The Skolem-Noether Problem for Albert Algebras [1]

Holger P. Petersson
Fachbereich Mathematik
FernUniversität in Hagen
D-58084 Hagen
Germany
Email: Holger.Petersson@FernUni-Hagen.de

Colloquium, University of Nottingham, March 4, 2005

It's certainly a pleasure having the opportunity to give a talk at the University of Nottingham, and I would like to thank the Department of Mathematics for the invitation.

For the purpose of my lecture today, it will be convenient to phrase the Skolem-Noether set-up in fairly general terms as follows. Consider two specimens, say S and T, of your favourate algebraic structure, and suppose you are given two embeddings, ι and ι', from T to S. Let us agree to say that these two embeddings are *equivalent* by definition if and only if there exists an automorphism φ of S moving the one to the other, in other words, making a commutative diagram as shown. The Skolem-Noether theorem asserts that all embeddings from T to S are equivalent. If the Skolem-Noether theorem holds, we are in good shape, but if it doesn't, we've got a problem, namely, the Skolem-Noether problem, which amounts to describing the obstructions that might prevent the embedding ι from becoming equivalent to the embedding ι'.

You are, of course, familiar with many classical instances, with many classical cases, for the Skolem-Noether set-up. For example, consider the case of simple algebras, which actually gave the name to the set-up. Here S is a central simple algebra, always assumed to be finite-dimensional associative over an arbitrary field, and T is a simple algebra. Then the classical Skolem-Noether theorem implies that the Skolem-Noether theorem holds. Or consider the case of quadratic forms. Here S is a non-singular quadratic form over an arbitrary base field, and T is any quadratic form. Then Witt's theorem implies that the Skolem-Noether theorem holds.

In what follows, I would like to discuss a number of other cases for the Skolem-Noether set-up, cases that are all related in one way or another to what I call cubic norm structures. What is a cubic norm structure?

By a *cubic norm structure* over an arbitrary base field k, I mean a triple $J = (V, N, 1)$ satisfying the following conditions.

- First of all, V is a finite-dimensional vector space over k.
- Secondly, $N : V \to k$ is a cubic form, called the *norm* of J.
- Thirdly, $1 \in V$ is a distinguished element, called the *base point* of J.

For these constituents to form a cubic norm structure it is necessary and sufficient that the following additional hypotheses be fulfilled.

[1] *Date*: February 21, 2005

- First of all, the norm is related to the base point by the normalizing condition $N(1)$ equal to the unit element of the base field.
- Secondly, the *trace* of J, denoted by T and defined up to a sign as the logarithmic Hessian of the norm evaluated at the base point, is a non-degenerate symmetric bilinear form,

allowing us to consider the gradient of the norm relative to the trace, which is called

- the *adjoint* $x \mapsto x^\sharp$ of J, determined by the condition $T(x^\sharp, y)$ equal to the total derivative of the norm evaluated at x in the direction y, and satisfies the *adjoint identity* $x^{\sharp\sharp} = N(x)x$.

This concludes the definition of a cubic norm structure. We proceed by introducing some standard terminology.

- *Homomorphisms* of cubic norm structures are linear maps of the underlying vector spaces preserving norms and base points.
- A cubic norm structure is said to be *reduced* if its norm is isotropic, hence represents 0 non-trivially.
- It is said to be *division* if it is not reduced.
- An *isotopy* of cubic norm structures is a bijective norm similarity, i.e., a linear bijection preserving norms up to a non-zero scalar factor.
- The totality of all isotopies from J onto itself forms a group, called the *structure group* of J.

It is important to note that

- the automorphism group identifies inside the structure group as the stabilizer subgroup of the base point.

Let me now specify the cubic norm structures that will take the place of S and T in the Skolem-Noether set-up.

I begin with S, which will always be an Albert algebra J, a special type of cubic norm structure to be described as follows. According to the above, we distinguish two cases.

- First of all, suppose J *is reduced.* Then $J \cong H_3(C, g)$, a cubic norm structure depending on two parameters. The first parameter is very easy. It's just a 3-by-3 diagonal matrix g with non-zero diagonal entries $\gamma_1, \gamma_2, \gamma_3$ in the base field. The second parameter is more involved. It's an octonion algebra C over k. Let me briefly remind you of its most important properties.
  - C is a non-associative k-algebra of dimension 8 containing a unit.
  - There is a unique non-singular quadratic from N_C on C, the *norm*, which is *multiplicative*, so $N_C(uv) = N_C(u)N_C(v)$.
  - The norm canonically induces a linear form T_C on C, the *trace*, which is *associative*, so $T_C((uv)w) = T_C(u(vw))$.
  - There is a unique algebra involution $x \mapsto \overline{x}$ of C, the *conjugation*, which is *central*, so $x\overline{x} = N_C(x)1_C$.

  Then $H_3(C, g) = (V, \mathrm{Det}_g, \mathbf{1}_3)$, where

* $V = \{x \in M_3(C) | x = g^{-1}\overline{x}^t g, \mathrm{Diag}(x) \in k^3\}$,
* Det_g is the *g-determinant* on V defined by

$$\mathrm{Det}_g\left(\begin{pmatrix} \xi_1 & u_3 & * \\ * & \xi_2 & u_1 \\ u_2 & * & \xi_3 \end{pmatrix}\right) = \xi_1\xi_2\xi_3 - \sum_{(ijl)} \gamma_j\gamma_l\xi_i N_C(u_i) + \gamma_1\gamma_2\gamma_3 T_C(u_1u_2u_3)\,,$$

* $\mathbf{1}_3$ is the 3-by-3 unit matrix.

Then we obviously have $\dim J = \dim H_3(C, g) = 27$. Also, an important invariant of J is supplied by the following theorem.

Theorem 1. (Albert-Jacobson 1957 [char $k \neq 2$], Faulkner 1970 [arbitrary]) $C = \text{Co-al}(J)$ *is an invariant of* J*, called the* co-ordinate algebra *of* J*.*

- Next, *let* J *be arbitrary.* Unfortunately, I don't have the time to say much about the structure of J in this case. Suffice it to quote the following theorem.

Theorem 2. (P.-Racine 1996) J *has a unique* reduced model, i.e., *there exists a unique reduced Albert algebra* $J_{\mathrm{red}} \cong H_3(C, g)$ *over* k *having the property that, for all field extensions* E/k *making the base change* $J \otimes E$ *reduced over* E*,* $J \otimes E$ *and* $J_{\mathrm{red}} \otimes E$ *will be isomorphic over* E*.*

In particular, $\text{Co-al}(J) := C$ is an invariant of J, called the *co-ordinate algebra* of J. For cohomologists, its basically the 3-invariant mod 2 of J.

Let me now turn to the cubic norm structures that will take the place of T in the Skolem-Noether set-up. There are the three possibilities $T \in \{E^+, A^+, H(B, \tau)\}$, where

- E is a cubic étale k-algebra, i.e., a k-form of $k \oplus k \oplus k$, and

$$\begin{aligned} E^+ &: = (V = E \text{ as a } k\text{-space}\,,\ N = N_E\,,\ 1 = 1_E)\,, \\ N_E(x) &: = \det(L_x : E \longrightarrow E\,,\ y \longmapsto xy)\,, \\ (E'^+ \xrightarrow{\sim} E^+) &= (E' \xrightarrow{\sim} E)\,, \\ \dim E^+ &= 3\,, \end{aligned}$$

- A is a central simple k-algebra of degree 3, i.e., a k-form of $M_3(k)$,

$$\begin{aligned} A^+ &: = (V = A \text{ as a } k\text{-space}\,,\ N = \mathrm{Nrd}_A\,,\ 1 = 1_A)\,, \\ (A'^+ \xrightarrow{\sim} A^+) &= (A' \xrightarrow{\sim} A \text{ or } A'^{\mathrm{op}} \xrightarrow{\sim} A)\,, \\ \dim A^+ &= 9\,, \end{aligned}$$

- (B, τ) is a central simple associative k-algebra of degree 3 with unitary involution, i.e., basically a k-form of $(M_3(k) \oplus M_3(k)^{\mathrm{op}}, \text{switch})$, more specifically,
  * K/k is a separable quadratic field extension,
  * B is a central simple K-algebra of degree 3,

* $\tau : B \to B$ is a k-involution,
* $\tau|_K \neq \mathbf{1}_K$.

and

$$H(B,\tau) := \big(V = \{x \in B | \tau(x) = x\} \text{ as a } k\text{-space}\,,\ N = \mathrm{Nrd}_B|_V\,,\ 1 = 1_B\big)\,,$$
$$\big(H(B',\tau') \xrightarrow{\sim} H(B,\tau)\big) \cong \big((B',\tau') \xrightarrow{\sim} (B,\tau)\big)\,,$$
$$\dim H(B,\tau) = 9\,.$$

The Skolem-Noether set-up for the case $S = J$ an Albert algebra, $T \in \{A^+, H(B,\tau)\}$. The significance of this case derives from the fact that every embedding from T to S induces an isomorphism from S onto a Tits process algebra arising from T. Thus the present case of the Skolem Noether set-up is intimately connected with the classification problem for Albert algebras. Fortunately, we have a very satisfactory answer.

Theorem 3. *The Skolem-Noether theorem holds for the special case $S = J$ an Albert algebra and $T \in \{A^+, H(B,\tau)\}$.*

This is a deep result that has many authors. The first contribution is due to Jacobson in 1968, who treated the case char $k \neq 2$ and T reduced, forcing J to be reduced as well. The case $T = A^+$ follows almost immediately from the existence of the invariant mod 3 for Albert algebras, due to Rost in 1991 for char $k \neq 2,3$ and Serre, P.-Racine in 1996-97 for k arbitrary, combined with the important fact that this invariant detects Albert division algebras, which follows from Merkurjev-Suslin theory. The case $T = H(B,\tau)$ is due to Parimala-Sridharan-Thakur in 1998 for char $k \neq 2,3$ and to P. in 2004 for k arbitrary.

The Skolem-Noether set-up for the case $S = J$ an Albert algebra, $T = E^+$. The significance of this case derives from the fact that every embedding $\iota : E^+ \to J$ determines a group scheme $\mathbf{G}_\iota$ over k via

$$\mathbf{G}_\iota(R) = \{\varphi \in \mathrm{Aut}(J \otimes R) | \varphi \circ (\iota \otimes \mathbf{1}_R) = \iota \otimes \mathbf{1}_R\}\,. \qquad (R \in k\text{-}\mathbf{alg})$$

Then

- $\mathbf{G}_\iota$ is a group of type D_4, i.e., a k-form of $\mathbf{Spin}_8$, and an exceptional one at that.
- (Jacobson 1957) If k has characteristic zero, then $\mathbf{G}_\iota$ and $\mathbf{G}_{\iota'}$ are isomorphic if and only if ι and ι' are equivalent.

The first crucial observation to be recorded here is that *the Skolem-Noether theorem does not hold.* Indeed, we have the following result.

Theorem 4. (Albert-Jacobson 1957) *If k has characteristic not 2 or 3 and E splits, then every embedding $\iota : E^+ \to J$ determines a classifying invariant*

$$[\iota] \in \big(k^\times / N_C(C^\times)\big)^3\,, \qquad \text{(norm class)}$$

so two embeddings ι and ι' from E^+ to J are equivalent if and only if they have the same norm class.

In spite of this, let us try anyway to develop a

Naive Strategy to prove the Skolem-Noether theorem (owing much to Thakur)**.** Given two embeddings $\iota, \iota' : E^+ \to J$, find

- a cubic norm structure $J_1 \in \{A^+, H(B,\tau)\}$,
- an embedding $\lambda : J_1 \to J$

such that ι and ι' both factor uniquely through λ, so there are unique embeddings $\iota_1, \iota_1' : E^+ \to J_1$ making a commutative diagram as shown. Then

- prove that ι_1 and ι_1' are equivalent, giving an automorphism φ_1 of J_1 as shown,
- apply Theorem 3 above to find an automorphism φ of J completing the diagram as shown.

Unfortunately, the naive strategy for proving the Skolem-Noether theorem cannot work, primarily for two reasons.

- Conceivably, it may happen that ι and ι' are equivalent but ι_1 and ι_1' are not.
- The analogue of Theorem 4 above holds for J_1 in place of J, so there do exist obstructions that might prevent the embedding ι_1 from becoming equivalent to the embedding ι_1'.

More specifically, define

- $\Delta(E) :=$ the discriminant of E, viewed as a quadratic étale k-algebra,
- $K := k \oplus k$ for $J_1 = A^+$,
- $L := \Delta(E) * K$, the product of $\Delta(E)$ and K as quadratic étale k-algebras.

Then we have

Theorem 5. (P.-Thakur 2004) *Every embedding $\iota_1 : E^+ \to J_1$ determines a classifying invariant*

$$[\iota_1] \in E^\times / N_L\big((E \otimes L)^\times\big)\,, \qquad \text{(norm class)}$$

so two embeddings ι_1 and ι_1' from E^+ to J_1 are equivalent if and only if they have the same norm class.

Surprisingly, the naive strategy can be brought back to life by combining it with the

Relaxation Strategy. The relaxation strategy is based upon the following modification.

- Enlarge the group by passing from the automorphism group to the structure group of J and relax the equivalence by passing from ordinary equivalence as defined earlier to weak equivalence defined as follows. Two embeddings $\iota, \iota' : E^+ \to J$ are said to be *weakly equivalent* by definition if and only if there exist an element η in the structure group of J and an invertible element $v \in E$ making a commutative diagram as shown.

Notice that, in the definition of weak equivalence above, the element v is a crucial ingredient. For $v = 1_E$, the element η of the structure group would fix the base point, hence would be an automorphism, and we were back to ordinary equivalence. Combining now the naive strategy with the relaxation strategy, we obtain the following result.

Theorem 6. (Weak Skolem-Noether theorem) *All embeddings from E^+ to J are weakly equivalent.*

The significance of the weak Skolem-Noether theorem is underscored by the fact that it has non-trivial applications to the ordinary Skolem-Noether problem. For example, one can derive the following corollary.

Corollary 1. *If the discriminant of E is a subalgebra of the co-ordinate algebra of J, then the Skolem-Noether theorem holds, so all embeddings from E^+ to J are equivalent.*

The key to the proof of this result is the observation that, thanks to the hypothesis on the discriminant of E, every embedding from E^+ to J factors through a substructure J_1 of J having the form $J_1 = H(B, \tau)$, for some central-simple associative algebra (B, τ) of degree 3 with *distinguished* in volution of the second kind.

Since first Tits construction Albert algebras can be characterized by the condition that their co-ordinate algebra is split, and since the split octonions contain all quadratic étale k-algebras, Corollary 1 immediately implies

Corollary 2. *If J is an Albert algebra arising from the first Tits construction, then the Skolem-Noether theorem holds, so all embeddings from E^+ to J are equivalent.*

Although I did not have the time so far to work out the details, I am convinced that the weak Skolem-Noether theorem will make it possible to identify the obstructions that might prevent two embeddings from E^+ to any Albert algebra J from becoming equivalent.