On development paths minimizing the aggregate labor-reallocation costs in the three-sector framework and an application to structural policy

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30th August 2018

Abstract. Cross-sector labor reallocation is associated with costs at the micro level ranging from the costs of geographical relocation and skill change/adaptation to unemployment. We show that monotonous reallocation paths minimize the aggregate reallocation costs in the three-sector framework (relating to agriculture, manufacturing, and services), where we assume that the aggregate reallocation costs are indicated by the strength of cross-sector labor reallocation. By using this result and the (qualitative) labor-reallocation 'laws' based on the theoretical/empirical literature consensus, we derive the cost-minimizing development strategy for a developing country. We use these results to discuss some well-known structural/trade strategies and compare the historical reallocation costs across countries.

Keywords: employment structure, long-run dynamics, costs, industrial/structural policy, uncertainty, dynamic optimization, calculus of variations, geometry, vector, simplex.

JEL: O14, O41, O21, O24, C61

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1. Introduction

Cross-sector labor reallocation is one of the major characteristics of the development process. While the (macroeconomic) long-run structural change modeling literature extensively studies the positive effects of cross-sector labor reallocation (e.g., welfare and GDP growth), it neglects the *costs* of labor reallocation arising at the micro level and, in particular, the costs associated with unemployment, geographical relocation, retraining/adaptation, and human capital (job-specific skills) depreciation among others. In this paper, we focus on the labor-reallocation costs in the three-sector framework relating to the agricultural, manufacturing, and services sector. [1] In particular, we assume that the aggregate reallocation costs (accumulated over a period) are indicated by the number of reallocated workers across sectors (over the period). This relatively general assumption seems meaningful, since each reallocated individual bears some of the aforementioned reallocation costs. In our paper, we derive two mathematical statements on monotonous labor-reallocation paths (Lemmas 1 and 2) and demonstrate how these statements can be used in the discussion of aggregate reallocation cost.

We start the analysis by showing that monotonous labor-reallocation paths minimize the aggregate reallocation costs in our framework (Lemma 1). This result is relatively robust/general, and we demonstrate that it can be directly used to study cross-country differences in aggregate reallocation costs (over very long periods of time) based on widely available labor-reallocation data.

Then, we elaborate on two policy related aspects of Lemma 1. First, empirical evidence and economic theory imply that (a) labor-reallocation paths (and, thus, the aggregate reallocation costs) differ significantly across countries, (b) policy makers can affect labor-reallocation paths (to some extent) via industrial and trade policy, and (c) there are limits to the capability of policy to affect labor reallocation. In particular, there are theoretically supported *empirical laws* of labor reallocation, i.e., common features of labor reallocation in "all" countries (despite the policy differences across the countries). Thus, it makes sense to study aggregate reallocation costs under the assumption that these empirical laws are obeyed. Lemma 2 proves the existence of monotonous labor-reallocation paths consistent with these laws. Second, as we show, Lemma 1 implies that the determination of the destination of a structural

[1] For an overview of the (macroeconomic) *structural change literature*, see, e.g., Schettkat and Yocarini (2006), Krüger (2008), Silva and Teixeira (2008), Stijepic (2011, Chapter IV), Herrendorf et al. (2014), and Neuss (2018). Recent contributions in this field dealing with the three-sector framework include, e.g., Kongsamut et al. (2001), Ngai and Pissarides (2007), Foellmi and Zweimüller (2008), Uy et al. (2013), and Stijepic (2015). Examples of the (microeconomic) *labor-reallocation costs literature* are Jakobson et al. (1993), Lee and Wolpin (2006), Artuç et al. (2010, 2015), and Bakas et al. (2017).

path is a major challenge in cost-minimal reallocation policy design. In particular, a policy planer cannot determine exactly the optimal labor allocation at the end of the planning horizon due to uncertainty regarding the future global environment in which the economy acts (e.g., trade conditions, global political situation,…) and, thus, faces *uncertainty regarding the path destination*. We prove that the mathematical results (Lemmas 1 and 2) derived in our paper imply that if the *empirical laws* and the path-destination uncertainty are considered, a cost-minimizing reallocation policy is characterized by an increasing (a decreasing) services (agricultural) employment share and a constant manufacturing employment share over the initial phase of development.

Finally, we discuss briefly how the rather theoretical results derived in our paper can be applied in the discussion of the aggregate reallocation costs associated with the standard development strategies (e.g., Washington Consensus and Kaldorian strategy) and observed labor-reallocation paths.

The modeling approach in our paper reflects the fact that we seek to derive rather general/robust results. In particular, we choose the assumptions that are common to most theoretical and empirical studies (meta-modeling), instead of choosing particular theoretical models/doctrines. This results in qualitative restrictions and, thus, requires a qualitative (and geometrical) approach to the respective calculus of variations problems.

The rest of the paper is set up as follows. In Section 2, we discuss briefly the relevant theoretical literature and evidence and derive the meta-theorems representing the consensus that can be derived from this literature and evidence. Section 3 is devoted to the derivation of Lemmas 1 and 2. Section 4 discusses the optimal labor-reallocation policy consistent with the previously derived meta-theorems and path-destination uncertainty, its duration, and its follow-up policies. Section 5 applies these results in the discussion of empirical data (in particular, it shows how structural change costs and the duration of optimal policy can be assessed on the basis of widely available labor-reallocation data) and standard development strategies. Concluding remarks are provided in Section 6.

2. Theoretical lessons and empirical evidence on labor-reallocation laws

The *theoretical* contributions on labor reallocation and, in particular, the models presented by Kongsamut et al. (1997, 2001), Ngai and Pissarides (2007), Foellmi and Zweimuller (2009), Uy et al. (2013), and Stijepic (2015) have the following characteristics:

(a) The shapes and the limit properties of the labor-reallocation trajectories generated by these models depend on the parameter settings (where the term 'limit properties' refers to the labor allocation to which the economy converges as time goes to infinity).
(b) We have no clear evidence/theory regarding these models' parameter values.
(c) The parameter sets determining the shape and the limit properties of the trajectories differ significantly across models. For example, the Ngai and Pissarides (2007) model predicts a curved trajectory (cf. Stijepic [2015, 80]) while the Kongsamut et al. (2001) model generates a linear trajectory (cf. Stijepic [2016a]); the Kongsamut et al. (2001) model predicts that in the limit, the manufacturing share is the same as in the initial state, while the Ngai and Pissarides (2007) model predicts a set of different limit manufacturing shares depending on the parameterization of the model; the parameters deciding for the labor-reallocation patterns in the Kongsamut et al. (2001) model (Ngai and Pissarides [2007] model) are preference parameters (technology parameters).
Despite these differences, we can formulate Meta-theorems 1 and 2, which represent the consensus statements of the theoretical literature, i.e., statements that are consistent with the predictions of all the models listed above under the parameter restrictions suggested by the authors of the corresponding papers.

Meta-theorem 1. *In a developing economy, the services employment share grows and the agricultural employment share decreases in the very long run. In other words, the models imply that in a more or less distant future ('long run perspective'), a developing country's services (agricultural) employment share will be greater (smaller) than it is today.*

Meta-theorem 2. *A developing country's manufacturing employment share may be growing, decreasing, or constant in the long run. Moreover, it may follow a non-monotonous pattern ('hump-shaped development') in the long run (as predicted by, e.g., Ngai and Pissarides [2007] and Uy et al. [2013]).*

For a discussion of the *empirically* observable shapes and limit properties of labor-reallocation trajectories, we refer to Stijepic (2017b), who depicts the labor-reallocation data from different sources covering a large set of countries on standard 2-simplexes. The following facts become immediately apparent when studying the figures presented by Stijepic (2017b). First, the shapes and the endpoints of the trajectories differ significantly across countries. Second, many trajectories are strongly curved; thus, depending on the planning horizon (h), the sector structure at the end of the planning horizon ($\mathbf{x}(h)$) varies strongly even when considering the trajectory of only one country. Third, the empirical evidence presented by Stijepic (2017b) supports Meta-theorems 1 and 2.

Overall, while economic theory and empirical evidence imply that there are significant differences in labor-reallocation paths across countries, Meta-theorems 1 and 2 seem to represent a theoretical-empirical consensus regarding labor-reallocation laws (of motion).

3. Two lemmas on monotonous labor-reallocation paths

In this section, we derive Lemmas 1 and 2, which seem quite useful in the study of labor-reallocation costs, as discussed extensively in Sections 4 and 5. In the rest of the paper, the mathematical notation is as follows: small italic letters (e.g., x) denote scalars, small bold letters (e.g., $\mathbf{x}$) denote vectors, capital italic letters (e.g., X) denote sets, and Greek small letters (e.g., δ) denote angles. R denotes the set of real numbers.

3.1 Monotonous paths as reallocation costs-minimizing paths

In this section, we show that aggregate reallocation costs-minimizing paths are monotonous. We start with a definition of structure and sectoral employment shares.

Definition 1. *The **sector structure** (indicated by the labor allocation) at time* $t \in [0, \infty)$ *is given by the vector* $\mathbf{x}(t) \equiv (x_1(t), x_2(t),\ldots, x_n(t)) \in R^n$*, where* $x_i(t)$ *denotes the **share of employment** devoted to sector i for* $i = 1, 2, \ldots n$*, and* R^n *is the n-dimensional real space.*

Thus, for example, if $l(t)$ is the aggregate employment (e.g., the number of employees in the economy) at time t and $l_i(t)$ is the employment in sector i (e.g., the number of employees in sector i) at time t, then $x_i(t) \equiv l_i(t)/l(t)$.

Definition 2. *The **development path** over the time interval* $[0, h]$ *is given by the curve* $\mathbf{x}(t)$, $0 \leq t \leq h$ *(cf. Definition 1), and the set* $P := \{\mathbf{x}(t) \in R^n: t \in [0, h]\}$.

Thus, we can imagine a development path as a curve/path connecting the points $\mathbf{x}(0)$ and $\mathbf{x}(h)$ in the n-dimensional Euclidean space. The time point h may be regarded as the (end of the) planning *h*orizon.

Definition 3. *A development path (cf. Definition 2) is* ***monotonous*** *on the time interval* $[0, h]$ *if* $\nexists i \in \{1, 2,\dots n\}$: $(\exists t_a \in [0, h]\ \ \exists t_b \in [0, h]: t_a \neq t_b \wedge dx_i(t_a)/dt < 0 \wedge dx_i(t_b)/dt > 0)$.

Definition 3 implies the following properties of a monotonous path. (1.) All x_i are behaving monotonously. Thus, for any $i \in \{1, 2,\dots n\}$, the following is true: either $\forall t \in [0, h]\ dx_i(t)/dt \geq 0$ or $\forall t \in [0, h]\ dx_i(t)/dt \leq 0$. (2.) Some x_i may be monotonously decreasing, while at the same time some x_i may be monotonously increasing or constant. That is, if the economy moves along a monotonous development path, the following scenario is possible among others: at the time point $t_a \in [0, h]$, $dx_1(t_a)/dt > 0$, $dx_2(t_a)/dt < 0$, and $dx_3(t_a)/dt = 0$.

Assumption 1. (a) *The* ***initial sector structure*** *(of the economy) is given, i.e.,* $\mathbf{x}(0) = \mathbf{x}^0 \equiv (x_{01}, x_{02},\dots, x_{0n}) \in R^n$. ***(b)*** *The planning horizon (h) and the path destination* ($\mathbf{x}^h$) *are given. In particular, at time* $h \in (0, \infty)$, $\mathbf{x}(h) = \mathbf{x}^h \equiv (x_{h1}, x_{h2},\dots, x_{hn}) \in R^n$. ***(c)*** *The economy moves along a* ***continuous path****, i.e.,* $\forall t\ \forall i\ x_i(t)$ *is continuous in t.* ***(d)*** *The (cumulative) aggregate costs* (c^{0h}) *of labor reallocation associated with the development path* $\mathbf{x}(t)$, $0 \leq t \leq h$, *are given by*

$$(1) \qquad c^{0h} := f(r^{0h}),\ \ r^{0h} := \int_0^h \sum_{i=1}^n |dx_i(t)/dt|dt,\ \ f: R \to R,\ \ df/dr^{0h} > 0.$$

Several aspects of Assumption 1a-c are noteworthy. First, it is obvious that the initial (or today's) labor allocation ($\mathbf{x}^0$) is given. Second, we assume here that the path destination ($\mathbf{x}^h$) is given only for purposes of presentation; later, we will discuss this restriction in detail and relax it. Third, the assumption of a continuous path is due to the long-run modeling horizon, i.e., we consider only the long-run dynamics and neglect shorter-run jumps and fluctuations. Again, this is a standard assumption in long-run growth modeling. For example, all the models listed in Section 2 are continuous-time models generating continuous paths. Moreover, allowing for discontinuous paths would imply that, in our optimization problem, the policymaker is able to reallocate (very) large numbers of workers (e.g., millions of workers) across sectors within very short periods of time (e.g., within seconds), which contradicts reality where cross-sector reallocation requires, in most cases, geographical reallocation and retraining among others.

The reallocation costs index (1), i.e., Assumption 1d, requires some explanation. Assume that l is the aggregate labor force. Furthermore, assume that l is constant. In this case, $r_i(t) := l dx_i(t)/dt$ is the change in employment in sector i at time t. If $r_i(t) > 0$, then $r_i(t)$ is the (net) number of workers reallocated to sector i at time t. If $r_i(t) < 0$, then $r_i(t)$ is the (net) number of workers reallocated (or: withdrawn) from sector i at time t. Thus, $r(t) := |r_1(t)| + |r_2(t)| +\ldots+ |r_n(t)|$ is an index of the number of reallocated workers at time t. We take here the absolute values of $r_i(t)$, since in most labor-reallocation models, $r_1(t) + r_2(t) +\ldots+ r_n(t)$ is equal to zero, as explained in Section 3.2. For the same reason, we could multiply $r(t)$ with 0.5, since in most models, 'reallocation of workers across sectors' means that a withdrawal of the workers from one sector is always associated with the hiring of these workers in another sector. However, since multiplying $r(t)$ with 0.5 does not change any of our results, we omit it here. Overall, $r(t)$ is the index of reallocation at time t. To obtain an index of reallocation over the time period $[0, h]$, we must sum up all $r(t)$ over this period, which in continuous time, corresponds to taking the integral over t. This integral is equal to r^{0h}. In fact, r^{0h} is an index of the magnitude of reallocation (or: an index of the number of reallocated workers). As noted in the introduction, we assume that the aggregate reallocation costs (c^{0h}) are a (strictly) monotonously increasing function (f) of this magnitude of reallocation (r^{0h}). Analogous results could be obtained if we used a measure of the magnitude of the changes in the sectoral GDP shares.

Overall, Assumption 1 can be used to define the following calculus-of-variations problem. Assume that Assumption 1 is satisfied and, thus, $\mathbf{x}(0) = \mathbf{x}^0$ is given and the path destination (at time h) is determined, i.e., $\mathbf{x}(h) = \mathbf{x}^h$ is given. There exist different paths that connect $\mathbf{x}^0$ and $\mathbf{x}^h$ in Euclidean space (cf. Figure 1). A path is 'admissible' if it is continuous (cf. Assumption 1c) and if it connects $\mathbf{x}^0$ and $\mathbf{x}^h$. The functional (1) associates each of these admissible paths with a certain magnitude of aggregate reallocation costs c^{0h}. We search for an answer to the following question: 'Which of the admissible paths is associated with minimal aggregate reallocation costs (c^{0h})?' That is, we want to find the (admissible) path that minimizes the aggregate reallocation costs c^{0h}. Lemma 1 provides the solution of this problem.

Figure 1. *The calculus-of-variations problem solved by Lemma 1.*

- insert Figure 1 here -

Lemma 1. *Let Assumption 1 be satisfied. Then, any* ***monotonous*** *(and continuous) development path (cf. Definition 2) that connects* $\mathbf{x}^0$ *and* $\mathbf{x}^h$ *(in Euclidean space) minimizes the aggregate reallocation costs* c^{0h} *(cf. (1)).*

For a **proof of Lemma 1**, we could apply the theorems of the calculus of variations (see, e.g., Gelfand and Fomin [1963, Chapter 15]). In APPENDIX A, we provide a more detailed (geometrical) proof, which uses the techniques familiar to the calculus of variations. This detailed proof provides us with lemmas and interpretations that are helpful for proving and understanding the properties of the minimal-costs paths that will be discussed later.

Figuratively speaking, Lemma 1 states that if we seek to minimize the aggregate reallocation costs arising on the way from $\mathbf{x}^0$ to $\mathbf{x}^h$, it does not matter which path we take from $\mathbf{x}^0$ to $\mathbf{x}^h$ as long as it is monotonous (and, per assumption, continuous).

Since Lemma 1 is valid for any $\mathbf{x}^h \in R^n$, we could formulate it more generally, i.e., we can omit the reference to $\mathbf{x}^0$ and $\mathbf{x}^h$, as follows: any monotonous path (in Euclidean space) is a cost-minimal connection between its point of departure and its destination.

Note that we assume throughout the paper that $\mathbf{x}(t)$ is C^1 (see, e.g., Definition 3 and Assumption 1d), i.e., the development path has a certain degree of smoothness. This assumption can be justified by the fact that we study only long-run trend paths, i.e., the smoothness of $\mathbf{x}(t)$ is per definition (of the term 'long run trend').

Obviously, if $\mathbf{x}^h = \mathbf{x}^0$, the aggregate reallocation costs-minimizing strategy (for 'moving' from $\mathbf{x}^0$ to $\mathbf{x}^h$) is: stay in $\mathbf{x}^0$ for all $t \in [0, h]$, i.e., no labor reallocation at all! Such a 'path' is per Definition 3 monotonous.

3.2 On the existence of monotonous labor-reallocation paths consistent with Meta-theorems 1 and 2

In this section, we prove Lemma 2, which postulates the existence of a path with certain characteristics, among others, monotonicity (cf. (4)) and consistency with Meta-theorems 1 and 2 (cf. (7)). In Section 4, we show that Lemmas 1 and 2 imply almost directly the properties of a cost-minimal reallocation policy/path consistent with Meta-theorems 1 and 2.

Most contributions on structural change assume that employment shares satisfy the conditions $\forall i \in \{1, 2,\dots, n\}$ $x_i(t) \geq 0$ and $x_1(t) + x_2(t) +\dots+ x_n(t) = 1$. That is, the sectoral employment shares are non-negative and add-up to one. Since the proof of Lemma 2 is more difficult when these requirements are satisfied, we construct the proof of Lemma 2 by assuming that they are satisfied (cf. (2)). The proof is analogous without these assumptions. Moreover, we study the three-sector framework and assume that initial sectoral shares (x_{01}, x_{02}, x_{03}) are in the interior of their domains (cf. (8)), i.e., in the initial state (e.g., today), the country under consideration employs labor in all three sectors (agriculture, manufacturing, and services). This assumption seems to make sense even for the most underdeveloped countries.

Lemma 2. *Consider the three-sector economy (*$n = 3$*) over the period* $[0, \infty)$ *and let Assumption 1 be valid.* ***(a)*** *There exists a path* $\mathbf{x}^*(t) \equiv (x_1^*(t), x_2^*(t), x_3^*(t))$, $t \in [0, h]$, *that has the following characteristics:*

(2) $\forall t \in [0, h]$ $\mathbf{x}^*(t) \in D := \{(x_1, x_2, x_3) \in R^3: \forall i \in \{1, 2, 3\}\ x_i \geq 0 \wedge x_1 + x_2 + x_3 = 1\}$

(3) $\forall t \in [0, h]$ $\mathbf{x}^*(t)$ *is continuous in t*

(4) $\forall t \in [0, h]$ $\mathbf{x}^*(t)$ *is monotonous in t (cf. Definition 3)*

(5) $\mathbf{x}^*(0) = \mathbf{x}^0 \equiv (x_{01}, x_{02}, x_{03}) \in D$

(6) $\mathbf{x}^*(h) = \mathbf{x}^h \equiv (x_{h1}, x_{h2}, x_{h3}) \in D$

(7) $x_{h1} < x_{01} \wedge x_{h3} > x_{03}$ *(cf. Meta-theorems 1 and 2)*

(8) $\forall i \in \{1, 2, 3\}\ x_{0i} > 0.$

(9) $\exists t^* \in (0, h): \forall t \in [0, t^*)\ x_2^*(t) = x_{02}.$

(b) *For some* $\mathbf{x}^h \in D$, *there does not exist a path* $\mathbf{x}^*(t) \equiv (x_1^*(t), x_2^*(t), x_3^*(t))$, $t \in [0, h]$, *satisfying the conditions (2)-(8) and (9'), where*

(9') $\exists t^* \in (0, h): \forall t \in [0, t^*)\ dx_2^*(t)/dt < 0.$

(c) *For some* $\mathbf{x}^h \in D$, *there does not exist a path* $\mathbf{x}^*(t) \equiv (x_1^*(t), x_2^*(t), x_3^*(t))$, $t \in [0, h]$, *satisfying the conditions (2)-(8) and (9''), where*

(9'') $\exists t^* \in (0, h): \forall t \in [0, t^*)\ dx_2^*(t)/dt > 0.$

Proof. We start with a proof of Lemma 2a. Define the path $P^* := \{\mathbf{x}^*(t) \in R^3: t \in [0, h]\}$ (cf. Definition 2) to which Lemma 2 refers. If (2) is satisfied, $P^* \subset D$, i.e., the optimal path is located on a standard 2-simplex D in R^3. If (3), (4), and (7) are satisfied, $\forall t \in [0, h]$ $x_1^*(t) \leq x_{01} \wedge x_3^*(t) \geq x_{03}$. Thus, if (2)-(4) and (7) are satisfied, $P^* \subset D^h := \{(x_1, x_2, x_3) \in D: x_1 \leq x_{01} \wedge x_3 \geq x_{03}\} \subset D$, i.e., P^* is located in D^h, which is a subset of the standard 2-simplex D. In particular, P^* connects $\mathbf{x}^0$ and $\mathbf{x}^h$ on D^h (cf. (5) and (6)), where $\mathbf{x}^0$ is in the interior of D (cf. (2), (5), and (8)). Assumption 1 and the wording of Lemma 2 do not impose any restrictions on the relation between x_{02} and x_{h2}. Thus, we can distinguish between three (alternative) cases:

(10) $x_{02} > x_{h2}$

(11) $x_{02} = x_{h2}$

(12) $x_{02} < x_{h2}$

According to these cases, we can partition D^h into three partitions (D^{ha}, D^{hb}, D^{hc}) as follows:

(13) $D^{ha} := \{(x_1, x_2, x_3) \in D: x_1 \leq x_{01} \wedge x_3 \geq x_{03} \wedge x_2 < x_{02}\} = \{(x_1, x_2, x_3) \in D^h: x_2 < x_{02}\}$

(14) $D^{hb} := \{(x_1, x_2, x_3) \in D: x_1 \leq x_{01} \wedge x_3 \geq x_{03} \wedge x_2 = x_{02}\} = \{(x_1, x_2, x_3) \in D^h: x_2 = x_{02}\}$

(15) $D^{hc} := \{(x_1, x_2, x_3) \in D: x_1 \leq x_{01} \wedge x_3 \geq x_{03} \wedge x_2 > x_{02}\} = \{(x_1, x_2, x_3) \in D^h: x_2 > x_{02}\}$

By now, we have shown that the path P^* connects $\mathbf{x}^0$, which is in the interior of D, and $\mathbf{x}^h$ on D^h if conditions (2)-(8) are satisfied. Moreover, we know that the path destination $\mathbf{x}^h$ is located in one and only one of the partitions D^{ha}, D^{hb}, D^{hc} (cf. (6) and (7)). The remaining question is whether condition (9) can be satisfied in addition to these facts, which represent the conditions (2)-(8). We can divide this question into the following three sub-questions, by using our partitioning (10)-(12) (which is represented by (13)-(15)): First, does a continuous path that (a) is in D^h, (b) has a path destination in D^{ha}, and (c) satisfies (9) exist (*case (10)/(13)*)? Second, does a continuous path that (a) is in D^h, (b) has a path destination in D^{hb}, and (c) satisfies (9) exist (*case (11)/(14)*)? Third, does a continuous path that (a) is in D^h, (b) has a path destination in D^{hc}, and (c) satisfies (9) exist (*case (12)/(15)*)?

To construct the corresponding existence proofs, we refer to the geometrical properties of paths and sets on the standard 2-simplex D. In particular, we assume in the following proof that the reader has knowledge of (a) the interpretation of tangential vector angles of trajectories on the standard 2-simplex in terms of the monotonicity characteristics of $x_1^*(t)$, $x_2^*(t)$, and $x_3^*(t)$ and (b) the implications of equality and inequality constraints (with respect to x_1, x_2, and x_3) for the geometrical properties of sets and, in particular, line-segments on the standard 2-simplex. For a detailed discussion of these properties (in the context of labor-reallocation modeling), see Stijepic (2015, 2017). In APPENDIX B, we provide a brief summary of them. Henceforth, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ denote the vertices of the standard 2-simplex D and, in particular, their Cartesian coordinates ($\mathbf{v}_1 := (1, 0, 0)$, $\mathbf{v}_2 := (0, 1, 0)$, $\mathbf{v}_3 := (0, 0, 1)$). Moreover, we define the vector angle $\delta(.)$ as follows: Let $\mathbf{x}$ be a (regular) point (of a trajectory/curve) on D and $\mathbf{r}(\mathbf{x})$ be a (tangential) vector indicating the direction of movement at the point $\mathbf{x}$. The vector angle $\delta(\mathbf{x})$ is the angle between $\mathbf{r}(\mathbf{x})$ and the simplex-edge $\mathbf{v}_1\mathbf{v}_2$, i.e., $\delta(\mathbf{x}) := \angle(\mathbf{r}(\mathbf{x}), \overline{\mathbf{v}_1\mathbf{v}_2})$ (see Figure 2).

Condition (9) states that there exists an initial period $[0, t^*) \subset [0, h]$ during which $x_2^*(t)$ is constant. For showing that condition (9) is consistent with conditions (2)-(8), we return to the cases (10)-(12). We start with the simplest case (11). In case (11), i.e., if $x_{02} = x_{h2}$, the path satisfying (2)-(8) must be characterized by $x_2^*(t)$ = constant over the period $[0, h]$ and, thus, over the period $[0, t^*) \subset [0, h]$, i.e., (16) is true.

(16) $\forall t \in [0, h]$ $x_2^*(t)$ = constant

In particular, if $x_2^*(0) = x_{02} = x_{h2} = x_2^*(h)$ (cf. (5), (6), and (11)) and $x_2^*(t)$ is monotonous over $[0, h]$ (cf. (4)), then (16) must be true. To prove that a path exists that is consistent with (2), (3), (5)-(8), and (16), we can simply show that it is always possible to construct it: Assume that the curve $\mathbf{x}^*(t)$, $t \in [0, h]$, satisfies (2), (3), (5)-(8) and (16). Then, (2) implies that the corresponding path P^* is located on D, and (16) implies that all the regular points $\mathbf{x}^*(t)$ on this path satisfy the vector angle condition $\delta(\mathbf{x}^*(t))$ = 60° (cf. APPENDIX B). This angle condition, (3), and the geometrical properties of the standard 2-simplex D (cf. APPENDIX B) imply that P^* is a line-segment (on D). (2) and (5)-(7) ensure that that the line-segment P^* exists and is of non-trivial length: (2), (5), and (6) ensure that $\mathbf{x}^0$ and $\mathbf{x}^h$ are on D; (5)-(7) ensure that $\mathbf{x}^0$ and $\mathbf{x}^h$ are two distinct points (on D); (2), (5), and (6) ensure that P^* connects $\mathbf{x}^0$ and $\mathbf{x}^h$ on D; D is a convex subset of a plane (in R^3) and, thus, two distinct points ($\mathbf{x}^0$ and $\mathbf{x}^h$) on D can always be connected by a line segment (P^*) of non-trivial length. If P^* is of non-trivial length (and continuous according to (3)), then the period $[0, h]$ over which P^* is traversed is non-trivial (if the velocity is finite). This completes the proof that there exists a path that is consistent with (2), (3), (5)-(8) and (16). Since (a) (16) implies that (4) and (9) are satisfied and (b) we have derived (16) under the assumption that (11) is true, we can also say that we have proven that there exists a path consistent with (2)-(9) under the assumption that (11) is true. That is, we have proven Lemma 2a for the *case (11)/(14)*. The proof of Lemma 2a for the cases (10)/(13) and (12)/(15) is analogous yet a little bit more difficult as shown in the following.

In *case (10)/(13)*, (4)-(7), and (10) imply that $x_1^*(t)$ decreases monotonously, $x_2^*(t)$ decreases monotonously, and $x_3^*(t)$ increases monotonously. In particular, (17) is true.

(17) $\forall t \in [0, h]$ $dx_1^*(t)/dt \leq 0 \wedge dx_2^*(t)/dt \leq 0 \wedge dx_3^*(t)/dt \geq 0$

Assume that the curve $\mathbf{x}^*(t)$, $t \in [0, h]$, and, thus, the corresponding path P^* are characterized by (2)-(8) and (17). Then, particularly (17) implies that all the regular points $\mathbf{x}^*(t)$ belonging to the path $P^* \subset D$ (cf. (2)) satisfy the vector angle condition (18) (cf. APPENDIX B).

(18) $60° \leq \delta(\mathbf{x}^*(t)) \leq 120°$.

Moreover, (5), (6), (13), (14), and (17) imply that P^* connects $\mathbf{x}^0$ and $\mathbf{x}^h$ on $D^{ha} \cup D^{hb}$. (8) ensures that $\mathbf{x}^0$ is in the interior of D. Thus, the geometrical properties of the simplex D (cf. APPENDIX B) and the definition of D^{ha} and D^{hb} (cf. (5), (13), and (14)) imply that $D^{ha} \cup D^{hb}$ is always a parallelogram and, thus, its boundary consists of four line-segments (L^1, L^2, L^3, L^4), where $L^1 := \{(x_1, x_2, x_3) \in D: x_1 = x_{01} \wedge x_3 \geq x_{03} \wedge x_2 \leq x_{02}\} = \{(x_1, x_2, x_3) \in D^{ha} \cup D^{hb}: x_1 =$

$x_{01}\}$, $L^2 := \{(x_1, x_2, x_3) \in D: x_1 \leq x_{01} \wedge x_3 \geq x_{03} \wedge x_2 = 0\} = \{(x_1, x_2, x_3) \in D^{ha}: x_2 = 0\}$, $L^3 := D^{hb}$ (cf. (14), and $L^4 := \{(x_1, x_2, x_3) \in D: x_1 = 0 \wedge x_3 \geq x_{03} \wedge x_2 \leq x_{02}\} = \{(x_1, x_2, x_3) \in D^{ha} \cup D^{hb}: x_1 = 0\}$ (see Figure 3). This implies that $\mathbf{x}^0 = L^1 \cap L^3$ (cf. (5)) is a vertex of $D^{ha} \cup D^{hb}$, and $\mathbf{x}^h \in D^{ha}/L^1$ (cf. (7) and (10)). Furthermore, note that condition (9) can be substituted by (19) in the case that we consider now (cf. APPENDIX B).

(19) $\exists t^* \in (0, h)\ \forall t \in [0, t^*)\ \ \delta(\mathbf{x}^*(t)) = 60°$

Note that (19) is consistent with (18). Moreover, if (19) is true, then the system moves along D^{hb} over the period $[0, t^*)$, as implied by (9) and (13) (see also APPENDIX B and Figure 3).

Overall, in case (10), we have to show that there exists a path that has the following characteristics: The dynamics start in $\mathbf{x}^0 = L^1 \cap L^3$ (at $t = 0$) and, initially, the system moves along $L^3 = D^{hb}$ (away from $\mathbf{x}^0$) over some period $[0, t^*)$ of non-trivial length. Afterwards, the system moves according to the law (18) and reaches $\mathbf{x}^h \in D^{ha}/L^1$ (cf. Figure 3). As we can see in Figure 3, the fact that such a path always exist (except in the limit case $\mathbf{x}^h \to L^1$, which corresponds to $x_{h1} \to x_{01}$ and is, thus, not of interest due to Meta-theorem 1) is obvious, and we can always construct such a path (within the parallelogram $D^{ha} \cup D^{hb}$) as follows: First draw a line on D that goes through $\mathbf{x}^h$ and is parallel to L^1. Denote the intersection of this line with L^3 by $\mathbf{z}$. Then construct a path (P^*) that describes the movement along the line-segment from $\mathbf{x}^0$ to $\mathbf{z}$ and then along the line-segment from $\mathbf{z}$ to $\mathbf{x}^h$. The first segment (connecting $\mathbf{x}^0$ and $\mathbf{z}$) of this path is characterized by $\delta(\mathbf{x}^*(t)) = 60°$ and is, thus, consistent with (18) and (19). The second segment (connecting $\mathbf{z}$ and $\mathbf{x}^h$) is characterized by $\delta(\mathbf{x}^*(t)) = 120°$ and is, thus, consistent with (18). Since the first segment is of non-trivial length (except in the limit-case $\mathbf{x}^h \to L^1$, i.e., $x_{h1} \to x_{01}$, which is not of interest, as already noted), and the path is continuous (cf. (3)), the period $[0, t^*)$ is of non-trivial length. Overall, we have shown that it is always possible to construct a path that (a) is continuous (thus, (3) is satisfied), (b) connects $\mathbf{x}^0$ satisfying (8) and $\mathbf{x}^h \in D^{ha}/L^1$ (thus, (5)-(8) and (10) are satisfied), (c) is located in $D^{ha} \cup D^{hb} \subset D$ (thus, (2) is satisfied), (d) is consistent with (18) and (19) (thus, (4) and (9) are satisfied). In other words, we have shown that in case (10), there exists a path consistent with (2)-(9). This completes the proof of Lemma 2a in the *case (10)/(14)*. Note that, in this proof, we have used the notion of derivative for describing the monotonicity characteristics (see, e.g., (17)), while the path used for proving Lemma 2a in case (10)/(14) is non-differentiable at the point $\mathbf{z}$ where the first and second path-segment meet (see Figure 3). This is, however, not problematic, since we could smoothen the path (i.e., making it differentiable) at the point $\mathbf{z}$ without restricting the validity of our proof. Moreover, we could choose a rule for assigning derivatives to the singularities of the type observed at the point **z** or avoid using the concept of derivative, which would make the proofs and the paper significantly longer without adding additional economic intuition.

The proof of Lemma 2a in the *case (12)/(15)* is analogous. In particular, in this case, the path satisfying (2)-(9) and (12) lies always within the triangle $D^{hb} \cup D^{hc}$, and a path can be constructed along which the system moves, first, along D^{hb} away from $\mathbf{x}^0$ and, then, along a line-segment parallel to the simplex edge $\mathbf{v}_1$-$\mathbf{v}_2$ towards $\mathbf{x}^h \in D^{hc}/L^5$, where L^5 is the edge of the triangle $D^{hb} \cup D^{hc}$ parallel to the simplex edge $\mathbf{v}_1$-$\mathbf{v}_2$. Moreover, in case (12)/(15), the length of the period $[0, t^*)$ does not depend on the difference $x_{h1} - x_{01}$ as in the previous case, but on the difference $x_{h3} - x_{03}$. This concludes the proof of Lemma 2a.

We turn now to the proof of Lemma 2b, which is relatively simple: Assume that (9') is satisfied, i.e., $x_2^*(t)$ decreases monotonously for all $t \in [0, t^*) \subset [0, h]$. Thus, due to (3) and (4), $x_2^*(t)$ decreases continuously and monotonously for all $t \in [0, h]$. Moreover, assume that $\mathbf{x}^h$ is such that $x_{h2} > x_{02}$ (which is not excluded by the wording of Lemma 2b) and, thus, $x_2^*(h) > x_2^*(0)$ (cf. (5) and (6)). Obviously, we have a contradiction here: if the function $x_2^*(t)$ is continuous and monotonously decreasing for all $t \in [0, h]$, then $x_2^*(h)$ cannot be greater than $x_2^*(0)$. This proves Lemma 2b. The proof of Lemma 2c is analogous: A function $x_2^*(t)$ that is continuous and monotonously increasing for all $t \in [0, h]$ due to (3), (4), and (9'') cannot reach a final state $\mathbf{x}^h$ that is characterized by $x_2^*(h) = x_{h2} < x_{02} = x_2^*(0)$ (cf. (5) and (6)). ■

Figure 2. *The vector angle δ associated with a point $\mathbf{x}(t)$ on a path P on the simplex D.*

- insert Figure 2 here -

Figure 3. *Examples of the set D^h, its partitioning, and a path representing case (10)/(13).*

- insert Figure 3 here -

Note. *$L_1 = \overline{\mathbf{x}_0\mathbf{a}}$; $L_2 = \overline{\mathbf{a}\mathbf{v}_3}$; $L_3 = D^{hb} = \overline{\mathbf{x}_0\mathbf{b}}$; $L_4 = \overline{\mathbf{v}_3\mathbf{b}}$; $L_5 = \overline{\mathbf{x}_0\mathbf{c}}$; $D^{ha} = A \backslash L_3$ and $D^{hc} = B \backslash L_3$, where A (B) is the closed parallelogram (triangle) with the vertices $\mathbf{x}_0$, $\mathbf{b}$, $\mathbf{v}_3$, and $\mathbf{a}$ ($\mathbf{x}_0$, $\mathbf{c}$, and $\mathbf{b}$).*

Meta-theorems 1 and 2 impose some (qualitative) restrictions on the employment shares of the agricultural and services sector (x_1 and x_3) but not on the employment share of the manufacturing sector (x_2). Thus, the set of all feasible path destinations (D^h) is relatively great. In other words, $\mathbf{x}^h$ is not determined uniquely but is an element of the set D^h, where the latter is determined by Meta-theorems 1 and 2 (cf. (7)) and the continuity and monotonicity constrains (3) and (4). In fact, Lemma 2a states that for an arbitrary path destination ($\mathbf{x}^h \in D^h$), we can always find a continuous and monotonous path that is characterized by a constant manufacturing employment share over some initial period [0, t^*) (and consistent with Meta-theorems 1 and 2). If, in contrast, we choose a path with an initially decreasing (cf. (9')) or an initially increasing (cf. (9'')) manufacturing share, then Lemmas 2b and 2c state that it is not possible to reach all feasible path destinations $\mathbf{x}^h$, while obeying the other restrictions listed in Lemma 2 and, in particular, the monotonicity requirement (4), which will be later of importance when discussing the cost-minimal paths.

Several further aspects of the proof of Lemma 2 are noteworthy:

First, there is a case (i.e., case (11)), where a path consistent with (2)-(9) is characterized by a constant manufacturing share over the whole period of consideration [0, h]. However, this is a knife-edge case where the manufacturing share at the path destination is equal to the initial manufacturing share (cf. (11)). In general, an optimal path (i.e., a path being consistent with (2)-(9)) consists of two path-segments, an *initial one*, where the manufacturing share x_2^* is constant, and a *final one*, where the manufacturing share is increasing or decreasing (while the agricultural share is decreasing or constant and the services share is increasing or constant according to Meta-theorems 1 and 2). In particular, while the first/initial path-segment is linear, the second/final path-segment need not being linear.

Second, the part of the proof of Lemma 2 that assumes that (10) is valid focuses on a linear final path-segment and, thus, represents a special case among the cases covered by (10). In particular, the movement along the *final* path-segment considered in the proof is characterized by a decreasing manufacturing share x_2, a growing services share x_3, and a constant agricultural share (cf. Figure 3 and APPENDIX B), while, in general, the agricultural share may be increasing along the (linear or non-linear) final path-segment in the cases covered by (10).

Third, analogous is true for the part of the proof of Lemma 2 that assumes that (12) is valid: The proof focuses on an extreme case of a linear final path-segment that is characterized by an increasing manufacturing share, a decreasing agricultural share, and a constant services share. Yet, in general, the services share may be increasing along a final path-segment in the cases covered by (12).

Fourth, in general, the length of the period [0, t^*), over which the system is on the initial path-segment (characterized by a constant manufacturing share), depends on the difference between the initial and destined employment shares. In case (10) (case (12)), the length of [0,

t^*) increases with the difference $x_{h1} - x_{01}$ ($x_{h3} - x_{03}$) ceteris paribus. In case (11), the length of [0, t^*) is equal to the length of the overall period of consideration [0, h].

Fifth, Lemma 2 and its proof refer to a finite planning horizon (h). The results and their proof are essentially the same if an infinite planning horizon is chosen, i.e., the period [0, ∞) is considered.

In the following sections, we discuss the implications and applications of Lemmas 1 and 2.

4. Cost-minimal policy consistent with Meta-theorems 1 and 2 under path-destination uncertainty

While Lemma 1 proves useful for the estimation of historical aggregate reallocation costs (cf. Section 5.1), it cannot be used without further ado for the discussion of the cost-minimal labor-reallocation policy in a nowadays developing country, since we face uncertainty regarding the path destination ($\mathbf{x}^h$) of such a country, as explained in Section 4.1. However, we show in Section 4.2 that a cost-minimal policy can be derived for a nowadays developing country, if we reduce the degree of path-destination uncertainty by assuming that Meta-theorems 1 and 2 are valid in future. In Sections 4.3 and 4.4, we briefly discuss the optimal duration of this cost-minimal policy and potential follow-up policies.

4.1 Path-destination uncertainty

As shown in Section 3.1, it is not difficult to determine the cost-minimizing labor-reallocation path if we know the path destination ($\mathbf{x}^h$), i.e., the labor allocation that is or should be realized at the end of the planning horizon (h). In particular, in this case, Lemma 1 and Definition 3 allow us to determine the qualitative restrictions that have to be imposed on the dynamics of the sectoral employment shares (see also Section 3.2). The path destination can be regarded as "known" in analyses of *historical* aggregate reallocation costs; in these analyses, the path destination is set to be the nowadays labor allocation, as demonstrated in Section 5.1.

Unfortunately, if we seek to derive the cost-minimal path/policy for a nowadays developing economy, we cannot regard the path destination as known. We do neither know the planning horizon (h) of the social planer nor the destination of a developing economy ($\mathbf{x}^h$); in particular, we do not know what the (optimal) labor allocation in a developed economy will be in, e.g., 20 years given all the thinkable and unthinkable exogenous determinants of the (optimal) labor allocation (in 20 years). Even Meta-theorems 1 and 2 do not allow us to precisely determine the path destination ($\mathbf{x}^h$); as shown in Section 3.2, Meta-theorems 1 and 2 allow for a set of potential path destinations (D^h). That is, in the derivation of the cost-minimal policy for a nowadays developing country, we face uncertainty regarding the path destination of the economy (even if we consider Meta-theorems 1 and 2). In Section 4.2, we show that a cost-minimal reallocation policy can be elaborated when there is path-destination uncertainty. For doing so, we use Lemmas 1 and 2.

4.2 Implications of Lemmas 1 and 2: Cost-minimizing development strategy (policy (a)-(c)) under path-destination uncertainty

For interpreting Lemmas 1 and 2, we assume that t = 0 denotes the present and, thus, h denotes a future time point and, in particular, the planning horizon (cf. Assumption 1b).

We can use Lemmas 1 and 2 to derive the optimal labor-reallocation policy consistent with Meta-theorems 1 and 2 as follows. *Lemma 1* states that monotonous development paths minimize the aggregate reallocation costs. *Lemma 2a* states that for any path destination $\mathbf{x}^h$ (cf. (6)) satisfying Meta-theorems 1 and 2 (cf. (7)), there exists a monotonous path (cf. (4)) that is characterized by a constant manufacturing employment share over some initial phase [0, t^*) (cf. (9)); moreover, Lemma 2a implies that this path is characterized by a monotonously growing (decreasing) services (agricultural) share (cf. (4), (7), and Definition 3). *Lemmas 2b and 2c* state that if the social planer does not choose a policy that ensures a constant manufacturing share over the initial development phase (cf. (9') and (9'')), then the economy may not be able to reach its destination along a monotonous path (cf. (4)).

Jointly, Lemmas 1 and 2 imply that an underdeveloped country not knowing the exact destination of its labor-reallocation path should choose the following **policy**: **(a)** decreasing agricultural share, **(b)** constant manufacturing share, and **(c)** increasing services share.

This policy, i.e., **policy (a)-(c)**, is consistent with the theoretical and empirical literature consensus on the path destination of a developing economy (cf. Meta-theorems 1 and 2) and minimizes the country's future aggregate reallocation costs (1).

4.3 On the optimal duration of policy (a)-(c)

Lemma 2 states that the reallocation policy (a)-(c) derived in Section 4.2 is only optimal over the initial phase of development, which is in our model denoted by the time interval [0, t^*). As discussed at the end of Section 3.2, the length of this depends on the differences between the initial and the destined agricultural, manufacturing, and services employment shares ($x_{h1} - x_{01}$, $x_{h2} - x_{02}$, and $x_{h3} - x_{03}$). Since, in general, these differences are relatively large in an underdeveloped yet developing country, it seems that policy (a)-(c) is optimal over a relatively long phase. For example, we demonstrate in Section 5.1 that policy (a)-(c) would have been optimal, i.e., reallocation costs-minimizing, over a period of ca. 170 years in the USA.

4.4 Optimal policies succeeding policy (a)-(c)

As discussed in Section 4.3, policy (a)-(c) may be optimal over a relatively long period, i.e., $[0, t^*)$ may be relatively long. The question arises, what the optimal policy after t^* is. The discussion at the end of Section 3.2 has shown that, in general, policy (a)-(c) must be followed by a de-industrialization ($dx_2^*(t)/dt < 0$) accompanied by a tertiarization ($dx_3^*(t)/dt > 0$) or an industrialization ($dx_2^*(t)/dt > 0$) accompanied by an agricultural decline ($dx_1^*(t)/dt < 0$) if we seek to minimize the aggregate reallocation costs. Thus, policy (a)-(c) does not only minimize the aggregate reallocation costs but also allows for a postponing of the industrialization/de-industrialization decision to a later phase of development, where additional information on the global environment may be available.

5. Applications of Lemmas 1 and 2 and policy (a)-(c)

5.1 Empirical application of Lemma 1 and policy (a)-(c)

Discussing and comparing the labor-reallocation paths and their costs across countries is a relatively extensive task and an interesting topic for further research. To demonstrate the direct and simple applicability of the cost index (1), Lemma 1, and their geometrical interpretation, we briefly discuss here the long-run data on labor reallocation in nowadays most developed countries (and Russia and China), which is depicted in Figure 4 (cf. Stijepic [2017b]). For an explanation of the construction of the empirical trajectories depicted in Figure 4, see Stijepic (2017b). We assume here that the reader is familiar with the interpretation of the trajectories on the standard 2-simplex, which has already been used in the proof of Lemma 2. For a detailed discussion of this interpretation (in the context of structural change modeling), see Stijepic (2015, 2017), and, for a brief summary, see APPENDIX B.

Figure 4 implies that (a) the countries' agricultural (services) shares decreased (increased) monotonously in the long run and (b) the dynamics of the manufacturing sector employment share are non-monotonous in all countries. The latter fact can be recognized immediately, since we can see that the initial segments of the trajectories point away from the simplex edge $\mathbf{v}_3$-$\mathbf{v}_1$, while the final trajectory segments point towards the simplex edge $\mathbf{v}_3$-$\mathbf{v}_1$. (China and Russia are exceptions, since only two data points are depicted in Figure 4 for each of them.

Thus, the monotonicity characteristics of the development paths of these countries cannot be studied by referring to Figure 4.) According to Definition 3, these facts imply that all the countries depicted in Figure 4 (except China and Russia) developed along non-monotonous labor-reallocation paths and, thus, their aggregate reallocation costs are not minimal according to Lemma 1.

Moreover, Figure 4 shows the following facts:

a) Germany, UK, and Netherlands have increased their manufacturing employment shares strongly in their early development phases and, thus, have developed the highest manufacturing shares in our country sample (cf. APPENDIX B).[2]

b) Between 1950 and 1992, China has followed the same strategy, i.e., it increased the manufacturing employment share strongly (while keeping the services share relatively low).

c) UK and Netherlands have significantly reduced their manufacturing employment shares again; thus, their labor-reallocation paths are strongly curved.

Our measure c^{0h} implies that, therefore, the aggregate reallocation costs in UK and Netherlands are relatively high in comparison to the other countries depicted in Figure 4. (See also the Proof of Lemma 1 in APPENDIX A, which implies that c^{0h} is relatively high if a strong increase in an employment share is followed by a strong decrease in it.) Whether Germany and China will face high overall reallocation costs depends on their future development (i.e., their future degrees of de-industrialization).

Figure 4. *Labor-reallocation trajectories for the USA, France, Germany, Netherlands, UK, Japan, China, and Russia.*

- insert Figure 4 here -

Notes. *Data source: Maddison (1995). The black dot represents the barycenter of the simplex. Abbreviations: C – China, F – France, G – Germany, J – Japan, N – Netherlands, R – Russia, US – United States, UK – United Kingdom. Data points (years in parentheses): USA (1820, 1870, 1913, 1950, 1992), France (1870, 1913, 1950, 1992), Germany (1870, 1913, 1950, 1992), Netherlands (1870, 1913, 1950, 1992), UK (1820, 1870, 1913, 1950, 1992), Japan (1913, 1950, 1992), China (1950, 1992), Russia (1950, 1992).*

Figure 4 can also be used to assess the optimal duration of policy (a)-(c) derived in Section 4.2. We choose the USA as an example. The USA accomplished their structural transformation from an agricultural to a services economy over a period of ca. 170 years, as

[2] The magnitude of the manufacturing employment share in Figure 4 is indicated by the closeness to vertex $\mathbf{v}_2$ (cf. APPENDIX B). As we can see, the trajectories of Germany and UK come very close to vertex $\mathbf{v}_2$.

illustrated by Figure 4, which depicts among others the US labor reallocation over the period 1820-1992. We can see there that it is possible to construct a linear line-segment (dashed line) that (a) is approximately parallel to the $\mathbf{v}_1\mathbf{v}_3$ edge of the simplex and (b) connects the initial point (representing 1820) and the last point (representing 1992) of the US trajectory.[3] According to our framework, this line-segment represents policy (a)-(c) (in the analysis of US structural change since 1820), i.e., a labor-reallocation path that is characterized by a constant manufacturing share (over the period 1820-1992). Thus, our results imply that in the case of the USA, policy (a)-(c) would have been optimal, i.e., reallocation costs-minimizing, over a period of ca. 170 years.

5.2 Application of Lemmas 1 and 2: Implications of policy (a)-(c) for the standard development strategies

Structural policy within the three-sector framework means fostering policies (e.g., choosing taxes, tariffs, subsidies, education system structure, infrastructure, research funding schemes, and legal entry barriers) that favor one sector over the others. The development literature provides different arguments for structural policy favoring one sector over the others. For an overview of such arguments see the manifold contributions (e.g., Harrison and Rodríguez-Clare [2010]) collected by Robinson (2009), Rodrik and Rosenzweig (2010), and Di Tommaso (2017). We will explain some of these arguments and the implications of our results in this context. We start with the arguments for agriculture.

The policy implications of the neoclassical growth and development literature, which are often summarized under the term ‘Washington Consensus’, favor a trade liberalization (see, e.g., Rodrik [2006]). In the context of north-south trade, where a (highly) underdeveloped country trades with more developed countries, trade liberalization implies that the underdeveloped country specializes in production and export of agricultural goods while importing manufactured goods because of comparative advantage (Ricardian argument) and resource constraints regarding, e.g., education required for manufacturing (Heckscher-Ohlin argument). Thus, according to these arguments (and the evidence on the trade structures of underdeveloped economies), an uncontrolled trade liberalization in (highly) underdeveloped countries is, de facto, a structural policy favoring the *agricultural sector*. Our results imply that the emphasis of the agricultural sector in the early stages of development is associated

[3] Note that Figure 4 depicts the development of the USA until 1992. Since 1992, the USA have come even closer to the simplex-edge $\mathbf{v}_1\mathbf{v}_3$ such that the line-segment connecting their nowadays location and their initial (i.e., 1820) location on the simplex is approximately parallel to the simplex-edge $\mathbf{v}_1\mathbf{v}_3$.

with high aggregate reallocation costs. It contradicts the policy aspect (a) derived in Section 4.2 ('decreasing agricultural share'). In particular, Meta-theorem 1 and Figure 4 imply that the agricultural share decreases over the development process (cf. Figure 4). That is, a policy favoring the agricultural share (e.g., the Washington Consensus strategy) contradicts the dynamic laws of structural change (Meta-theorem 1) and, thus, causes 'unnecessary' reallocation costs.

In the literature, we can identify two major lines of argument for protection/subsidizing of the *manufacturing sector*. The first line ('Prebisch-Singer thesis') argues that the terms-of-trade development is such that the agricultural goods exporting countries (the South) have disadvantages in comparison to the manufacturing goods exporting countries (the North) (see, e.g., Hadass and Williamson [2003]). The second line of argument ('Kaldorian strategy') emphasizes the importance of the manufacturing sector for the long-run growth of a country, since the manufacturing sector is a source of technological progress (see, e.g., Greenwald and Stiglitz [2006] and Stiglitz et al. [2013]). The emphasis of the manufacturing sector (in early stages of development) contradicts the policy aspect (b) derived in Section 4.2 ('constant manufacturing share'). Many nowadays highly developed countries experienced high costs of industrialization in their early phases of development (e.g., the costs associated with hasted urbanization as documented in the case of the USA in the 19th century and later) and went through severe phases of de-industrialization in later stages of development, which were characterized by unemployment, urban decline, and political/social instabilities (as in the case of UK). These crises can be avoided if an overshooting of the manufacturing sector is avoided and, in particular, the manufacturing share is kept approximately constant (over the early phases of development) as suggested by policy (a)-(c). However, our results do not prohibit a restructuring of the manufacturing sector towards more modern products and technologies, while keeping the employment share of the manufacturing sector constant. Thus, policy (a)-(c) is rather a policy of restructuring the manufacturing sector than a policy of increasing its share/size disproportionately.

Finally, we can find several arguments in favor of the *services sector* in the development literature. First, in the less developed countries that have some specific structural characteristics, a policy favoring the (modern) services sector may be effectively implementable and growth-enhancing while avoiding the negative effects of industrialization mentioned above. The major example for this argument is India, which is characterized by a relatively high share of highly educated English-speaking population that can be employed in IT branches (exporting IT services to the USA and UK). Second, the development of the financial (services) sector seems to be essential for generating economic growth (see, e.g., Demirgüç-Kunt and Levine [2004]). Third, the services sector seems to be less volatile in comparison to the manufacturing sector; thus, a greater services share implies lower volatility of the economy (see, e.g., Moro [2012]). The favoring of the services sector at the early stages of development, which implies a transformation from an agricultural to a services economy, is consistent with policy (a)-(c) derived in Section 4.2. Note, however, that there is one major argument against undifferentiated services sector emphasis, which has been pioneered by Baumol (1967) and Baumol et al. (1985): It seems that it is relatively difficult to generate innovation and productivity growth in the (personal) services sector (due to the personal nature of services, among others). Thus, economies dominated by services may face difficulties in generating significant growth (in the long run).

6. Concluding remarks

The growth and development process is characterized by massive labor reallocation, which does not only generate positive effects (e.g., income growth) but also high costs at the individual level (e.g., costs of geographical relocation, retraining, unemployment, and sunk human capital). In this paper, we focused on the reallocation costs in the three-sector framework, where we assumed that the aggregate reallocation costs depend on the strength of labor reallocation. We have elaborated on three aspects. First, we have shown that monotonous labor-reallocation paths minimize the aggregate reallocation costs in our framework and demonstrated the application of this result in the assessment of historical aggregate reallocation costs in nowadays highly developed countries. Second, we discussed the application of this result in structural policy (in less developed countries). In particular, we have shown that, when considering path-destination uncertainty and the standard labor-reallocation laws, our results imply that the cost-minimizing reallocation policy is characterized by a decreasing agricultural employment share, a constant manufacturing employment share, and a growing services employment share. Third, we have applied this theoretical result for evaluating the standard development strategies regarding the aggregate reallocation costs they generate. This application shows among others that the standard development strategies (in particular, the ‘Washington Consensus strategy’ and the ‘Kaldorian strategy’) generate relatively high aggregate reallocation costs and that, e.g., UK, Germany, and China have chosen labor-reallocation paths that are (potentially) associated with high aggregate reallocation costs. In contrast, India’s recent development strategy of emphasizing the role of the services sector seems to minimize the aggregate reallocation costs. Overall, our results imply that the strategy of manufacturing sector restructuring (towards more modern industries/branches) is preferable to the strategy of increasing the manufacturing's share in employment over the initial phases of development.

While we have focused on the mathematical derivations of the theorems and a brief demonstration of their applicability, future research could focus on more elaborate empirical studies of the topics raised in our paper. For example, countries could be grouped into groups with relatively high and relatively low aggregate reallocation costs and the properties of these groups (e.g., prevalence of crises, political regime, etc.) could be analyzed. Moreover, the importance of the aggregate reallocation costs in relation to the other (rather positive) effects of structural policies and labor reallocation for welfare and growth could be estimated.

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APPENDIX A (Proof of Lemma 1)

Lemma 1 refers to the solution of the following problem*: min* c^{0h}*, where* c^{0h} *is given by (1) and* $\mathbf{x}(0) = \mathbf{x}^0$ *and* $\mathbf{x}(h) = \mathbf{x}^h$*!* Since, among others, c^{0h} is monotonous in r^{0h}, we can rewrite this problem as follows:

(A1) *Min* r^{0h}*, where* $r^{0h} := \sum_{i=1}^{n} r_i^{0h}$, $r_i^{0h} := \int_0^h |dx_i(t)/dt|dt$*, and* $\mathbf{x}(0) = \mathbf{x}^0$ *and* $\mathbf{x}(h) = \mathbf{x}^h$*!*

First, we solve the following problem, which is simpler:

(A2) Min r_i^{0h}*, where* $r_i^{0h} := \int_0^h |dx_i(t)/dt|dt$ *and* $x_i(0) = x_{0i}$ *and* $x_i(h) = x_{hi}$ *are given.*

Note that $\forall t\ \forall i$, $x_i(t)$ must be continuous in t (see Assumption 1c and Lemma 1). First, assume that $x_{hi} > x_{0i}$. Problem (A2) is about finding the path $x_i^*(t)$, $0 \le t \le h$, that minimizes r_i^{0h}, where we must search among all the (continuous) paths that connect x_{0i} and x_{hi} on the real line (R), as depicted in Figure A1. Obviously, if $x_{hi} > x_{0i}$, a monotonously decreasing path ($\forall t\ dx_i(t)/dt \le 0$) cannot connect x_{0i} and x_{hi} (see Figure A2). Thus, we can formulate Property A1.

Property A1. *If* $x_{hi} > x_{0i}$*, only two classes of paths are admissible in the solution of problem (A2): (A) monotonously increasing paths (*$\forall t \; dx_i(t)/dt \geq 0$*) and (B) non-monotonous paths.*

Figure A1.

- insert Figure A1 here -

Figure A2.

- insert Figure A2 here -

First, consider class A. The geometrical interpretation of a monotonously increasing path (connecting x_{0i} and x_{hi}) on R is relatively straight forward: it is a path on the real line along which the economy moves from x_{0i} to x_{hi} monotonously, i.e., the movement (from x_{0i} to x_{hi}) is *unidirectional* (see Figure A3). The length of this path is equal to the length ($|x_{hi} - x_{0i}|$) of the real line-segment x_{0i}-x_{hi}.[4]

Figure A3.

- insert Figure A3 here -

In contrast, a non-monotonous path (class B) is characterized by at least one *change in direction*. A non-monotonous path (on R) is associated with at least one point in time $t_1 \in [0, h]$ at which the economy does not move towards x_{hi} but away from x_{hi}, i.e., there is a 'backward step' or an 'overshooting step' (see Figures A4 and A5 for illustrative examples). Furthermore, we know that the economy must turn towards x_{hi} again at some later point in time $t_2 \in (t_1, h)$, since the economy must arrive at x_{hi} at time h. Obviously, such a path (i.e., a path with at least one change in direction) is longer than a monotonous path: the length of the path with a 'backward/overshooting step' is equal to the length of the monotonous path ($|x_{hi} - x_{0i}|$) plus two times the length of the 'backward/overshooting step' (cf. Figures A4 and A5). Overall, these facts imply the following statement:

[4] Recall that the (Euclidean) length of an interval (or line-segment) on the real line is given by the absolute value of the difference between its endpoints. Most introductory books on analysis discuss this fact. For a discussion of the length of paths in two-dimensional space, where the (Euclidean) length of the path is measured by a quadratic formula, see, e.g., Gelfand and Fomin (1963). In one-dimensional space this quadratic formula becomes the absolute value function that we use.

Property A2. *If* $x_{hi} > x_{0i}$*, the length of a non-monotonous path connecting* x_{0i} *and* x_{hi} *on the real line is greater than the length of a monotonous path connecting* x_{0i} *and* x_{hi} *on the real line.*

Figure A4.

- insert Figure A4 here -

Figure A5.

- insert Figure A5 here -

The proof of the following two properties is analogous to the proof of Properties A1 and A2. Thus, we omit it here.

Property A3. *If* $x_{hi} < x_{0i}$*, only two classes of paths are admissible in the solution of problem (A2): (I) monotonously decreasing paths (*$\forall t\ dx_i(t)/dt \leq 0$*) and (II) non-monotonous paths.*

Property A4. *If* $x_{hi} < x_{0i}$*, the length of a non-monotonous path connecting* x_{0i} *and* x_{hi} *on the real line is greater than the length of a monotonous path connecting* x_{0i} *and* x_{hi} *on the real line.*

Now, we show that the length of a path is equal to the r_i^{0h} associated with this path. Let t_1, t_2, …, t_z denote the points of time at which the economy changes its direction on the real line (see Figure A6). A direction change at time $t_B \in (0, h)$ is given if there exists a $t_A \in (0, h)$ and a $t_C \in (0, h)$ such that **either** $(\forall t \in (t_A, t_B]\ dx_i(t)/dt \geq 0) \wedge (\exists t \in (t_A, t_B]\ dx_i(t)/dt > 0) \wedge (\forall t \in (t_B, t_C]\ dx_i(t)/dt < 0)$ **or** $(\forall t \in (t_A, t_B]\ dx_i(t)/dt \leq 0) \wedge (\exists t \in (t_A, t_B]\ dx_i(t)/dt < 0) \wedge (\forall t \in (t_B, t_C]\ dx_i(t)/dt > 0)$. These facts imply (A3).

(A3) $r_i^{0h} := \int_0^h |dx_i(t)/dt|dt = \int_0^{t_1} |dx_i(t)/dt|dt + \int_{t_1}^{t_2} |dx_i(t)/dt|dt + \ldots + \int_{t_z}^{h} |dx_i(t)/dt|dt.$

Since there are no changes in direction within the intervals $[0, t_1]$, $(t_1, t_2]$, $(t_2, t_3]$, …, $(t_z, h]$ per definition of the $t_1, t_2, \ldots, t_z$, $x_i(t)$ is monotonous within these intervals and we can rewrite (A3) as follows:

(A4) $r_i^{0h} = |\int_0^{t_1} dx_i(t)/dtdt| + |\int_{t_1}^{t_2} dx_i(t)/dtdt| + \ldots + |\int_{t_z}^{h} dx_i(t)/dtdt| = |x_i(t_1) - x_i(0)| + |x_i(t_2) - x_i(t_1)| + \ldots + |x_i(h) - x_i(t_z)|$

In fact, our definition of the points t_1, t_2, …, t_z implies a partitioning of the path (connecting x_{0i} and x_{hi}) into sections/partitions of monotonous dynamics (see Figure A6). (A4) implies that r_i^{0h} is equal to the sum of the lengths of the partitions of monotonous dynamics on R (see Figure A6 for an example). This is consistent with the natural/standard definition of path length (in R) used in Properties A2 and A4. Thus, we can state the following property:

Property A5. *r_i^{0h} is equal to the length of the path connecting x_{0i} and x_{hi} on the real line (R).*

Figure A6.

- insert Figure A6 here -

Obviously, if $x_{hi} = x_{0i}$, r_i^{0h} is minimized if the economy stays in x_{0i} for all t, i.e., $\forall t\ dx_i(t)/dt = 0$, which corresponds per Definition 3 to a monotonous path. Thus:

Property A6. *If $x_{hi} = x_{0i}$, the solution of the problem (A2) is given by a monotonous path ($\forall t\ dx_i(t)/dt = 0$) on R. In this case, the minimal r_i^{0h} is equal to 0.*

Furthermore, if $x_{hi} \neq x_{0i}$, all monotonous paths connecting x_{0i} and x_{hi} on R have the same length and, thus, the same value of r_i^{0h}, since if the path is monotonous we can write

(A5) $\quad r_i^{0h} := \int_0^h |dx_i(t)/dt|dt = |\int_0^h dx_i(t)/dtdt| = |x_{hi} - x_{0i}| > 0.$

Property A7. *Any monotonous path connecting x_{0i} and x_{hi} on R is characterized by $r_i^{0h} = |x_{hi} - x_{0i}|$.*

Properties A1-A7 imply the following lemma:

Lemma A1. *The solution of problem (A2) is given by a monotonous path. In particular, any monotonous path connecting x_{0i} and x_{hi} on R is associated with minimal r_i^{0h}. If $x_{hi} \neq x_{0i}$, the minimal r_i^{0h} is equal to $|x_{hi} - x_{0i}| > 0$. If $x_{hi} = x_{0i}$, the minimal r_i^{0h} is equal to 0. Here, the path connecting x_{0i} and x_{hi} on R is monotonous if either $\forall t \in [0, h]\ dx_i(t)/dt \geq 0$ or $\forall t \in [0, h]\ x_i(t)/dt \leq 0$.*

Now, we can turn to the solution of problem (A1). Since the functions $x_i(t)$ are independent of each other (cf. Definition 1), r_i^{0h} are independent of each other (cf. (A1)). Furthermore, as implied by (A1), $\forall i\ r_i^{0h} \geq 0$. Thus, the cost-index $r^{0h} = r_1^{0h} + r_2^{0h} + \ldots + r_n^{0h}$ is separable. That is, minimizing r^{0h} is equivalent to minimizing $r_i^{0h}\ \forall i$. The minimum of r^{0h} is attained if and only if all r_i^{0h} are minimal. This fact and Lemma A1 imply that r^{0h} is minimal if and only if all the functions $x_i(t)$ are monotonous. In other words, r^{0h} is minimal if and only if there does not exist any $x_i(t)$ that is non-monotonous. Exactly speaking, r^{0h} is minimal if and only if

(A6) $\nexists i \in \{1, 2, \ldots, n\}: (\exists t_a \in [0, h]\ \exists t_b \in [0, h]: t_a \neq t_b \wedge dx_i(t_a)/dt < 0 \wedge dx_i(t_b)/dt > 0)$.

(A6) corresponds to the definition of a monotonous development path (see Definition 3). Finally, note that c^{0h} is monotonously increasing in r^{0h}. These facts prove Lemma 1.

APPENDIX B (Geometrical properties of the standard 2-simplex)

It is a well-known fact (see, e.g., Stijepic (2015, 2017)) that the standard 2-simplex $D \equiv \{(x_1, x_2, x_3) \in R^3: \forall i \in \{1, 2, 3\}\ x_i \geq 0 \wedge x_1 + x_2 + x_3 = 1\}$ is a subset of a plane in R^3 and, in particular, a triangle with the following Cartesian coordinates of its vertices: (1, 0, 0) =: $\mathbf{v}_1$, (0, 1, 0) =: $\mathbf{v}_2$, and (0, 0, 1) =: $\mathbf{v}_3$ (see Figure B1). In general, we depict D without the coordinate system (cf. Figures 2-4).

Figure B1. *The standard 2-simplex (D) in the Cartesian coordinate system.*

- insert Figure B1 here -

Labor-reallocation paths, e.g., the path P^* discussed in the Proof of Lemma 2, are simply directed curves on D (cf. Figures 2-4). As stated in the Proof of Lemma 2. It makes sense to define the tangential vector angles of such curves as follows: *Let* $\mathbf{x}$ *be a (regular) point (of a path/curve) on D and* $\mathbf{r}(\mathbf{x})$ *be a (tangential) vector indicating the direction of movement at the point* $\mathbf{x}$. *The vector angle* $\delta(\mathbf{x})$ *is the angle between* $\mathbf{r}(\mathbf{x})$ *and the simplex-edge* $\mathbf{v}_1\mathbf{v}_2$, *i.e.,* $\delta(\mathbf{x}) := \angle(\mathbf{r}(\mathbf{x}), \overline{\mathbf{v}_1\mathbf{v}_2})$. The positioning of D in R^3 (see Figure B1) implies the following interpretation of tangential vectors associated with regular points ($\mathbf{x}$) of a trajectory/path on D (cf. Figure B2):

Property B1. *a) If and only if* $\delta(\mathbf{x}) = 0°$, *the movement indicated by vector* $\mathbf{r}(\mathbf{x})$ *is characterized by a decrease in* x_1, *an increase in* x_2, *and a constant* x_3.

b) If and only if $0 < \delta(\mathbf{x}) < 60°$, *the movement indicated by vector* $\mathbf{r}(\mathbf{x})$ *is characterized by a decrease in* x_1*, an increase in* x_2*, and an increase in* x_3*.*

c) If and only if $\delta(\mathbf{x}) = 60°$, *the movement indicated by vector* $\mathbf{r}(\mathbf{x})$ *is characterized by a decrease in* x_1*, a constant* x_2*, and an increase in* x_3*.*

d) If and only if $60° < \delta(\mathbf{x}) < 120°$, *the movement indicated by vector* $\mathbf{r}(\mathbf{x})$ *is characterized by a decrease in* x_1*, a decrease in* x_2*, and an increase in* x_3*.*

e) If and only if $\delta(\mathbf{x}) = 120°$, *the movement indicated by vector* $\mathbf{r}(\mathbf{x})$ *is characterized by a constant* x_1*, a decrease in* x_2*, and an increase in* x_3*.*

f) If and only if $\delta(\mathbf{x}) > 120°$, *the movement indicated by vector* $\mathbf{r}(\mathbf{x})$ *is characterized by an increase in* x_1 *or a decrease in* x_3*.*

Figure B2. *Examples of vectors characterized by Property B1.*

- insert Figure B2 here -

In our paper, we use Property B1 to characterize paths as follows. The path P assigns to each $t \in [0, h]$ an $\mathbf{x}(t)$ (cf. Definition 2). We can assign to each $\mathbf{x}(t)$ a directional vector $\mathbf{r}(\mathbf{x}(t))$ indicating the direction of movement along the path P at the point $\mathbf{x}(t)$. (In case of differentiable functions, i.e., if $\mathbf{x}(t)$ is differentiable with respect to t, $\mathbf{r}(\mathbf{x}(t))$ can be interpreted as the tangential (or directional) vector at point $\mathbf{x}(t)$ of the curve $\mathbf{x}(t)$, $t \in [0, h]$, associated with the path P.) Moreover, we can measure the vector angle $\delta(\mathbf{x}(t))$ and identify the changes in x_1, x_2, and x_3 at the point $\mathbf{x}(t)$ by using Property B1, i.e., we can identify the signs of $dx_1(t)/dt$, $dx_2(t)/dt$, and $dx_3(t)/dt$ at each point of P (if $\mathbf{x}(t)$ is differentiable). This approach for interpreting paths on D is the same, irrespective of whether we refer to P (as in this example), or to P^* (as in the Proof of Lemma 2), or to the empirical trajectories discussed in Section 5.1. For a detailed discussion and numerous examples, see Stijepic (2015, 2017).

Figure 1

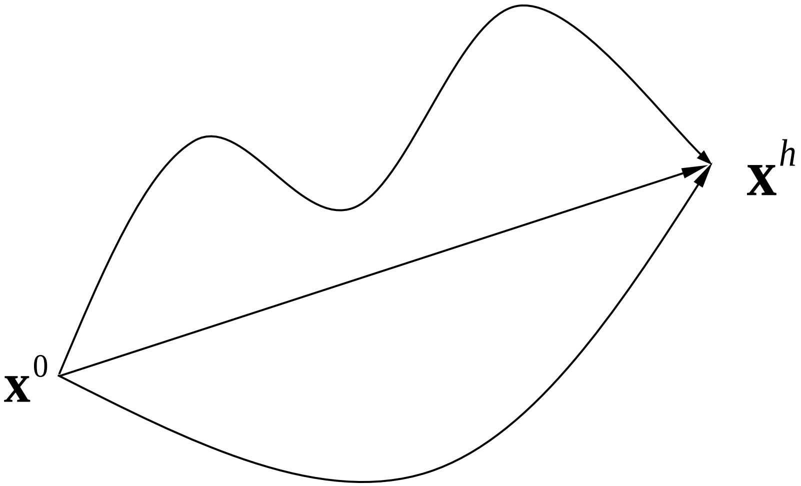
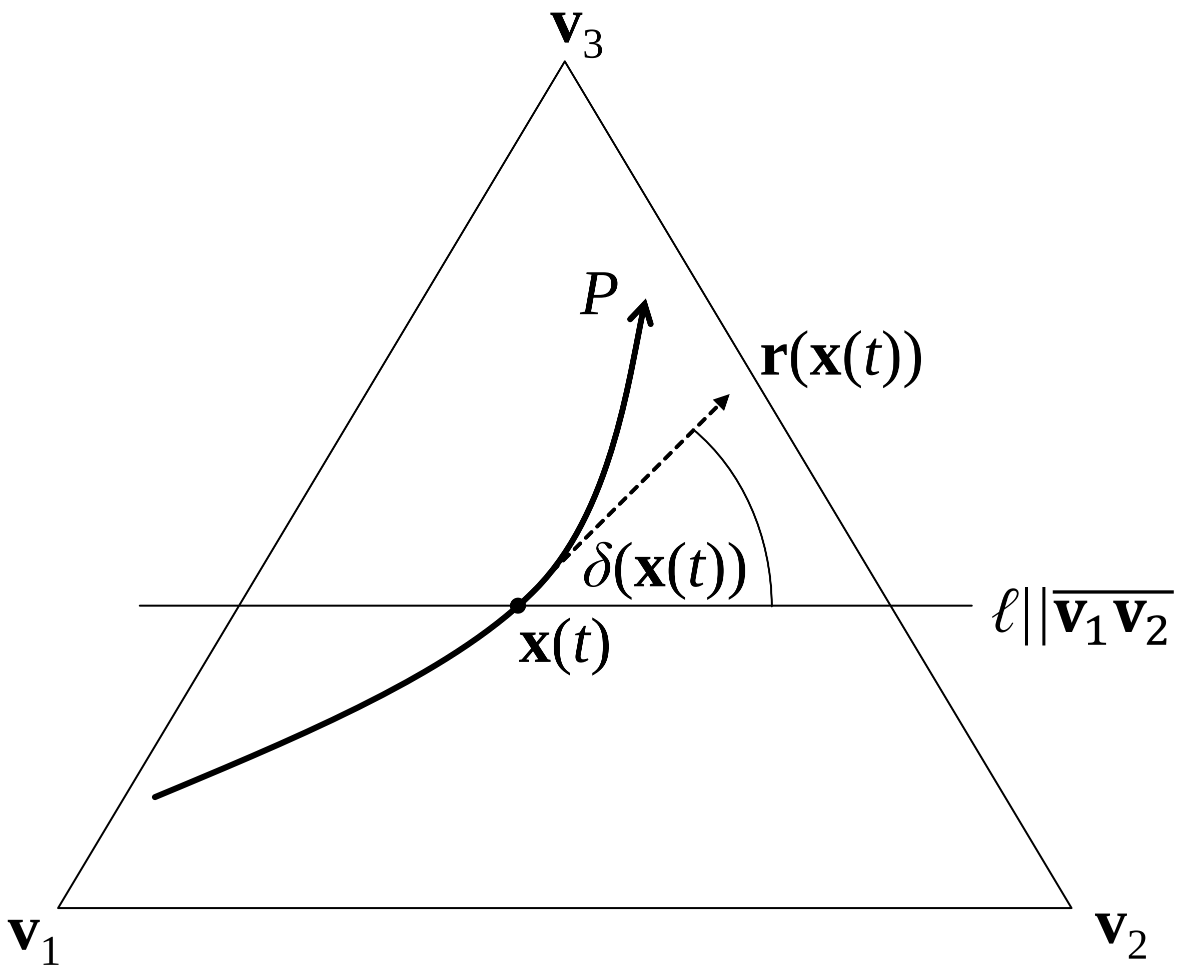


Figure 2



$\mathbf{v}_3$
P
$\mathbf{r}(\mathbf{x}(t))$
$\delta(\mathbf{x}(t))$
$\ell || \overline{\mathbf{v}_1\mathbf{v}_2}$
$\mathbf{x}(t)$
$\mathbf{v}_1$
$\mathbf{v}_2$

Figure 3

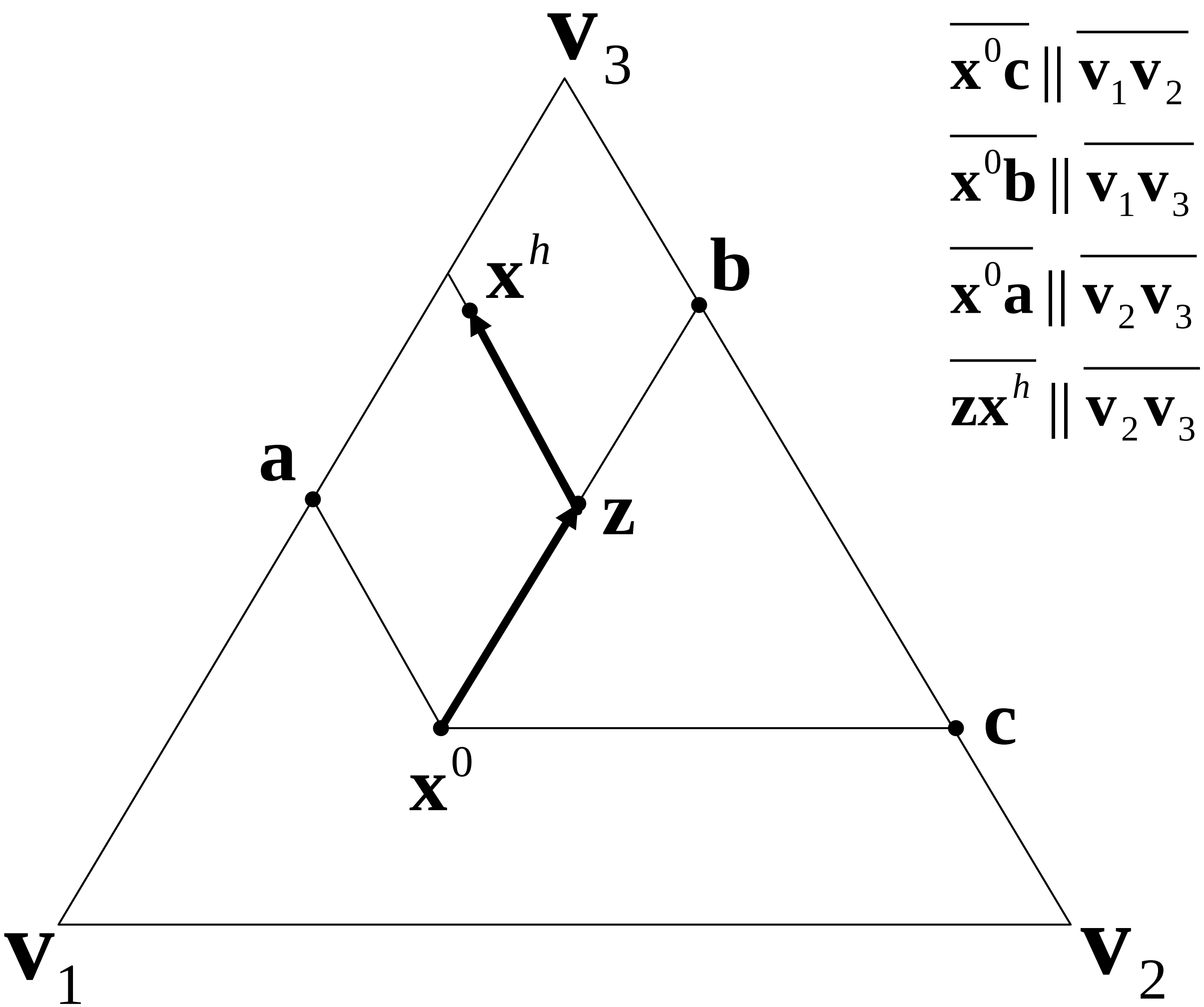


Figure 4

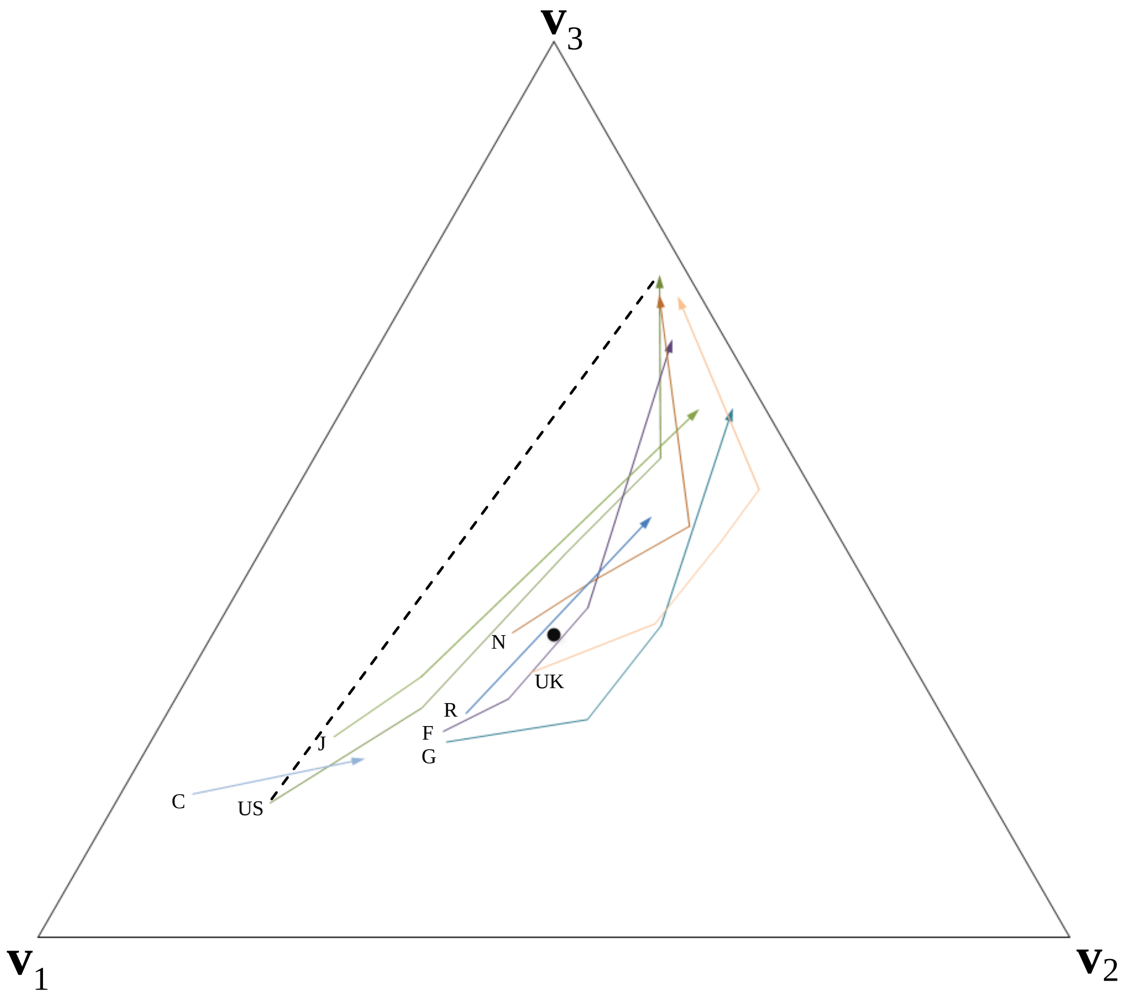


Figure A1

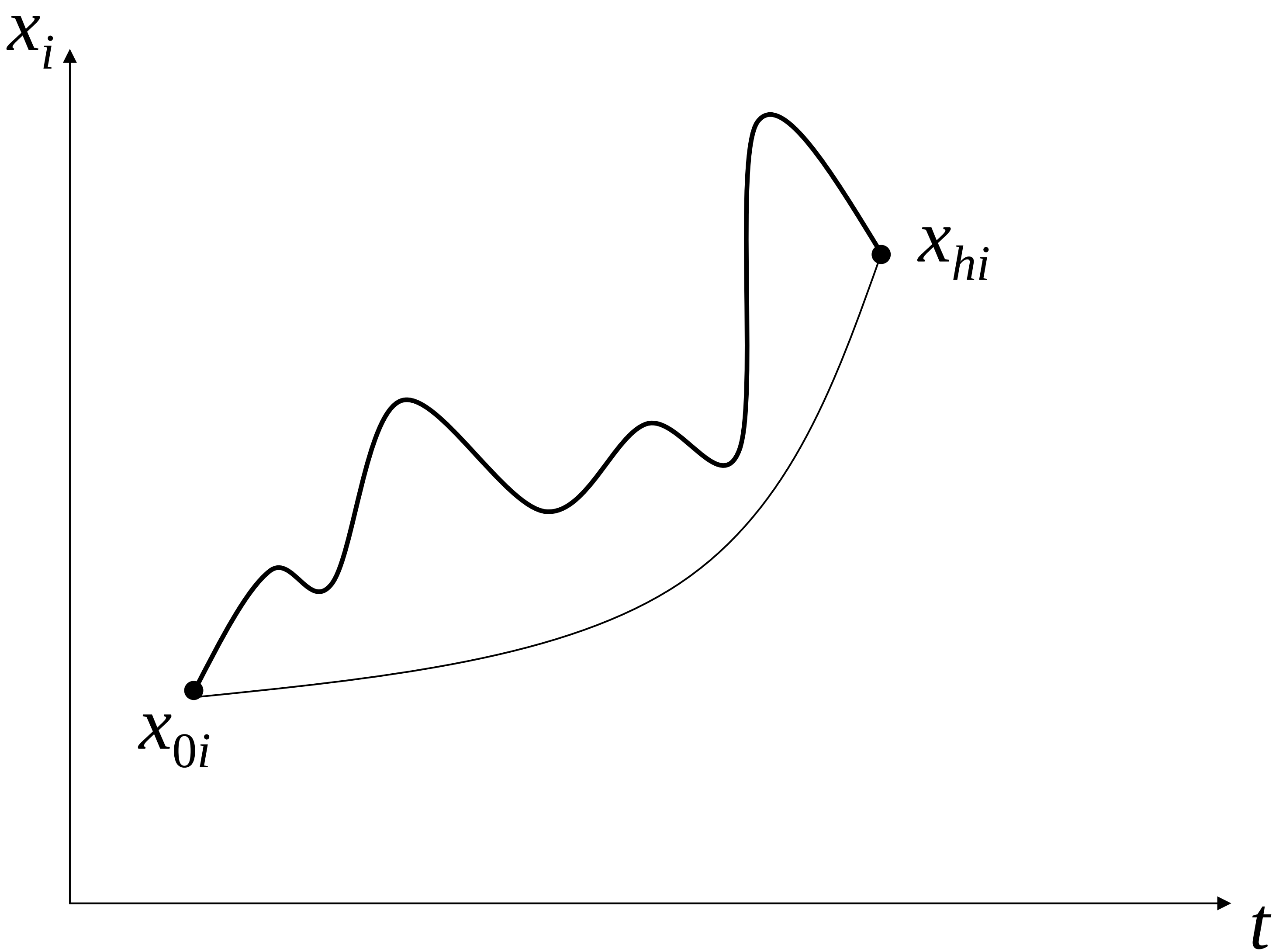


Figure A2

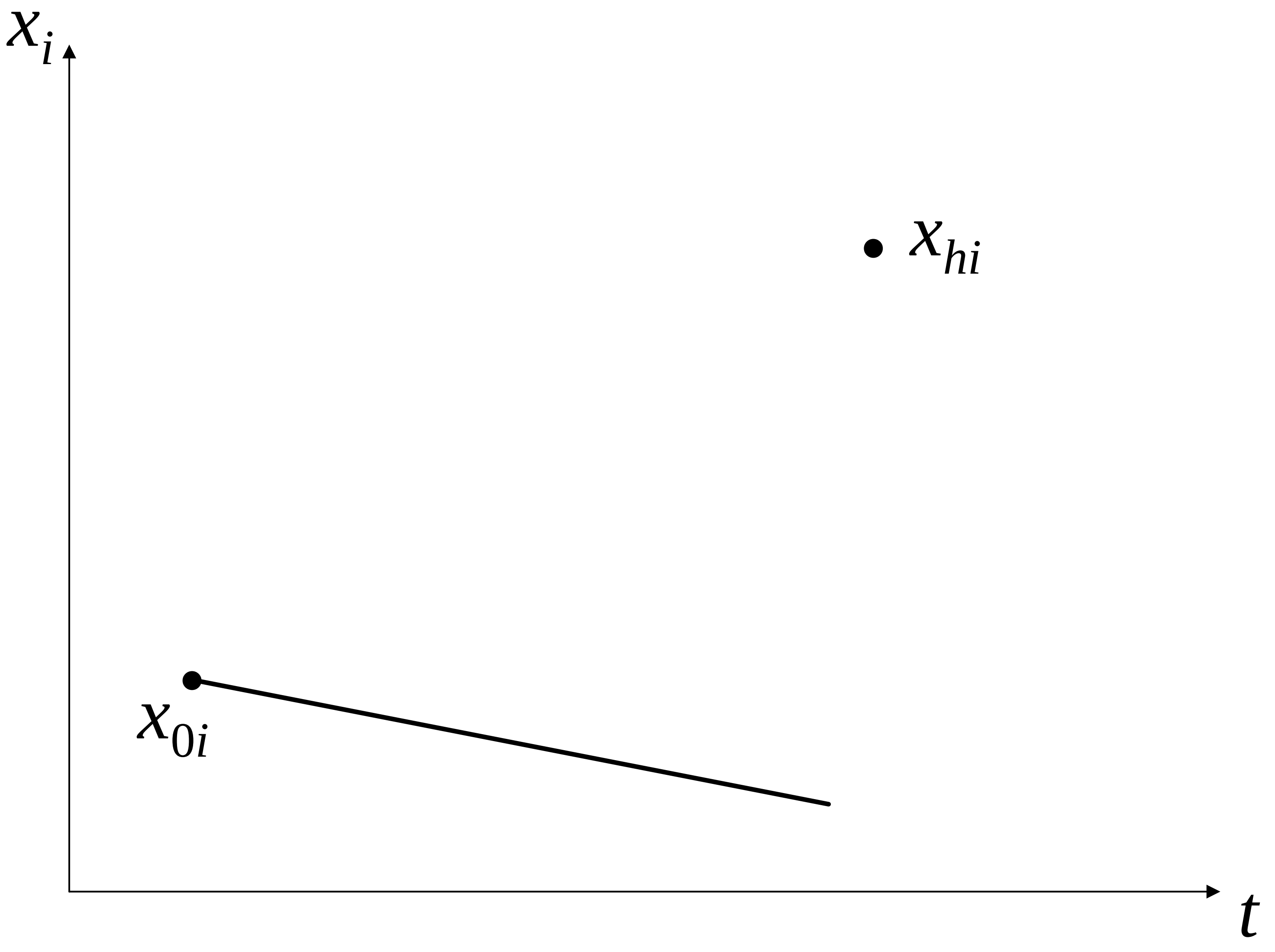


Figure A3

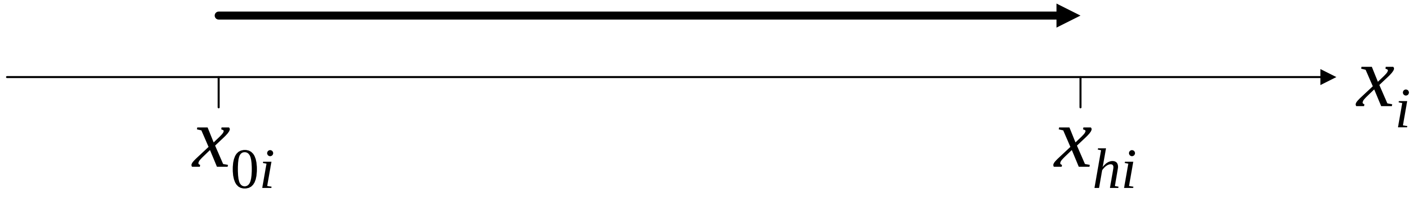


Figure A4

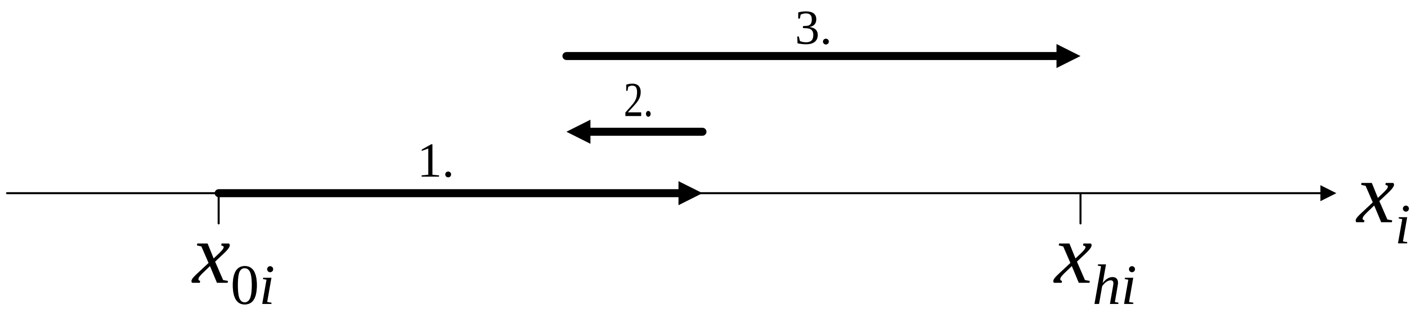


Figure A5

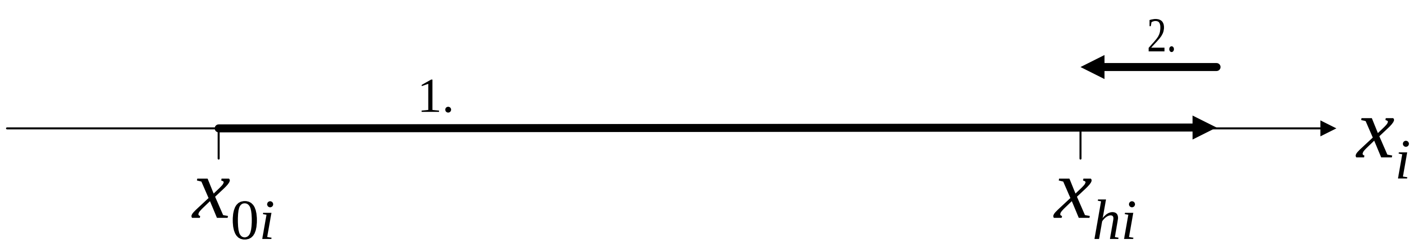


Figure A6

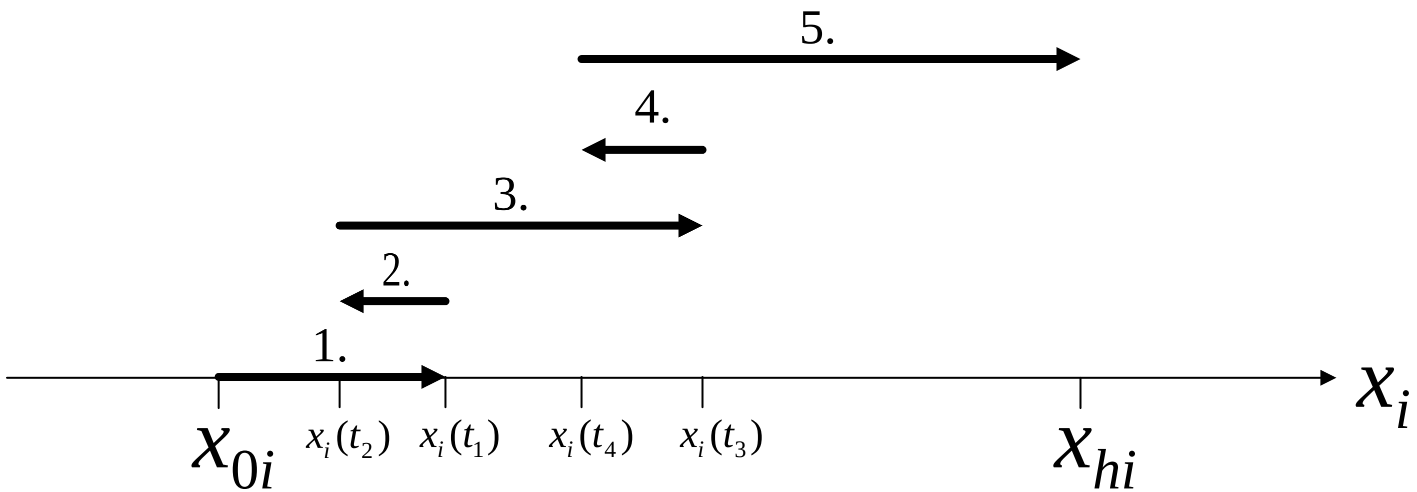


Figure B1

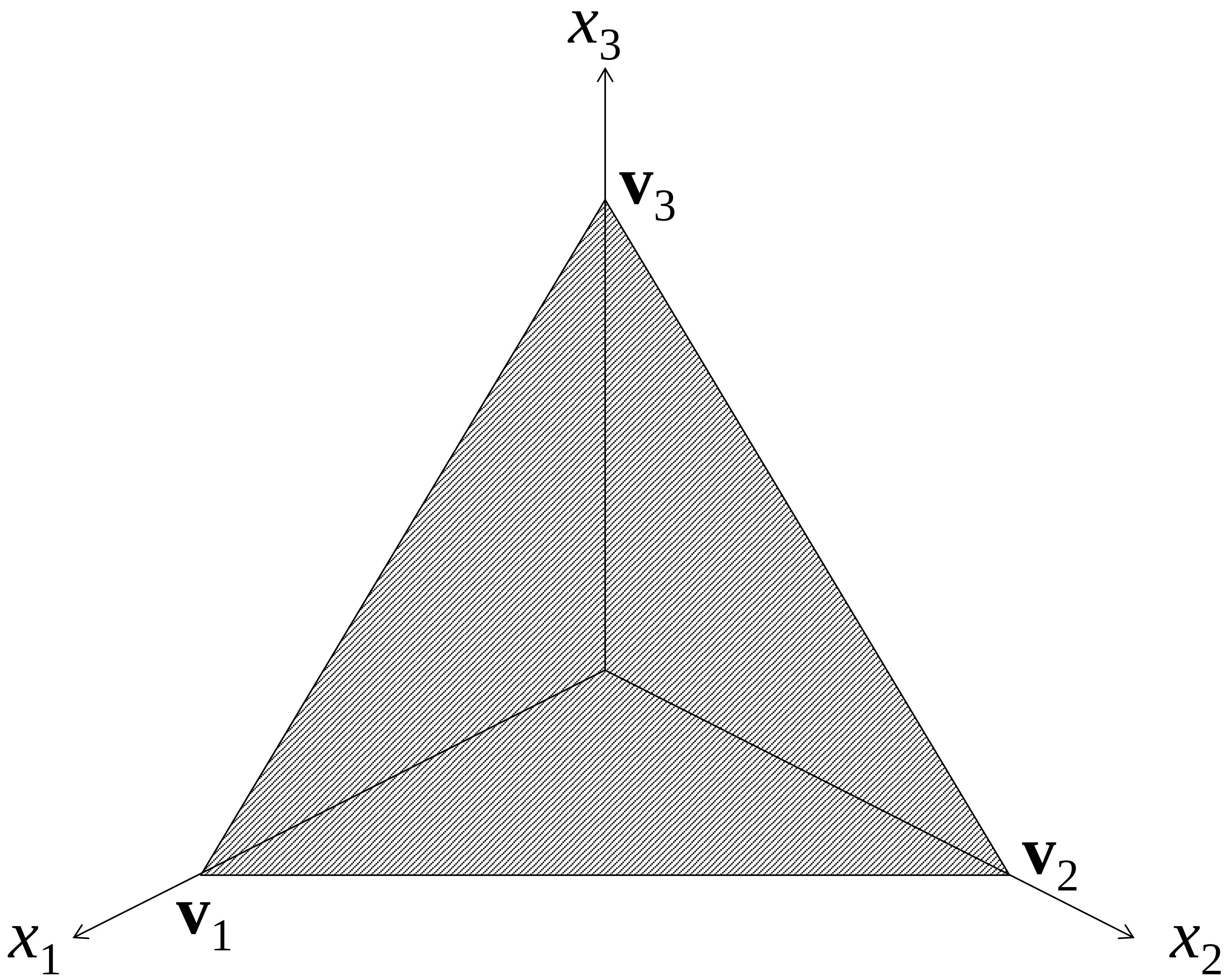
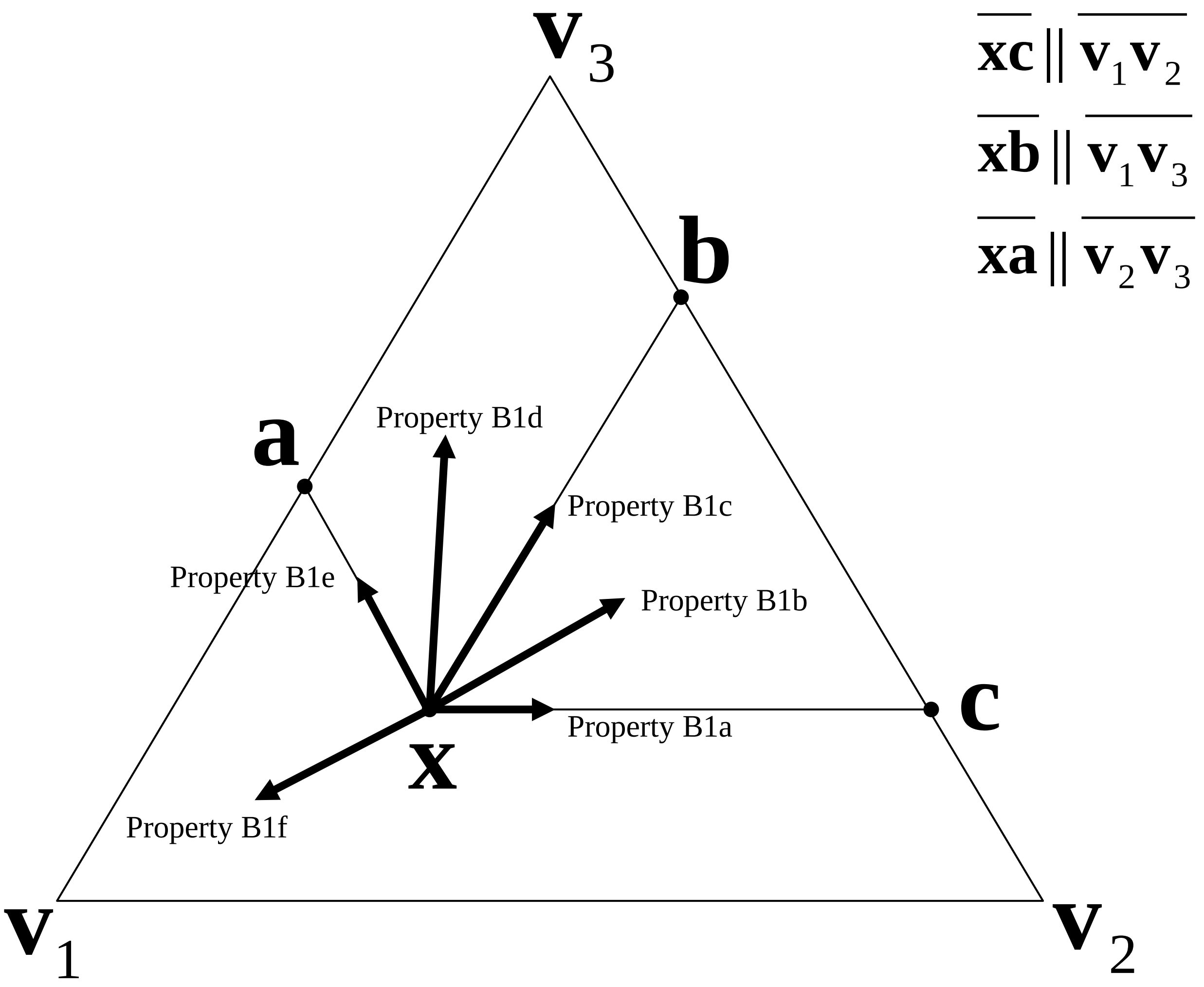


Figure B2



$\overline{\mathbf{xc}} \parallel \overline{\mathbf{v}_1\mathbf{v}_2}$
$\overline{\mathbf{xb}} \parallel \overline{\mathbf{v}_1\mathbf{v}_3}$
$\overline{\mathbf{xa}} \parallel \overline{\mathbf{v}_2\mathbf{v}_3}$
$\mathbf{v}_3$
b
a
Property B1d
Property B1c
Property B1e
Property B1b
c
Property B1a
x
Property B1f
$\mathbf{v}_1$
$\mathbf{v}_2$