Two Views on Building Conceptual Hierarchies

Rainer Osswald

Intelligente Informations- and Kommunikationssysteme
Fachbereich Informatik
FernUniversität in Hagen

http://pi7.fernuni-hagen.de/osswald
Formal Concept Analysis (FCA) [Ganter & Wille 1999]

- **Formal context** \( K = \langle U, \Sigma, \models_K \rangle \)
  
  - \( U \) set of objects, \( \Sigma \) set of attributes, \( \models_K \subseteq U \times \Sigma \) satisfaction relation
  
  - \( V^\triangleright = \{ p \in \Sigma \mid x \models_K p \text{ for all } x \in V \} \) intent of \( V \subseteq U \)
  
  - \( X^\triangleleft = \{ x \in U \mid x \models_K p \text{ for all } p \in X \} \) extent of \( X \subseteq \Sigma \)

- **Formal concept**: \( \langle V, X \rangle \) with \( V^\triangleright = X \), \( X^\triangleleft = V \)

- **Concept lattice**: ordered set of formal concepts

  \[
  \langle V, X \rangle \leq \langle W, Y \rangle \text{ iff } V \subseteq W, \ X \supseteq Y
  \]

- **Example**

<table>
<thead>
<tr>
<th></th>
<th>human</th>
<th>biped</th>
<th>featherless</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>x</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Mary</td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Fido</td>
<td></td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Alex</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Complete implicational theory of context:

\( \text{human} \rightarrow \text{biped} \land \text{featherless}, \ \text{biped} \land \text{featherless} \rightarrow \text{human} \)

- **Basic Theorem of FCA**

  Every concept lattice is a complete lattice and every complete lattice is representable as the concept lattice of a formal context.
Building Conceptual Lattices – Version IIa


- Logico-algebraic language for specifying taxonomies/ontologies
  
  \[ p \in \Sigma \text{ (set of basic concepts), null, univ,} \]
  \[ \phi + \psi \text{ (conceptual sum), } \phi \times \psi \text{ (conceptual product),} \]
  \[ \phi \leq \psi \text{ (subsumption), } \phi = \psi \text{ (equality)} \]

- Distributive lattice of concepts via equational specification
  
  Example  human = biped \times featherless

- Attributive descriptions (not taken into account here)
  
  \[ r : \phi \text{ (Peirce Product)} \]

- Relation to FCA?

  “In some sense this approach can be compared to Formal Concept Analysis, since both are mathematical methods for generating concept lattices. However, our approach is fundamentally different, because ONTOLOG is a language for specifying taxonomies while the Formal Concept Analysis is a mathematical theory for analysis of conceptual data (the relations between objects and their attributes).” [Oldager 2000]
Building Conceptual Lattices – Version IIb

Oles (2000): *An Application of Lattice Theory to Knowledge Representation*

- **Knowledge base (KB):** set of terminological axioms
  - new concept *I* (*I* primitive identifier),
  - subconcept *C*₁ *C*₂ (*Cᵢ* built by ∧, ∨ from primitives, ⊤, ⊥)

- **Example**
  - new concept *human*;
  - new concept *biped*;
  - new concept *featherless*;
  - subconcept *human* (*biped* ∧ *featherless*);
  - subconcept (*biped* ∧ *featherless*) *human*;

- **Lattice of semantic concepts:**
  - distributive lattice presented by generators and relations given by KB
  - “Birkhoff implementation” by ordered set of ∨-irreducibles

- **Relation to FCA?**
  - “Right from the start, let us say what this paper is not about. It is not about the formal concept analysis of Wille, which deals with the derivation, from tables of attributes for individuals, of hierarchies of what are also called concepts. Wille’s hierarchies are lattices, but they are not necessarily distributive. There are no clear connections between Wille’s work and the contents of this paper.” [Oles 2000]
Overview

- A Simple Propositional Framework
- Theories and Information Domains
- Formal Concept Analysis Reanalyzed
- The Lindenbaum Algebra of a Theory
- The Predicational Viewpoint
- Perspectives
A Simple Propositional Framework

Working hypothesis: concepts are subsets of a set $\Sigma$ of attributes

- Regard $p \in \Sigma$ as atomic proposition (propositional variable)
- Boolean formulas $B[\Sigma]$ over $\Sigma$: inductively built by $\land, \lor, \neg, \top, \bot$
  $\phi \rightarrow \psi$ for $\neg \phi \lor \psi$, $\phi \leftrightarrow \psi$ for $\phi \rightarrow \psi \land \psi \rightarrow \phi$

Positive/affirmative formula: no $\neg$

Implication: $\land P \rightarrow \land Q$, $P, Q \subseteq \Sigma$ finite ($\land \emptyset = \top$, $\lor \emptyset = \bot$)
Conditional (biconditional) form: $\phi \rightarrow \psi$ ($\phi \leftrightarrow \psi$), $\phi, \psi$ positive
Conditional normal form: $\land P \rightarrow \lor Q$

- Theory over $\Sigma$: set of Boolean formulas over $\Sigma$
- Interpretation: function $m : \Sigma \rightarrow \{0, 1\}$, canonical extension to $B[\Sigma]$ equivalent: subset $X \subseteq \Sigma$, via characteristic function $\chi_X$

  $X$ satisfies $\phi$ iff $\chi_X(\phi) = 1$, notation: $X \models \phi$
  $(X \models p$ iff $p \in X$, $X \models \neg \phi$ iff $X \not\models \phi$, $\ldots$)

- $X \subseteq \Sigma$ model of theory $\Gamma$ iff $X \models \phi$ for all $\phi \in \Gamma$
Theories and Information Domains (I)

- **Information domain** $C(\Gamma)$ of theory $\Gamma$: models of $\Gamma$ ordered by set inclusion

  $$C(\Gamma) = \{X \subseteq \Sigma \mid X \vDash \phi \text{ for all } \phi \in \Gamma\}$$

- **Examples**

  $$\Sigma = \{h, b, f\} \quad \Gamma = \{h \leftrightarrow b \land f\}$$

  $$\emptyset \quad \{f\} \quad \{b\} \quad \{h, f, b\}$$

  $$\Sigma = \{a, b, c, d\} \quad \Gamma = \{\top \rightarrow a \lor b, c \land d \rightarrow \bot, a \land b \leftrightarrow c \lor d\}$$

  $$\emptyset \quad \{a\} \quad \{b\} \quad \{a, b\} \quad \{a, b, c\} \quad \{a, b, d\}$$

  $$\Sigma = \{a_1, a_2, \ldots\} \quad \Gamma = \bigcup_{n \geq 1} \{a_n \rightarrow a_1 \lor a_{n+1}, a_{n+1} \rightarrow a_n\}$$

  $$X_n = \{a_m \mid m \leq n\}, \ Y = \{a_n \mid n > 1\}$$
Classification of information domains

<table>
<thead>
<tr>
<th>Class of formulas of $\Gamma$</th>
<th>Closure properties of $C(\Gamma)$</th>
<th>$C(\Gamma)$ as ordered set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\land P \rightarrow \lor Q$</td>
<td>local membership</td>
<td></td>
</tr>
<tr>
<td>$\land P \rightarrow q, \land P \rightarrow \bot$</td>
<td>nonempty intersection, directed union</td>
<td>Scott domain</td>
</tr>
<tr>
<td>$(\text{Horn})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\land P \rightarrow q$</td>
<td>intersection, directed union</td>
<td>complete algebraic lattice</td>
</tr>
<tr>
<td>(implication)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p \rightarrow q$</td>
<td>intersection, union</td>
<td>complete coprime-algebraic lattice</td>
</tr>
<tr>
<td>(simple implication)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\land P \rightarrow \bot$</td>
<td>subsets, finitely bounded union</td>
<td>bounded-complete atomic dcpo with completely coprime atoms</td>
</tr>
<tr>
<td>(contradiction)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Theories and Information Domains (III)

- Canonical $\mathcal{C}$-theory $Th_\mathcal{C}(\mathcal{U})$ associated with subset system $\mathcal{U} \subseteq \wp(\Sigma)$

$$Th_\mathcal{C}(\mathcal{U}) = \{ \phi \in \mathcal{C} \mid X \models \phi \text{ for all } X \in \mathcal{U} \}$$

$\mathcal{C} \in \{ \text{Boolean formulas, implications, } \ldots \}$

- $\langle Th_\mathcal{C}, \mathcal{C} \rangle$ is Galois connection between $\wp(\wp(\Sigma))$ and $\wp(\mathcal{C})$ induced by $\models$

  Consequence: $C \circ Th_\mathcal{C}$ is closure operator on $\wp(\Sigma)$

- $\Gamma$ is complete $\mathcal{C}$-theory of $\mathcal{U}$ iff $\Gamma \subseteq Th_\mathcal{C}(\mathcal{U})$ and $\Gamma \vdash Th_\mathcal{C}(\mathcal{U})$

- Example

  Complete implicational theory of $\mathcal{U}$:
  
  \[ d \rightarrow c, \ c \rightarrow a, \ e \rightarrow a \land b, \ b \land c \rightarrow d \]
Formal Concept Analysis Reanalyzed

► Formal context \( K = \langle U, \Sigma, \models_K \rangle \)

\[
V^\updownarrow = \{ p \in \Sigma \mid x \models_K p \text{ for all } x \in V \} = \bigcap \{ x^\updownarrow \mid x \in V \} \quad \text{(with } x^\updownarrow \text{ for } \{x\}^\updownarrow \text{)}
\]

► Complete implicational theory of \( K \):

complete implicational theory of object intents \( \mathcal{U}_K = \{ x^\updownarrow \mid x \in U \} \)

► Theorem For finite attribute sets, there is an order-reversing one-to-one correspondence between the concept lattice of a formal context and the information domain of any complete implicational theory of that context.

► Example

\[
\begin{array}{c|ccccc}
& a & b & c & d & e \\
\hline
x_1 & x & x & x & x \\
x_2 & x & & & x \\
x_3 & & & & x \\
x_4 & x & x & x & x \\
x_5 & x & x & & \\
x_6 & x & x & x & x \\
x_7 & x & x & x & x \\
\end{array}
\]

\[
\begin{array}{c}
\mathcal{U}_K \\
\{a,c,d\} \\
\{a,c\} \\
\{a\} \\
\{a,b,c,d\} \\
\{a,b,e\} \\
\{b\} \\
\end{array}
\]

\[
\begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
\end{array}
\]

Rainer Osswald
The Lindenbaum Algebra of a Theory

Lindenbaum algebra $L(\Gamma)$ of theory $\Gamma$

$$L(\Gamma) = A[\Sigma]/\simeq_\Gamma \quad \text{(with } \phi \simeq_\Gamma \psi \text{ iff } \Gamma \vdash \phi \leftrightarrow \psi)$$

$L(\Gamma)$ is a distributive lattice with zero and unit.

Theorem (Variant of Birkhoff’s Representation Theorem) Suppose $\Sigma$ is finite.

(i) There is an order-reversing one-to-one correspondence between $C(\Gamma)$ and the ordered set of $\lor$-irreducible elements of $L(\Gamma)$.

(ii) There is a one-to-one correspondence between $L(\Gamma)$ and the antichains in $C(\Gamma)$, where an antichain $\mathcal{I}$ corresponds to the equivalence class of $\bigvee\{\land X \mid X \in \mathcal{I}\}$.

Example

$$C(\Gamma)$$

\[
\begin{array}{c}
\{a, c, d\} & \{a, b, c, d\} & \{a, b, e\} \\
\{a, c\} & \{a\} & \{b\} \\
\{a\} & \{b\}
\end{array}
\]

$$\Gamma = \{d \rightarrow c, \ c \rightarrow a, \ e \rightarrow a \land b, \ b \land c \rightarrow d \}$$

\[c \land e \rightarrow \bot, \ a \land b \rightarrow c \lor e, \ \top \rightarrow a \lor b\]
The Predicational Viewpoint

- Natural formulation within monadic predicate logic
  attributes are monadic predicates
  statements are universally quantified (e.g. $\forall x (\text{human } x \rightarrow \text{biped } x)$)

- Formalization
  $\Sigma$ set of (atomic) monadic predicates
  Boolean predicates by $\land$, $\lor$, $\neg$ via abstraction (i.e. $(\phi \land \psi) x$ is $\phi x \land \psi x$, etc)
  theory = set of universal statements $\forall \phi$ (for $\forall x (\phi x)$)

- Interpretation of $\Sigma$: universe $U$, function $M : \Sigma \rightarrow \mathcal{P}(U)$
  Model of theory $\Gamma$: interpretation of $\Sigma$ such that $\Gamma$ is true

- Information domain of theory $\Gamma$:
  $C(\Gamma) = \{ X \subseteq \Sigma \mid X \models \phi \text{ for all } \forall \phi \in \Gamma \}$
  Canonical model of $\Gamma$ with universe $C(\Gamma)$:
  $p \in \Sigma \mapsto \{ X \in C(\Gamma) \mid p \in X \}$

- Theorem The canonical model of $\Gamma$ is universal in the sense that it embeds every other model of $\Gamma$ satisfying identity of indiscernibles.
Summary

▶ *Take Home Message I*

The lattices of FCA are the information domains of the complete implicational theories of the formal contexts (with finite attribute set).

Their “latticehood” is due to the restriction to implicational theories.

▶ *Take Home Message II*

The information domain of a theory (of universally quantified Boolean predicates) is the (ordered) universe of a universal model of that theory.

In terms of propositional logic, the information domain is the set of all models of the theory.

▶ *Take Home Message III*

The Lindenbaum algebra (of affirmative terms) of a theory is a distributive lattice with zero and unit.

The Lindenbaum algebra and the information domain of a theory are dual to each other in the sense of Birkhoff’s Representation Theorem.
Perspectives

► Infinite set of atomic propositions/predicates

    Lindenbaum algebra not uniquely determined by information domain

► Attribute-value theories

    Attributive descriptions $\pi : \phi$ (for $\lambda x \exists y (\pi xy \land \phi y)$)

► Default statements

    Default extensions as elements of information domains

► Categorical issues

    Category of theories, functors $C$ and $L$