_d)$ classifies pairs $[L, h]$ up to isomorphism.

We say that homogeneous polynomial laws related by the isomorphism in Lemma 3.4(2) are *projectively similar*, imitating the language from [AuBB, Section 1.2] for the case of quadratic forms ($d = 2$). (This relationship was called "lax-similarity" in [BC].) We say that homogeneous degree d laws f and $[L, h] \cdot f$ for $[L, h] \in H^1(R, \mu_d)$ as in the preceding example are *similar*. If $\mathrm{Pic}(R)$ is d-torsion, the two notions coincide.

For $f \in \mathscr{P}^d(M, N)$, we define $\mathrm{Aut}(f)$ to be the subgroup of $\mathrm{GL}(M)$ consisting of elements g such that $fg = f$ as polynomial laws. In case M and N are finitely generated projective, so is $\mathscr{P}^d(M, N)$, whence the functor $\mathbf{Aut}(f)$ from R-**alg** to groups defined by $\mathbf{Aut}(f)(T) = \mathrm{Aut}(f_T)$ is a closed subgroup-scheme of $\mathbf{GL}(M)$.

Lemma 3.6. *Let f and f' be homogeneous polynomial laws on finitely generated projective modules. If f and f' are projectively similar, then their automorphism groups are isomorphic.*

Proof. By hypothesis, $f \in \mathscr{P}^d(M, N)$ and $f' \in \mathscr{P}^d(M \otimes L, N \otimes L^{\otimes d})$ for some modules M and N, line bundle L, and $d \geq 0$. The group scheme $\mathbf{Aut}(f)$ is the closed sub-group-scheme of $\mathbf{GL}(M)$ stabilizing the element f in $\mathsf{S}^d(M^*) \otimes N$. Now, any element of $\mathbf{GL}(M)$ acts on $\mathsf{S}^d((M \otimes L)^*) \otimes (N \otimes L^{\otimes d})$ by defining it to act as the identity on L. In this way, we find a homomorphism $\mathbf{Aut}(f) \to \mathbf{Aut}(f')$. Viewing M as $(M \otimes L) \otimes L^*$ and N as $(N \otimes L^{\otimes d}) \otimes (L^*)^{\otimes d}$, and repeating this construction, we find an inverse mapping $\mathbf{Aut}(f') \to \mathbf{Aut}(f)$. □

4. Background on composition algebras

A not-necessarily-associative R-algebra C is an R-module with an R-linear map $C \otimes_R C \to C$, which we view as a multiplication and write as juxtaposition. Such a C is *unital* if it has an element $1_C \in C$ such that $1_C c = c 1_C = c$ for all $c \in C$ (see, for example, [Sch]). A *composition R-algebra* as in [Pe93] is such a C that is finitely generated projective as an R-module, is unital, and has a quadratic form $n_C\colon C \to R$ that allows composition (that is, such that $n_C(xy) = n_C(x)n_C(y)$ for all $x, y \in C$), satisfies $n_C(1_C) = 1$, and whose bilinearization defined by $n_C(x, y) := n_C(x + y) - n_C(x) - n_C(y)$ gives an isomorphism $C \to C^*$ via $x \mapsto n_C(x, \cdot)$. We say that a symmetric bilinear form with this property is *regular*. The quadratic form n_C (which is unique by Proposition 4.3 below) is called the *norm* of C.

Remark 4.1. In the definition above, one can swap the condition $n_C(1_C) = 1$ with the requirement that the rank of C is nowhere zero, that is, $C \otimes F \neq 0$ for every field $F \in R$-**alg**.

We put $\mathrm{Tr}_C(x) := n_C(x, 1_C)$, a linear map $C \to R$, called the *trace* of C. Trivially, $\mathrm{Tr}_C(1_C) = 2$. Lemma 3.1 gives that 1_C is unimodular, so we may identify R with $R1_C$, and C is a faithful R-module.

The unimodularity of 1_C is equivalent to the existence of some $\lambda \in C^*$ such that $\lambda(1_C) = 1$, that is, some $x \in C$ such that $\mathrm{Tr}_C(x) = 1$, whence $\mathrm{Tr}_C : C \to R$ is surjective.

The class of composition algebras is stable under base change. That is, if C is a composition R-algebra with norm n_C, then for every $S \in R$-**alg**, $C \otimes S$ is a composition S-algebra with norm $n_C \otimes S$. The following two results are essentially well known [Pe93, 1.2–1.4]. For convenience, we include their proof.

Lemma 4.2 ("Cayley-Hamilton"). *Let C be a composition algebra with norm n_C, and define* Tr_C *as above. Then*

$$x^2 - \mathrm{Tr}_C(x)x + n_C(x)1_C = 0$$

for all $x \in C$.

Proof. Linearizing the composition law $n_C(xy) = n_C(x)n_C(y)$, we find

$$n_C(xy, x) = n_C(x)\,\mathrm{Tr}_C(y) \quad \text{and} \tag{4.1}$$

$$n_C(xy, wz) + n_C(wy, xz) = n_C(x, w)n_C(y, z) \tag{4.2}$$

for all $x, y, z, w \in C$. Setting $z = x$ and $w = 1_C$ in (4.2), we find:

$$n_C(xy, x) + n_C(y, x^2) = \mathrm{Tr}_C(x)n_C(x, y).$$

Combining these with (4.1), we find:

$$n_C(x^2 - \mathrm{Tr}_C(x)x + n_C(x)1_C, y) = 0 \quad \text{for all } x, y \in C.$$

Since the bilinear form n_C is regular, the claim follows. ◻

A priori, a composition algebra is a unital algebra together with a quadratic form, the norm. The next result shows that these data are redundant.

Proposition 4.3. *If C is a composition algebra, then the norm n_C is uniquely determined by the algebra structure of C.*

Proof. Let $n' : C \to R$ be any quadratic form making C a composition algebra, and write Tr' for the corresponding trace $\mathrm{Tr}'(x) := n'(x + 1_C) - n'(x) - n'(1_C)$. Then $\lambda := \mathrm{Tr}_C - \mathrm{Tr}'$ (respectively, $q := n_C - n'$) is a linear (respectively, quadratic) form on C and the Cayley-Hamilton property yields

$$\lambda(x)x = q(x)1_C \quad \text{for all } x \in C. \tag{4.3}$$

We aim to prove that $q = 0$. Because 1_C is unimodular, it suffices to prove $\lambda = 0$. This can be checked locally, so we may assume that R is local and, in particular, $C = R1_C \oplus M$ for a free module M. Now, $\mathrm{Tr}_C(1_C) = 2 = \mathrm{Tr}'(1_C)$, so $\lambda(1_C) = 0$. For $m \in M$ a basis vector, $\lambda(m)m$ belongs to $M \cap R1_C$ by (4.3), so it is zero, whence $\lambda(m) = 0$, proving the claim. ◻

Corollary 4.4. *Let C be a unital R-algebra. If there is a faithfully flat $S \in R$-**alg** such that $C \otimes S$ is a composition S-algebra, then C is a composition algebra over R.*

Proof. Because the norm $n_{C\otimes S}$ of $C \otimes S$ is uniquely determined by the algebra structure, one obtains by faithfully flat descent a quadratic form $n_C : C \to R$ such that $n_C \otimes S = n_{C\otimes S}$. Because $n_{C\otimes S}$ satisfies the properties required to make $C \otimes S$ a composition algebra and S is faithfully flat over R, it follows that the same properties hold for n_C. ◻

The following facts are standard, see, for example [Kn, Section V.7]: Composition algebras are alternative algebras. The map $\bar{\ } : C \to C$ defined by $\bar{x} := \mathrm{Tr}_C(x)1_C - x$ is an involution, that is, an R-linear antiautomorphism of period 2.

Composition algebras of constant rank

In case R is connected — that is, $R \not\cong R_1 \times R_2$, where neither R_1 nor R_2 are the zero ring — a composition R-algebra has rank 2^e for $e \in \{0, 1, 2, 3\}$ [Kn, p. 206, Theorem V.7.1.6]. Therefore, specifying a composition R-algebra C is equivalent to writing

$$R = \prod_{e=0}^{3} R_e \quad \text{and} \quad C = \prod_{e=0}^{3} C_e, \tag{4.4}$$

where C_e is a composition R_e-algebra of constant rank 2^e.

If C is a composition algebra of rank 1, then since 1_C is unimodular, C is equal to R. The bilinear form $n_C(\cdot, \cdot)$ gives an isomorphism $C \to C^*$ and $n_C(1_C, \alpha 1_C) = 2\alpha$, and we deduce that 2 is invertible in R. Conversely, if 2 is invertible, then R is a composition algebra by setting $n_C(\alpha) = \alpha^2$; in this case, we say that R is a split composition algebra.

A composition algebra whose rank is 2 is not just an associative and commutative ring, it is an étale algebra [Kn, p. 43, Theorem I.7.3.6]. Conversely, every rank 2 étale algebra is a composition algebra. Among rank 2 étale algebras, there is a distinguished one, $R \times R$, which is said to be split.

A composition algebra whose rank is 4 is associative and is an Azumaya algebra, commonly known as a *quaternion algebra*. (Note that our notion of quaternion algebra is more restrictive than the one in the books [Kn, see p. 43] and [Vo].) Among quaternion R-algebras, there is a distinguished one, the 2-by-2 matrices $\mathrm{Mat}_2(R)$, which is said to be split.

A composition algebras whose rank is 8 is known as an *octonion algebra*. Among octonion R-algebras, there is a distinguished one that is said to be split, called the Zorn vector matrices and denoted $\mathrm{Zor}(R)$ (see [LoPR, 4.2]). As a module, we view it as $\left(\begin{smallmatrix} R & R^3 \\ R^3 & R \end{smallmatrix}\right)$ with multiplication

$$\begin{pmatrix} \alpha_1 & u \\ x & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & v \\ y & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 - u^\top y & \alpha_1 v + \beta_2 u + x \times y \\ \beta_1 x + \alpha_2 y + u \times v & -x^\top v + \alpha_2\beta_2 \end{pmatrix},$$

where $\times$ is the ordinary cross product on R^3. The quadratic form is

$$n_{\mathrm{Zor}(R)} \begin{pmatrix} \alpha_1 & u \\ x & \alpha_2 \end{pmatrix} = \alpha_1\alpha_2 + u^\top x.$$

One says that a composition R-algebra C is *split* if, when we write R and C as in (4.4), C_e is isomorphic to the split composition R_e-algebra for $e \geq 1$.

It is well known in the case where $R = \mathbb{R}$, the real numbers, that a composition algebra is determined up to isomorphism by its dimension and whether it is split. That is, there are only seven isomorphism classes of composition $\mathbb{R}$-algebras, consisting of four split ones and four division algebras, namely, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$; note that both collections of four contain $\mathbb{R}$.

Example 4.5. The real octonions $\mathbb{O}$ are a composition $\mathbb{R}$-algebra with basis $1_{\mathbb{O}}, e_1, e_2, \ldots, e_7$ which is orthonormal with respect to the quadratic form $n_{\mathbb{O}}$ with multiplication table

$$e_r^2 = -1 \quad \text{and} \quad e_r e_{r+1} e_{r+3} = -1$$

for all r with subscripts taken modulo 7, and the displayed triple product is associative.

The $\mathbb{Z}$-sublattice $\mathcal{O}$ of $\mathbb{O}$ spanned by $1_{\mathbb{O}}$, the e_r, and

$$h_1 = (1 + e_1 + e_2 + e_4)/2, \quad h_2 = (1 + e_1 + e_3 + e_7)/2,$$
$$h_3 = (1 + e_1 + e_5 + e_6)/2 \quad \text{and} \quad h_4 = (e_1 + e_2 + e_3 + e_5)/2$$

is a composition $\mathbb{Z}$-algebra. It is a maximal order in $\mathcal{O} \otimes \mathbb{Q}$, and all such are conjugate under the automorphism group of $\mathcal{O} \otimes \mathbb{Q}$. (As a consequence, there is some choice in the way one presents this algebra. We have followed [ElkiesGr].) As a subring of $\mathbb{O}$, it has no zero divisors. For more on this, see [Di, Section 19], [Cox], [ConwS, Section 9], or [Conrad, Section 5]. The nonuniqueness of this choice of maximal order and its relationship to other orders like $\mathbb{Z} \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_7$ can be understood in terms of the Bruhat-Tits building of the group $\mathrm{Aut}(\mathrm{Zor}(\mathbb{Q}_2))$ of type G_2 over the 2-adic numbers, compare [GanY, Section 9].

5. Background on Jordan algebras

Para-quadratic algebras

A (unital) *para-quadratic algebra* over a ring R is an R-module J together with a quadratic map $U \colon J \to \mathrm{End}_R(J)$ — that is, U is an element of $\mathscr{P}^2(J, \mathrm{End}_R(J))$ — called the *U-operator*, and a distinguished element $1_J \in J$ such that $U_{1_J} = \mathrm{Id}_J$. A *homomorphism* $\phi \colon J \to J'$ of para-quadratic R-algebras is an R-linear map such that $\phi(1_J) = 1_{J'}$ and $U'_{\phi(x)}\phi(y) = \phi(U_x y)$ for all $x, y \in J$, where U' denotes the U-operator in J'.

Jordan algebras

As a notational convenience, we define a linear map $J \otimes J \otimes J \to J$ denoted $x \otimes y \otimes z \mapsto \{xyz\}$ via

$$\{xyz\} := (U_{x+z} - U_x - U_z)y. \tag{5.1}$$

Evidently, $\{xyz\} = \{zyx\}$ for all $x, y, z \in J$. A para-quadratic R-algebra J is a *Jordan R-algebra* if the identities

$$U_{U_x y} = U_x U_y U_x \quad \text{and} \quad U_x\{yxz\} = \{(U_x y)zx\} \tag{5.2}$$

hold for all $x, y, z \in J \otimes S$ for all $S \in R$-**alg**. (Alternatively, one can define a Jordan R-algebra entirely in terms of identities concerning elements of J, avoiding the “for all $S \in R$-**alg**”, at the cost of requiring a longer list of identities (see [McC66, Section 1]).) Note that if J is a Jordan R-algebra, then $J \otimes T$ is a Jordan T-algebra for every $T \in R$-**alg** (“Jordan algebras are closed under base change”). If J is a para-quadratic algebra and $J \otimes T$ is Jordan for some faithfully flat $T \in R$-**alg**, then J is Jordan.

For x in a Jordan algebra J and $n \geq 0$, we define the n-th power x^n via

$$x^0 := 1_J, \quad x^1 := x, \quad x^n = U_x x^{n-2} \text{ for } n \geq 2. \tag{5.3}$$

An element $x \in J$ is *invertible with inverse y* if $U_x y = x$ and $U_x y^2 = 1$ [McC66, Section 5]. It turns out that x is invertible if and only if U_x is invertible if and only if 1 is in the image of U_x; when these hold, the inverse of x is $y = U_x^{-1}x$, which we denote by x^{-1}. It follows from (5.2) that $x, y \in J$ are both invertible if and only if $U_x y$ is invertible, and in this case, $(U_x y)^{-1} = U_{x^{-1}} y^{-1}$.

Example 5.1. Let A be an associative and unital R-algebra. Define $U_x y := xyx$ for $x, y \in A$. Then $\{xyz\} = xyz + zyx$ and A endowed with this U-operator is a Jordan algebra denoted by A^+. Note that for $x \in A$ and $n \geq 0$, the n-th powers of x in A and A^+ are the same.

Relations with other kinds of algebras

Suppose for this paragraph and the next that 2 is invertible in R. Given a para-quadratic algebra J as in the preceding paragraph, one can define a commutative (bilinear) product $\bullet$ on J via

$$x \bullet y := \frac{1}{2}\{x 1_J y\} \quad \text{for } x, y \in J. \tag{5.4}$$

(In the case where J is constructed from an associative algebra as in Example 5.1, one finds that $x \bullet y = \frac{1}{2}(xy + yx)$. If, additionally, the associative algebra is commutative, $\bullet$ equals the product in that associative algebra.) If J is Jordan, then $\bullet$ satisfies

$$(x \bullet y) \bullet (x \bullet x) = x \bullet (y \bullet (x \bullet x)), \tag{5.5}$$

which is the axiom classically called the "Jordan identity".

In the opposite direction, given an R-module J with a commutative product $\bullet$ with identity element 1_J, we obtain a para-quadratic algebra by setting

$$U_x y := 2x \bullet (x \bullet y) - (x \bullet x) \bullet y \quad \text{for } x, y \in J. \tag{5.6}$$

If the original product satisfied the Jordan identity, then the para-quadratic algebra so obtained satisfies (5.2), that is, is a Jordan algebra in our sense (see, for example [J69, Section 1.4]).

Definition 5.2 (hermitian matrix algebras). Let C be a composition R-algebra and $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle \in \mathrm{GL}_3(R)$. We define $\mathrm{Her}_3(C, \Gamma)$ to be the R-submodule of $\mathrm{Mat}_3(C)$ consisting of elements fixed by the involution $x \mapsto \Gamma^{-1}\bar{x}^{\mathsf{T}}\Gamma$ and with diagonal entries in R. Note that, as an R-module, $\mathrm{Her}_3(C, \Gamma)$ is a sum of three copies of C and three copies of R, so it is finitely generated projective.

In the special case where 2 is invertible in R, one can define a multiplication $\bullet$ on $\mathrm{Her}_3(C, \Gamma)$ via $x \bullet y := \frac{1}{2}(xy + yx)$, where juxtaposition denotes the usual product of matrices in $\mathrm{Mat}_3(C)$. It satisfies the Jordan identity [J68, p. 61, Corollary], and therefore, the U-operator defined via (5.6) makes $\mathrm{Her}_3(C, \Gamma)$ into a Jordan algebra.

6. Cubic Jordan algebras

In this section, we define cubic Jordan algebras and the closely related notion of cubic norm structure. They provide a useful alternative language for computation.

Definition 6.1. Following [McC69] (see [PeR86a, p. 212] for the terminology), we define a *cubic norm R-structure* as a quadruple $\mathbf{M} = (M, 1_{\mathbf{M}}, \sharp, N_{\mathbf{M}})$ consisting of an R-module M; a distinguished element $1_{\mathbf{M}} \in M$ (the *base point*); a quadratic map $\sharp \colon M \to M$, written $x \mapsto x^\sharp$ (the *adjoint*) with (symmetric bilinear) polarization $x \times y := (x+y)^\sharp - x^\sharp - y^\sharp$; and a cubic form $N_{\mathbf{M}} \colon M \to R$ (the *norm*) such that the following axioms are fulfilled. Define a bilinear form $T_{\mathbf{M}} \colon M \times M \to R$ by

$$T_{\mathbf{M}}(x, y) := (\nabla_x N_{\mathbf{M}})(1_{\mathbf{M}})(\nabla_y N_{\mathbf{M}})(1_{\mathbf{M}}) - (\nabla_x \nabla_y N_{\mathbf{M}})(1_{\mathbf{M}}) \tag{6.1}$$

(the *bilinear trace*), which is symmetric since the directional derivatives ∇_x, ∇_y commute [Roby, p. 241, Proposition II.5], and a linear form $\mathrm{Tr}_{\mathbf{M}} \colon M \to R$ by

$$\mathrm{Tr}_{\mathbf{M}}(x) := T_{\mathbf{M}}(x, 1_{\mathbf{M}}) \tag{6.2}$$

(the *linear trace*). For $\mathbf{M}$ to be a cubic norm structure, we require that the identities

$$1_{\mathbf{M}}^\sharp = 1_{\mathbf{M}}, \quad N_{\mathbf{M}}(1_{\mathbf{M}}) = 1, \tag{6.3}$$

$$1_{\mathbf{M}} \times x = \mathrm{Tr}_{\mathbf{M}}(x) 1_{\mathbf{M}} - x, \;\; (\nabla_y N_{\mathbf{M}})(x) = T_{\mathbf{M}}(x^\sharp, y), \;\; x^{\sharp\sharp} = N_{\mathbf{M}}(x)x \tag{6.4}$$

hold in all scalar extensions $M \otimes S$, $S \in R$-**alg**.

For such a cubic norm structure $\mathbf{M}$, we then define a U-operator by

$$U_x y := T_{\mathbf{M}}(x, y)x - x^\sharp \times y, \tag{6.5}$$

which together with $1_{\mathbf{M}}$ converts the R-module M into a Jordan R-algebra $J = J(\mathbf{M})$ [McC69, Theorem 1]. In the sequel, we rarely distinguish carefully between the cubic norm structure $\mathbf{M}$ and the Jordan algebra $J(\mathbf{M})$. By abuse of notation, we write $1_J = 1_{\mathbf{M}}$, $N_J = N_{\mathbf{M}}$, $T_J = T_{\mathbf{M}}$, and $\mathrm{Tr}_J := \mathrm{Tr}_{\mathbf{M}}$ if there is no danger of confusion, even though, in general, J does not determine $\mathbf{M}$ uniquely [PeR86a, p. 216].

Definition 6.2. A Jordan R-algebra J is said to be *cubic* if there exists a cubic norm R-structure $\mathbf{M}$ as in Definition 6.1 such that (i) $J = J(\mathbf{M})$ and (ii) $J = M$ is a finitely generated projective R-module. With the quadratic form $S_J : M \to R$ defined by $S_J(x) := \mathrm{Tr}_J(x^\sharp)$ for $x \in J$ (the *quadratic trace*), the cubic Jordan algebra J satisfies the identities

$$(U_x y)^\sharp = U_{x^\sharp} y^\sharp, \quad N_J(U_x y)U_x y = N_J(x)^2 N_J(y) U_x y, \tag{6.6}$$

$$U_x x^\sharp = N_J(x)x, \quad U_x (x^\sharp)^2 = N_J(x)^2 1_J, \tag{6.7}$$

$$x^\sharp = x^2 - \mathrm{Tr}_J(x)x + S_J(x)1_J \quad \text{and} \tag{6.8}$$

$$x^3 - \mathrm{Tr}_J(x)x^2 + S_J(x)x - N_J(x)1_J = 0 = x^4 - \mathrm{Tr}_J(x)x^3 + S_J(x)x^2 - N_J(x)x \tag{6.9}$$

for all $x \in J$. For (6.6)–(6.8) and the first equation of (6.9), see [McC69, p. 499], while the second equation of (6.9) follows from the first, (6.7), and (6.8) via $x^4 = U_x x^2 = U_x x^\sharp + \mathrm{Tr}_J(x) U_x x - S_J(x) U_x 1_J = \mathrm{Tr}_J(x)x^3 - S_J(x)x^2 + N_J(x)x$.

Remark 6.3. Note that the second equality of (6.9) derives from the first through formal multiplication by x. But, due to the para-quadratic character of Jordan algebras, this is not a legitimate operation unless 2 is invertible in R. In fact, cubic Jordan algebras exist that contain elements x satisfying $x^2 = 0 \neq x^3$ [J69, 1.31–1.32].

Example 6.4 (3-by-3 matrices). We claim that $\mathrm{Mat}_3(R)^+$ is a cubic Jordan algebra, in particular, it is $J(\mathbf{M})$ for $\mathbf{M} := (\mathrm{Mat}_3(R), \mathrm{Id}, \sharp, \det)$, where $\sharp$ denotes the classical adjoint. We first verify that $\mathbf{M}$ is a cubic norm structure. Computing directly from the definition (6.1), we find that $T_{\mathbf{M}}(x, y) = \mathrm{Tr}_{\mathrm{Mat}_3(R)}(xy)$, where the juxtaposition on the right is usual matrix multiplication. The formulas in (6.3) are obvious. For (6.4), the first two equations can be verified directly and the third equation is a standard property of the classical adjoint, completing the proof that $\mathbf{M}$ is a cubic norm structure. Similarly, one can check directly that the U-operator defined from the cubic norm structure by (6.5) equals the U-operator defined from the usual matrix product in Example 5.1, that is, $J(\mathbf{M}) = \mathrm{Mat}_3(R)^+$.

Lemma 6.5. *Let J be a cubic Jordan R-algebra and $x, y \in J$.*

1. *x is invertible in J if and only if $N_J(x)$ is invertible in R. In this case*

$$x^{-1} = N_J(x)^{-1} x^\sharp \quad \textit{and} \quad N_J(x^{-1}) = N_J(x)^{-1}.$$

2. *Invertible elements of J are unimodular.*
3. *$N_J(U_x y) = N_J(x)^2 N_J(y)$ and $N_J(x^2) = N_J(x)^2 = N_J(x^\sharp)$.*

Proof. (1): If $N_J(x)$ is invertible in R, then (6.7) shows that so is x, with inverse $x^{-1} = N_J(x)^{-1} x^\sharp$. Conversely, assume x is invertible in J. Then $y := (x^{-1})^2$ satisfies $U_x y = 1_J$, and (6.6) yields $1_J = N_J(U_x y)U_x y = N_J(x)^2 N_J(y) 1_J$, hence

$$N_J(x)^2 N_J(y) = 1$$

since 1_J is unimodular by Lemma 3.1 and (6.3). Thus, $N_J(x) \in R^\times$. Before proving the final formula of (1), we deal with (2), (3).

(2) follows immediately from Lemma 3.1 combined with the first part of (1).

(3): Applying Lemma 3.2 to the polynomial law $g\colon J \times J \to \mathrm{End}_R(J)$ defined by $g(x,y) := U_{U_x y}$ in all scalar extensions, we may assume that $U_x y$ is invertible. By (2), therefore, $U_x y$ is unimodular, and the first equality follows from (6.6). The second equality follows from the first for $y = 1_J$, while in the third equality, we may again assume that x is invertible, hence, unimodular. Then (6.7) combines with the first equality to imply $N_J(x)^4 = N_J(N_J(x)x) = N_J(U_x x^\sharp) = N_J(x)^2 N_J(x^\sharp)$, as desired.

Now the second equality of (1) follows from the first and (3) via

$$N_J(x^{-1}) = N_J(x)^{-3} N_J(x^\sharp) = N_J(x)^{-1}. \qquad \square$$

Without the assumption that J is finitely generated projective as an R-module, Lemma 6.5 would be false [PeR85, Theorem 10].

Example 6.6. We endow the R-module $M := \mathrm{Her}_3(C,\Gamma)$ from Definition 5.2 with a cubic norm R-structure $\mathbf{M} = (M, 1_{\mathbf{M}}, \sharp, N_{\mathbf{M}})$, where $1_{\mathbf{M}}$ is the 3-by-3 identity matrix. An element $x \in \mathrm{Her}_3(C,\Gamma)$ may be written as

$$x = \begin{pmatrix} \alpha_1 & \gamma_2 c_3 & \gamma_3 \bar{c}_2 \\ \gamma_1 \bar{c}_3 & \alpha_2 & \gamma_3 c_1 \\ \gamma_1 c_2 & \gamma_2 \bar{c}_1 & \alpha_3 \end{pmatrix}$$

for $\alpha_i \in R$ and $c_i \in C$. Because three of the entries are determined by symmetry, we may denote such an element by

$$x := \sum_{i=1}^{3} \Big(\alpha_i \varepsilon_i + \delta_i^\Gamma(c_i)\Big), \tag{6.10}$$

where ε_i has a 1 in the (i,i) entry and zeros elsewhere, and $\delta_i^\Gamma(c)$ has $\gamma_{i+2}c$ in the $(i+1, i+2)$ entry — where the symbols $i+1$ and $i+2$ are taken modulo 3 — and zeros in the other entries not determined by symmetry. In the literature on Jordan algebras, one finds the notation $c[(i+1)(i+2)]$ for what we denote $\delta_i^\Gamma(c)$. We define the adjoint $\sharp$ by

$$x^\sharp := \sum_{i=1}^{3} \Big(\big(\alpha_{i+1}\alpha_{i+2} - \gamma_{i+1}\gamma_{i+2} n_C(c_i)\big)\varepsilon_i + \delta_i^\Gamma\big(-\alpha_i c_i + \gamma_i \overline{c_{i+1} c_{i+2}}\big) \Big)$$

with indices modulo (mod) 3, and the norm $N_{\mathbf{M}}$ by

$$N_{\mathbf{M}}(x) := \alpha_1\alpha_2\alpha_3 - \sum_{i=1}^{3} \gamma_{i+1}\gamma_{i+2}\alpha_i n_C(c_i) + \gamma_1\gamma_2\gamma_3 \mathrm{Tr}_C(c_1 c_2 c_3) \tag{6.11}$$

in all scalar extensions, where the last summand on the right of (6.11) is unambiguous since $\mathrm{Tr}_C((c_1c_2)c_3) = \mathrm{Tr}_C(c_1(c_2c_3))$ [McC85, Theorem 3.5]. By [McC69, Theorem 3], $\mathbf{M}$ is indeed a cubic norm structure. The corresponding cubic Jordan algebra will again be denoted by $J := \mathrm{Her}_3(C,\Gamma) := J(\mathbf{M})$.

(In case 2 is invertible in R, the commutative product $\bullet$ on $\mathrm{Her}_3(C,\Gamma)$ defined from the U-operator by (5.4) equals the product $x \bullet y := \frac{1}{2}(xy + yx)$ from Definition 5.2. In order to see this, it suffices to note that the square of $x \in \mathrm{Her}_3(C,\Gamma)$ as defined in (5.3) is the same as the square of x in the matrix algebra $\mathrm{Mat}_3(C)$. This in turn follows immediately from (6.8), (6.11), and the definition of the adjoint.)

For x as above and $y = \sum(\beta_i\varepsilon_i + \delta_i^\Gamma(d_i))$, with $\beta_i \in R$, $d_i \in C$, evaluating the bilinear trace at x, y yields

$$T_J(x,y) = \sum_{i=1}^{3} \big(\alpha_i\beta_i + \gamma_{i+1}\gamma_{i+2} n_C(c_i, d_i)\big). \tag{6.12}$$

Since the bilinearization of n_C is regular, so is T_J.

Here is an important special case.

Definition 6.7. For the special case where $\Gamma = \mathrm{Id}$, we define $\mathrm{Her}_3(C) := \mathrm{Her}_3(C, \mathrm{Id})$ and write δ_i for δ_i^Γ. It can be useful to write elements of $\mathrm{Her}_3(C)$ as

$$\begin{pmatrix} \alpha_1 & c_3 & \cdot \\ \cdot & \alpha_2 & c_1 \\ c_2 & \cdot & \alpha_3 \end{pmatrix},$$

where $\cdot$ denotes an entry that is omitted because it is determined by symmetry. As an example of the triple product defined from (5.1) and (6.5), we mention that for $x = \sum \alpha_i \varepsilon_i$ diagonal, we have

$$\{\delta_i(a)\delta_{i+1}(b)x\} = \delta_{i+2}(\overline{ab})\alpha_i \quad \text{and} \quad \{\delta_{i+1}(b)\delta_i(a)x\} = \delta_{i+2}(\overline{ab})\alpha_{i+1} \tag{6.13}$$

for $i \in 1, 2, 3$ taken mod 3 and $a, b \in C$.

Note that, for the Jordan algebra $\mathrm{Her}_3(C, \Gamma)$, if we multiply Γ by an element of $R^\times$ or any entry in Γ by the square of an element of $R^\times$, we obtain an algebra isomorphic to the original. Therefore, replacing Γ by $\langle (\det\Gamma)^{-1}\gamma_1, (\det\Gamma)^{-1}\gamma_2, (\det\Gamma)\gamma_3 \rangle$ does not change the isomorphism class of $\mathrm{Her}_3(C, \Gamma)$, and we may assume that $\gamma_1\gamma_2\gamma_3 = 1$.

Example 6.8. When studying the Jordan R-algebras $\mathrm{Her}_3(C, \Gamma)$ in the special case $R = \mathbb{R}$, the preceding paragraph shows that it is sufficient to consider two choices for Γ, namely, $\langle 1, s, s \rangle$ for $s = \pm 1$. We compute $T_{\mathrm{Her}_3(C,\Gamma)}$ for each choice of C and Γ. Regular symmetric bilinear forms over $\mathbb{R}$ are classified by their dimension and signature (an integer), so it suffices to specify the signature. If $C = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, the signature of n_C is 2^r for $r = 0$, 1, 2, 3, respectively. By (6.12), T_J has signature $3(1 + 2^r)$ for $J = \mathrm{Her}_3(C)$ and $3 - 2^r$ for $J = \mathrm{Her}_3(C, \langle 1, -1, -1 \rangle)$. For C the split composition algebra of rank 2^r for $r = 1$, 2, or 3 and any Γ, the signature of n_C is 0 and the signature of $T_{\mathrm{Her}_3(C,\Gamma)}$ is 3.

Remark 6.9. Alternatively, one could define the Jordan algebra structure on $\mathrm{Her}_3(C, \Gamma)$ for an arbitrary ring R without referring to cubic norm structures as follows. Writing out the formulas for the U-operator from Definition 5.2 in case $R = \mathbb{Q}$, one finds that the formulas do not involve any denominators other than γ_i terms and therefore make sense for any R regardless of whether 2 is invertible. This makes $\mathrm{Her}_3(C, \Gamma)$ a para-quadratic algebra. Because it is a Jordan algebra in case $R = \mathbb{Q}$ as in Definition 5.2, we conclude that $\mathrm{Her}_3(C, \Gamma)$ is a Jordan algebra with no hypothesis on R by extension of identities [BouA2, Section IV.2.3, Theorem 2]. This alternative definition gives the same objects but is much harder to work with.

7. Albert algebras are Freudenthal algebras are Jordan algebras

Definition 7.1. A *split Freudenthal R-algebra* is a Jordan algebra $\mathrm{Her}_3(C)$ as in Definition 6.7 for some split composition R-algebra C. Because split composition algebras are determined up to isomorphism by their rank function, so are split Freudenthal algebras.

A para-quadratic R-algebra J is a *Freudenthal* algebra if $J \otimes S$ is a split Freudenthal S-algebra for some faithfully flat $S \in R$-**alg**. It is immediate that every Freudenthal algebra is a Jordan algebra. Since every split Freudenthal R-algebra is finitely generated projective as an R-module for every R, the same is true for every Freudenthal R-algebra J, and by the same reasoning, we see that the identity element 1_J is unimodular. Because the rank of a composition algebra takes values in $\{1, 2, 4, 8\}$, the rank of a Freudenthal algebra takes values in $\{6, 9, 15, 27\}$.

We are now prepared to define the objects named in the title of this paper.

Definition 7.2. An *Albert R-algebra* is a Freudenthal R-algebra of rank 27.

We continue to prove results about Freudenthal algebras, rather than merely Albert algebras. The extra generality comes at a low cost.

Proposition 7.3. *For every composition R-algebra C and every* $\Gamma \in \mathrm{GL}_3(R)$, $\mathrm{Her}_3(C, \Gamma)$ *is a Freudenthal algebra.*

Proof. Replacing R with R_e as in (4.4), we may assume that C has constant rank. There is a faithfully flat $S \in R$-**alg** such that $C \otimes S$ is a split composition algebra.

Consider $T := S[t_1, t_2, t_3]/(t_1^2 - \gamma_1, t_2^2 - \gamma_2, t_3^2 - \gamma_3)$. It is a free S-module, so faithfully flat. Then $\mathrm{Her}_3(C, \Gamma) \otimes T$ is isomorphic to $\mathrm{Her}_3(C \otimes T)$ as Jordan algebras, and the latter is a split Freudenthal algebra. □

A Freudenthal algebra is said to be *reduced* if it is isomorphic to $\mathrm{Her}_3(C, \Gamma)$ for some C and Γ.

Example 7.4. Let J be a Freudenthal R-algebra. *If $x \in J$ has $U_x = \mathrm{Id}_J$, then $x = \zeta 1_J$ for* some $\zeta \in R$ *such that* $\zeta^2 = 1$. To see this, first suppose that J is $\mathrm{Her}_3(C)$ for some composition algebra C and write $x = \sum(\alpha_i \varepsilon_i + \delta_i(c_i))$ for $\alpha_i \in R$ and $c_i \in C$. We find

$$U_x \varepsilon_i = \alpha_i^2 \varepsilon_i + \delta_{i+2}(\alpha_i c_{i+2}) + \cdots$$

for each i, so $\alpha_i^2 = 1$ and $c_{i+2} = 0$ for all i. Then

$$U_x \delta_i(1_C) = \delta_i(\alpha_{i+1}\alpha_{i+2} 1_C).$$

Since 1_C is unimodular, $\alpha_{i+1}\alpha_{i+2} = 1$ for all i, proving the claim for this J.

For general J, let $S \in R$-**alg** be faithfully flat such that $J \otimes S$ is split. Then $x \in J$ maps to an element of $R1_J \otimes S \subseteq J \otimes S$ and so belongs to $R1_J \subseteq J$. Since $U_{\zeta 1_J} = \zeta^2 \,\mathrm{Id}_J$ for $\zeta \in R$, the claim follows.

The following result is well known when R is a field or perhaps a local ring (see, for example [Pe19, Proposition 20]). We impose no hypothesis on R.

Proposition 7.5. *Suppose C is a split composition R-algebra of constant rank at least 2, that is, C is $R \times R$, $\mathrm{Mat}_2(R)$, or $\mathrm{Zor}(R)$. Then $\mathrm{Her}_3(C, \Gamma) \cong \mathrm{Her}_3(C)$ for all Γ.*

Proof. Define γ_i via $\Gamma = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$. We may assume $\gamma_1\gamma_2\gamma_3 = 1$. Since n_C is universal, there are invertible $p, q \in C$ such that $\gamma_2 = n_C(q^{-1})$ and $\gamma_3 = n_C(p^{-1})$, so $\gamma_1 = n_C(pq)$. We define $C^{(p,q)}$ to be a not-necessarily-associative R-algebra with the same underlying R-module structure and with multiplication $\cdot_{(p,q)}$ defined by

$$x \cdot_{(p,q)} y := (xp)(qy),$$

where the multiplication on the right is the multiplication in C. Certainly $(pq)^{-1}$ is an identity element in $C^{(p,q)}$. The algebra $C^{(p,q)}$ is called an isotope of C and is studied in [McC71a], where it is proved to be alternative. One checks that it is a composition algebra with quadratic form $n_{C^{(p,q)}} = n_C(pq) n_C$ (see [McC71a, Proposition 5] for a more general statement in case R is a field).

Define $\phi \colon \mathrm{Her}_3(C^{(p,q)}) \to \mathrm{Her}_3(C, \Gamma)$ via $\phi(\sum x_i \varepsilon_i + \delta_i(c_i)) = \sum x_i \varepsilon_i + \delta_i^\Gamma(c_i')$, where

$$c_1' = (pq)c_1(pq), \quad c_2' = c_2 p, \quad \text{and} \quad c_3' = q c_3.$$

It is evidently an isomorphism of R-modules, and one checks that it is an isomorphism of Jordan algebras, compare [McC71a, Theorem 3]. Therefore, we are reduced to verifying that $C^{(p,q)}$ is split.

If C is associative, then the R-linear map

$$L_{pq} \colon C^{(p,q)} \to C \text{ such that } \quad L_{pq}(x) = pqx$$

is an isomorphism of R-algebras. So assume $C = \mathrm{Zor}(R)$.

At the beginning, when we chose p and q, we were free to pick $\xi_i, \eta_i \in R^\times$ such that $p = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$ and $q = \begin{pmatrix} \eta_1 & 0 \\ 0 & \eta_2 \end{pmatrix}$. Let $A \in \mathrm{Mat}_3(R)$ be any matrix such that $\det A = (\xi_1 \xi_2^2 \eta_1^2 \eta)^{-1}$ and put $B := \xi_2 \eta_1 (A^\sharp)^\intercal$, where $\sharp$ denotes the classical adjoint. With $\zeta_i := (\xi_i \eta_i)^{-1}$, one checks, using the formula $(Sx) \times (Sy) = (S^\sharp)^\top (x \times y)$ for $\times$ the usual cross product in R^3, that the assignment

$$\begin{pmatrix} \alpha_1 & u_1 \\ u_2 & \alpha_2 \end{pmatrix} \mapsto \begin{pmatrix} \zeta_1 \alpha_1 & A u_1 \\ B u_2 & \zeta_2 \alpha_2 \end{pmatrix}$$

defines an isomorphism $C \xrightarrow{\sim} C^{(p,q)}$. □

8. The ideal structure of Freudenthal algebras

It is a standard exercise to show that every (two-sided) ideal in the matrix algebra $\mathrm{Mat}_n(R)$ is of the form $\mathrm{Mat}_n(\mathfrak{a})$ for some ideal $\mathfrak{a}$ in R. More generally, every ideal in an Azumaya R-algebra A is of the form $\mathfrak{a}A$ some ideal $\mathfrak{a}$ of R [KnO, p. 95, Corollary III.5.2].

A similar result holds for every octonion R-algebra C: *Every one-sided ideal in C is a two-sided ideal that is stable under the involution on C. The maps $I \mapsto I \cap R$ and $\mathfrak{a}C \leftarrow\!\shortmid \mathfrak{a}$ are bijections between the set of ideals of C and ideals in R.* See [Pe21, Section 4] for a proof in this generality and the references therein for earlier results of this type going back to [Ma].

We now prove a similar result for Freudenthal algebras.

Definition 8.1. An *ideal* in a para-quadratic R-algebra J is the kernel of a homomorphism, that is, an R-submodule I such that

$$U_I J + U_J I + \{JJI\} = I,$$

where we have written $U_I J$ for the R-span of $U_x y$ with $x \in I$ and $y \in J$. (This is sometimes written with a $\subseteq$ instead of $=$, but the two are equivalent since $U_J I \supseteq U_{1_J} I = I$.) An R-submodule I is an *outer ideal* if

$$U_J I + \{JJI\} = I. \tag{8.1}$$

Here are some observations about outer ideals:

1. Every ideal is an outer ideal.
2. If 2 is invertible in R, then for every $x \in I$ and $y \in J$, $U_x y = \frac{1}{2}\{xyx\} \in \{JJI\}$, so the notions of ideal and outer ideal coincide, and both agree with the notion of ideal for the commutative bilinear product $\bullet$ defined in (5.4).
3. For every ideal $\mathfrak{a}$ in R, the R-submodule $\mathfrak{a}J$ is an ideal of J.
4. If 1_J is unimodular, then for every outer ideal I of J, $I \cap R1_J$ is an ideal in R, for the trivial reason that I is an R-module.
5. If $\mathfrak{a}$ is an ideal in R and 1_J is unimodular, then $\mathfrak{a}1_J = (\mathfrak{a}J) \cap R1_J$. The containment $\subseteq$ is clear. To see the opposite containment, suppose $\alpha 1_J \in \mathfrak{a}J \cap R1_J$ for some $\alpha \in R$ and write $\alpha 1_J = \sum \alpha_i y_i$ with $\alpha_i \in \mathfrak{a}$ and $y_i \in J$. There is some R-linear $\lambda\colon J \to R$ such that $\lambda(1_J) = 1$. Then $\alpha = \lambda(\alpha 1_J) = \sum \alpha_i \lambda(y_i)$ is in $\mathfrak{a}$.

Theorem 8.2. *Let J be a Freudenthal R-algebra. Every outer ideal of J is an ideal. The maps $I \mapsto I \cap R1_J$ and $\mathfrak{a}J \leftarrow\!\shortmid \mathfrak{a}$ are bijections between the set of outer ideals of J and the set of ideals of R.*

Proof. It suffices to show that the stated maps are bijections, because then observation (3) implies that every outer ideal is of the form $\mathfrak{a}J$ and therefore an ideal. In view of (5) (noting that 1_J is unimodular), it suffices to verify that $(I \cap R1_J)J = I$ for every outer ideal I. First suppose that $J = \mathrm{Her}_3(C)$ for some composition R-algebra C and write $\mathfrak{a} := I \cap R1_J$. The Peirce projections relative to the diagonal frame of J, that is, U_{ε_i} and $x \mapsto \{\varepsilon_j x \varepsilon_l\}$ for $i, j, l = 1, 2, 3$ [McC66, p. 1074] stabilize I, and we find

$$I = \sum_i (I \cap R\varepsilon_i) + (I \cap \delta_i(C)).$$

Set $B := \{c \in C \mid \delta_1(c) \in I\}$. We claim that B is an ideal in C. Note that $U_{\delta_1(1_C)}\delta_1(b) = \delta_1(\bar{b})$, so B is stable under the involution.

We leverage (6.13). Repeatedly applying this with $a = 1_C$ and using that B is stable under the involution, we conclude that $\delta_i(B) = I \cap \delta_i(C)$ for all i. For $c \in C$ and $b \in B$, I contains $\{1_J \delta_2(\bar{c})\delta_1(\bar{b})\} = \delta_3(cb)$, so $cB \subseteq B$, that is, B is an ideal in C and therefore $B = \mathfrak{a}C$ for some ideal $\mathfrak{a}$ of R.

For $c \in C$, I contains $\{\delta_i(1_C)\varepsilon_{i+1}\delta_i(\mathfrak{a}c)\} = \mathrm{Tr}_C(\mathfrak{a}c)\varepsilon_{i+2}$. Since Tr_C is surjective, $\mathfrak{a}\varepsilon_j \subseteq I$ for all j.

In the other direction, if $\alpha_i\varepsilon_i \in I$, then so is

$$\{\delta_{i+1}(1_C)1_J(\alpha_i\varepsilon_i)\} = \delta_{i+1}(\alpha_i 1_C).$$

It follows that $I \cap R\varepsilon_i = \mathfrak{a}R$ for all i and, in particular, $I \cap R1_J = \mathfrak{a}R$ and $I = \mathfrak{a}J$.

We now treat the general case. Suppose I is an outer ideal in a Freudenthal R-algebra J. There is a faithfully flat $S \in R$-**alg**, such that $J \otimes S$ is a split Freudenthal algebra. We have

$$((I \cap R1_J)J) \otimes S = ((I \otimes S) \cap S1_J)(J \otimes S) = I \otimes S,$$

where the first equality is because S is flat and the second is by the previous case, since $I \otimes S$ is an outer ideal. It follows that $I = (I \cap R1_J)J$ as desired. □

Remark 8.3. In the proof above, the inclusion $(I \cap R1_J)J \subseteq I$ could instead have been argued as follows. Define $\mathrm{Sq}(J)$ as the R-submodule of J generated by x^2 for $x \in J$. Since 1_J is unimodular, one finds that $(I \cap R1_J)\,\mathrm{Sq}(J) \subseteq I$. Then, one argues that $\mathrm{Sq}(J) = J$ for a split Freudenthal algebra, and that $\mathrm{Sq}(J \otimes S) = \mathrm{Sq}(J) \otimes S$ for all flat $S \in R$-**alg**.

Corollary 8.4. *Let $\phi\colon J \to A$ be a homomorphism of para-quadratic R algebras. If J is a Freudenthal algebra and 1_A is unimodular in A, then ϕ is injective.*

In particular, the corollary applies to every homomorphism between Freudenthal R-algebras.

Proof. The kernel of ϕ is an ideal of J and therefore $\mathfrak{a}J$ for some ideal $\mathfrak{a}$ of R. For $\alpha \in \mathfrak{a}$, we have $0 = \phi(\alpha 1_J) = \alpha\phi(1_J) = \alpha 1_A$, so $\alpha = 0$ because 1_A is unimodular. □

Because a Freudenthal algebra J is a projective R-module of rank ≥ 3, the corollary says that there is no homomorphism of para-quadratic R-algebras $J \to R^+$. This might be stated as *J is not augmented* or *J has no counit*.

A para-quadratic algebra J is said to be *simple* if the underlying module is not the zero module and if every ideal in J equals 0 or J. Theorem 8.2 immediately gives:

Corollary 8.5. *If J is a Freudenthal R-algebra and R is a field, then J is simple.* □

Remark 8.6. There is also the notion of an *inner* ideal in a Jordan algebra (see [McC71b, Theorem 8] for a description of them for $\mathrm{Her}_3(\mathrm{Zor}(R))$). The inner ideals are related to the projective homogeneous varieties associated with the group of isometries described in Section 15 and "outer automorphisms" relating these varieties (see [Rac] and [CarrG]).

9. Groups of type F_4 and C_3

In the following, for a Jordan R-algebra J, we write $\mathrm{Aut}(J)$ for the ordinary group of R-linear automorphisms of J and $\mathbf{Aut}(J)$ for the functor from R-**alg** to groups such that $S \mapsto \mathrm{Aut}(J \otimes S)$. Recall that for every simple root datum, there is a unique simple group scheme over $\mathbb{Z}$ called a *Chevalley group* [DemG, Corollary XXIII.5.4], and every split simple algebraic group over a field is obtained from a unique Chevalley group by base change [Milne, Section 23g].

Lemma 9.1. *Let J be a Freudenthal algebra of rank 15 or 27 over a ring R. Then* $\mathbf{Aut}(J)$ *is a semisimple R-group scheme that is adjoint (i.e., its center is the trivial group scheme). Its root system has type C_3 if*

J has rank 15 and type F_4 if J has rank 27. If J is the split Freudenthal algebra, then the group scheme $\mathbf{Aut}(J)$ *is obtained from the Chevalley group over $\mathbb{Z}$ by base change.*

Proof. First suppose that $R = \mathbb{Z}$ and J is split. If J has rank 15, then the proof of 14.19 in [Sp73] shows that the automorphisms of $J \otimes F$ for every field F are exactly the automorphisms of the algebra $\mathrm{Mat}_6(F)$ with the split symplectic involution, which is the split adjoint group PGSp_6 of type C_3. For J of rank 27, $\mathbf{Aut}(J) \times F$ is split of type F_4 by [J71, Section 6] (written for Lie algebras), [Fr85, Satz 4.11] (written for $\mathbb{R}$), [SpV, Theorem 7.2.1] (if $\mathrm{char}\, F \neq 2, 3$), or [Sp73, 14.24] in general.

Note that $\mathbf{Aut}(J) \times F$ is connected and smooth as a group scheme over F, and $\mathbf{Aut}(J)$ is finitely presented (because $\mathbb{Z}$ is noetherian and J is a finitely generated module), so it follows by [GanY, Proposition 6.1] or [AlsG, Lemma B.1] that $\mathbf{Aut}(J)$ is smooth as a scheme over the Dedekind domain $\mathbb{Z}$. In summary, $\mathbf{Aut}(J)$ is semisimple and adjoint of the specified type. Moreover, because $\mathbf{Aut}(J) \times \mathbb{Q}$ is split, $\mathbf{Aut}(J)$ is a Chevalley group [Conrad, Theorem 1.4].

In the case of general R and J, let $S \in R$-**alg** be faithfully flat such that $J \otimes S$ is split. Then $\mathbf{Aut}(J) \times S$ is semisimple adjoint of the specified type. Certainly, $\mathbf{Aut}(J)$ is also smooth. Moreover, for each $\mathfrak{p} \in \mathrm{Spec}\, R$, there is a $\mathfrak{q} \in \mathrm{Spec}\, S$ such that $\mathfrak{q} \cap R = \mathfrak{p}$. Then the field of fractions $R(\mathfrak{p})$ of $R/\mathfrak{p}$ embeds in the field $S(\mathfrak{q})$, so the algebraic closure $\overline{R(\mathfrak{p})}$ includes in the algebraic closure $\overline{S(\mathfrak{q})}$. Because $\mathbf{Aut}(J) \times \overline{S(\mathfrak{q})}$ is adjoint semisimple of the specified type and this property is unchanged by replacing one algebraically closed field by a smaller one, the same holds over $\overline{R(\mathfrak{p})}$. Since this holds for every $\mathfrak{p}$, the claim is verified. □

Remark 9.2. In case R is a field, the automorphism group of the split Freudenthal algebra of rank 6 or 9 can be deduced in a similar manner, referring to 14.17 and 14.16 in [Sp73]. The automorphism group of the split Freudenthal algebra of rank 9 is $\mathrm{PGL}_3 \rtimes \mathbb{Z}/2$. The automorphism group of the split Freudenthal algebra of rank 6 is the special orthogonal group of the quadratic form $x^2 + y^2 + z^2$, that is, the group commonly denoted $\mathrm{SO}(3)$. In particular, it is not smooth when R is a field of characteristic 2, and indeed, one can give examples of Freudenthal algebras of rank 6 over a field F of characteristic 2 that are not split by any separable field extension of F. (From this, it follows that such an algebra is not split by any étale F-algebra R.)

Suppose J, J_0 are Jordan R-algebras and there is an fppf $S \in R$-**alg** and an isomorphism $f\colon J \otimes S \to J_0 \otimes S$. The isomorphism f gives a class in $H^1(S/R, \mathbf{Aut}(J_0))$ that depends only on the isomorphism class of J as a Jordan R-algebra. Analogous statements apply when the role of Jordan algebras is replaced with affine group schemes. This is the source of the diagonal arrows in the theorem below.

Theorem 9.3. *Let J_0 be a Freudenthal R-algebra of rank $r = 15$ or 27. In the diagram*

$$\begin{tikzcd}
\boxed{\begin{array}{l}\text{Isomorphism classes}\\ \text{of rank } r \text{ Freudenthal}\\ R\text{-algebras}\end{array}} \arrow[rr, "J \mapsto \mathbf{Aut}(J)"] \arrow[dr] & & \boxed{\begin{array}{l}\text{Isomorphism classes of}\\ \text{adjoint semisimple } R\text{-}\\ \text{group schemes of type } \mathsf{C}_3\\ (r = 15) \text{ or } \mathsf{F}_4\ (r = 27)\end{array}} \arrow[dl] \\
& H^1(R, \mathbf{Aut}(J_0)) &
\end{tikzcd}$$

all arrows are bijections that are functorial in R.

Proof. The top arrow has the claimed codomain by Lemma 9.1.

Every Freudenthal algebra is split by some faithfully flat R-algebra by definition, so the discussion preceding the theorem of the statement yields the lower left arrow.

As another consequence of Lemma 9.1, the conjugation map $\mathbf{Aut}(J_0) \to \mathbf{Aut}(\mathbf{Aut}(J_0))$ is an isomorphism that we use to identify $H^1(R, \mathbf{Aut}(J_0)) \xrightarrow{\sim} H^1(R, \mathbf{Aut}(\mathbf{Aut}(J_0)))$. Every semisimple group scheme is split by some faithfully flat R-algebra (even an étale cover) [DemG, Corollary XXIV.4.1.6], and the remarks before the statement of the theorem now define the lower right arrow.

The facts that the diagram commutes and that the diagonal arrows are bijective are general features of the machinery of descent as in [Wa, Theorem 17.6] or see [CalF, 2.2.4.5]. □

The machinery of descent shows a more refined statement, where each of the boxes in the theorem are replaced by groupoids and the arrows are equivalences of groupoids. In that statement, the bottom box is replaced by the groupoid of $\mathbf{Aut}(J_0)$-torsors and the diagonal arrows are provided by [Gir, p. 151, Theorem III.2.5.1].

In the theorem, the set $H^1(R, \mathbf{Aut}(J_0))$ is naturally a pointed set and the bijections are actually of pointed sets, where the distinguished elements are J_0 in the upper left and $\mathbf{Aut}(J_0)$ in the upper right.

In case R is a field of characteristic different from 2, 3 and $r = 27$, the theorem goes back to [Hij]. Or see [KnMRT, 26.18]. The statement of the theorem is similar to various statements over a field that can be found in [Serre, Section III.1].

Corollary 9.4. *For each Freudenthal R-algebra J of rank 15 or 27, there is an étale cover $S \in R$-**alg** such that $J \otimes S$ is a split Freudenthal algebra.*

Proof. Let J_0 be the split Freudenthal R-algebra of the same rank as J. The image $\mathbf{Iso}(J, J_0)$ of J in $H^1(R, \mathbf{Aut}(J_0))$ is an $\mathbf{Aut}(J_0)$-torsor. Since $\mathbf{Aut}(J_0)$ is smooth (Lemma 9.1), there is an étale cover of R that trivializes $\mathbf{Iso}(J, J_0)$. □

Note that exactly the same kind of argument gives analogues of Lemma 9.1 and Theorem 9.3 for composition algebras, where $r = 4$ or 8, and the group is of type A_1 or G_2, respectively.

10. Generic minimal polynomial of a Freudenthal algebra

Polynomials with polynomial-law coefficients

Let J be a Jordan R-algebra, $\mathscr{P}(J, R)$ the R-algebra of polynomial laws from J to R, and t a variable. Consider a polynomial $\mathbf{p}(t) = \sum_{i=0}^{n} f_i t^i$ with $f_i \in \mathscr{P}(J, R)$ for $0 \leq i \leq n$. For $S \in R$-**alg**, $x \in J \otimes S$, we have $\mathbf{p}(t, x) := \sum_{i=0}^{n} f_{iS}(x) t^i \in S[t]$, and we define

$$\mathbf{p}(x, x) := \sum_{i=0}^{n} f_{iS}(x) x^i \qquad \in J \otimes S.$$

The algebra J is said to *satisfy* $\mathbf{p}$ if $\mathbf{p}(x, x) = 0 = (t\mathbf{p})(x, x)$ for all $S \in R$-**alg** and $x \in J \otimes S$. Note that the second equation follows from the first if 2 is invertible in R but not in general (see Remark 6.3).

The generic minimal polynomial

Let $J := \mathrm{Her}_3(C, \Gamma)$ as in Example 6.6. With a variable t, we recall from (6.9) that J satisfies the monic polynomial

$$\mathbf{m}_J = t^3 - \mathrm{Tr}_J \cdot t^2 + S_J \cdot t - N_J \qquad \in \mathscr{P}(J, R)[t]. \tag{10.1}$$

More precisely, by [Lo, 2.4(b)], J is generically algebraic of degree 3 in the sense of [Lo, 2.2] and $\mathbf{m}_J$ is the *generic minimal polynomial* of J, that is, the unique monic polynomial in $\mathscr{P}(J, R)[t]$ of minimal degree satisfied by J [Lo, 2.7]. It follows that the Jordan algebra J determines the polynomial $\mathbf{m}_J$ uniquely. In particular, the *generic norm* N_J, the *generic trace* T_J (or Tr_J) and, in fact, the cubic norm structure underlying J in the sense of Definition 6.2 are uniquely determined by J as a Jordan algebra. By faithfully flat descent, every Freudenthal algebra J has a uniquely determined generic minimal polynomial of the form (10.1), and a uniquely determined underlying cubic norm structure. We conclude:

Proposition 10.1. *Every Freudenthal algebra J is a cubic Jordan algebra and the bilinear trace form T_J is regular.* □

The preceding discussion shows that, for a Freudenthal R-algebra J, the Jordan algebra structure of J alone (ignoring that J is a *cubic* Jordan algebra) determines the bilinear form T_J. (For example, the 11 Freudenthal $\mathbb{R}$-algebras discussed in Example 6.8 have distinct trace forms and therefore are distinct.) When R is a field of characteristic $\neq 2, 3$ and J and J' are reduced Freudenthal algebras, Springer proved that the converse also holds, that is, $J \cong J'$ if and only if $T_J \cong T_{J'}$ [SpV, Theorem 5.8.1]. We do not use Springer's result in this paper.

The following result can also be found in [Pe19, Corollary 18(b)], based on the definition of Albert algebra appearing there.

Lemma 10.2. *Let J and J' be Freudenthal R-algebras. An R-linear map $\phi\colon J \to J'$ is an isomorphism of para-quadratic algebras if and only if ϕ is surjective, $\phi(1_J) = 1_{J'}$, and $N_{J'} = N_J\phi$ as polynomial laws.*

Proof. The "only if" direction follows from the uniqueness of the generic minimal polynomial as in (10.1), so we show "if". The equality $N_{J'} = N_J\phi$ of polynomial laws and the definition of the directional derivative in Section 3 gives formulas, such as

$$\nabla_y N_J(x) = \nabla_{\phi(y)} N_{J'}(\phi(x)).$$

Since $\phi(1_J) = 1_{J'}$, the definition of the bilinear forms T_J and $T_{J'}$ in (6.1) give:

$$T_{J'}(\phi(x), \phi(y)) = T_J(x, y)$$

for all x, y. Therefore, on the one hand we have

$$\nabla_y N_J(x) = T_J(x^\sharp, y) = T_{J'}(\phi(x^\sharp), \phi(y)).$$

On the other hand, we have

$$\nabla_y N_J(x) = \nabla_{\phi(y)} N_{J'}(\phi(x)) = T_{J'}((\phi(x))^\sharp, \phi(y)).$$

Therefore, $\phi(x^\sharp) = \phi(x)^\sharp$ for all x. In summary, ϕ commutes with $\sharp$ and preserves T_J. Therefore, by (6.5), ϕ is a homomorphism of Jordan algebras.

Suppose that x is in $\ker\phi$. Then for all $y \in J$, $T_J(x, y) = T_{J'}(\phi(x), \phi(y)) = 0$, so $x = 0$ since the bilinear form T_J is regular. Since ϕ is both surjective and injective, it is an isomorphism. □

Example 10.3. We claim that $\mathrm{Her}_3(R \times R)$, the split Freudenthal algebra of rank 9, is isomorphic to $\mathrm{Mat}_3(R)^+$. To see this, define $\pi_i : R \times R \to R$ to be the projection on the i-th coordinate and define $\phi\colon \mathrm{Her}_3(R \times R) \to \mathrm{Mat}_3(R)^+$ by sending

$$\begin{pmatrix} \alpha_1 & c_3 & \cdot \\ \cdot & \alpha_2 & c_1 \\ c_2 & \cdot & \alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_1 & \pi_1(c_3) & \pi_2(c_2) \\ \pi_2(c_3) & \alpha_2 & \pi_1(c_1) \\ \pi_1(c_2) & \pi_2(c_2) & \alpha_3 \end{pmatrix} \quad \text{for } \alpha_i \in R \text{ and } c_i \in R \times R.$$

This map is obviously R-linear and surjective and sends the identity to the identity. One checks directly that ϕ preserves norms, that is, that $\det(\phi(x))$ equals $N(x)$ according to (6.11). Because $\mathrm{Her}_3(R \times R)$ and $\mathrm{Mat}_3(R)^+$ are both cubic Jordan algebras with regular trace bilinear forms, the proof of the "if" direction of Lemma 10.2 shows that φ is an isomorphism of Jordan algebras.

11. Basic classification results for Albert algebras

In the case where R is a field, such as the real numbers, a finite field, a local field, or a global field, one can find in many places in the literature classifications of Albert algebras proved using techniques involving algebras as in [SpV, Section 5.8]. For such an R, groups of type F_4 can be classified using techniques from algebraic groups, such as in [PlRap, Chapter 6] or [Gil19]. The two approaches are equivalent by Theorem 9.3.

Example 11.1 (Albert algebras over $\mathbb{R}$). Up to isomorphism, there are three Albert $\mathbb{R}$-algebras, namely, the split one $\mathrm{Her}_3(\mathrm{Zor}(\mathbb{R}))$, $\mathrm{Her}_3(\mathbb{O}, \langle 1, -1, -1\rangle)$, and $\mathrm{Her}_3(\mathbb{O})$. Rather than proving this in the language of Jordan algebras as in [AlbJ, Theorem 10], one may leverage Theorem 9.3 as follows. The three algebras are pairwise nonisomorphic because their trace forms are (Example 6.8). At the same time, a computation in the Weyl group of F_4 as in [Serre, Section III.4.5], [BorE, 14.1], or [AdT, Table 3] shows that $H^1(\mathbb{R}, \mathbf{Aut}(\mathrm{Her}_3(\mathrm{Zor}(\mathbb{R}))))$ has three elements. That is, there are exactly three isomorphism classes of simple affine group schemes over $\mathbb{R}$ of type F_4, so we have found all of them.

Example 11.2 (Albert algebras over global fields). Let A be an Albert F-algebra for F a global field. Put Ω for the (finite) set of inequivalent embeddings $\omega\colon F \hookrightarrow \mathbb{R}$. Since $\mathbf{Aut}(A)$ is a simple and simply connected affine group scheme, the natural map

$$H^1(F, \mathbf{Aut}(A)) \xrightarrow{\prod \omega} \prod_{\omega\in\Omega} H^1(\mathbb{R}, \mathbf{Aut}(A))$$

is a bijection by [Ha66] or [PlRap, p. 286, Theorem 6.6]. Because $H^1(\mathbb{R}, \mathbf{Aut}(A))$ has three elements by the preceding example, there are exactly $3^{|\Omega|}$ isomorphism classes of Albert F-algebras. This result, for number fields and with a proof in the language of Albert algebras, dates back to [AlbJ, p. 417, Corollary of Theorem 12].

The same argument goes through for octonion algebras, and one finds that there are $2^{|\Omega|}$ isomorphism classes of octonion F-algebras. This result, for number fields and with a proof in the language of octonion algebras, dates back to [Z, p. 400].

In the special case where F has a unique real embedding (e.g., $F = \mathbb{Q}$), the two isomorphism classes of octonion algebras are $\mathrm{Zor}(F)$ and $\mathcal{O} \otimes F$, and the three isomorphism classes of Albert algebras are $\mathrm{Her}_3(\mathrm{Zor}(F))$ and $\mathrm{Her}_3(\mathcal{O} \otimes F, \langle 1, s, s\rangle)$ for $s = \pm 1$.

Below, we will focus our attention on classification results in the case where R is not a field. We translate known results about cohomology of affine group schemes into the language of Albert algebras.

Proposition 11.3. *If R is: (1) a complete discrete valuation ring whose residue field is finite or (2) a finite ring, then every Freudenthal R-algebra of rank 15 or 27 and every quaternion or octonion R-algebra is split.*

Proof. In view of Theorem 9.3 and its analogue for composition algebras, it suffices to prove that $H^1(R, \mathbf{G}) = 0$ for $\mathbf{G}$ a simple R-group scheme of type F_4 or C_3 obtained by base change from a Chevalley group over $\mathbb{Z}$. In case (1), this is [Conrad, Proposition 3.10]. In case (2), we apply the following lemma. □

Lemma 11.4. *If R is a finite ring and $\mathbf{G}$ is a smooth connected R-group scheme, then $H^1(R, \mathbf{G}) = 0$.*

Proof. If R is not connected, then it is a finite product $R = \prod R_i$, where each ring R_i is finite, so $H^1(R, \mathbf{G}) = \prod H^1(R_i, \mathbf{G} \times R_i)$. Therefore, it suffices to assume that R is connected.

Suppose $\mathbf{X}$ is a $\mathbf{G}$-torsor. Our aim is to show that $\mathbf{X}$ is the trivial torsor, that is, $\mathbf{X}(R)$ is nonempty. Put $\mathfrak{a}$ for the nil radical $\mathrm{Nil}(R)$ of R. Because R is finite, there is some minimal $m \geq 1$ such that $\mathfrak{a}^m = 0$. We proceed by induction on m. If $m = 1$, then R is reduced and connected, so it is a finite field and $H^1(R, \mathbf{G}) = 0$ by Lang's theorem. For the case $m \geq 2$, put $I := \mathfrak{a}^{m-1}$. The ring R/I has $\mathrm{Nil}(R/I)^{m-1} = (\mathrm{Nil}(R)/I)^{m-1} = 0$, so by induction, $\mathbf{X}(R/I)$ is nonempty. On the other hand, $I^2 = \mathfrak{a}^{2m-2} = \mathfrak{a}^m \cdot \mathfrak{a}^{m-2} = 0$ and $\mathbf{X}$ is smooth, so the natural map $\mathbf{X}(R) \to \mathbf{X}(R/I)$ is surjective. □

Example 11.5. Suppose R is a Dedekind domain, and write F for its field of fractions. For $\mathbf{G}$ a Chevalley group of type G_2, F_4, or E_8, the map $H^1(R, \mathbf{G}) \to H^1(F, \mathbf{G})$ has zero kernel [Ha67, Satz 3.3]. Consequently, *if A is an Albert or octonion R-algebra and $A \otimes F$ is split, then the R-algebra A is split.*

In particular, if F is a global field with no real embeddings, then every Albert or octonion F-algebra is split, so every Albert or octonion R-algebra is split.

In the case where F is a number field with a real embedding, we provide the following partial result, which relies on Example 11.1.

Proposition 11.6. *Suppose F is a number field and R is a localization of its ring of integers at finitely many primes. If A is an Albert (respectively, octonion) F-algebra such that $A \otimes \mathbb{R}$ is not isomorphic to* $\mathrm{Her}_3(\mathbb{O})$ *(respectively,* $\mathbb{O}$*) for every embedding $F \hookrightarrow \mathbb{R}$, then there is an Albert (respectively, octonion) R-algebra B such that $B \otimes F \cong A$ and B is uniquely determined up to R-isomorphism.*

Proof. Write $\mathbf{G}$ for the automorphism group of the split Albert (respectively, octonion) F-algebra. Write $H^1_{\mathrm{ind}}(R, \mathbf{G}) \subseteq H^1(R, \mathbf{G})$ for the isomorphism classes of R-algebras B such that $B \otimes F_v$ is not $\mathrm{Her}_3(\mathbb{O})$ (respectively, $\mathbb{O}$), that is, such that $\mathbf{Aut}(B) \times F_v$ is not compact, for all real places v of F. Since $\mathbf{G}$ is simply connected, Strong Approximation gives that the natural map $H^1_{\mathrm{ind}}(R, \mathbf{G}) \to H^1_{\mathrm{ind}}(F, \mathbf{G})$ is an isomorphism [Ha67, Satz 4.2.4], which is what is claimed. □

12. The number of generators of an Albert algebra

The goal of this section is to prove Proposition 12.1, which is inspired by the work of First-Reichstein [FiR] generalizing the Forster-Swan theorem. Let J be a para-quadratic R-algebra. By a *(para-quadratic) subalgebra* of J, we mean a submodule $J' \subseteq J$ containing 1_J and closed under the U-operator, that is, such that $U_x y \in J'$ for all $x, y \in J'$. For any subset $S \subseteq J$, the smallest subalgebra of J containing S is called the subalgebra *generated by* S; if this subalgebra is all of J, we say that S *generates* J.

Proposition 12.1. *For every noetherian ring R, every Albert R-algebra can be generated in the sense of the preceding paragraph by* $3 + \dim \operatorname{Max} R$ *elements.*

In the statement, $\operatorname{Max} R$ is the topological space whose points are the maximal ideals of R, endowed with the subspace topology inherited from $\operatorname{Spec} R$. It is evident that $\dim \operatorname{Max} R \leq \dim \operatorname{Spec} R$, also known as the Krull dimension. Beyond this inequality, the two numbers may be quite different (e.g., for a local ring, $\dim \operatorname{Max} R = 0$ and $\dim \operatorname{Spec} R$ can be any number).

If 2 is invertible in R, the bound in the proposition is Corollary 4.2c in [FiR]. The contribution here is to remove the hypothesis on 2. In the special case where R contains an infinite field of characteristic $\neq 2$, a different (and possibly smaller) upper bound is given in [FiRW, Section 13].

The results in [FiR] reduce the proof of the proposition to the case where R is a field. For a field of characteristic not 2, a proof may be read off from [McC04, p. 112]; we give a characteristic-free proof using the first Tits construction of cubic Jordan algebras [McC69, pp. 507–509]. We briefly recall its details.

The first Tits construction

Let A be a (finite-dimensional) separable associative algebra of degree 3 over a field F, so its generic norm N_A is a cubic form on A and its trace $T_A \colon A \times A \to F$ defined as in (6.1) is a nondegenerate symmetric bilinear form. Given any nonzero scalar $\mu \in F$, we obtain a cubic norm structure

$$\mathbf{M} := (A \times A \times A, 1_{\mathbf{M}}, \sharp, N_{\mathbf{M}})$$

by defining

$$1_{\mathbf{M}} := (1_A, 0, 0), \tag{12.1}$$

$$x^\sharp := (x_0^\sharp - x_1 x_2, \mu^{-1} x_2^\sharp - x_0 x_1, \mu x_1^\sharp - x_2 x_0) \quad \text{and} \tag{12.2}$$

$$N_{\mathbf{M}}(x) := N_A(x_0) + \mu N_A(x_1) + \mu^{-1} N_A(x_2) - T_A(x_0 x_1 x_2) \tag{12.3}$$

for all $x = (x_0, x_1, x_2)$ in all scalar extensions of $A \times A \times A$. By [McC69, Theorem 6], $\mathbf{M}$ is a cubic norm structure, so $J(\mathbf{M})$ is a cubic Jordan F-algebra which we denote by $J(A, \mu)$.

Example 12.2. (a): Let E be a cubic étale F-algebra. If E is either split — that is, isomorphic to $F \times F \times F$ — or a cyclic cubic field extension of F, then $J(E, 1) \cong \mathrm{Her}_3(F \times F, \langle 1, -1, -1 \rangle)$ [PeR84b, Theorem 3], that is, is the split Freudenthal algebra of rank 9 (Proposition 7.5), which is $\mathrm{Mat}_3(F)^+$ by Example 10.3.

(b): Let A be a central simple associative F-algebra of degree 3. Then $J(A, 1)$ is a split Albert algebra [PeR84a, Corollary 4.2].

Lemma 12.3. *Let A be a separable associative F-algebra of degree 3 and $\mu \in F^\times$. Then the first Tits construction $J(A, \mu)$ is generated by A^+ (identified in $J(A, \mu)$ through the initial summand) and $(0, 1_A, 0)$.*

Proof. Let J' be the subalgebra of $J(A, \mu)$ generated by A^+ and $w := (0, 1_A, 0)$. As a subalgebra, it is closed under $\sharp$ by (6.8), that is, $x^\sharp \in J'$ and $x \times y \in J'$ for all $x, y \in J'$. Since $w^\sharp = \mu(0, 0, 1_A)$ and $x_0 \times (0, x_1, x_2) = (0, -x_0 x_1, -x_2 x_0)$ for all $x_i \in A$ by (12.2), it follows that J' must be all of $J(A, \mu)$. □

Proof of Proposition 12.1. Theorem 1.2 in [FiR] reduces the proof to the case where R is a field F, in which case $\dim \operatorname{Max} R = 0$, so the task is to prove that three elements suffice to generate an Albert algebra J. If F is infinite, then Proposition 4.1 in [FiR] reduces us to considering the case where J is split. If F is finite, then J is split by Proposition 11.3.

If $F \neq \mathbb{F}_2$, then the split cubic étale F-algebra $E := F \times F \times F$ can be generated by a single element x as an associative algebra. On the other hand, if $F = \mathbb{F}_2$, let $E := \mathbb{F}_8$ be the cyclic cubic extension of F, which is again generated by one element, call it x. In either case, x also generates the Jordan algebra E^+, because the powers of x in E^+ and E are the same.

Hence, Example 12.2 (a) and Lemma 12.3 show that $\mathrm{Mat}_3(F)^+$ is generated by two elements. Lemma 12.3 combined with Example 12.2 (b) shows that the split Albert algebra $J(\mathrm{Mat}_3(F), 1)$ is generated by three elements. □

Remark 12.4. That the Jordan algebra $\mathrm{Mat}_3(F)^+$, for any field F, is generated by two elements doesn't seem too surprising, but one should keep in mind that *the analogous result for 2-by-2 matrices is false*: the minimal number of generators for the Jordan algebra $\mathrm{Mat}_2(F)^+$ is three.

Remark 12.5 (dichotomy of fields and the Tits construction). The classification of Albert algebras over a field F of characteristic $\neq 3$ has a fundamentally different flavor depending on whether or not $H^3(F, \mathbb{Z}/3)$ is zero, as indicated by [Rost], [PeR96], or [Gar09, Section 8]. If $H^3(F, \mathbb{Z}/3) = 0$ — as is the case for global fields, p-adic fields, and the real numbers — every Albert F-algebra is reduced, that is, of the form $\mathrm{Her}_3(C, \Gamma)$ for some C and Γ, and is not a division algebra. (It is natural to speculate that this is the reason it took many years after Albert algebras were defined — all the way until 1958 — for the first Albert division algebra to be exhibited in [Alb58].) In the other case, when $H^3(F, \mathbb{Z}/3) \neq 0$, as happens when $F = \mathbb{Q}(t)$, for example, one can construct an Albert division algebra via the first Tits construction described above as $J(A \otimes \mathbb{Q}(t), t)$ for A an associative division algebra of dimension 9 over $\mathbb{Q}$.

It is known that every Albert algebra over a field is obtained by the first Tits construction or second Tits construction (which we have not described here) (see [McC70, Theorem 10] or [PeR86b, Theorem 3.1(i)]). Both constructions have been extended from the case of algebras over a field to an arbitrary base ring [PeR86a]. However, in this more general setting, the Tits constructions do not produce all Albert algebras [PaST].

13. Isotopy

The aim of this section is to discuss the notion of isotopy of Jordan algebras, which will pay off later in the paper when we discuss groups of type E_6 in Section 15 and E_7 in Section 17. We include this material at this point in the paper because Corollary 13.6 is needed in the following section.

Definition 13.1. Let J be a Jordan R-algebra, and suppose $u \in J$ is invertible. We define a Jordan algebra $J^{(u)}$ with the same underlying R-module, with U-operator $U_x^{(u)} := U_x U_u$ (where the unadorned U on the right denotes the U-operator in J), and with identity element $1^{(u)} := u^{-1}$. One checks that $J^{(u)}$ is indeed a Jordan algebra, and for u, v invertible, we have $(J^{(u)})^{(v)} = J^{(U_u v)}$. A Jordan R-algebra J' is an *isotope* of J if it is isomorphic to $J^{(u)}$ for some invertible $u \in J$; equivalently, one says that J and J' are *isotopic*. This defines an equivalence relation on Jordan algebras that is a priori weaker than isomorphism.

We have presented the notion of isotopy, for Jordan algebras. However, there are analogous notions for other classes of algebras, which go back at least to [Alb42]. For associative algebras, isotopy is the same as isomorphism. For octonion algebras, isotopy amounts to norm equivalence [AlsG, Corollary 6.7], which is a weaker condition than isomorphism (see [Gil14] and [AsHW]).

Isotopes of cubic Jordan algebras

If J is a cubic Jordan R-algebra and $u \in J$ is invertible, then [McC69, Theorem 2] and its proof show that the isotope $J^{(u)}$ is a cubic Jordan algebra as well whose identity element, adjoint and norm are given by

$$1_{J^{(u)}} = u^{-1}, \quad x^{\sharp(u)} = N_J(u) U_u^{-1} x^\sharp, \quad N_{J^{(u)}}(x) = N_J(u) N_J(x). \tag{13.1}$$

Moreover, the (bi-)linear and quadratic trace of $J^{(u)}$ have the form

$$T_{J^{(u)}}(x, y) = T_J(U_u x, y), \quad \mathrm{Tr}_{J^{(u)}}(x) = T_J(u, x), \quad S_{J^{(u)}}(x) = T_J(u^\sharp, x^\sharp). \tag{13.2}$$

The first equation of (13.2) is in [McC69, p. 500], while the second one follows from (13.1), the first, and Lemma 6.5 (1) via $\mathrm{Tr}_{J^{(u)}}(x) = T_{J^{(u)}}(u^{-1}, x) = T_J(U_u u^{-1}, x) = T_J(u, x)$. Similarly,

$$S_{J^{(u)}}(x) = \mathrm{Tr}_{J^{(u)}}(x^{\sharp(u)}) = T_J(u, N_J(u) U_u^{-1} x^\sharp) = T_J(N_J(u) U_u^{-1} u, x^\sharp) = T_J(u^\sharp, x^\sharp).$$

Example 13.2. $\mathrm{Her}_3(C, \Gamma)$ *is isotopic to* $\mathrm{Her}_3(C)$ *for every* Γ. Indeed, for

$$u := \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix} \quad \in \mathrm{Her}_3(C, \Gamma),$$

the map $\phi \colon \mathrm{Her}_3(C, \Gamma)^{(u)} \to \mathrm{Her}_3(C)$ defined by

$$\phi \begin{pmatrix} \alpha_1 & \gamma_2 c_3 & \gamma_3 \bar{c}_2 \\ \gamma_1 \bar{c}_3 & \alpha_2 & \gamma_3 c_1 \\ \gamma_1 c_2 & \gamma_2 \bar{c}_1 & \alpha_3 \end{pmatrix} = \begin{pmatrix} \gamma_1 \alpha_1 & \gamma_1 \gamma_2 c_3 & \cdot \\ \cdot & \gamma_2 \alpha_2 & \gamma_2 \gamma_3 c_1 \\ \gamma_1 \gamma_3 c_2 & \cdot & \gamma_3 \alpha_3 \end{pmatrix}$$

is an isomorphism of Jordan algebras. One can also turn this around:

$$\mathrm{Her}_3(C, \Gamma) = (\mathrm{Her}_3(C, \Gamma)^{(u)})^{(u^{-2})} \cong \mathrm{Her}_3(C)^{(\phi(u^{-2}))} = \mathrm{Her}_3(C)^{(u^{-1})}.$$

Jordan algebras isotopic to a reduced Freudenthal algebra

In the special case where R is a field, a Jordan algebra that is isotopic to the split Albert algebra $\mathrm{Her}_3(\mathrm{Zor}(R))$ is necessarily isomorphic to it (see, for example [J71, p. 53, Theorem 9]). This need not hold for general R: Alsaody has shown in [Als21, Theorem 2.7] that there exists a ring R finitely generated over $\mathbb{C}$ and an Albert R-algebra that is isotopic to the split Albert R-algebra but is not isomorphic to it. Here we show that it is sufficient to assume that R is a semilocal ring (Corollary 13.4), as a consequence of a more general result (Theorem 13.3) from which we also obtain the key Corollary 13.6.

We work in a slightly more general context than semilocal rings. For the following statements, see [EsG] and [McDW]. We say that R is *an LG ring* if whenever a polynomial $f \in R[x_1, \ldots, x_n]$ represents a unit over $R_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R, then f represents a unit over R. Every semilocal ring is an LG ring. It is easy to see that rings R_1, R_2 are both LG if and only if their product $R_1 \times R_2$ is LG. The ring of all algebraic integers and the ring of all real algebraic integers are LG rings. Every integral extension of an LG ring is LG [EsG, Corollary 2.3].

Theorem 13.3. *Suppose J is a Jordan R-algebra that is isotopic to* $\mathrm{Her}_3(C,\Gamma)$ *for some composition R-algebra C and some Γ. If R is an LG ring, then J is isomorphic to* $\mathrm{Her}_3(C,\Gamma')$ *for some* Γ'.

Proof. As in previous proofs, we reduce to the case where C has constant rank.

In view of Example 13.2, J is isotopic to $\mathrm{Her}_3(C)$, that is, $J \cong \mathrm{Her}_3(C)^{(u^{-1})}$ for some invertible $u \in \mathrm{Her}_3(C)$. The same example shows we are done if u is diagonal.

Write N for the cubic form on $\mathrm{Her}_3(C)$. In case u is not diagonal, we will apply successive elements $\eta \in \mathrm{GL}(\mathrm{Her}_3(C))$ such that $N\eta = N$ as polynomial laws. (In the notation of Section 15 below, $\eta \in \mathbf{Isom}(\mathrm{Her}_3(C))(R)$.) Note that each such η defines an isomorphism of R-modules

$$\eta\colon\ \mathrm{Her}_3(C)^{(u^{-1})} \to \mathrm{Her}_3(C)^{(\eta(u)^{-1})}. \tag{13.3}$$

We have

$$N(\eta(u)^{-1}) = N(\eta(u))^{-1} = N(u)^{-1} = N(u^{-1}),$$

so we have by (13.1) that

$$N_{\mathrm{Her}_3(C)^{(u^{-1})}} = N(u)^{-1}N = N(\eta(u)^{-1})N\eta = N_{\mathrm{Her}_3(C)^{(\eta(u)^{-1})}}\eta.$$

Since η is a norm isometry that maps the identity element u^{-1} in the domain of (13.3) to the identity element in the codomain, it is an isomorphism of algebras by Lemma 10.2. Thus, if successive elements η transform u into a diagonal element, the proof will be complete.

We employ the transformation $\tau_{st}(q)$ for $1 \le s \ne t \le 3$ and $q \in C$ defined by

$$\tau_{st}(q)\colon\ A \mapsto (I_3 + qE_{st})A(I_3 + \bar{q}E_{ts}),$$

where I_3 is the identity matrix, E_{st} is the 3-by-3 matrix with a 1 in the (s,t)-entry and 0 elsewhere, and juxtaposition defines naive multiplication of 3-by-3 matrices with entries in C. For example,

$$\tau_{12}(q)\begin{pmatrix} \alpha_1 & c_3 & \cdot \\ \cdot & \alpha_2 & c_1 \\ c_2 & \cdot & \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 + n_C(q,c_3) + \alpha_2 n_C(q) & c_3 + \alpha_2 q & \cdot \\ \cdot & \alpha_2 & c_1 \\ c_2 + \bar{c}_1\bar{q} & \cdot & \alpha_3 \end{pmatrix}.$$

These transformations appear in [J61, Section 5] and [Kr02, Section 2]; the argument in either reference shows that $\tau_{st}(q)$ preserves N for all choices of s, t, and q. For $e = 2, 3$, define polynomial functions ν_e from C^e to the group scheme $\mathbf{G}$ of linear transformations stabilizing the norm N via

$$\nu_3(q_1,q_2,q_3) = \tau_{31}(q_3)\tau_{21}(q_2)\tau_{12}(q_1) \quad \text{and} \quad \nu_2(q_1,q_2) = \tau_{32}(q_2)\tau_{23}(q_1).$$

Additionally, for every permutation π of $\{1,2,3\}$, there is a linear transformation of $\mathrm{Her}_3(C)$ that we denote also by π, for example, the transposition $(1\,2)$ acts via

$$\begin{pmatrix} \alpha_1 & c_3 & \cdot \\ \cdot & \alpha_2 & c_1 \\ c_2 & \cdot & \alpha_3 \end{pmatrix} \mapsto \begin{pmatrix} \alpha_2 & \bar{c}_3 & \cdot \\ \cdot & \alpha_1 & \bar{c}_2 \\ \bar{c}_1 & \cdot & \alpha_3 \end{pmatrix} \tag{13.4}$$

(see, for example, [Kr02, p. 282]). The other transpositions are constructed analogously and each evidently preserves the norm. In this way, we obtain a representation of the permutation group on $\mathrm{Her}_3(C)$.

<u>*Case: R is local*</u>: We collect some observations in the case where R is a local ring. Write

$$u = \begin{pmatrix} \alpha_1 & c_3 & \cdot \\ \cdot & \alpha_2 & c_1 \\ c_2 & \cdot & \alpha_3 \end{pmatrix}.$$

By hypothesis, $N(u)$ is invertible, that is, does not lie in the maximal ideal $\mathfrak{m}$ of R.

If $\alpha_1 \in R^\times$ or $\alpha_2, \alpha_3 \notin R^\times$, then after modifying u by a transformation in the image of ν_3, we may assume that $\alpha_1 \in R^\times$ and $c_2 = c_3 = 0$. Indeed, if $\alpha_1 \notin R^\times$, then by (6.11) we have

$$N(u) \equiv \mathrm{Tr}_C(c_1 c_2 c_3) \bmod \mathfrak{m},$$

whence $c_3 \notin \mathfrak{m}C$. Since n_C continues to be regular when changing scalars to $R/\mathfrak{m}$, some $q \in C$ has $n_C(q, c_3) \notin \mathfrak{m}$. Applying $\tau_{12}(q)$, we may arrange $\alpha_1 \in R^\times$. Then note that $\tau_{21}(q)\left(\begin{smallmatrix} \alpha_1 & c_3 & \cdot \\ \cdot & \alpha_2 & c_1 \\ c_2 & \cdot & \alpha_3 \end{smallmatrix}\right)$ has top row entries $\alpha_1, c_3 + \alpha_1\bar{q}, \bar{c_2}$. Taking $q = -\bar{c}_3\alpha_1^{-1}$ shows that we may assume $c_3 = 0$. The argument that we may assume $c_2 = 0$ is similar, with the role of τ_{21} replaced by τ_{31}.

Now suppose that $\alpha_1 \in R^\times$ and $c_2 = c_3 = 0$. If $\alpha_2 \in R^\times$ or $\alpha_3 \notin R^\times$, then after modifying u by a transformation in the image of ν_2, we may assume that u is diagonal. Indeed, since u has norm $\alpha_1(\alpha_2\alpha_3 - n_C(c_1)) \notin \mathfrak{m}$, at least one of α_2, α_3, or $n_C(c_1)$ is not in $\mathfrak{m}$. The same argument as in the preceding paragraph, with τ_{12} replaced by τ_{23}, shows that we may assume that $\alpha_2 \notin \mathfrak{m}$. The same argument as in the preceding paragraph, with τ_{21} replaced by τ_{32}, shows that we may assume that $c_1 = 0$. Thus, we have transformed u into a diagonal element, as required.

<u>*General case*</u>: Return to the setting of R as in the statement of the theorem. We combine the transformations ν_3, ν_2, and permutations together into a polynomial function $C^{21} \to \mathbf{G}$, namely

$$(\nu_2\,(2\,3)\,\nu_2\nu_3\,(1\,3))(\nu_2\,(2\,3)\,\nu_2\nu_3\,(1\,3))(\nu_2\,(2\,3)\,\nu_2\nu_3), \tag{13.5}$$

where the arguments to the various ν_2, ν_3 are assigned independently. Combining this with the polynomial function on $\mathbf{G}$ that sends $g \in \mathbf{G}(R)$ to the product of the diagonal entries of gu, we obtain a polynomial law in $\mathscr{P}(C^{21}, R)$. But more is true. Because R is LG and C is projective of constant rank, C is a free module (see [EsG, Theorem 2.10] or [McDW, p. 457]). Choosing a basis for C expresses this polynomial law as a polynomial with coefficients in R.

We claim that this polynomial represents a unit over $R_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$ of R. For a given $\mathfrak{m}$, here is how to pick the element of C^{21} that produces a unit. If $\alpha_1 \in R_\mathfrak{m}^\times$ or $\alpha_2, \alpha_3 \notin R_\mathfrak{m}^\times$, applying ν_3 to u, with arguments chosen as in the second paragraph of the local case, we obtain an element with $\alpha_1 \in R_\mathfrak{m}^\times$ and $c_3 = c_2 = 0$. We take this to be the rightmost term in (13.5). If that element has $\alpha_2 \in R_\mathfrak{m}^\times$ or $\alpha_3 \notin R_\mathfrak{m}^\times$, the next ν_2 term can be chosen to produce a diagonal u; one takes the remaining ν terms in (13.5) to have argument 0. Otherwise, α_3 is invertible in $R_\mathfrak{m}$, and we plug 0 into the rightmost ν_2, pick the argument for the next ν_2 as in the proof of the local case, and plug 0 into the remaining ν terms to the left in (13.5). The claim is verified if $\alpha_1 \in R_\mathfrak{m}^\times$ or $\alpha_2, \alpha_3 \notin R_\mathfrak{m}^\times$.

The next case of the claim is where $\alpha_2 \in R_\mathfrak{m}^\times$. In that case, we plug 0 into the rightmost three ν terms in (13.5). After applying the permutation $(2\,3)$ and then $(1\,3)$, we obtain an element of $\mathrm{Her}_3(C)$ with α_1 invertible, and a well-chosen argument for the next ν_3 term will assure that $c_3 = c_2 = 0$. As in the preceding paragraph, choosing the arguments for the leftmost two ν_2 terms in the middle product in (13.5) suffices to transform u into a diagonal element, verifying the claim in this case.

The last case of the claim is when $\alpha_3 \in R_\mathfrak{m}^\times$. Plug 0 in the ν terms in the middle and right parenthetical expressions in (13.5). After applying all permutations in (13.5) besides the leftmost transposition to u, we obtain an element of $\mathrm{Her}_3(C)$ with $\alpha_1 \in R_\mathfrak{m}^\times$ and the argument in the preceding paragraph, again, transforms u into a diagonal element, completing the proof of the claim.

Since R is an LG ring, the claim provides an element $g \in \mathbf{G}(R)$ such that gu has $(1,1)$-entry a unit. That is, we may assume that in the element u, α_1 is invertible. Applying now $\tau_{21}(q)$ and $\tau_{31}(q)$ to u for values of q chosen as in the local case, we may assume that $c_3 = c_2 = 0$.

Applying now an argument as in the preceding five paragraphs, with the function

$$\nu_2\,(2\,3)\,\nu_2 \colon C^4 \to \mathbf{G},$$

we conclude that we may transform u to further assume that α_2 is invertible, and therefore, apply a transformation $\tau_{32}(q)$ to transform it into a diagonal element, as required. □

Corollary 13.4. *Suppose J is a Jordan R-algebra over an LG ring R. If J is isotopic to a split Freudenthal algebra whose rank does not take the value 6, then J is itself a split Freudenthal algebra.*

Proof. As in previous proofs, one is reduced to the case where J has constant rank, which is not 6. The theorem and Proposition 7.5 give the claim. □

The hypothesis that the rank is not 6 is necessary, because $\mathrm{Her}_3(\mathbb{R}, \langle 1, -1, -1\rangle)$ is isotopic to the split Freudenthal algebra $\mathrm{Her}_3(\mathbb{R})$ (Example 13.2) but is not isomorphic to it (Example 6.8).

Example 13.5 (isotopy over global fields). For $F = \mathbb{R}$ or a global field, there is a bijection between the isomorphism classes of octonion algebras and isotopy classes of Albert algebras given by $C \leftrightarrow \mathrm{Her}_3(C)$. Indeed, every Albert F-algebra is reduced (Example 11.2), so $C \mapsto \mathrm{Her}_3(C)$ touches every isotopy class. For injectivity, if C, C' are distinct octonion algebras, there is a real embedding $F \hookrightarrow \mathbb{R}$ such that $C \otimes \mathbb{R} \not\cong C' \otimes \mathbb{R}$, and Corollary 13.4 shows that $\mathrm{Her}_3(C) \otimes \mathbb{R}$ and $\mathrm{Her}_3(C') \otimes \mathbb{R}$ are not isotopic.

Corollary 13.6. *Every isotope of a Freudenthal algebra is itself a Freudenthal algebra.*

Proof. Suppose J is an isotope of a Freudenthal algebra. After base change to a faithfully flat extension, J is an isotope of a split Freudenthal algebra.

The R-algebra $S := \prod_{\mathfrak{m}} R_{\mathfrak{m}}$, where $\mathfrak{m}$ ranges over maximal ideals of R, is faithfully flat. For each $\mathfrak{m}$, $J \otimes R_{\mathfrak{m}}$ is $\mathrm{Her}_3(C, \Gamma)$ for C a split composition $R_{\mathfrak{m}}$-algebra and some Γ by Theorem 13.3. By Proposition 7.3, there is a faithfully flat $R_{\mathfrak{m}}$-algebra T such that $J \otimes T$ is a split Freudenthal algebra. The product of these T's is a faithfully flat R-algebra over which J is the split Freudenthal algebra. □

We close this section by making explicit the relationship between isotopy and norm similarity between Freudenthal algebras, extending Lemma 10.2.

Proposition 13.7. *Let J and J' be Freudenthal R-algebras. For an R-linear map $\phi : J \to J'$, the following are equivalent:*

1. *ϕ is an isomorphism $J \to (J')^{(u)}$ for some invertible $u \in J'$ ("ϕ is an isotopy").*
2. *$N_{J'}\phi = \alpha N_J$ as polynomial laws for some $\alpha \in R^\times$, and ϕ is surjective ("ϕ is a norm similarity").*

Proof. Since $(J')^{(u)}$ is a Freudenthal algebra by Corollary 13.6, condition (2) follows from (1) by Lemma 10.2 and (13.1). Conversely, we assume (2) and prove (1). Because $N_{J'}(\phi(1_J)) = \alpha$, the element $\phi(1_J)$ is invertible in J'. We set $u := \phi(1_J)^{-1}$ and $J'' := (J')^{(u)}$. We have

$$\phi(1_J) = u^{-1} = 1_{J''}.$$

Also, $N_{J'}(u) = N_{J'}(\phi(1_J))^{-1} = \alpha^{-1}$. Then

$$N_{J''}\phi = N_{J'}(u) N_{J'}\phi = N_J$$

as polynomial laws. Lemma 10.2 implies that ϕ is an isomorphism $J \xrightarrow{\sim} J''$, as desired. □

14. Classification of Albert algebras over $\mathbb{Z}$

In this section, we study Albert algebras over the integers.

Definition 14.1. In the notation of Example 4.5, consider the element

$$\beta := (-1 + e_1 + e_2 + \cdots + e_7)/2 = h_1 + h_2 + h_3 - (2 + e_1) \quad \in \mathcal{O},$$

as was done in [ElkiesGr, (5.2)]. That element has

$$\mathrm{Tr}_{\mathcal{O}}(\beta) = -1, \quad n_{\mathcal{O}}(\beta) = 2 \quad \text{and} \quad \beta^2 + \beta + 2 = 0.$$

Put

$$v := \begin{pmatrix} 2 & \beta & \cdot \\ \cdot & 2 & \beta \\ \beta & \cdot & 2 \end{pmatrix} \quad \in \mathrm{Her}_3(\mathcal{O}).$$

Since $\mathrm{Tr}_{\mathcal{O}}(\beta^3) = 5$, we find that $N_{\mathrm{Her}_3(\mathcal{O})}(v) = 1$. In particular, v is invertible with inverse $v^\sharp$. We define $\Lambda := \mathrm{Her}_3(\mathcal{O})^{(v)}$; it is an Albert algebra by Corollary 13.6.

Proposition 14.2. $\mathrm{Her}_3(\mathcal{O}) \not\cong \Lambda$ *as Jordan* $\mathbb{Z}$*-algebras, but* $\mathrm{Her}_3(\mathcal{O}) \otimes \mathbb{Q} \cong \Lambda \otimes \mathbb{Q}$ *as Jordan* $\mathbb{Q}$*-algebras.*

Proof. We first prove the claim over $\mathbb{Z}$, which amounts to a computation from [ElkiesGr]. The isomorphism class of a Freudenthal algebra determines its cubic norm form and also its trace linear form. From (13.1), we deduce for $x \in \mathrm{Her}_3(\mathcal{O})$ that $x^{\sharp(v)} = 0$ if and only if $x^\sharp = 0$. Hence, [ElkiesGr, Proposition 5.5] says that $\mathrm{Her}_3(\mathcal{O})$ contains exactly three elements x such that $x^\sharp = 0$ and $\mathrm{Tr}_{\mathrm{Her}_3(\mathcal{O})}(x) = 1$, whereas Λ has no elements x such that $x^{\sharp(v)} = 0$ and

$$T_{\mathrm{Her}_3(\mathcal{O})}(v, x) = 1,$$

where the left side is $\mathrm{Tr}_\Lambda(x)$ by (13.2). This proves that $\mathrm{Her}_3(\mathcal{O}) \not\cong \Lambda$.

Now consider $\mathrm{Her}_3(\mathcal{O}) \otimes \mathbb{R}$. It is called a "euclidean" Jordan algebra or, in older references, a "formally real" Jordan algebra, because every sum of nonzero squares is not zero [BrK, p. 331]. The element v has generic minimal polynomial, in the sense of (10.1), $(t-1)(t^2 - 5t + 1)$, which has three positive real roots. Therefore, there is some $u \in \mathrm{Her}_3(\mathcal{O}) \otimes \mathbb{R}$ such that $u^2 = v$ [BrK, Section XI.3, S. 3.6 and 3.7]. From this, it is trivial to see that

$$\Lambda \otimes \mathbb{R} \cong (\mathrm{Her}_3(\mathcal{O}) \otimes \mathbb{R})^{(v)} \cong \mathrm{Her}_3(\mathcal{O}) \otimes \mathbb{R},$$

and Example 11.2 gives that $\Lambda \otimes \mathbb{Q} \cong \mathrm{Her}_3(\mathcal{O}) \otimes \mathbb{Q}$. □

Theorem 14.3. *Over* $\mathbb{Z}$*:*

a. *There are exactly two isomorphism classes of octonion algebras:* $\mathrm{Zor}(\mathbb{Z})$ *and* $\mathcal{O}$.
b. *There are exactly four isomorphism classes of Albert algebras:* $\mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))$, $\mathrm{Her}_3(\mathcal{O}, \langle 1, -1, -1\rangle)$, $\mathrm{Her}_3(\mathcal{O})$*, and* Λ.
c. *There are exactly two isotopy classes of Albert algebras:* $\mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))$ *and* $\mathrm{Her}_3(\mathcal{O})$.

Proof. We first prove (a) and (b). No pair of the algebras listed are isomorphic to each other. For $\mathrm{Her}_3(\mathcal{O})$ and Λ, this is Proposition 14.2. For any other pair, base change to $\mathbb{Q}$ yields nonisomorphic $\mathbb{Q}$-algebras. To complete the proof, it suffices to show that every octonion or Albert $\mathbb{Z}$-algebra B is isomorphic to one of the ones listed.

If B is indefinite — that is, $B \otimes \mathbb{R}$ is not isomorphic to $\mathbb{O}$ nor $\mathrm{Her}_3(\mathbb{O})$ — then the isomorphism class of B is determined by $B \otimes \mathbb{Q}$ as a $\mathbb{Q}$-algebra (Proposition 11.6). Since the indefinite octonion or Albert $\mathbb{Q}$-algebras are $\mathrm{Zor}(\mathbb{Q})$, $\mathrm{Her}_3(\mathrm{Zor}(\mathbb{Q}))$, and $\mathrm{Her}_3(\mathcal{O} \otimes \mathbb{Q}, \langle 1, -1, -1\rangle)$ by Example 11.2, B is isomorphic to one of the algebras listed in the statement.

If B is definite, then $\mathbf{Aut}(B)$ is a $\mathbb{Z}$-form of the compact real group of type G_2 or F_4. Gross's mass formula [Gr, Proposition 5.3] shows that, up to $\mathbb{Z}$-isomorphism, there is only one group of type G_2 and two groups of type F_4 with this property. Using the equivalence between these groups and octonion or

Albert algebras (Theorem 9.3), we conclude that up to isomorphism $\mathcal{O}$ is the unique definite octonion $\mathbb{Z}$-algebra and $\mathrm{Her}_3(\mathcal{O})$ and Λ are the two isomorphism classes of definite Albert $\mathbb{Z}$-algebras, completing the proof of (a) and (b).

For (c), note that the three algebras in (b) that are not $\mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))$ are all isotopic, see Example 13.2, so the two algebras listed in (c) represent all of the isotopy classes of Albert $\mathbb{Z}$-algebras. The base change of these two algebras to $\mathbb{Q}$ are not isotopic (Example 13.5), so they are not isotopic as $\mathbb{Z}$-algebras. □

Note that part (a) of the theorem can be proved entirely in the language of octonion algebras (see [vdBS]).

In view of Theorem 9.3, part (b) is equivalent to a classification of the group schemes of type F_4 over $\mathbb{Z}$, which was done in Sections 6 and 7 of [Conrad], especially Examples 6.7 and 7.4. The innovation, here, is that we can use the language of Albert algebras also in the case of $\mathbb{Z}$ where 2 is not invertible. Because of this extra flexibility, we can substitute results from the literature over algebraically closed fields (including characteristic 2) for some of the computations over $\mathbb{Z}$ done in [Conrad].

Part (c) corresponds to the classification of groups of type E_6 over $\mathbb{Z}$ up to isogeny (see Section 18).

15. Groups of type E_6

Roundness of the norm

We note that the cubic norm of a Freudenthal algebra has the following special property. A quadratic form with this property is called "round" (see [ElKM, Section 9.A]).

Lemma 15.1 (roundness). *For every Freudenthal R-algebra J,*

$$\{\alpha \in R^\times \mid \alpha N_J \cong N_J\} = \{N_J(x) \in R^\times \mid x \text{ invertible in } J\}.$$

Proof. If $\alpha \in R^\times$ and $\phi \in \mathrm{GL}(J)$ are such that $\alpha N_J = N_J\phi$, then for $x := \phi(1_J)$, we have $N_J(x) = \alpha$. Conversely, if x is invertible in J, put $\alpha := N_J(x)$ and define $\phi := \alpha U_{x^{-1}}$. Then $N_J\phi = \alpha^3 N_J(x^{-1})^2 N_J$ by Lemma 6.5(3), so $N_J\phi = \alpha N_J$. □

Example 15.2. For $J = \mathrm{Her}_3(C, \Gamma)$, the sets displayed in Lemma 15.1 equal $R^\times$. To see this for the right side, take $\alpha \in R^\times$ and note that $N_J(\alpha\varepsilon_1 + \varepsilon_2 + \varepsilon_3) = \alpha$. For the left side, consider $\phi \in \mathrm{GL}(J)$ defined by

$$\phi(\varepsilon_i) = \alpha\varepsilon_i \quad \text{and} \quad \phi(\delta_i(c)) = \delta_i(c) \quad \text{for } i = 1, 2,$$
$$\phi(\varepsilon_3) = \alpha^{-1}\varepsilon_3 \quad \text{and} \quad \phi(\delta_3(c)) = \delta_3(\alpha c).$$

Then $N_J\phi = \alpha N_J$ as polynomial laws.

Example 15.3. In contrast to the preceding example, we now show that the sets displayed in Lemma 15.1 may be properly contained in $R^\times$. Suppose F is a field and J is a Freudenthal F-algebra such that N_J is anisotropic, that is, $N_J(x) = 0$ if and only if $x = 0$. (For example, such a J exists if F is Laurent series or rational functions in one variable over a global field, see Remark 12.5.) We claim that, for t an indeterminate, every nonzero element in the image of $N_{J\otimes F((t))}$ has lowest term of degree divisible by 3. Because the norm is a homogeneous form, it suffices to prove this claim for $J \otimes F[[t]]$.

Let $x \in J \otimes F[[t]]$ be nonzero, so $x = \sum_{j\ge j_0} x_j t^j$ for some $j_0 \ge 0$ with $x_{j_0} \ne 0$. Since N_J is anisotropic, $N_J(x_{j_0}) \ne 0$. If $j_0 = 0$, then the homomorphism $F[[t]] \to F$ such that $t \mapsto 0$ sends $x \mapsto x_0$ and $N_{J\otimes F[[t]]}(x) \mapsto N_J(x_0) \ne 0$, therefore $N_{J\otimes F[[t]]}(x)$ has lowest degree term $N_J(x_0)t^0$. If $j_0 > 0$, then

$$N_{J\otimes F[[t]]}(x) = N_{J\otimes F[[t]]}(t^{j_0}(xt^{-j_0})) = t^{3j_0}(N_J(x_{j_0})t^0 + \cdots),$$

proving the claim.

Corollary 15.4. *For Freudenthal R-algebras J and J', the following are equivalent:*

1. *J and J' are isotopic.*
2. *$N_J \cong \alpha N_{J'}$ for some $\alpha \in R^\times$.*
3. *$N_J \cong N_{J'}$.*

Proof. The equivalence of (1) and (2) is Proposition 13.7.

Supposing (2), let $\phi\colon J' \to J$ be an R-module isomorphism such that $\alpha N_{J'} = N_J\phi$. Take $x := \phi(1_{J'})$. Since $N_J(x) = \alpha$, Lemma 15.1 gives that $\alpha N_J \cong N_J$. As N_J is also isomorphic to $\alpha N_{J'}$, we conclude (3). The converse is trivial. □

In the corollary, the inclusion of (3) seems to be new, even in the case where R is a field. Omitting that, in the special case where R is a field of characteristic $\neq 2, 3$, the equivalence of (1) and (2) and Proposition 15.6 below can be found as Theorems 7 and 10 in [J71].

Albert algebras and groups of type E_6

The stabilizer of the cubic form N_J in $\mathbf{GL}(J)$ is a closed sub-group-scheme denoted $\mathbf{Isom}(J)$. It contains $\mathbf{Aut}(J)$ as a natural sub-group-scheme, namely, it is the stabilizer of 1_J by Lemma 10.2. Arguing as in the proof of Lemma 9.1, one finds that $\mathbf{Isom}(J)$ is a simple affine group scheme that is simply connected of type E_6. (In the case where R is an algebraically closed field, this claim is verified in [Sp73, 11.20, 12.4], or see [SpV, Theorem 7.3.2] for the case where R is a field of characteristic different from 2, 3.) Compare [Als21, Lemma 2.3] or [Conrad, Appendix C]. Moreover, $\mathbf{Isom}(J)$ is a “pure inner form” in the sense of [Conrad, Section 3], respectively, “strongly inner” in [CalF, Definition 2.2.4.9], meaning that it is an inner twist of $\mathbf{Isom}(J_0)$ for the split Albert algebra J_0. We note that the center of $\mathbf{Isom}(J)$ is the group scheme μ_3 of cube roots of unity operating on J by scalar multiplication.

Faithfully flat descent shows that the set $H^1(R, \mathbf{Isom}(J))$ is in bijection with isomorphism classes of pairs (M, f), where M is a projective module of the same rank as J and f is a cubic form on M — that is, an element of $\mathsf{S}^3(M^*)$ — such that $f \otimes S$ is isomorphic to the norm on $\mathrm{Her}_3(\mathrm{Zor}(S))$ for some faithfully flat $S \in R$-**alg**. For every Albert R-algebra J and every $\alpha \in R^\times$, $(J, \alpha N_J)$ is such a pair by Example 15.2. In the special case where R is a field, every such pair (M, f) — that is, every element of $H^1(R, \mathbf{Isom}(J))$ — is of the form $(J, \alpha N_J)$ for some J and $\alpha \in R^\times$ (see [Gar09, 9.12] in general or [Sp62] for the case of characteristic $\neq 2, 3$).

Outer automorphism of $\mathbf{Isom}(J)$

Suppose J and J' are Freudenthal R-algebras and $\phi\colon J \to J'$ is an isomorphism of R-modules. Since the bilinear form $T_{J'}$ is regular, there is a unique R-linear map $\phi^\dagger\colon J \to J'$ such that $T_{J'}(\phi x, \phi^\dagger y) = T_J(x, y)$ for all $x, y \in J$. Because T_J and $T_{J'}$ are symmetric, we have $(\phi^\dagger)^\dagger = \phi$ for all ϕ. If J'' is another Freudenthal R-algebra and $\psi\colon J' \to J''$ is an R-linear bijection, then $(\phi\psi)^\dagger = \phi^\dagger\psi^\dagger$.

Proposition 15.5. *Let J be a Freudenthal R-algebra.*

1. *If $\phi \in \mathrm{GL}(J)$ is such that $N_J\phi = \alpha N_J$ for some $\alpha \in R^\times$, then $N_J\phi^\dagger = \alpha^{-1}N_J$.*
2. *The map $\phi \mapsto \phi^\dagger$ is an automorphism of $\mathbf{Isom}(J)$ of order 2 that is not an inner automorphism.*
3. *For ϕ as in (1) or in $\mathbf{Isom}(J)(R)$, $\phi^\dagger = \phi$ if and only if ϕ is an automorphism of J.*

Proof. (1): Put $u := \phi(1_J)^{-1}$. On the one hand,

$$T_J(x, y) = T_{J^{(u)}}(\phi(x), \phi(y))$$

for all $x, y \in J$, because ϕ is an isomorphism $J \to J^{(u)}$ by Proposition 13.7. On the other hand, (13.2) yields

$$T_{J^{(u)}}(\phi(x), \phi(y)) = T_J(U_u\phi(x), \phi(y)).$$

Therefore,

$$\phi^\dagger = U_{\phi(1_J)^{-1}}\phi. \tag{15.1}$$

To complete the proof of (1), we note by Lemma 6.5(3) that

$$N_J\phi^\dagger = N_J U_u \phi = N_J(u)^2 N_J\phi = \alpha^{-1}N_J.$$

For (2), we only have to check that the map is not an inner automorphism. Let $S \in R$-**alg** be such that there exists $\zeta \in \mu_3(S)$ such that $\zeta \neq 1$. Then $\zeta^\dagger = \zeta^{-1} \neq \zeta$ and ζ is in the center of $\mathbf{Iso}(J)(R)$, proving that the automorphism is not inner (and not the identity).

For (3), suppose $\phi^\dagger = \phi$. Then $N_J\phi = N_J$. By (15.1), $U_{\phi(1_J)^{-1}} = \mathrm{Id}_J$, so $\phi(1_J) = \zeta 1_J$ for some $\zeta \in R$ with $\zeta^2 = 1$ (Example 7.4). Yet $1 = N_J(1_J) = N_J\phi(1_J)$, so ζ^3 also equals 1, whence $\phi(1_J) = 1_J$. Lemma 10.2 shows that ϕ is an automorphism of J. Conversely, if ϕ is an automorphism of J, then $u = 1_J$, so $\phi^\dagger = \phi$ by (15.1). □

Proposition 15.6. *Let J and J' be Albert R-algebras. Among the statements*

1. $\mathbf{Isom}(J) \cong \mathbf{Isom}(J')$,
2. *There is a line bundle L and isomorphism $h\colon L^{\otimes 3} \to R$ such that $(J', N_{J'}) \cong [L, h] \cdot (J, N_J)$ for $\cdot$ as defined in Section 3,*
3. *J and J' are isotopic,*

we have the implications (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3). If Pic R *has no 3-torsion other than zero, then all three statements are equivalent.*

Proof. Suppose (1); we prove (2). We may assume R is connected.

The conjugation action gives a homomorphism $\mathbf{Isom}(J) \to \mathbf{Aut}(\mathbf{Isom}(J))$, which gives a map of pointed sets

$$H^1(R, \mathbf{Isom}(J)) \to H^1(R, \mathbf{Aut}(\mathbf{Isom}(J))), \tag{15.2}$$

where the second set is in bijection with isomorphism classes of R-group schemes that become isomorphic to $\mathbf{Isom}(J)$ after base change to an fppf R-algebra. By hypothesis, the class of $N_{J'} \in H^1(R, \mathbf{Isom}(J))$ is in the kernel of (15.2).

There is an exact sequence

$$1 \to \mathbf{Isom}(J)/\mu_3 \to \mathbf{Aut}(\mathbf{Isom}(J)) \to \mathbb{Z}/2 \to 1$$

of fppf sheaves by [DemG, Theorem XXIV.1.3]. Since R is connected, $(\mathbb{Z}/2)(R)$ has one nonidentity element, and it is the image of the map $\dagger$ from Lemma 15.5. That is, in the exact sequence

$$\mathbf{Aut}(\mathbf{Isom}(J))(R) \to (\mathbb{Z}/2)(R) \to H^1(R, \mathbf{Isom}(J)/\mu_3) \to H^1(R, \mathbf{Aut}(\mathbf{Isom}(J))),$$

the first map is surjective, so the third map has zero kernel, and we deduce that the image of $N_{J'}$ in $H^1(R, \mathbf{Isom}(J)/\mu_3)$ is the zero class. It follows that $N_{J'}$ is in the image of the map

$$H^1(R, \mu_3) \to H^1(R, \mathbf{Isom}(J)),$$

which is the orbit of the zero class N_J under the action of the group $H^1(R, \mu_3)$, which is (2).

That (2) implies (1) is Lemma 3.6. The claimed implications between (3) and (2) are Corollary 15.4. □

16. Freudenthal triple systems

In this section, we define Freudenthal triple systems, also known as FT systems. We will see in Theorem 17.4 in the next section that they play the same role relative to groups of type E_7 that forms of the norm on an Albert algebra play for groups of type E_6.

For any Albert R-algebra J, define $Q(J)$ to be the rank 56 projective R-module $R \oplus R \oplus J \oplus J$ endowed with a 4-linear form Ψ and an alternating bilinear form b, defined as follows.

We write an element of $Q(J)$ as $\left(\begin{smallmatrix} \alpha & x \\ x' & \alpha' \end{smallmatrix}\right)$ for $\alpha, \alpha' \in R$ and $x, x' \in J$. Define

$$b_J\left(\left(\begin{smallmatrix} \alpha & x \\ x' & \alpha' \end{smallmatrix}\right), \left(\begin{smallmatrix} \beta & y \\ y' & \beta' \end{smallmatrix}\right)\right) := \alpha\beta' - \alpha'\beta + T_J(x, y') - T_J(x', y). \tag{16.1}$$

As an intermediate step to defining Ψ, define a quartic form

$$q_J\left(\begin{smallmatrix} \alpha & x \\ x' & \alpha' \end{smallmatrix}\right) = -4T_J(x^\sharp, x'^\sharp) + 4\alpha N_J(x) + 4\alpha' N_J(x') + (T_J(x, x') - \alpha\alpha')^2, \tag{16.2}$$

compare [Brown, p. 87] or [Kr07, p. 940].

To define the 4-linear form, consider first the case $R = \mathbb{Z}$ and $J := \mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))$. (The following definitions are inspired by [Lur, Section 6].) Putting X_i for an element of $Q(J)$ and t_i for an indeterminate, the coefficient of $t_1t_2t_3t_4$ in $q(\sum t_iX_i)$, equivalently, the 4-linear form

$$(X_1, X_2, X_3, X_4) \mapsto \nabla_{X_1}\nabla_{X_2}\nabla_{X_3} q(X_4)$$

on $Q(J)$, equals 2Θ for a symmetric 4-linear form Θ. Define 4-linear forms Φ_i via

$$\begin{aligned} \Phi_1(X_1, X_2, X_3, X_4) &= b(X_1, X_2)\, b(X_3, X_4) \\ \Phi_2(X_1, X_2, X_3, X_4) &= b(X_1, X_3)\, b(X_4, X_2) \\ \Phi_3(X_1, X_2, X_3, X_4) &= b(X_1, X_4)\, b(X_2, X_3). \end{aligned} \tag{16.3}$$

Then $\Theta + \sum \Phi_i$ is divisible by 2 as a 4-linear function on $Q(\mathrm{Zor}(\mathbb{Z}))$, and we set

$$\Psi_{\mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))} := \frac{1}{2}(\Theta + \sum \Phi_i). \tag{16.4}$$

As Θ is symmetric, Ψ is evidently stable under even permutations of its arguments, and we have:

$$\Psi(X_1, X_2, X_3, X_4) - \Psi(X_2, X_1, X_3, X_4) = \sum \Phi_i.$$

For any ring R, we define $\Psi_{\mathrm{Her}_3(\mathrm{Zor}(R))} := \Psi_{\mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))} \otimes R$, and we define Ψ_J for an Albert R-algebra J by descent.

Definition 16.1. A *Freudenthal triple system*[1] or *FT system* (M, Ψ, b) is an R-module M endowed with a 4-linear form Ψ and an alternating bilinear form b such that $(M, \Psi, b) \otimes S$ is isomorphic (in an obvious sense) to $Q(J)$ for some faithfully flat $S \in R$-**alg** and some Albert S-algebra J.

Comparison with other definitions

Suppose for this paragraph that 6 is invertible in R. Given an FT system (M, Ψ, b), we may define 4-linear forms Φ_i on M via (16.3) and recover Θ and q via

$$\Theta := 2\Psi - \sum \Phi_i \quad \text{and} \quad \Theta(X, X, X, X) = 12q(X) \tag{16.5}$$

as polynomial laws in X. (This last is a special case of the general fact that going from a homogeneous form of degree d to a d-linear form and back to a homogeneous form of degree d equals multiplication

[1]See p. 273 of [Sp06] for remarks on the history of this term.

by $d!$ [BouA2, Section IV.5.8, Proposition 12(i)].) Since the form b is regular and Θ is symmetric, the equation

$$\Theta(X_1, X_2, X_3, X_4) = b(X_1, t(X_2, X_3, X_4))$$

implicitly defines a symmetric 3-linear form $t\colon M \times M \times M \to M$, and $\mathbf{Aut}(M, \Psi, b)$ equals $\mathbf{Aut}(M, t, b)$. That is, under the hypothesis that 6 is invertible in R, we would obtain an equivalent class of objects if we replaced the asymmetric 4-linear form Ψ in the definition of FT systems with the quartic form q (the version studied in [Brown]) or with the trilinear form t (the version studied in [Mey]).

Similarity of FT systems

For a d-linear form f on an R-module M, that is, an R-linear map $f\colon M^{\otimes d} \to R$, and a d-trivialized line bundle $[L, h] \in H^1(R, \mu_d)$, we define a d-linear form $[L, h] \cdot f$ on $M \otimes L$ via the composition

$$(M \otimes L)^{\otimes d} \xrightarrow{\sim} M^{\otimes d} \otimes L^{\otimes d} \xrightarrow{f \otimes h} R.$$

For $Q := (M, \Psi, b)$, an FT system and a discriminant module $[L, h] \in H^1(R, \mu_2)$, we define $[L, h] \cdot Q$ to be the triple consisting of the module $M \otimes L$, the 4-linear form $[L, h^{\otimes 2}] \cdot \Psi$ for $[L, h^{\otimes 2}] \in H^1(R, \mu_4)$, and the bilinear form $[L, h] \cdot b$. Since $\langle 1 \rangle \cdot Q$ is Q itself, we deduce that $[L, h] \cdot Q$ is also an FT system. We say that FT systems Q, Q' are *similar* if $Q' \cong [L, h] \cdot Q$ for some $[L, h] \in H^1(R, \mu_2)$. For example, for any FT system (M, Ψ, b) and any $\alpha \in R^\times$, (M, Ψ, b) and $(M, \alpha^2\Psi, \alpha b)$ are similar.

Example 16.2. Suppose $(M, \Psi, b) = Q(J)$ for some Albert R-algebra J. Then for every $\mu \in R^\times$, the map

$$\left(\begin{smallmatrix} \alpha & x \\ x' & \alpha' \end{smallmatrix}\right) \mapsto \begin{pmatrix} \alpha/\mu & \mu x \\ x' & \mu^2 \alpha' \end{pmatrix}$$

is an isomorphism $\langle \mu \rangle \cdot Q(J) \xrightarrow{\sim} Q(J)$. One checks this for $R = \mathbb{Z}$ and $J = \mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))$ using (16.1) and (16.2). It follows for general R and J by base change and twisting.

17. Groups of type E_7

We will now relate FT systems as defined in the previous section to affine group schemes of type E_7. Here is a tool that allows us to work with the quartic form q as in (16.2) rather than the less convenient 4-linear form Ψ, while still getting results that hold when 6 is not invertible.

Lemma 17.1. *Let (M, Ψ, b) be an FT system over $\mathbb{Z}$, let $\mathbf{G}$ be a closed subgroup of $\mathbf{GL}(M)$, and let F be a field of characteristic zero. If $\mathbf{G}(F)$ is dense in $\mathbf{G}$ (which holds if $\mathbf{G}$ is connected) and $\mathbf{G}(F)$ preserves $b \otimes F$ and the quartic form q over F defined by (16.5), then $\mathbf{G}$ is a closed sub-group-scheme of $\mathbf{Aut}(M, \Psi, b)$.*

Proof. Since $\mathbf{G}(F)$ is dense in $\mathbf{G}$, the group scheme $\mathbf{G} \times F$ preserves $b \otimes F$ and q, whence also $\Psi \otimes F$. Viewing b and Ψ as elements of the representation $V := (M^*)^{\otimes d}$ of $\mathbf{G}$ for $d = 2$ or 4, the natural map $V^{\mathbf{G}} \otimes F \to (V \otimes F)^{\mathbf{G} \times F}$ is an isomorphism because F is flat over $\mathbb{Z}$ [Ses, Lemma 2], so $\mathbf{G}$ preserves b and Ψ. ◻

Corollary 17.2. *For every Freudenthal R-algebra J, there is an inclusion $f\colon \mathbf{Isom}(J) \hookrightarrow \mathbf{Aut}(Q(J))$ via*

$$f(\phi)\left(\begin{smallmatrix} \alpha & x \\ x' & \alpha' \end{smallmatrix}\right) = \begin{pmatrix} \alpha & \phi(x) \\ \phi^\dagger(x') & \alpha' \end{pmatrix}.$$

Proof. Consider the case $J = \mathrm{Her}_3(\mathrm{Zor}(\mathbb{Z}))$. For $\phi \in \mathbf{Isom}(J)(\mathbb{Q})$, it follows from the definition of $\phi^\dagger$ and Proposition 15.5(1) that $f(\phi)$ is an isomorphism of the bilinear and quartic forms $b \otimes \mathbb{Q}$ and q defined by (16.2) for $J \otimes \mathbb{Q}$. The lemma gives the claim in this case. Base change and twisting give the claim for every R and every Albert R-algebra J. ◻

Corollary 17.3. *Suppose J and J' are Albert R-algebras. If J and J' are isotopic, then* $\mathbf{Aut}(Q(J)) \cong \mathbf{Aut}(Q(J'))$.

Proof. The inclusions $\mathbf{Aut}(J) \hookrightarrow \mathbf{Isom}(J) \hookrightarrow \mathbf{Aut}(Q(J))$ induce maps

$$H^1(R, \mathbf{Aut}(J)) \to H^1(R, \mathbf{Isom}(J)) \to H^1(R, \mathbf{Aut}(Q(J))),$$

where the last set classifies FT systems over R. The class of J' in $H^1(R, \mathbf{Aut}(J))$ maps to the class of $N_{J'}$ in $H^1(R, \mathbf{Isom}(J))$, and by hypothesis and by Proposition 15.6, this is the trivial class. Therefore, the image of J' in $H^1(R, \mathbf{Aut}(Q(J)))$, which is $Q(J')$, is the trivial class. □

In case R is a field of characteristic $\neq 2, 3$, the converse of Corollary 17.3 is true by [Fe72, Corollary 6.9]. That is, if $Q(J) \cong Q(J')$, then J and J' are isotopic. The paper [Als22] provides an example related to this construction over rings.

Theorem 17.4. *The group scheme* $\mathbf{Aut}(Q(\mathrm{Her}_3(\mathrm{Zor}(R))))$ *over R is obtained from the simply connected Chevalley group of type E_7 over $\mathbb{Z}$ by base change. Every strongly inner and simply connected simple R-group scheme of type E_7 over R is of the form $\mathbf{Aut}(Q)$ for some FT system Q. For FT systems Q and Q', $\mathbf{Aut}(Q) \cong \mathbf{Aut}(Q')$ if and only if Q and Q' are similar.*

Proof. Put $J_R := \mathrm{Her}_3(\mathrm{Zor}(R))$ and $Q_R := Q(J_R)$. We will show that $\mathbf{Aut}(Q_R)$ is isomorphic to the base change to R of the simply connected Chevalley group E_7 over $\mathbb{Z}$.

In addition to the sub-group-scheme $\mathbf{Isom}(J_R)$ of $\mathbf{Aut}(Q_R)$ provided by Corollary 17.2, we consider a rank 1 torus $\mathbb{G}_{\mathrm{m}}$ defined by

$$\beta\left(\begin{smallmatrix} \alpha & x \\ x' & \alpha' \end{smallmatrix}\right) = \begin{pmatrix} \beta^{-3}\alpha & \beta x \\ \beta^{-1}x' & \beta^3\alpha' \end{pmatrix} \quad \text{for } \beta \in R^\times$$

and two copies of J_R (as group schemes under addition) through which an element $y \in J_R$ acts via

$$y\left(\begin{smallmatrix} \alpha & x \\ x' & \alpha' \end{smallmatrix}\right) = \begin{pmatrix} \alpha+b(x',y) & x+\alpha' y \\ x'+x\times y & \alpha' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & x+x'\times y \\ x'+\alpha y & \alpha'+b(x',y) \end{pmatrix}.$$

These preserve b and q, see, for example, [Brown, p. 95] or [Kr07, p. 942], and so by Lemma 17.1 do belong to $\mathbf{Aut}(Q_R)$. Considering the Lie algebras of $\mathbf{Isom}(J_R)$, $\mathbb{G}_{\mathrm{m}}$, and the two copies of J, as subalgebras of $\mathrm{Lie}(\mathbf{GL}(Q_R))$, one can identify the subalgebra L_R they generate with the Lie algebra of $E_7 \times R$ by picking out specific root subalgebras and so on as in [Fr54] or [Sel], or see [Gar01, Section 7] for partial information. Note that $\mathrm{Lie}(\mathbf{Aut}(Q_R)) \supseteq L_R$. For F any algebraically closed field, we may identify the smooth closed subgroup of $\mathbf{Aut}(Q_F)$ generated by $\mathbf{Isom}(J_F)$, $\mathbb{G}_{\mathrm{m}}$, and the two copies of J_F with $E_7 \times F$.

In [Lur], Lurie begins with $L_{\mathbb{Z}}$ and defines $L_{\mathbb{Z}}$-invariant 4-linear forms Θ^L, Φ_i^L, and Ψ^L and alternating bilinear form b^L on the 56-dimensional Weyl module of $L_{\mathbb{Z}}$. Over $\mathbb{C}$, $\mathbf{Aut}(Q_{\mathbb{C}})$ is simply connected of type E_7 by the references in the previous paragraph, so it preserves the base change of Lurie's forms $\Theta^L \otimes \mathbb{C}$, etc. Because $\mathbf{Aut}(Q_{\mathbb{C}})(\mathbb{C})$ is dense in $\mathbf{Aut}(Q_{\mathbb{C}})$, Lemma 17.1 shows that $\mathbf{Aut}(Q_{\mathbb{Z}})$ preserves Θ^L, the Φ_i^L, Ψ^L, and b^L. By the uniqueness of E_7 invariant bilinear and symmetric 4-linear forms on M (which follows from the uniqueness over $\mathbb{C}$ as in the proof of Lemma 17.1), we find that $b^L = \pm b$ and $\Theta^L = \pm\Theta$. Note that regardless of the sign on b in the preceding sentence, we find $\Phi_i^L = \Phi_i$ for all i and $\mathbf{Aut}(Q_F)$ preserves b^L. Now let F be an algebraically closed field. If F has characteristic different from 2, then $\mathbf{Aut}(Q_F)$ preserves $2\Psi = \Theta + \sum \Phi_i$ and the Φ_i, so it preserves Θ, hence Θ^L, hence Ψ^L. If F has characteristic 2, then although $\Psi^L = \frac{1}{2}(\pm\Theta + \sum \Phi_i)$ for some choice of sign as polynomials over $\mathbb{Z}$, we have $\Psi^L \otimes F = \Psi \otimes F$. In either case, $\mathbf{Aut}(Q_F)$ preserves $b^L \otimes F$ and $\Psi^L \otimes F$, whence so does its Lie algebra, so $\dim \mathrm{Lie}\, \mathbf{Aut}(Q_F) \leq \dim L_F$ by [Lur, Theorem 6.2.3]. Putting this together with the previous paragraph, we see that $\mathbf{Aut}(Q_F)$, an affine group scheme over the field F, is smooth with identity component $E_7 \times F$.

We claim that $\mathbf{Aut}(Q_F)$ is connected. Since its identity component E_7 has no outer automorphisms, every element of $\mathbf{Aut}(Q_F)(F)$ is a product of an element of $E_7(F)$ and a linear transformation centralizing E_7. The action of $E_7 \times F$ on Q_F is irreducible (it is the 56-dimensional minuscule representation), so the centralizer of E_7 consists of scalar transformations. Finally, we note that the intersection of $\mathbf{Aut}(Q_F)$ and the scalar transformations is the group scheme μ_2 of square roots of unity, which is contained in E_7. In summary, $\mathbf{Aut}(Q_F) = E_7 \times F$ for every algebraically closed field F.

As in the proof of Lemma 9.1, it follows that $\mathbf{Aut}(Q_\mathbb{Z})$ is a simple affine group scheme that is simply connected of type E_7, and we deduce from the fact that $\mathbf{Aut}(Q_\mathbb{R})$ is split that $\mathbf{Aut}(Q_\mathbb{Z})$ is in fact the Chevalley group.

The second claim now follows by descent.

The third claim is proved in the same manner as Proposition 15.6, although the current situation is somewhat easier due to the absence of nontrivial automorphisms of the Dynkin diagram of E_7 and therefore the absence of outer automorphisms for semisimple groups of that type. The sequence

$$H^1(R, \mu_2) \to H^1(R, \mathbf{Aut}(Q)) \to H^1(R, \mathbf{Aut}(\mathbf{Aut}(Q))) \tag{17.1}$$

is exact, where μ_2 is the center of $\mathbf{Aut}(Q)$ and $\mathbf{Aut}(\mathbf{Aut}(Q)) \cong \mathbf{Aut}(Q)/\mu_2$ is the adjoint group. We have $\mathbf{Aut}(Q') \cong \mathbf{Aut}(Q)$ if and only if the element Q' in $H^1(R, \mathbf{Aut}(Q))$ is in the kernel of the second map in (17.1), if and only if Q' is in the image of the first map. To complete the proof, it suffices to calculate by descent that the action of $H^1(R, \mu_2)$ on $H^1(R, \mathbf{Aut}(Q))$ is exactly by the similarity action defined in Section 16. □

A partial rephrasing of the second statement of the theorem is that, for any FT system Q, the set $H^1(R, \mathbf{Aut}(Q))$ is in bijection with the set of isomorphism classes of FT systems over R.

Corollary 17.5. *If R is: (1) a complete discrete valuation ring whose residue field is finite; (2) a finite ring; or (3) a Dedekind domain whose field of fractions F is a global field with no real embeddings, then the split FT system is the only one over R, up to isomorphism.*

Proof. Imitate the arguments in Proposition 11.3 or Example 11.5, where $\mathbf{G}$ is the base change to R of the simply connected Chevalley group $\mathbf{Aut}(Q(\mathrm{Zor}(\mathbb{Z})))$. □

Remarks 17.6. A previous work that considered groups of type E_7 over rings is [Luz]. Aschbacher [Asch] studied the 4-linear form in the case where R is a field of characteristic 2. The paper [MüW] studied the case of fields of any characteristic, organized around a polynomial law $\Theta \in \mathscr{P}(Q, R)$ that is not homogeneous. For a field F of characteristic $\neq 2, 3$, FT systems have been studied in this century in [Cl], [Hel], [Kr07], [Sp06], and [BDFMR] to name a few. They arise naturally in the context of the bottom row of the magic triangle from [DelG, Table 2], in connection with the existence of extraspecial parabolic subgroups as in [Röh] or [Gar09, Section 12], or from groups with a BC_1 grading [GrG, p. 995]. For every Albert F-algebra J, the group scheme $\mathbf{Aut}(Q(J))$ is isotropic (see, for example [Sp06, Lemma 5.6(i)]). Yet there exist strongly inner groups of type E_7 that are anisotropic, see [T, 3.1] or [Gar09, Appendix A], and therefore, there exist FT systems Q that are not isomorphic to $Q(J)$ for any J. A construction that produces all FT systems can be obtained by considering a subgroup $\mathbf{Isom}(J) \rtimes \mu_4$ of $\mathbf{Aut}(Q(J))$, which leads to a surjection $H^1(F, \mathbf{Isom}(J) \rtimes \mu_4) \to H^1(F, \mathbf{Aut}(Q(J)))$ (see [Gar09, 12.13], [Gar01, Lemma 4.15] or [Sp06, Section 4]).

18. Exceptional groups over $\mathbb{Z}$

We now record explicit descriptions of the isomorphism classes of semisimple affine group schemes over $\mathbb{Z}$ of types F_4, G_2, E_6, and E_7.

There are four such group schemes of type F_4, namely, $\mathbf{Aut}(J)$ for each of the four Albert $\mathbb{Z}$-algebras listed in Theorem 14.3(b). The proof of this fact is intertwined with the proof of that theorem. Similarly, there are two such group schemes of type G_2, namely, $\mathbf{Aut}(C)$ for $C = \mathrm{Zor}(\mathbb{Z})$ or $\mathcal{O}$.

Proposition 18.1. *For $R = \mathbb{Z}$, $\mathbb{R}$ or a number field with a unique real embedding, and $J_i = \mathrm{Her}_3(C_i)$ for $C_0 = \mathrm{Zor}(R)$ and $C_1 = \mathcal{O} \otimes R$:*

1. *there are exactly two isomorphism classes of R-forms of the cubic norm on J_0, namely, N_{J_i} for $i = 0, 1$.*
2. *there are exactly two isomorphism classes of FT systems over R, namely, $Q(J_i)$ for $i = 0, 1$.*

Proof. For $n = 6$ or 7, put E_n for the semisimple simply connected Chevalley group scheme over $\mathbb{Z}$ of type E_n. The set $H^1(\mathbb{R}, E_n)$ has two elements (see [BorE], [BorT, esp. Section 15] or [AdT, Table 3]). For F a number field with a unique real embedding, the map $H^1(F, E_n) \to H^1(\mathbb{R}, E_n)$ is a bijection, a fact we have already used in Example 11.2.

By [Conrad, Remark 4.8], $\mathbb{Z}$ forms of absolutely simple and simply connected $\mathbb{Q}$-group schemes are purely inner forms, that is, in this case they are obtained by twisting E_n by a class $\xi \in H^1(\mathbb{Z}, E_n)$. Now the compact real form of type E_n is not a pure inner form, so $(E_n)_\xi \times \mathbb{R}$ is not compact for all $\xi \in H^1(\mathbb{Z}, E_n)$. Therefore, the natural map $H^1(\mathbb{Z}, E_n) \to H^1(\mathbb{Q}, E_n)$ is a bijection by [Ha67, Satz 4.2.4].

We have observed that the set $H^1(R, E_n)$ has two elements for each choice of R, and we have already noted that this set is in bijection with the isomorphism classes in (1) for $n = 6$ and (2) for $n = 7$ (Theorem 17.4). It suffices to prove that the two exhibited elements are distinct, for which it suffices to consider the case $R = \mathbb{R}$.

In case (1), $\mathrm{Her}_3(\mathcal{O})$ is not split (Example 11.1), so it is not isotopic to $\mathrm{Her}_3(\mathrm{Zor}(\mathbb{R}))$ (Cor. 13.4) and the cubic norms on the two algebras are not isomorphic (Corollary 15.4). In case (2), one can invoke Ferrar's converse to Corollary 17.3. Alternatively, one can use the methods used to calculate $H^1(\mathbb{R}, E_n)$ to observe that the nontrivial element of $H^1(\mathbb{R}, E_7)$ is in the image of $H^1(\mathbb{R}, E_6)$. □

(Apart from the case $R = \mathbb{Z}$, the proposition is well known. Analogous statements for any number field can be deduced from the result over $\mathbb{R}$ via Harder's local-global principle as in Example 11.2, see, for example [Fe76] and [Fe78].)

The proof provides the following corollary.

Corollary 18.2. *Regarding isomorphism classes of semisimple and simply connected affine group schemes over $\mathbb{Z}$:*

1. *there are two of type E_6, namely, $\mathbf{Isom}(\mathrm{Her}_3(C))$ and*
2. *there are two of type E_7, namely, $\mathbf{Aut}(Q(\mathrm{Her}_3(C)))$*

for $C = \mathrm{Zor}(\mathbb{Z})$ or $\mathcal{O}$. □

We have addressed now all the simple types that are usually called "exceptional", apart from E_8. A classification of $\mathbb{Z}$-groups of type E_8 like Proposition 18.2 appears currently out of reach, because among those group schemes $\mathbf{G}$ over $\mathbb{Z}$ such that $\mathbf{G} \times \mathbb{R}$ is the compact group of type E_8, there are at least 13,935 distinct isomorphism classes [Gr, Proposition 5.3]. Among those $\mathbf{G}$ over $\mathbb{Z}$ of type E_8 such that $\mathbf{G} \times \mathbb{R}$ is not compact, the same argument as in the proof of Proposition 18.2 shows that there are two isomorphism classes.

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