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Abstract

A new method for pricing contingent claims, which is particularly well suited for options with complex barrier and volatility structures, is introduced. The approach is based on a high precision approximation of the *Feynman-Kac*-equation with distributed approximating functionals (DAFs). The method under consideration is most elegant from a computational point of view, and it is shown to be faster and more accurate than conventional solution schemes.

Keywords: Backward-Operator; *Feynman-Kac*-Equation; Distributed Approximating Functionals.

JEL Class.: C61, G13
MSC Code: 60G, 65C, 65M

1. Introduction

Since the CBOE started trading standardized call options in 1973, derivatives evolved into a fundamental constituent of modern financial markets. Their role as most sophisticated risk transfer instruments is reflected in the enormous variety of standard and exotic contract types. Today most options are priced numerically, because the *Black-Scholes*-equation does not provide analytical solutions when more realistic models for the dynamics of the underlying are chosen. This concerns primarily the *Black-Scholes*-assumption of constant volatilities, which is falsified with overwhelming empirical proof. Thus, it is advisable to use models with stochastic or deterministic volatility term structure, especially when pricing *vega*-sensitive contracts[1]. A certain class of options, considered particularly volatility sensitive by practitioners, are barrier options.

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[1]An excellent treatment of volatility term structure models is provided in Gatheral (2006).

The trading volume of barrier options steadily increased over the last twenty years and they are barely considered exotics nowadays. Their popularity is partly due to their weak path dependency (cf. Wilmott, 2006, chap. 22 & 23), resulting in a low dimensional partial differential equation problem with boundary conditions. Even vanilla barrier options come in many flavors, making them interesting contracts for a variety of hedging- or risk transfer requirements. Most basic types can be valuated analytically in the *Black-Scholes*-framework (see for example Merton, 1973; Reiner and Rubinstein, 1991a; Haug, 2007, chap. 4.17–20). For complex barrier structures and more sophisticated models for the underlying, the corresponding partial differential equation problem has to be solved numerically. Standard numerical methods are usually preferred over Monte Carlo simulation, because the partial differential equation is low dimensional and thus numerical schemes are more efficient regarding computational resources.

There are generally two types of numerical schemes available, tree based schemes and finite differences. Tree structures (Cox et al., 1979; Boyle, 1986) often suffer from poor accuracy because the barrier is not necessarily located on the nodes. Refinements have been suggested by Kamrad and Ritchken (1991), Derman et al. (1995) and Ahn et al. (1999). Alternatively, finite difference schemes are available; especially the unconditionally stable scheme of Crank and Nicolson (1996) is the working horse of modern derivative pricing. Recently, finite element approaches, permitting variably dense polygon meshing, have gained some popularity. For example Zhu and de Hoog (2010) suggested a fully coupled scheme, based on the *Galerkin* method.

In this article a new method is introduced, based on so called distributed approximating functionals (DAFs). The DAF-formalism was originally applied to quantum mechanical wave propagation problems as numerical solution method for the time-dependent *Schrödinger*-equation (Hoffman et al., 1991; Hoffman and Kouri, 1992). Later it was used to solve the *Fokker-Planck*-equation (Wei et al., 1997; Zhang et al., 1997a,b). Interpolating properties of DAFs are explored in Hoffman et al. (1998). Recently, distributed approximating functionals were successfully applied to nonlinear filtering problems (Mazzoni, 2011).

The remainder of the manuscript is organized as follows: Section 2 introduces the DAF-formalism and illustrates the choice of an optimal bandwidth. In section 3 the DAF-method is applied to the *Feynman-Kac*-equation. It is also shown how to incorporate different boundary conditions. Section 4 provides a plain vanilla example and analyzes numerical properties of the DAF-approximation. In section 5 several test scenarios are analyzed, including problems without explicit solutions and complex barrier and volatility term structures. The results are benchmarked against the classical *Crank-Nicolson*-method. Section 6 summarizes the findings, including a discussion of pros and cons of the DAF-method.

2. Distributed Approximating Functionals

This section provides a brief introduction to the distributed approximating functional formalism. A more rigorous treatment on this subject can be found in Hoffman et al. (1991); Hoffman and Kouri (1992) and Zhang et al. (1997a,b).

2.1. Definition and Properties of DAFs

A distributed approximating functional or DAF is characterized as approximate mapping of a particular subset of continuous functions in the *Hilbert*-space L^2 to itself (Zhang et al., 1997a,b). Consider the definition of *Dirac*s δ-function

$$f(x) = \int \delta(x - x')f(x')dx'. \tag{1}$$

A particular class of DAF-functions, the *Hermite*-DAFs, can be used to approximate the δ-function in a very convenient way

$$f(x) \approx \int \delta_M(x - x'; h)f(x')dx', \tag{2a}$$

with

$$\delta_M(x; h) = \frac{1}{h}\phi\left(\frac{x}{h}\right)\sum_{m=0}^{M/2}\frac{1}{m!}\left(\frac{-1}{4}\right)^m H_{2m}\left(\frac{x}{\sqrt{2}h}\right). \tag{2b}$$

In (2b), $\phi(x)$ denotes the standard normal probability density function and $H_m(x)$ is the m-th *Hermite*-polynomial, orthogonal with respect to the weight function e^{-x^2}. Notice that only even *Hermite*-polynomials are used because the δ-function is symmetric in its argument. Furthermore, M is the highest degree polynomial involved in the construction of the DAF and h is its bandwidth. Both parameters control the accuracy of the approximation. By fixing one or the other, one obtains

$$\lim_{M\to\infty}\int \delta_M(x - x'; h)f(x')dx' = \lim_{h\to 0}\int \delta_M(x - x'; h)f(x')dx' = f(x), \tag{3}$$

which is an alternative way of defining the δ-function as limit of a sequence of functions (Lighthill, 1966, chap. 2.2).

The DAF mapping can be used to sample an arbitrary function at discrete points. If these nodes form an equispaced grid, equation (2a) can be approximated by

$$f(x) \approx \Delta x \sum_{j=1}^{N} \delta_M(x - x_j; h) f(x_j), \tag{4}$$

with $\Delta x = x_j - x_{j-1}$. Equation (4) is a DAF-based interpolation formula (cf. Hoffman et al., 1998). But there is another important implication of the DAF-formalism. Consider the definition of the l-th derivative of *Dirac*s δ-function

$$f^{(l)}(x) = \int \delta^{(l)}(x - x') f(x') dx'. \tag{5}$$

Usually, this is a purely formal expression because the derivative of the δ-function is not defined and the operation has to be rolled over to the test function $f(x)$ by partial integration. If the *Hermite*-DAF approximation is used, the derivative can be evaluated immediately. Considering the usual relations between *Hermite*-polynomials and their derivatives (see e.g. Abramowitz and Stegun, 1970, p. 783), one obtains the differentiating *Hermite*-DAF

$$\delta_M^{(l)}(x; h) = \frac{(-1)^l}{2^{l/2} h^{l+1}} \phi\left(\frac{x}{h}\right) \sum_{m=0}^{M/2} \frac{1}{m!} \left(\frac{-1}{4}\right)^m H_{2m+l}\left(\frac{x}{\sqrt{2}h}\right). \tag{6}$$

Now the derivative (5) can be approximated by

$$f^{(l)}(x) \approx \Delta x \sum_{j=1}^{N} \delta_M^{(l)}(x - x_j; h) f(x_j). \tag{7}$$

Thus, the operation of differentiation has turned into an algebraic operation. Furthermore, the derivative is approximated at the same level of approximation as the function itself. This property of *Hermite*-DAFs is referred to as 'well tempered'.

By discretizing the left hand side of (7) on the same spatial grid, one obtains

$$f^{(l)}(x_i) \approx \Delta x \sum_{j=1}^{N} \delta_M^{(l)}(x_i - x_j; h) f(x_j). \tag{8}$$

Obviously (8) can be written most conveniently in matrix/vector form, $f^{(l)} = Lf$, by identifying the components of the operator matrix $L(x_i, x_j) =$

$\Delta x \delta_M^{(l)}(x_i - x_j; h)$. Thus, an arbitrary differential operator of the form

$$L(x) = f(x)\frac{\partial}{\partial x} + g(x)\frac{\partial^2}{\partial x^2} \tag{9a}$$

has the *Hermite*-DAF matrix representation

$$L(x_i, x_j) = \Delta x f(x_i)\delta_M^{(1)}(x_i - x_j; h) + \Delta x g(x_i)\delta_M^{(2)}(x_i - x_j; h), \tag{9b}$$

which is exactly the structure of the *Kolmogoroff*-backward-operator. Notice that the operator matrices (9b) can be computed most efficiently, because the differentiating DAF matrices have *Toeplitz*-structure.

2.2. Choosing the Optimal Bandwidth

The limit relation (3) implies a connection between the number of expansion terms M and the bandwidth h of the *Hermite*-DAF. It is natural to think of the expansion order M as a measure of pre-defined accuracy so one might ask how to choose h optimally with respect to a given M. One possible approach is to make the approximation as accurate as possible on the discrete grid. For $l = 0$ equation (8) is exact, if

$$\delta_M(x_i - x_j; h) = \frac{1}{\Delta x}\delta_{ij} \tag{10}$$

holds, with the *Kronecker*-δ on the right-hand side of (10). To the first order this implies $\delta_M(0, h) = 1/\Delta x$ and one obtains from the definition of the *Hermite*-DAF (2b)

$$h = \frac{\Delta x}{\sqrt{2\pi}} \sum_{m=0}^{M/2} \frac{1}{m!}\left(\frac{-1}{4}\right)^m H_{2m}(0). \tag{11}$$

In the remainder of the article the bandwidth is chosen according to (11). Therefore, the dependence on h is suppressed to simplify notation. Figure 1 illustrates

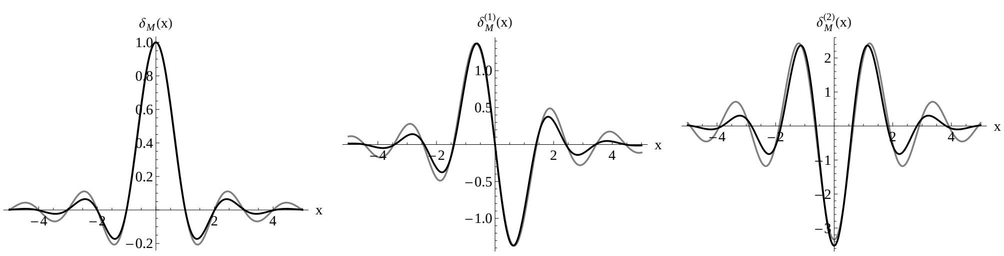

Figure 1: Differentiating *Hermite*-DAFs with $l = 0, 1, 2$ (Left, Center, Right) and $\Delta x = 1$ – Degrees of Approximation are $M = 20$ (Black) and $M = 100$ (Gray)

the (differentiating) *Hermite*-DAFs for different degrees of approximation. Notice that $\delta_M(x)$ is approximately zero at all integer multiples of Δx, except at $x = 0$, where it is one, as required.

3. Backward-Operator and Time Evolution

This section details the approximation of the *Feynman-Kac*-equation by using the DAF-formalism. Furthermore, it is shown how the appropriate boundary conditions can be integrated into the DAF-representation of the corresponding differential operator.

3.1. Backward Time Evolution

Assume that the stock price S_t is a P-measurable random variable on the probability space $(\Omega, \mathcal{F}, P)$, and that a natural filtration $\mathcal{F}_0 \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ is induced by the time-evolution of S_t, with all null sets contained in $\mathcal{F}_0$. Further, assume that the dynamics of S_t are governed most generally by the *Itô*-process

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t, \tag{12}$$

where dW_t is the increment of the *Wiener*-process, and all regularity conditions are fulfilled as required. Now, consider the *Feynman-Kac*-picture of the option pricing problem

$$V(S,t) = E^Q\left[e^{-\int_t^T r(S,t')dt'}V(S,T)\middle|\mathcal{F}_t\right], \tag{13}$$

where the expectation is to be taken with respect to the risk-neutral probability measure Q, conditioned on the information set $\mathcal{F}_t$, available at time t. This can be converted into

$$V(S,\tau) = \int_{-\infty}^{\infty}\left(\overleftarrow{T}e^{\int_0^\tau L(S',\tau')d\tau'}V(S',0)\right)\delta(S-S')dS', \tag{14a}$$

where the backward-operator is given by

$$L(S,\tau) = q(S,\tau)\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2(S,\tau)\frac{\partial^2}{\partial S^2} - r(S,\tau), \tag{14b}$$

with time to maturity $\tau = T - t$, and risk neutral drift $q(S,\tau)$. The *Dyson*-operator $\overleftarrow{T}$ enforces the correct time order of operations, if the exponential is expanded into a power series. Equation (14a) is usually of little practical use, but considering its DAF-representation, this changes dramatically.

Assume for the moment that the backward-operator is constant with respect to time, then one obtains

$$V(S,\tau) = \int_{-\infty}^{\infty} \left(e^{L(S')\tau} V(S',0) \right) \delta(S-S') dS'. \tag{15}$$

Equation (15) is very appealing with respect to the DAF-approximation in two ways. First, the DAF-formalism transforms the differential operator $L(S)$ into a matrix, which means, the terms in brackets translate to a matrix/vector multiplication involving a matrix exponential. This is a much simpler problem than the original abstract operator exponential. Second, the integral involving *Dirac*s δ-function can be approximated by a DAF-functional, resulting in an interpolation between the grid nodes, maintaining the same order of accuracy as the approximation of the differential operator.

Define an equally spaced grid $S_1, \ldots, S_N$, with $\Delta S = S_i - S_{i-1}$. From the terms in brackets in (15) and the DAF-formalism of section 2 one obtains

$$V(S_i,\tau) = \sum_{j=1}^{N} e^{L(S_i,S_j)\tau} V(S_j,0), \tag{16a}$$

with

$$\begin{aligned} L(S_i,S_j) = \Delta S \bigg(& q(S_i)\delta_M^{(1)}(S_i - S_j) \\ & + \frac{1}{2}\sigma^2(S_i)\delta_M^{(2)}(S_i - S_j) - r(S_i)\delta_M(S_i - S_j) \bigg). \end{aligned} \tag{16b}$$

For all practical purposes it can be assumed that the $(N \times N)$ operator matrix L is not defective, which means that it has linearly independent eigenvectors. Then the eigenvalue decomposition of L is

$$e^{L\tau} = P e^{\Lambda\tau} P^{-1}, \tag{17}$$

where P^{-1} is the inverse matrix of P, and Λ is the diagonal matrix of eigenvalues λ_i of L. Thus, $\left(e^{\Lambda\tau}\right)_{ij} = e^{\lambda_i\tau}\delta_{ij}$ holds, which can be computed easily. In case of a defective matrix L see for example Moler and van Loan (2003) for alternative ways to calculate the matrix exponential. Summarizing these results, equation (15) can be approximated by

$$V(S,\tau) \approx \Delta S \sum_{i=1}^{N} \sum_{j=1}^{N} e^{L(S_i,S_j)\tau} V(S_j,0) \delta_M(S - S_i), \tag{18}$$

where the matrix exponential is computed according to (17). Notice that the operator matrix and its eigenvalue decomposition has to be computed only once to cover the complete time interval $[0, \tau]$.

3.2. Integrating Boundary Conditions

For every approximation scheme, operating on a finite discrete grid, it is necessary to impose conditions on the boundaries. In this sense barrier options are optimal candidates for numerical schemes because their payoff structure includes natural boundaries. This is not necessarily true for more complex contracts like partial barrier options with intermittent barriers.

Whatever the boundary conditions are, they have to be absorbed into the operator matrix $L(S_i, S_j)$. This can be done by injecting rows from another operator matrix $L_B(S_i, S_j)$, which governs the dynamics on and beyond a specific boundary. These dynamics are much simpler than those in the regular region of the problem, because they are usually restricted by *Dirichlet*- or *Neumann*-conditions, rendering the original PDE-problem to an ODE-problem on the boundary. One result of this simplification is that the operator matrix L_B is at least diagonal and in most cases time-independent, which means that it is commutative.

Let $V(\tau)$ be the vector containing all values $V(S_i, \tau)$ at nodes $i = 1, \ldots, N$. Then for very small $\Delta\tau$

$$\begin{aligned} V(\tau + \Delta\tau) &\approx \exp\left[L_B(\tau)\Delta\tau + \int_0^\tau L_B(s)ds\right] V(0) \\ &= e^{L_B(\tau)\Delta\tau} V(\tau) \\ &\approx (I + L_B(\tau)\Delta\tau)V(\tau) \end{aligned} \tag{19}$$

holds on and beyond the boundary, with the $(N \times N)$ identity matrix I. From this expression the boundary operator can be constructed for a variety of boundary conditions. To illustrate the procedure, lets assume that the risk neutral *Itô*-diffusion is given by a geometrical *Brown*ian motion

$$dS_t = rS_t dt + \sigma S_t dW_t. \tag{20}$$

Then the DAF-Representation of the backward-operator is a matrix with entries

$$L(S_i, S_j) = \Delta S \left(rS_i \delta_M^{(1)}(S_i - S_j) + \frac{1}{2}\sigma^2 S_i^2 \delta_M^{(2)}(S_i - S_j) - r\delta_M(S_i - S_j) \right). \tag{21}$$

First consider the case of an upper knockout barrier S_u when no rebate is granted. The necessary condition on and beyond the boundary is

$$0 = V(\tau + \Delta\tau) \approx (I + L_B \Delta\tau) V(\tau), \tag{22}$$

with $V(\tau) = 0$. It follows immediately that $L_B(S_i, S_j) = 0$ is a trivial solution. If a rebate R is granted immediately when the barrier is hit, the boundary operator is also zero everywhere, only the initial value along the boundary is R.

If the rebate is payed at maturity, in case the underlying has hit the barrier during its lifetime, then the (componentwise) relation

$$V(S_i, \tau + \Delta\tau) = e^{-r(\tau+\Delta\tau)} R = e^{-r\Delta\tau} V(S_i, \tau) \approx (1 - r\Delta\tau) V(S_i, \tau) \tag{23}$$

has to hold. Obviously, the boundary operator is $L_B(S_i, S_j) = -r\delta_{ij}$ in this case.

Now consider the case where no barrier is present. For example, a plain vanilla call option with strike K has an approximate value of $V(S, \tau) = S - e^{-r\tau}K$ for $S \gg K$. By *Taylor*-expanding $e^{-r(\tau+\Delta\tau)} = e^{-r\tau}(1 - r\Delta\tau + \ldots)$ and neglecting terms of order $(r\Delta\tau)^2$, it follows that

$$V(S, \tau + \Delta\tau) \approx V(S, \tau) + re^{-r\tau} K \Delta\tau = \left(1 + \frac{re^{-r\tau}K}{S - e^{-r\tau}K} \Delta\tau\right) V(S, \tau) \tag{24}$$

has to hold. Equation (24) entails the inconvenient result that the elements of the operator matrix are no longer time-independent. However, *Taylor*-expanding the bracket around $\tau = 0$ and collecting terms by orders of $r\Delta\tau$ yields

$$1 + \frac{re^{-r\tau}K}{S - e^{-r\tau}K} \Delta\tau = 1 + \frac{rK}{S - K} \Delta\tau + \mathcal{O}(r^2 \Delta\tau). \tag{25}$$

Because r can be assumed small, terms of orders higher than $r\Delta\tau$ can be neglected without significant loss of accuracy, and thus the boundary operator is approximately

$$L_B(S_i, S_j) \approx \frac{rK}{S_i - K} \delta_{ij}. \tag{26}$$

It turns out that (26) is also the correct boundary operator for a deep in-the-money vanilla put option.

In general, the operator matrix L assembles rows of the regular operator inside the real or artificially imposed boundaries, and rows of the boundary operator on and beyond the boundary. Due to the interpolating properties of the *Hermite*-DAFs, it is generally not sufficient to impose the boundary condition only on the first or last row of the operator matrix, as illustrated in the next section. This is a slight disadvantage over conventional methods like *Crank*-

Nicolson.

4. Numerical Analysis

In this section a plain vanilla call option is valuated to demonstrate the procedure. Because an analytical vanilla call price is available, accuracy and performance of the method can be analyzed easily. Furthermore, because no barrier is present, conditions on and outside an artificially imposed boundary can be investigated.

4.1. Plain Vanilla Call Option

Because the artificially imposed boundary conditions for a plain vanilla option are the most delicate ones, it is appropriate to check the properties of the DAF-approximation under these conditions first. The test scenario is a European plain vanilla call option with exercise price $K = 100$, time to maturity $\tau = 100$ days, a fixed interest rate of 5% p.a. and a daily volatility of $\sigma = 2\%$. The model for the underlying is a geometrical *Brown*ian motion, which implies time-independence of the backward-operator. The discrete grid is chosen to cover the range $S_{\min} = 30$ to $S_{\max} = 180$ with different spacings ΔS. The resulting operator matrix is given by (21), with the last ten rows replaced by the corresponding rows of the boundary operator (26).

Figure 2 illustrates the results for the particular choice $\Delta S = 1$ and $M = 30$. The DAF-approximation literally coincides with the analytical solution (figure 2 left). In the artificial boundary region, the DAF-approximation can no longer be trusted unconditionally, even though it holds well at the discrete nodes (figure 2 right). This is due to the interpolating properties of the *Hermite*-DAFs as

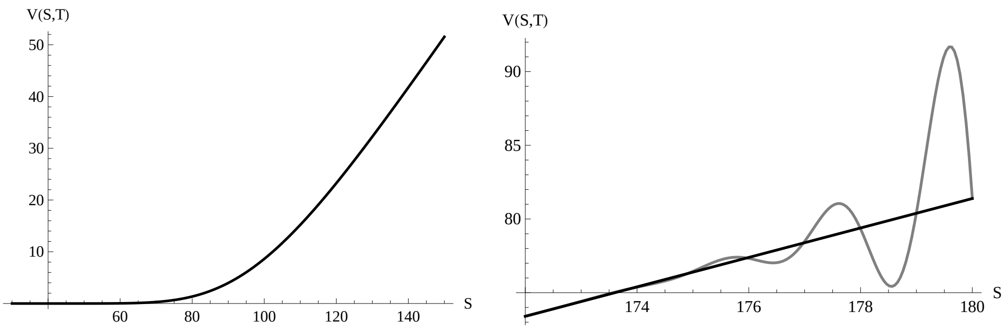

Figure 2: DAF-Approximation of a European Plain Vanilla Call Option with Exercise Price $K = 100$ (Left) and Illustration of Loss of Accuracy in the Boundary Region (Right) – Numerical Settings: $M = 30$ and $\Delta S = 1$

DAF-Approximation Performance						
	$M = 20$		$M = 30$		$M = 50$	
ΔS	Timing	Rel. Error	Timing	Rel. Error	Timing	Rel. Error
5	0.0087 sec.	0.8621 %	0.0145 sec.	0.8609 %	0.0318 sec.	0.8650 %
2	0.0250 sec.	0.1078 %	0.0396 sec.	0.1073 %	0.0830 sec.	0.1096 %
1	0.0608 sec.	0.0278 %	0.0894 sec.	0.0269 %	0.1716 sec.	0.0263 %
0.5	0.2044 sec.	0.0170 %	0.2621 sec.	0.0090 %	0.4181 sec.	0.0076 %
0.2	1.8814 sec.	0.0771 %	1.9001 sec.	0.0040 %	2.1528 sec.	0.0025 %
0.1	12.574 sec.	0.3167 %	12.402 sec.	0.0032 %	12.418 sec.	0.0017 %

Table 1: Numerical Results for Different Grid Spacings ΔS and Expansion Orders M

suggested in the last paragraph of the previous section.

Table 1 reports computation times and average relative errors for different combinations of grid spacing ΔS and expansion order M. The execution time includes the calculation of the operator matrix and its eigenvalue decomposition. Owing to the time-independence of the backward-operator, this calculation has to be conducted only once to provide solutions for arbitrary tenors. Approximation errors are reported as average absolute relative errors, evaluated at the discrete nodes,

$$\text{Rel. Error} = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{V(S_i,\tau) - V_{BS}(S_i,\tau)}{V_{BS}(S_i,\tau)}\right|, \tag{27}$$

with $V_{BS}(S_i,\tau)$ indicating the *Black-Scholes*-value of the call option at S_i. To avoid numerical problems due to very small option prices, only nodes with $V(S_i,\tau) \geq 0.05$ are included in the average.

Obviously, the method is extremely accurate, even if the grid resolution is rather coarse. A moderate choice of parameters, e.g. $\Delta S = 1$ and $M = 30$, already results in a negligible error and an execution time below 0.1 seconds. All computations are conducted on an usual personal computer, equipped with a 6-core AMD Phenom II X6 1090T processor, running at 3.6 GHz, and 4 Gb RAM.

5. Barrier Option Test Scenarios

In this section several test problems are analyzed, including barrier options with complex barrier structures. Barrier options are particularly well suited objects because of their predetermined finite value along the barrier. Results are compared with conventional numerical schemes and analytical solutions, as far as available.

5.1. Cash-or-Nothing Option

The first example is a European binary cash-or-nothing knockout call, with exercise price $K = 5$ and upper barrier $S_u = 10$. This particular contract is valued for different times to maturity. The annualized volatility is set to $\sigma = 0.2$ and the risk free interest rate is $r = 0.1$. This kind of barrier option is particularly well suited for conventional numerical schemes like *Crank-Nicolson*, because boundary conditions can be incorporated very efficiently. On the other hand binary options have discontinuous payoffs and thus numerical schemes require a fine-grained discrete grid in order to maintain accuracy. An analytical solution to this problem is available, see Reiner and Rubinstein (1991b), and also Haug (2007, pp. 176).

Table 2 summarizes the results of the benchmark. Because the *Crank-Nicolson*-scheme is globally accurate to the order $\mathcal{O}(\Delta t^2, \Delta S^2)$, a balanced space and time discretization $\Delta = \Delta t = \Delta S$ is used. The relative error is calculated according to (27) and option prices smaller than 0.05 are excluded again to avoid numerical problems. Obviously, the DAF-method is always more accurate than

Cash-or-Nothing Barrier Option					
		Computation Time		Average Relative Error	
Δ	Maturity	Crank-Nicolson	DAF	Crank-Nicolson	DAF
0.1	2 years	0.0078 sec.	0.0530 sec.	0.8712 %	0.8178 %
0.05	2 years	0.0312 sec.	0.1373 sec.	0.4315 %	0.4081 %
0.02	2 years	0.6048 sec.	0.6864 sec.	0.1737 %	0.1635 %
0.01	2 years	8.7828 sec.	3.8376 sec.	0.0877 %	0.0817 %
		Computation Time		Average Relative Error	
Δ	Maturity	Crank-Nicolson	DAF	Crank-Nicolson	DAF
0.1	5 years	0.0109 sec.	0.0530 sec.	1.0447 %	0.8563 %
0.05	5 years	0.0608 sec.	0.1404 sec.	0.5241 %	0.4230 %
0.02	5 years	1.3104 sec.	0.6864 sec.	0.2095 %	0.1719 %
0.01	5 years	19.048 sec.	3.8532 sec.	0.1049 %	0.0860 %
		Computation Time		Average Relative Error	
Δ	Maturity	Crank-Nicolson	DAF	Crank-Nicolson	DAF
0.1	10 years	0.0171 sec.	0.0546 sec.	1.3648 %	1.1380 %
0.05	10 years	0.1201 sec.	0.1373 sec.	0.6768 %	0.5670 %
0.02	10 years	2.5896 sec.	0.6864 sec.	0.2702 %	0.2270 %
0.01	10 years	36.239 sec.	3.8828 sec.	0.1351 %	0.1135 %

Table 2: Numerical Results of Cash-or-Nothing Option Valuation for Different Grid Spacings Δ

Crank-Nicolson, although it is slightly more time consuming at coarse-grained grid resolutions. Nevertheless, in high precision or long lifetime situations the DAF-method is clearly preferable from the computational point of view. The approximation order was chosen $M = 30$ uniformly.

5.2. American Binary Knock-out Option

The American binary knock-out call of Hui (1996) is a one-touch double barrier binary option that pays a predetermined rebate R immediately when the lower (upper) barrier is hit, and knocks out if the upper (lower) barrier is crossed. If the option does not hit any barrier during its lifetime, it expires worthlessly. The option is also known as double-barrier binary asymmetrical call option (Haug, 2007, pp. 181), and can be valued with the help of *Fourier*-series expansion. If S_u and S_l labels the upper and lower barrier, respectively, then the fair value of the call option is

$$V(S,\tau) = R\left(\frac{S}{S_l}\right)^{\alpha}\left[\sum_{k=1}^{\infty}\frac{2}{k\pi}\left(\frac{\beta - \left(\frac{k\pi}{Z}\right)^2 e^{-\frac{1}{2}\left[\left(\frac{k\pi}{Z}\right)^2-\beta\right]\sigma^2\tau}}{\left(\frac{k\pi}{Z}\right)^2-\beta}\right) \times \sin\left[\frac{k\pi}{Z}\log[S/S_l]\right] + \left(1-\frac{\log[S/S_l]}{Z}\right)\right], \tag{28a}$$

where

$$Z = \log[S_u/S_l]\,, \quad \alpha = \frac{1}{2} - \frac{r}{\sigma^2} \quad \text{and} \quad \beta = -\alpha^2 - \frac{2r}{\sigma^2}. \tag{28b}$$

For short times to maturity, the value of the American knock-out call is extremely nonlinear and hence the following analysis focuses on the precision of the numerical approximation.

In this scenario the barriers are chosen $S_u = 120$ and $S_l = 80$. The annual risk free interest rate is $r = 0.1$ and the volatility is $\sigma = 0.2$. The rebate is set to $R = 1$ and the remaining lifetime of the option is three months, $\tau = 0.25$. The analytical reference price is calculated according to (28a) and (28b), with the first 100 terms of the *Fourier*-series evaluated. In order to provide sufficient accuracy within the barrier region, the DAF-operator matrix includes 10 rows of the boundary operator at and beyond S_u, respectively. The DAF-approximation order is chosen $M = 30$.

Table 3 shows the results of the analysis for varying choices of ΔS and Δt. Even though the conventional *Crank-Nicolson*-scheme is roughly ten times faster than the DAF-method, it does not achieve its accuracy. The reported average relative errors are again calculated according to (27), with reference to the approximated analytical solution (28a) and (28b). The superior timing of

Double-Barrier Binary Asymmetrical Option					
		Computation Time		Average Relative Error	
ΔS	Δt	Crank-Nicolson	DAF	Crank-Nicolson	DAF
1	0.1	0.0020 sec.	0.0234 sec.	6.3572 %	0.9499 %
1	0.05	0.0020 sec.	0.0234 sec.	1.1399 %	0.9499 %
0.5	0.05	0.0042 sec.	0.0468 sec.	1.9181 %	0.4358 %
0.5	0.02	0.0045 sec.	0.0468 sec.	0.8418 %	0.4358 %
0.2	0.02	0.0123 sec.	0.1482 sec.	1.3001 %	0.1674 %
0.2	0.01	0.0133 sec.	0.1482 sec.	0.1960 %	0.1674 %
0.1	0.01	0.0343 sec.	0.4649 sec.	0.3637 %	0.0829 %

Table 3: Numerical Results of Double-Barrier Binary Asymmetrical Option Valuation for Different Grid Spacings ΔS and Δt

the *Crank-Nicolson*-scheme in this scenario is due to several favorable conditions. Owing to the short residual time to maturity, only a limited number of iterations has to be conducted. Furthermore, there is no volatility- or interest rate term structure involved. Thus, the corresponding matrices are time independent and need not to be recomputed with every time step. The DAF-approach can support such term structures without recomputation of the operator matrix, as will be shown in the next example.

5.3. Partial Reverse Barrier Option with Volatility Term Structure

This scenario investigates a more complex setup in order to emphasize the full potential of the DAF-method. First, instead of the geometrical *Brown*ian motion, the more general class of (risk-neutral) CEV-diffusions (Cox and Ross, 1976; Schroder, 1989), with time-dependent volatility (e.g. Lo et al., 2009) is used,

$$dS_t = rS_t dt + \sigma_t S_t^{\beta/2} dW_t\,, \quad 0 \leq \beta < 2, \tag{29}$$

which is considered more appropriately from an empirical point of view (cf. Campbell, 1987; Glosten et al., 1993; Brandt, 2004). Second, the term structure of volatility is modeled by a deterministic mean reversion process

$$d\sigma_t^2 = \lambda(\bar{\sigma}^2 - \sigma_t^2)dt, \tag{30}$$

where $\lambda > 0$ is the mean reversion speed, and $\bar{\sigma}^2$ is the long term mean reversion level. The respective parameters may be estimated from a local volatility surface or from the expectation of a stochastic volatility model.

To see how this fits into the DAF-formalism, observe that the time-dependent operator matrix can be decomposed into

$$L(S_i, S_j, t) = L_1(S_i, S_j) + L_2(S_i, S_j)\sigma^2(t), \tag{31a}$$

with the operator sub-matrices

$$L_1(S_i, S_j) = \Delta S \left(rS_i\delta_M^{(1)}(S_i - S_j) - r\delta_M(S_i - S_j) \right) \tag{31b}$$

and

$$L_2(S_i, S_j) = \frac{\Delta S}{2} S_i^\beta \delta_M^{(2)}(S_i - S_j). \tag{31c}$$

Thus, using the previous notation $V(\tau)$ for the vector of values at $S_1, \ldots, S_N$ at time τ, one obtains

$$V(\tau) = \exp\left[L_1\tau + L_2 \int_0^\tau \tilde{\sigma}^2(s)ds\right] V(0), \tag{32}$$

with the time reversed volatility $\tilde{\sigma}_s^2 = \sigma_{T-s}^2$.The solution to the mean reversion problem (30) is

$$\sigma_T^2 = e^{-\lambda(T-t)}(\sigma_t^2 - \bar{\sigma}^2) + \bar{\sigma}^2, \tag{33}$$

and thus one obtains for the volatility integral on the r.h.s. of (32) after changing variables from t to τ

$$\int_0^\tau \tilde{\sigma}^2(s)ds = \bar{\sigma}^2\tau + \frac{\tilde{\sigma}_0^2 - \bar{\sigma}^2}{\lambda}\left(e^{\lambda\tau} - 1\right). \tag{34}$$

This situation is very similar to the previous examples. The operator sub-matrices L_1 and L_2 have to be calculated only once to cover the interval $[0, \tau]$. Furthermore, the computation time still does not depend on the length of the time step.

The following settings are used in this scenario: the elasticity of variance is set to $\beta - 2 = -1$, resulting in the model of Cox and Ross (1976). Mean reversion speed and level of variance are $\lambda = 2.5$ and $\bar{\sigma}^2 = 0.04$, respectively. The initial values of stock price and (squared) volatility are chosen $S_0 = 5$ and $\sigma_0^2 = 0.1$, and the annual risk free interest rate is 8 %. The partial up-and-out barrier is located at $S_u = 10$ and is active for the first year of the tenor. The option expires at $T = 2$ years.

Notice that two different time-dependent operator matrices are involved in the problem. During the first year the operator $\bar{L}$ is used, which incorporates the active barrier condition, and subsequently the operator L, equipped with natural boundary conditions, is used until maturity of the contract. Therefore,

Partial Reverse Barrier Option							
		Computation Time		$V(5,2)$		σ-Deviation	
ΔS	Δt	CN	DAF	CN	DAF	CN	DAF
2.5	0.1	0.0073 sec.	0.0080 sec.	0.4780 $	0.7274 $	154.60	28.372
1	0.1	0.0181 sec.	0.0126 sec.	0.6761 $	0.7614 $	54.338	11.145
0.5	0.1	0.0357 sec.	0.0207 sec.	0.7616 $	0.7792 $	11.015	2.1069
0.2	0.1	0.0936 sec.	0.0452 sec.	0.7817 $	0.7837 $	0.8463	0.1571
0.1	0.1	0.1997 sec.	0.0952 sec.	0.7841 $	0.7843 $	0.3479	0.4792
0.05	0.05	0.8689 sec.	0.2512 sec.	0.7846 $	0.7845 $	0.6254	0.5597
0.02	0.02	7.6596 sec.	6.2868 sec.	0.7848 $	0.7845 $	0.7027	0.5822
0.01	0.01	56.846 sec.	52.806 sec.	0.7848 $	0.7846 $	0.7137	0.5854

Table 4: Numerical Results of Partial Reverse Barrier Option Valuation for Different Grid Spacings ΔS and Δt

the previously detailed procedure has to be applied twice and the compound solution to the problem is

$$\begin{aligned} V(2) &= \exp\left[\int_1^2 \bar{L}(s)ds + \int_0^1 L(s)ds\right] V(0) \\ &= \exp\left[\bar{L}_1 + \bar{L}_2 \int_1^2 \tilde{\sigma}^2(s)ds + L_1 + L_2 \int_0^1 \tilde{\sigma}^2(s)ds\right] V(0). \end{aligned} \tag{35}$$

Because there is no closed-form solution available to this problem, the option value has to be simulated at a high level of precision in order to obtain a reliable Monte Carlo reference value. To this end a *Euler-Maruyama*-scheme (Kloeden and Platen, 1992, pp. 340) with time discretization $\Delta t = 10^{-4}$ and $R = 100\,000$ replications is used. The resulting estimate and its standard deviation is

$$\hat{V}_{MC} = 0.7834 \quad \text{and} \quad \frac{\hat{\sigma}_{MC}}{\sqrt{R}} = 1.9753 \times 10^{-3}.$$

The computation time for the Monte Carlo estimate is roughly 40 minutes, including full parallelization.

Table 4 shows the results for different spatial and temporal resolutions. Again the DAF-approximation order is $M = 30$ and there are 10 rows of the respective boundary operator included in the operator matrix for $0 \leq \tau \leq 1$. The upper boundary is set to $S_N = 15$ for both numerical schemes. The last column reports the absolute deviations from the Monte Carlo estimated value in terms of standard deviations

$$\sigma\text{-Deviation} = \sqrt{R}\left|\frac{V_{Num.} - \hat{V}_{MC}}{\hat{\sigma}_{MC}}\right|, \tag{36}$$

where $V_{Num.}$ represents either the value for the DAF- or *Crank-Nicolson*-method. Because the standard deviation of the Monte Carlo estimate is of order $\mathcal{O}(10^{-3})$, a σ-deviation ≈ 2 can roughly be regarded as sufficiently accurate with 95% confidence. As illustrated in table 4, this level of accuracy is already reached by the DAF-method at a spacing of $\Delta S = 0.5$. The computation time is about 20 milliseconds.

6. Summary

A new method for pricing barrier options, based on distributed approximating functionals was introduced. The advantage of the method is that an arbitrary differential operator can be transformed into a matrix, provided an appropriate equidistant spacial grid was chosen beforehand. Thus, the operation of differentiation becomes an algebraic operation, a matrix/vector multiplication. The operator matrix itself can be computed very efficiently because of its *Toeplitz*-structure. Once the eigenvalue decomposition of the matrix exponential of the operator matrix is computed, the solution can be calculated for arbitrary time steps. This is particularly beneficial in situations involving contracts with long maturities, because the computation times of conventional methods is related to the tenor of the option.

Several test scenarios were analyzed, including situations where no closed-form solution is available. The main results concern both, efficiency and accuracy. The long life cash-or-nothing option scenario clearly indicates that the DAF-method is faster than a conventional *Crank-Nicolson*-scheme for long maturities and/or high precision. The double-barrier binary asymmetrical scenario suggests that the DAF-approach is considerably more accurate, if the same spacial grid resolution is used. However, this can be compensated up to a certain degree by choosing finer time discretizations for the *Crank-Nicolson*-scheme. But this enhancement comes at the cost of computation time. By construction, the DAF-method does not suffer from discretization errors in the time domain. Finally, a partial reverse barrier option scenario was investigated, where the model for the underlying belongs to the class of CEV-diffusions. Additionally, the volatility was given a term structure. This situation is far more complex than the previous examples, and the implementation of a standard *Crank-Nicolson*-scheme becomes more cumbersome. In the DAF-framework, constructing the operator matrix is still straight forward and the computation time is still independent of the time step. The results of this scenario suggest that the DAF-method is superior with respect to both, efficiency and accuracy.

Even though DAF-based numerical approximation seems superior in all analyzed scenarios, it shares some of the drawbacks of conventional methods. For example, it is difficult to extend the method to higher dimensions in order to price strongly path dependent or second order contracts like asians or parisian options, or more generally basket options. Furthermore, the treatment of boundary conditions is slightly more involved than in the classical framework. Nevertheless, it is an elegant method and the results are promising.

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