Grids and the Arithmetics of Jordan Pairs

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Abstract

We prove that every 2-finitedimensional covering standard division grid of a Jordan pair V over a Henselian field canonically determines a norm on V. This is used to classify maximal orders which contain a covering division grid of V. We show that weakly separable orders always satisfy this condition and link their existence to ramification properties of V.

0. Introduction

In this paper, we use Neher's theory of grids [12, 13, 14] to derive new results on orders in finitedimensional Jordan pairs over local fields, generalizing our earlier approach [19] to the same topic. In particular, we classify saturated maximal orders (5.3), where an order is said to be saturated if it contains a covering division grid of the ambient Jordan pair. Though not all maximal orders are saturated (5.5), our results seem to indicate that the saturated ones are a natural object to study in the local arithmetics of Jordan pairs. For example, by an unpublished result of Neher (5.11), whose proof we will include here with its author's kind permission, every order which is weakly separable in the sense that it becomes semi-simple after reduction modulo the valuation ideal of the base field is automatically saturated. The existence of weakly separable orders, as well as of separable ones in the sense of Loos [8], is linked to ramification properties in 7.6. As an application we prove in 7.12 that a finitedimensional Jordan pair over the function field of a regular

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integral scheme X of dimension 1 extends to a separable Jordan pair over all of X if and only if it is everywhere unramified.

Beside grids, some of whose main features will be summarized and, occasionally, expanded in Section 3, our main tools to establish these results are the general valuation theory of Jordan division rings due to Niggemann [16], which we adapt to our purposes in Section 1, and its extension to Jordan division pairs carried out in Section 2. The technically most demanding result is the Norm Theorem 4.3, according to which every 2-finitedimensional covering standard division grid of a Jordan pair V over a Henselian field induces a norm on V in a canonical way. Once the Norm Theorem has been established, most of our arithmetic results reduce to the case of Jordan division pairs where methods from valuation theory can be applied. In particular, after having defined the anisotropic part of a nondegenerate simple Artinian Jordan pair in Section 6, this reduction lends itself to an intrinsic treatment of the ramification properties we are interested in.

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1. Valuations of Jordan division rings.

A valuation theory of Jordan division rings, its conceptual foundations already implicit in the work of Knebusch [6], was developed by the author [17] for valuations of height 1 and by Niggemann [16] in full generality. In the present section, we will briefly describe those features of the theory that are relevant for the intended applications. The reader is referred to Jacobson [5] for notations and standard facts about quadratic Jordan algebras; he may consult Bourbaki [1] and Ribenboim [26] for results on classical valuation theory.

1.1 The concept of a valuation. Let $\triangle$ be a totally ordered additive abelian group and J a Jordan division ring. Following Niggemann [16], a *valuation of J with values in* $\triangle$ is a map $\nu : J \longrightarrow \triangle_\infty = \triangle \cup \{\infty\}$ satisfying the following conditions for all $x, y \in J$.

(VA1) $$\nu(x) = \infty \Longleftrightarrow x = 0.$$

(VA2) $$\nu(x+y) \geq \min(\nu(x), \nu(y)).$$

(VA3) $$\nu(U_x y) = 2\nu(x) + \nu(y).$$

Then $\nu(1) = 0$;

(1.1.1) $$\mathfrak{O} = \mathfrak{O}(J, \nu) = \{x \in J : \nu(x) \geq 0\} \subset J$$

is a unital subring, called the *valuation ring of* (J, ν);

(1.1.2) $$\mathfrak{P} = \mathfrak{P}(J, \nu) = \{x \in J : \nu(x) > 0\} \subset \mathfrak{O}$$

is an ideal, called the *valuation ideal of* (J, ν), and $\kappa(J, \nu) = \mathfrak{O}/\mathfrak{P}$ is a Jordan division ring, called the *residue class ring of* (J, ν). The set of all valuations of J with values in $\triangle$ will be denoted by $\mathrm{Val}(J, \triangle)$. For $\delta \in \triangle$ we generalize (1.1.1), (1.1.2) to define

(1.1.3) $$\mathfrak{O}(J, \nu)^{(\delta)} = \{x \in J : \nu(x) \geq \delta\},$$

(1.1.4) $$\mathfrak{P}(J, \nu)^{(\delta)} = \{x \in J : \nu(x) > \delta\}.$$

1.2 Isotopes and valuations. It is a fundamental observation, though straightforward to prove, that a valuation on a Jordan division ring J canonically induces valuations on all its isotopes. More specifically, for an invertible (= nonzero) element $y \in J$ and a valuation ν of J with values in $\triangle$, the map $\nu^{(y)} : J^{(y)} \longrightarrow \triangle_\infty$ given by $\nu^{(y)}(x) = \nu(x) + \nu(y)$ for $x \in J$ is a valuation of the y-isotope $J^{(y)}$ of J [16, 2.1.4]; in addition, given another invertible element $z \in J$, we have $(\nu^{(y)})^{(z)} = \nu^{(U_y z)}$ (loc. cit). From this we immediately derive the following conclusion.

1.3 Proposition. *Let* $\triangle$ *be a totally ordered additive abelian group,* J *a Jordan division ring and* y *an invertible element of* J*. Then the assignment* $\nu \longmapsto \nu^{(y)}$ *gives a bijection from* $\mathrm{Val}(J,\triangle)$ *onto* $\mathrm{Val}(J^{(y)},\triangle)$*, with inverse* $\rho \longmapsto \rho^{(z)}, z = y^{-2}$. $\square$

1.4 Extensions of valuations. Now suppose J is a Jordan division algebra over a field K. Then K identifies with $K1 \subset J$, and every $\nu \in \mathrm{Val}(J,\triangle)$ restricts to map $\nu^0 : K \longrightarrow \triangle_\infty$, *which is actually a valuation in the usual sense* [16, 2.2.1, 2.2.2]. Conversely, given a subgroup Γ of $\triangle$ and a valuation $\lambda : K \longrightarrow \Gamma_\infty$, a valuation $\nu : J \to \triangle_\infty$ is said to be an *extension of* λ in case $\nu^0 = \lambda$ on $K = K1$. It then follows

$$\nu(ax) = \lambda(a) + \nu(x) \qquad (a \in K,\ x \in J) \tag{1.4.1}$$

by [16, (2.1)].

1.5 r-finitedimensionality for Jordan algebras. In analogy to Cohn [3] and Schilling [29], a (unital) quadratic Jordan algebra over a field is said to be *locally finitedimensional of level* r (or *r-finitedimensional* for short) ($r \in \mathbb{Z},\ r > 0$) if every unital subalgebra on r generators is finitedimensional.

1.6 Henselian fields. Recall that a Henselian field is a triple (K,Γ,λ), where K is a field, Γ is a totally ordered additive abelian group and $\lambda : K \to \Gamma_\infty$ is a valuation such that the various equivalent properties known as Hensel's Lemma are fulfilled; see Ribenboim [26] for details. In keeping with the terminology of 1.1, we write $\mathfrak{o} = \mathfrak{o}(K,\lambda)$ for the valuation ring, $\mathfrak{p} = \mathfrak{p}(K,\lambda)$ for the valuation ideal and $\kappa = \kappa(K,\lambda) = \mathfrak{o}/\mathfrak{p}$ for the residue class field of (K,Γ,λ).

We are now ready to state the most important single result in the valuation theory of Jordan division rings.

1.7 Extension Theorem for Jordan division algebras. *Let* (K,Γ,λ) *be a Henselian field and* J *a* 2*-finitedimensional Jordan division algebra* *over K. Then λ has a unique extension to a valuation of J with values in $\triangle = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. Since J is 2-finitedimensional and the defining properties of valuations ((VA1) - (VA3) of 1.1) involve at most two elements, we are immediately reduced to the finitedimensional case, which has been solved by Niggemann [16, 4.2.4].

2. Valuations of Jordan division pairs.

We now carry out the transition from the valuation theory of Jordan division rings to that of Jordan division pairs. Since Jordan division pairs up to isomorphism are virtually the same as Jordan division rings up to isotopy, this transition is neither surprising nor difficult. Yet it entails an increase in overall flexibility which is very much worth the effort and, in fact, of critical importance later on. In the sequel, the basic theory of Jordan pairs will be taken for granted; the reader is referred to Loos [7] for details. Given a Jordan pair $V = (V^+, V^-)$ and $\varepsilon = \pm$, the set of elements in V^ε that are invertible in V will be denoted by $V^{\varepsilon\times}$. Observe that we may have $V^{+\times} = V^{-\times} = \emptyset$. In this paper, ε always stands for one of the symbols $\pm$. Unspecified statements involving ε are always meant to hold for both $+$ and $-$. We denote by $\mathrm{Rad}\, V$ the Jacobson radical of V. Throughout this section, we fix a totally ordered additive abelian group $\triangle$ as in 1.1.

2.1 The concept of a valuation. Let V be a Jordan division pair. By a *valuation of V with values in* $\triangle$, symbolized as $\mu : V \longrightarrow \triangle_\infty$, we mean a pair $\mu = (\mu^+,\ \mu^-)$ of mappings $\mu^\varepsilon : V^\varepsilon \longrightarrow \triangle_\infty$ satisfying the following conditions for all $x, x' \in V^\varepsilon,\ y \in V^{-\varepsilon}$.

(VP1) $$\mu^\varepsilon(x) = \infty \Longleftrightarrow x = 0.$$

(VP2) $$\mu^\varepsilon(x + x') \geq \min(\mu^\varepsilon(x), \mu^\varepsilon(x')).$$

(VP3) $$\mu^\varepsilon(Q(x)y) = 2\mu^\varepsilon(x) + \mu^{-\varepsilon}(y).$$

The set of all valuations of V with values in $\triangle$ will be denoted by $\mathrm{Val}(V, \triangle)$. Always containing the *trivial valuation* given by the maps $V^\varepsilon \longrightarrow \triangle_\infty,\ 0 \longmapsto$

$\infty,\ x \longmapsto 0$ for $x \in V^{\varepsilon\times}$, $\mathrm{Val}(V, \triangle)$ is not empty. Given $\mu \in \mathrm{Val}(V, \triangle)$, we put

$$\mathfrak{O}(V,\mu) = (\mathfrak{O}(V,\mu)^+, \mathfrak{O}(V,\mu)^-),\ \mathfrak{P}(V,\mu) = (\mathfrak{P}(V,\mu)^+, \mathfrak{P}(V,\mu)^-), \tag{2.1.1}$$

where

$$\begin{aligned} \mathfrak{O}(V,\mu)^\varepsilon &= \{x \in V^\varepsilon : \mu^\varepsilon(x) \geq 0\}, \\ \mathfrak{P}(V,\mu)^\varepsilon &= \{x \in V^\varepsilon : \mu^\varepsilon(x) > 0\}. \end{aligned} \tag{2.1.2}$$

More generally, for $\delta \in \triangle$ we put

$$\begin{aligned} \mathfrak{O}(V,\mu)^{(\delta)} &= (\mathfrak{O}(V,\mu)^{(\delta)+}, \mathfrak{O}(V,\mu)^{(\delta)-}), \\ \mathfrak{P}(V,\mu)^{(\delta)} &= (\mathfrak{P}(V,\mu)^{(\delta)+}, \mathfrak{P}(V,\mu)^{(\delta)-}), \end{aligned} \tag{2.1.3}$$

where

$$\begin{aligned} \mathfrak{O}(V,\mu)^{(\delta)\varepsilon} &= \{x \in V^\varepsilon : \mu^\varepsilon(x) \geq \varepsilon\delta\}, \\ \mathfrak{P}(V,\mu)^{(\delta)\varepsilon} &= \{x \in V^\varepsilon : \mu^\varepsilon(x) > \varepsilon\delta\}. \end{aligned} \tag{2.1.4}$$

2.2 Examples. a) Let J be a Jordan division ring and $V = (J, J)$ the associated Jordan division pair. If $\nu : J \longrightarrow \triangle_\infty$ is a valuation, then so is $\mu = (\nu, \nu) : V \longrightarrow \triangle_\infty$, and we have

$$\mathfrak{O}(V,\mu) = (\mathfrak{O}(J,\nu), \mathfrak{O}(J,\nu)),\ \mathfrak{P}(V,\mu) = (\mathfrak{P}(J,\nu), \mathfrak{P}(J,\nu)).$$

See 2.8 below for generalization.
b) Let V be a Jordan division pair and $\mu = (\mu^+,\ \mu^-)$ a valuation of V with values in $\triangle$. Then $\mu^{\mathrm{op}} = (\mu^-,\ \mu^+)$ is a valuation of V^{op} with values in $\triangle$.

2.3 Proposition. *Let V be a Jordan division pair and $\mu \in \mathrm{Val}(V, \triangle)$. Then*
a) $\mu^{-\varepsilon}(x^{-1}) = -\mu^\varepsilon(x)$ *for all* $x \in V^{\varepsilon\times}$.
b) $\mathfrak{O}(V,\mu)$ *is a subpair of V, called the* valuation pair of (V,μ).
c) $\mathfrak{O}(V,\mu)^{\varepsilon\times} = \mathfrak{O}(V,\mu)^\varepsilon - \mathfrak{P}(V,\mu)^\varepsilon$.
d) $\mathfrak{P}(V,\mu)$ *is an ideal in* $\mathfrak{O}(V,\mu)$, *called the* valuation ideal of (V,μ).
e) *Calling* $\kappa(V,\mu) = \mathfrak{O}(V,\mu)/\mathfrak{P}(V,\mu)$ *the* residue class pair of (V,μ), *the*

following statements are equivalent .
(i) $\mathfrak{O}(V,\mu)$ *is a local Jordan pair.*
(ii) $\kappa(V,\mu)$ *is a Jordan division pair.*
(iii) $\kappa(V,\mu) \neq 0$.
(iv) $\mu^+(x) = 0$ *for some* $x \in V^+$.
(v) $\mu^-(y) = 0$ *for some* $y \in V^-$.
In this case, Rad $\mathfrak{O}(V,\mu) = \mathfrak{P}(V,\mu)$.

Proof. a) follows immediately from (VP3) and $Q(x)x^{-1} = x$.
b) follows immediately from (VP2), (VP3).
c) follows immediately from a) and the fact that V is a division pair.
d) By (VP2), (VP3), $\mathfrak{P} = \mathfrak{P}(V,\mu)$ is additively closed in $\mathfrak{O} = \mathfrak{O}(V,\mu)$ and satisfies $Q(\mathfrak{O}^\varepsilon)\mathfrak{P}^{-\varepsilon} + Q(\mathfrak{P}^\varepsilon)\mathfrak{O}^{-\varepsilon} \subset \mathfrak{P}^\varepsilon$. Hence it remains to prove $\{xyz\} \in \mathfrak{P}^\varepsilon$ for $x \in \mathfrak{O}^\varepsilon,\ y \in \mathfrak{O}^{-\varepsilon},\ z \in \mathfrak{P}^\varepsilon$. By c) we may assume $x \in \mathfrak{O}^{\varepsilon\times}$. But then $x^{-1} \in \mathfrak{O}^{-\varepsilon\times}$, and from [7, 2.1.2, p. 22] we conclude

$$\begin{aligned} Q(x^{-1})\{xyz\} &= Q(x^{-1})Q(x,z)y = D(x^{-1},z)y \\ &= \{x^{-1}zy\} \in Q(\mathfrak{O}^{-\varepsilon})\mathfrak{P}^\varepsilon \subset \mathfrak{P}^{-\varepsilon}, \end{aligned}$$

whence $\{xyz\} \in Q(x)\mathfrak{P}^{-\varepsilon} \subset \mathfrak{P}^\varepsilon$.
e) By c), $\mathfrak{P}(V,\mu)$ is the set of noninvertible elements of $\mathfrak{O}(V,\mu)$, and all nonzero elements of $\kappa(V,\mu)$ are invertible. Hence (i) - (iii) are equivalent. (iii) in turn is equivalent to $\mu^\varepsilon(x) = 0$ for some $\varepsilon = \pm$ and some $x \in V^\varepsilon$. By a) this last condition is equivalent to (iv) and to (v) as well. The last statement follows from [7, 4.4a)]. $\square$

2.4 Translates of valuations. The main difference between valuations of pairs and algebras, which also accounts for the increase in flexibility mentioned earlier, derives from the fact that, in the situation of 2.1, $\triangle$ acts on Val$(V,\triangle)$ by (right) translations: Let $\mu : V \longrightarrow \triangle_\infty$ be a valuation and $\delta \in \triangle$. Then $\mu + \delta : V \longrightarrow \triangle_\infty$, with $(\mu+\delta)^\varepsilon : V^\varepsilon \longrightarrow \triangle_\infty$ given by

$$(\mu+\delta)^\varepsilon(x) = \mu^\varepsilon(x) + \varepsilon\delta \qquad (x \in V^\varepsilon),$$

is again a valuation, called the δ-*translate of* μ. From (2.1.1-4) we conclude

$$\mathfrak{O}(V,\mu+\delta) = \mathfrak{O}(V,\mu)^{(-\delta)},\ \mathfrak{P}(V,\mu+\delta) = \mathfrak{P}(V,\mu)^{(-\delta)}. \tag{2.4.1}$$

Though translates have no direct analogue in the setting of Jordan algebras, the do relate to the possibility of passing to the valuation induced on an isotope (1.2). The precise nature of this relationship will be described in 2.5, 2.8 below.

2.5 Proposition. *Let V be a Jordan division pair and $y \in V^{-\varepsilon\times}$.*
a) *If $\mu : V \longrightarrow \triangle_\infty$ is a valuation of V, then*

$$\mu_y^\varepsilon : V_y^\varepsilon \longrightarrow \triangle_\infty,\ x \longmapsto \mu_y^\varepsilon(x) = \mu^\varepsilon(x) + \mu^{-\varepsilon}(y)$$

is a valuation of the Jordan division ring V_y^ε.
b) *If $\nu : V_y^\varepsilon \longrightarrow \triangle_\infty$ is a valuation, so is $\nu_y : V \longrightarrow \triangle_\infty$, where $(\nu_y)^\varepsilon = \nu, (\nu_y)^{-\varepsilon} = \nu \circ Q(y^{-1})$.*
c) *For valuations $\mu : V \longrightarrow \triangle_\infty,\ \nu : V_y^\varepsilon \longrightarrow \triangle_\infty$ we have*

$$(\mu_y^\varepsilon)_y = \mu + \varepsilon\mu^{-\varepsilon}(y),\ (\nu_y)_y^\varepsilon = \nu.$$

d) *The assignments $\mu \longmapsto \mu_y^\varepsilon$, $\nu \longmapsto \nu_y$ give inverse bijections from* $\mathrm{Val}(V, \triangle)/\triangle$ *onto* $\mathrm{Val}(V_y^\varepsilon, \triangle)$ *and vice versa.*

The proof consists in straightforward verifications of the various defining conditions and is omitted.

2.6 Separated valuations. Let V be a Jordan division pair. A valuation $\mu : V \longrightarrow \triangle_\infty$ is said to be *separated* if it satisfies the equivalent conditions (i) - (v) of 2.3e). *A valuation μ, though it need not be separated itself, always has a separated translate.* Indeed, for any $\delta \in \triangle$, $\mu - \delta$ is separated if and only if δ is a value of μ^+ (or, equivalently, of μ^- (2.3e)). We denote by $\mathrm{Val}_{\mathrm{sep}}(V, \triangle)$ the set of separated valuations of V with values in $\triangle$. The trivial valuation is always separated, as are the valuations of 2.2a).

2.7 Proposition. *Let J be a Jordan division ring and $V = (J, J)$ the associated Jordan division pair. Then the projection $\mu = (\mu^+, \mu^-) \longmapsto \mu^+$ yields a bijective map*

$$\mathrm{Val}_{\mathrm{sep}}(V, \triangle) \xrightarrow{\sim} \bigcup_{y \in J^\times} \mathrm{Val}(J^{(y)}, \triangle)$$

whose inverse sends a valuation $\nu : J^{(y)} \longrightarrow \triangle_\infty$ $(y \in J^\times)$ *to* $(\nu, \nu \circ U_{y^{-1}}) \in \mathrm{Val}_{\mathrm{sep}}(V, \triangle)$.

Proof. If $\mu : V \longrightarrow \triangle_\infty$ is a separated valuation, 2.3e) yields an element $y \in V^- = J$ satisfying $\mu^-(y) = 0$. Then $\mu^+ = \mu_y^+$ belongs $\mathrm{Val}(J^{(y)}, \triangle)$ (2.5a)). Conversely, suppose $\nu \in \mathrm{Val}(J^{(y)}, \triangle)$ for some $y \in J^\times$. Then $\nu_y = (\nu, \nu \circ U_{y^{-1}}) = (\nu, \nu \circ Q(y^{-1}))$ belongs to $\mathrm{Val}(V, \triangle)$ (2.5b)) and is obviously separated. We claim that ν_y does not depend on the choice of y. To see this, let also $y' \in J^\times$ satisfy $\nu \in \mathrm{Val}(J^{(y')}, \triangle)$. Then $\nu(y'^{-1}) = \nu(1^{(y')}) = 0$, and for all $x \in J$ we obtain

$$\begin{aligned} \nu \circ U_{y'^{-1}}(x) &= \nu(U^{(y)}_{y'^{-1}} U_{y^{-1}} x) = 2\nu(y'^{-1}) + \nu(U_{y^{-1}} x) \\ &= \nu \circ U_{y^{-1}}(x), \end{aligned}$$

whence $\nu_y = \nu_{y'}$. Using 2.5c), it is now easily checked that the assignments $\mu \longmapsto \mu^+$, $\nu \longmapsto \nu_y$ yield inverse bijections of the desired kind. □

2.8 Proposition. *Let* J *be a Jordan division ring and put* $V = (J, J)$. *For* $\mu = (\mu^+, \mu^-)$ *to be a separated valuation of* V *with values in* $\triangle$ *it is necessary and sufficient that there exist a valuation* $\nu : J \longrightarrow \triangle_\infty$ *and* $y \in J^\times$ *satisfying* $\mu^+ = \nu^{(y)}, \mu^- = \nu^{(y^{-1})}$ *in the sense of* 1.2. *In this case,* ν *and* y *are unique; moreover, the isomorphism*

$$(\mathbf{1}_J,\ U_y) : (J^{(y)}, J^{(y)}) \xrightarrow{\sim} (J, J) = V$$

of [7, 1.11] *is actually one of valued Jordan pairs in the sense that* $\mu \circ (\mathbf{1}_J, U_y) = (\nu^{(y)},\ \nu^{(y)})$ *and hence canonically induces isomorphisms*

$$\begin{aligned} &(\mathfrak{O}(J^{(y)}, \nu^{(y)}),\ \mathfrak{O}(J^{(y)}, \nu^{(y)})) \cong \mathfrak{O}(V, \mu), \\ &(\mathfrak{P}(J^{(y)}, \nu^{(y)}, \mathfrak{P}(J^{(y)}, \nu^{(y)})) \cong \mathfrak{P}(V, \mu), \\ &(\kappa(J^{(y)}, \nu^{(y)}), \kappa(J^{(y)}, \nu^{(y)})) \cong \kappa(V, \mu). \end{aligned}$$

Proof. Sufficiency being straightforward to check, let us suppose that $\mu : V \longrightarrow \triangle_\infty$ is a separated valuation. Then, for some invertible element $y \in J$, $\mu^+ : J^{(y)} \longrightarrow \triangle_\infty$ is a valuation (2.7), forcing $\mu^+(y) = 0 = \mu^-(y^{-1})$

by 2.3a). Choose $\nu \in \mathrm{Val}(J, \triangle)$ satisfying $\nu^{(y)} = \mu^+$ (1.3). Since $\mu^- = \mu^+ \circ U_z$, $z = y^{-1}$, again by 2.7, we conclude, for all $v \in J$:

$$\begin{aligned}\mu^-(v) &= \nu^{(y)}(U_z v) = \nu(U_z v) + \nu(y) \\ &= 2\nu(z) + \nu(v) + \nu(y) = \nu(v) + \nu(z) = \nu^{(z)}(v),\end{aligned}$$

i.e., $\mu^- = \nu^{(z)}$. Uniqueness of ν, y as well as the relation $\mu \circ (\mathbf{1}_J, U_y) = (\nu^{(y)}, \nu^{(y)})$ are obvious; by 2.2a), the rest follows. □

2.9 Extensions of valuations. Let V be a Jordan division pair over a field K. We assume that Γ is a subgroup of $\triangle$ and $\lambda : K \longrightarrow \Gamma_\infty$ is a valuation. In keeping with (1.4.1), a valuation $\mu : V \longrightarrow \triangle_\infty$ is said to be an *extension of* λ if

$$\mu^\varepsilon(ax) = \lambda(a) + \mu^\varepsilon(x) \qquad (a \in K,\ x \in V^\varepsilon). \tag{2.9.1}$$

Note that all the previous constructions preserve the property of being an extension of a fixed valuation of K.

2.10 r-finitedimensionality for Jordan pairs. Generalizing on 1.5, a Jordan pair V over a field is said to be *locally finitedimensional of level* r (or r-*finitedimensional* for short) ($r \in \mathbb{Z},\ r > 0$) if all subpairs on r generators of the form $(x_1, y), \dots, (x_r, y)$ or $(x, y_1), \dots, (x, y_r)$ $(x, x_1, \dots, x_r \in V^+; y, y_1, \dots, y_r \in V^-)$ are finitedimensional.

2.11 Extension Theorem for Jordan division pairs. *Let* (K, Γ, λ) *be a Henselian field and* V *a 2-finitedimensional Jordan division pair over* K. *Then* λ *extends to a valuation of* V *with values in* $\triangle = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. *Moreover, this extension is unique up to translates by elements of* $\triangle$.

Proof. For $0 \neq y \in V^-$, V_y^+ is a 2-finitedimensional Jordan division algebra over K. Hence 1.7 applies and, together with 2.5d), completes the proof. □

2.12 Corollary. *In the situation of* 2.11, *let V' be another* 2-*finitedimensional Jordan division pair over K and $\eta : V \longrightarrow V'$ a nonzero homomorphism. If $\mu : V \longrightarrow \triangle_\infty$, $\mu' : V' \longrightarrow \triangle_\infty$ are valuations extending λ, then there exists an element $\delta \in \triangle$ such that $\mu' \circ \eta = \mu + \delta$.* □

2.13 Corollary. *Notations being as in* 2.11, *let c be a nontrivial idempotent of V. Then there exists a unique extension of λ to a valuation μ of V with values in $\triangle$ satisfying $\mu^\varepsilon(c^\varepsilon) = 0$.*

Proof. By 2.11, there exists a unique element $\mu \in \mathrm{Val}(V, \triangle)$ extending λ and satisfying $\mu^+(c^+) = 0$. But this implies $\mu^-(c^-) = 0$ by 2.3a). □

2.14 Remark. If $\mu \in \mathrm{Val}(V, \triangle)$ extends λ as in 2.11, its translate $\mu' = \mu + \mu^-(c^-)$ for a nontrivial idempotent $c \in V$ has $\mu'^\varepsilon(c^\varepsilon) = 0$ and hence does not depend on the choice of μ (2.13).

2.15 The triple product inequality. Given a Jordan division pair V and a valuation $\mu : V \longrightarrow \triangle_\infty$, it seems difficult to decide whether

$$\text{(TPI)} \qquad \mu^\varepsilon(\{xyz\}) \geq \mu^\varepsilon(x) + \mu^{-\varepsilon}(y) + \mu^\varepsilon(z)$$

holds for all $x, z \in V^\varepsilon$, $y \in V^{-\varepsilon}$. Observe, however, that (TPI) is *translation invariant*, i.e., if $\mu \in \mathrm{Val}(V, \triangle)$ satisfies (TPI), so does $\mu + \delta$ for every $\delta \in \triangle$. Hence we may combine 2.5 with the proof of [20, Lemma 1] *to derive* (TPI) *under the assumption that $\triangle$ has height* 1 [1, VI §4 Definition 2], i.e., may be regarded as a totally ordered subgroup of the additive group of real numbers equipped with the natural ordering [1, VI §4 Proposition 8]. Further sufficient conditions for the validity of (TPI) will be discussed in Section 4 below.

3. Grids.

Our approach to the arithmetics of Jordan pairs relies heavily on Neher's theory of grids, whose principal features, in so far as they are needed to understand the subsequent development, will be summarized and, occasionally, expanded in this section. The main references are [12] (most notably §§1-4 of Chap.I) and [13] (regarding Jordan pairs as polarized Jordan triple systems [7, 1.14] in both cases) as well as [14]. Throughout this section, we let V be a Jordan pair over any commutative associative ring of scalars. As before, Rad V stands for the Jacobson radical of V.

3.1 Elementary relations of idempotents. For elements $r, s, t \in V^+ \times V^-$ with V^ε-components $r^\varepsilon,\ s^\varepsilon,\ t^\varepsilon$, respectively, we use the conventions

$$Q(r)s = (Q(r^+)s^-, Q(r^-)s^+),\ \{rst\} = (\{r^+s^-t^+\}, \{r^-s^+t^-\}).$$

These extend in a straightforward manner to pairs X, Y, Z of subsets $X^\varepsilon, Y^\varepsilon, Z^\varepsilon$, respectively, of V^ε. Idempotents $c, d \in V$ are called *associated* (in V) (written as $c \approx d$) if they have the same Peirce components. They are called *collinear* (written as $c \top d$) if $c \in V_1(d)$ and $d \in V_1(c)$. We say that c *governs* (or *dominates)* d (written as $c \vdash d$ or $d \dashv c$) if $c \in V_1(d)$ and $d \in V_2(c)$. As usual, we write $c \perp d$ in case c and d are orthogonal.

3.2 Lemma. *Let U be a subpair* of V.
a) *If $U^\varepsilon \cap V^{\varepsilon\times} \neq \emptyset$ for some $\varepsilon = \pm$, then $U^{\varepsilon\times} \subset V^{\varepsilon\times}$.*
b) Idempotents c, d of U are associated in U if and only if they are so in V.

Proof. a) Let $z \in U^\varepsilon$ be invertible in V. Given $x \in U^{\varepsilon\times}$, there exists $y \in U^{-\varepsilon}$ such that $Q(x)y = z$. Since $Q(z) = Q(x)Q(y)Q(x)$ is a bijective map $V^{-\varepsilon} \longrightarrow V^\varepsilon$, so is $Q(x)$.
b) If c, d are associated in U, we have $U_2(d) = U_2(c) \subset V_2(c)$, and $c^\varepsilon \in U_2(d)^\varepsilon$ is invertible in $V_2(c)$. Hence so is d^ε by a), forcing c, d to be associated in V [12, I.2.3]. The converse is obvious. $\square$

3.3 Cogs. A nonempty subset E of V is called a *cog in* V if it consists of nontrivial idempotents and any two distinct elements $c, d \in E$ satisfy one of the relations $c \perp d,\ c \top d,\ c \vdash d,\ d \vdash c$.Then the Peirce projections of arbitrary elements of E commute by pairs, forcing a direct sum decomposition

$$\bigoplus_{I \in \mathbb{Z}^E} V_I(E) \subset V, \tag{3.3.1}$$

where, for each $I \in \mathbb{Z}^E$, the corresponding Peirce component relative to E is given by

$$V_I = V_I(E) = \bigcap_{c \in E} V_{I(c)}(c),$$

adopting the convention $V_i(c) = 0$ for $c \in E,\ i \in \mathbb{Z}$ unless $i \in \{0,1,2\}$. The ordinary Peirce decomposition relative to a single idempotent immediately implies

$$Q(V_I)V_J \subset V_{2I-J},\ \{V_I V_J V_K\} \subset V_{I-J+K} \tag{3.3.2}$$

for $I, J, K \in \mathbb{Z}^E$. Note that we have equality in (3.3.1) if E is finite but not in general. Also, every $c \in E$ is contained in exactly one $V_I(E),\ I \in \mathbb{Z}^E$, and conversely, for every $I \in \mathbb{Z}^E,\ V_I(E)$ contains at most one element of E. Calling

$$\operatorname{supp} E = \{I \in \mathbb{Z}^E : V_I(E) \cap E \neq \emptyset\} \tag{3.3.3}$$

the *support of* E, the *cover of* E *in* V is defined by

$$C_V(E) = \sum_{I \in \operatorname{supp} E} V_I(E) \tag{3.3.4}$$

and we say that E *covers* V if $C_V(E) = V$. If all Peirce components $V_I(E), I \in \operatorname{supp} E$, are division (resp. local) pairs, we call E a *division* (resp. *local) cog* in V.

3.4 Closed cogs and 3-graded root systems. Let E be a cog in V. For $c, d \in E$ satisfying $c \vdash d,\ Q(c)d$ is a nonzero idempotent in V, as is $\{e_1e_2e_3\}$ for $e_1, e_2, e_3 \in E$ satisfying $e_1 \top e_2 \top e_3 \perp e_1$ or $e_1 \vdash e_2 \dashv e_3 \top e_1$. If each one of the idempotents thus constructed is associated with some element of E, then E is said to be *closed.* We recall from [13, 2.22, 3.2] the fundamental fact that closed cogs are in a one-to-one correspondence with 3-graded root systems and refer to [13, §1] for a summary on this concept. Accordingly, let (R, R_1) be the 3-graded root system associated with a closed cog E in V. Then each $\alpha \in R_1$ uniquely determines an element $c_\alpha \in E$ and conversely [13, 2.7], which in turn uniquely determines $I \in \mathbb{Z}^E$ such that $c_\alpha \in V_I(E)$

(3.3). We are therefore allowed to write $V_\alpha(E) = V_I(E)$. Hence, if E covers V, the Peirce decomposition of V relative to E (3.3.4) attains the form

$$V = \sum_{\alpha \in R_1} V_\alpha(E) \tag{3.4.1}$$

Setting $V_\alpha(E) = 0$ for $\alpha \in X - R_1$, X being the ambient real vector space of (R, R_1), and abbreviating $V_\alpha = V_\alpha(E)$ for $\alpha \in X$, the multiplication rules (3.3.2) may be rewritten as

$$Q(V_\alpha)V_\beta \subset V_{2\alpha-\beta},\ \{V_\alpha V_\beta V_\gamma\} \subset V_{\alpha-\beta+\gamma} \tag{3.4.2}$$

for $\alpha, \beta, \gamma \in R_1$.

3.5 Quadrangles, triangles, diamonds. A *quadrangle* in V is a quadruple (e_1, e_2, e_3, e_4) of idempotents satisfying

$$e_i \top e_{i+1},\ e_i \perp e_{i+2} \qquad \text{(i mod 4)}, \tag{3.5.1}$$

$$\{e_i e_{i+1} e_{i+2}\} = e_{i+3} \qquad \text{(i mod 4)}. \tag{3.5.1$'$}$$

A *triangle* in V is a triple (e_0, e_1, e_2) of idempotents satisfying

$$e_1 \dashv e_0 \vdash e_2 \perp e_1, \tag{3.5.2}$$

$$\{e_1 e_0 e_2\} = e_0,\ Q(e_0)e_i = e_{3-i} \qquad \text{(i = 1, 2)}. \tag{3.5.2$'$}$$

A *diamond* in V is a quadruple (e_0, e_1, e_2, e_3) of idempotents satisfying

$$e_1 \vdash e_0 \dashv e_3,\ e_0 \perp e_2,\ e_1 \top e_2 \top e_3 \top e_1, \tag{3.5.3}$$

$$\{e_0 e_1 e_2\} = e_3. \tag{3.5.3$'$}$$

Following [13, 2.1, 3.1] we speak of *root quadrangles* (resp. *root triangles*, *root diamonds*) if only (3.5.1) (resp. (3.5.2), (3.5.3)) is fulfilled.

3.6 Grids and standard grids. A *grid* G in V is a closed cog satisfying the following two conditions, for all $e_1, e_2, e_3 \in G$.

(3.6.1) If e_1, e_2, e_3 are mutually collinear having $\{e_1e_2e_3\} \neq 0$, there exists $c \in G$ such that $e_1 \vdash c \dashv e_3$ and $c \perp e_2$.

(3.6.2) If $e_1 \dashv e_2 \vdash e_3 \top e_1$, then $\{e_1e_2e_3\} = 0$.

A grid G in V is said to be *standard* if

(SG1) every root triangle in G is a triangle,
(SG2) every root diamond in G is a diamond,
(SG3) every root quadrangle (e_1, e_2, e_3, e_4) in G satisfies $\{e_1e_2e_3\} = \pm e_4$.

The following elementary observation is already implicit in the work of Neher [12].

3.7 Proposition. *Every covering cog is a grid.*

Proof. Let $E \subset V$ be a covering cog and denote by E_c the closure of E, which is a closed cog containing E and having the same Peirce components as E [12, I.4.11]. But since E covers V, this is easily seen to imply $E = E_c$, i.e., E is closed and hence a grid [12, I.4.14]. $\square$

3.8 Ortho-collinear systems and root lengths. Let $E \subset V$ be a cog. We denote by $E^{(2)}$ the set of elements $c \in E$ dominating some $d \in E$ and put $E^{(1)} = E - E^{(2)}$. For $i = 1, 2$, $E^{(i)}$ is an ortho-collinear system, i.e., any two distinct members of $E^{(i)}$ are either orthogonal or collinear. If E is closed, the elementary configurations $\perp, \top, \vdash$ of E correspond to elementary configurations, also denoted $\perp, \top, \vdash$, respectively, of the associated 3-graded root system (R, R_1). Suppose in addition that E is connected, so any two elements of E can be joined by a finite chain in E no two successive members of which are orthogonal. Then the elements of R_1 have at most two different lengths, which occur both if and only if $E^{(2)}$ is not empty, $E^{(2)}$ canonically matching with the short roots of R_1 in this case.

3.9 Lemma. *Let G be a covering grid of V and $c \in G^{(1)}$. Then $V_2(c)$ is a Peirce component of V relative to G.*

Proof. Write (R, R_1) for the 3-graded root system associated with G, put $V_\beta = V_\beta(G)$ for $\beta \in R_1$ and let $\alpha \in R_1$ satisfy $c = c_\alpha$. As G covers V,

$$V_2(c) = V_\alpha + \sum_{\substack{\beta \in R_1 \\ \beta \dashv \alpha}} V_\beta,$$

and $c \in G^{(1)}$ implies that the second term on the right vanishes. $\square$

3.10 Connected components. Let $G \subset V$ be a covering grid and G' a *connected component,* i.e., a maximal connected subcog, of G. It is easy to see that G' is a grid whose cover in V is an ideal and that V is the direct sum of these ideals as G' varies over the connected components of G.

3.11 Theorem. *Let G be a covering grid of V and (R, R_1) the corresponding 3-graded root system. Then*

$$\text{Rad } V = \sum_{\alpha \in R_1} \text{Rad } V_\alpha(G).$$

Note. 3.11 in particular implies

$$\text{Rad } V_\alpha(G) = V_\alpha(G) \cap \text{Rad } V \qquad (\alpha \in R_1).$$

Observe, however, for any cog $E \subset V$ that the relations

$$\text{Rad } V_I(E) = V_I(E) \cap \text{Rad } V \qquad (I \in \mathbb{Z}^E),$$

though valid if E is finite [12, I.6.1], fail to hold in general [12, I.6.5].

Proof. Setting $V_\alpha = V_\alpha(G)$ for $\alpha \in R_1$, we have

$$\text{Rad } V = \sum_{\alpha \in R_1} (V_\alpha \cap \text{Rad } V)$$

since Rad V is an ideal in V. As the Bergman operator $B(x, y)$ for $(x, y) \in V_\alpha$ stabilizes the Peirce components relative to G, $V_\alpha \cap \text{Rad } V$ is contained in Rad V_α. Hence it remains to prove

$$\text{Rad } V_\alpha \subset \text{Rad } V \qquad (\alpha \in R_1). \tag{1}$$

To this end, we may assume that G is connected (3.10). If $c_\alpha \in G^{(1)}$, then $V_\alpha = V_2(c_\alpha)$ (3.9), and (1) follows from [7, 5.8]. Hence we may assume $c_\alpha \in G^{(2)}$, forcing

$$G_\alpha = \{c \in G : c_\alpha \vdash c\}$$

to be a nonempty orthocollinear systems [12, I.4.9]. Then there is an involutorial map $c \longmapsto c^*$ of G_α such that $c, d \in G_\alpha$ are orthogonal if and only if $d = c^*$ (loc. cit.). This implies $c \dashv c_\alpha \vdash c^* \perp c$ and $c^* \approx Q(c_\alpha)c$, $c_\alpha \approx c + c^*$ [12, I.2.5], hence

$$V' = V_2(c_\alpha) = V_2(c) + (V_1(c) \cap V_1(c^*)) + V_2(c^*). \tag{2}$$

Suppose first that $G_\alpha = \{c, c^*\}$ consists of two elements. Then $V_1(c) \cap V_1(c^*)$ is a Peirce component of G [12, I.4.9] which contains c_α and so agrees with V_α. Hence, by (2) and [7, 5.8],

$$\text{Rad } V_\alpha = \text{Rad } V_1'(c) \subset \text{Rad } V' \subset \text{Rad } V.$$

We are left with the case that G_α contains more than two elements, forcing $G = \{c_\alpha\} \cup G_\alpha$ since G is connected [12, I.4.9]. Let $F \subset G_\alpha$ be a maximal collinear system, which exists by Zorn's Lemma. Then

$$G_\alpha = F \cup F^*, \ F \cap F^* = \emptyset,$$

and, by 3.9, the Peirce decomposition relative to G reads

$$V = V_\alpha + \sum_{f \in F} (V_2(f) + V_2(f^*)). \tag{3}$$

For arbitrary $F' \subset F$ we now claim

(4) *that*

$$V_{F'} = V_\alpha + \sum_{f \in F'} (V_2(f) + V_2(f^*))$$

is a subpair of V.

To see this, one checks that $G' = \{c_\alpha\} \cup F' \cup F'^*$ is a closed cog in V, uses [12, I.3.9, I.4.14] to establish $C_V(G')$ as a subpair of V, and proves

$$V_{F'} = C_V(G') \cap \bigcap_{f \in F - F'} (V_1(f) \cap V_1(f^*));$$

details are left to the reader. Returning to the proof of (1), we note that quasi-invertibility in a subpair extends to quasi-invertibility in all of V [7, 3.2] and that every element of V belongs to $V_{F'}$ for some finite subset F' of F. It therefore suffices to prove $\mathrm{Rad}\, V_{F'} \subset \mathrm{Rad}\, V_{F''}$ for all finite sets $F' \subset F'' \subset F$ (observe $V_\emptyset = V_\alpha$). Arguing by induction on $|F'' - F'|$, we may assume $F'' = F' \cup \{g\}$ for some $g \notin F'$, forcing

$$V_{F''} = V_2(g) + V_{F'} + V_2(g^*)$$

by (4), hence $V_{F'} = (V_{F''})_1(g)$, and the assertion follows from [7, 5.8]. $\square$

3.12 Simple Cogs. A cog $E \subset V$ is said to be *simple* if each Peirce component $V_I(E)$ ($I \in \mathrm{supp}\ E$) is a simple Jordan pair. There is a useful elementary connection between simplicity of a grid and simplicity of the ambient Jordan pair.

3.13 Proposition. *Let* G *be a covering simple grid of* V. *Then* V *is simple if and only if* G *is connected.*

Proof. If V is simple, G is connected, by [12, IV.1.1] or 3.10. Conversely, suppose G is connected and let U be a nonzero ideal of V. Writing (R, R_1) for the 3-graded root system associated with G and putting $V_\alpha = V_\alpha(G)$ for $\alpha \in R_1$, we obtain

$$U = \sum_{\alpha \in R_1} (V_\alpha \cap U).$$

Hence it suffices to show $G \subset U$. To do so, we pick $\alpha \in R_1$ such that $c_\alpha \in U$ and let $\beta \in R_1,\ \beta \neq \alpha$. To prove $c_\beta \in U$, we may assume, as G is connected, that one of the relations $c_\alpha \top c_\beta,\ c_\alpha \dashv c_\beta,\ c_\alpha \vdash c_\beta$ holds. The first two of these yield $c_\beta = \{c_\alpha c_\alpha c_\beta\} \in U$, whereas the last one implies $c^\varepsilon_\beta = Q(c^\varepsilon_\alpha)Q(c^{-\varepsilon}_\alpha)c^\varepsilon_\beta \in U^\varepsilon$. $\square$

3.14 Example. A covering grid of a simple Jordan pair need not be simple: Let A be a simple unital associative algebra, $n \in \mathbb{Z},\ n > 1$, and $M_n(A)$ the algebra of n-by-n matrices with entries in A. Writing $e_{ij} \in M_n(A)$ $(1 \leq i, j \leq n)$ for the ordinary matrix units and $V = (M_n(A),\ M_n(A)^{\mathrm{op}})^J$ for the simple Jordan pair associated with $M_n(A)$, we put $G = \{c_{ij}; 1 \leq i \leq j \leq n\}$, where $c_{ii} = (e_{ii}, e_{ii})$ $(1 \leq i \leq n)$, $c_{ij} = (e_{ij} + e_{ji}, e_{ij} + e_{ji})$ $(1 \leq i < j \leq n)$. Then G is a covering grid of V but not simple since the Peirce-12-component of V relative to the standard diagonal frame is a Peirce component of G and isomorphic to the direct sum of two copies of $(A, A^{\mathrm{op}})^J$.

4. Norms.

Given a Jordan pair V (not necessarily division or finitedimensional) over a Henselian field (K, Γ, λ) as in 1.6, we will describe here certain $\mathfrak{o}(K, \lambda)$-subpairs of V derived from 2-finitedimensional covering standard division grids and the concept of a norm. These subpairs will turn out in the next section to be maximal orders in the sense of [19] if V has finite dimension and (K, Γ, λ) is a local field.

4.1 The concept of a norm. Let (K, Γ, λ) be a valued field, so K is a field, Γ is a totally ordered additive abelian group, and $\lambda : K \longrightarrow \Gamma_\infty$ is a valuation. In dealing with Jordan pairs over K containing zero divisors, the concept of a valuation is no longer appropriate and has to be replaced by that of a norm. Let V be a Jordan pair over K and $\triangle$ a totally ordered additive abelian group containing Γ as a totally ordered subgroup. By a λ-*norm* (or just a *norm) of V with values in* $\triangle$ we mean a pair $\rho = (\rho^+,\ \rho^-)$ of mappings $\rho^\varepsilon : V^\varepsilon \longrightarrow \triangle_\infty$ satisfying the following conditions for all $x, x' \in V^\varepsilon,\ y \in V^{-\varepsilon},\ a \in K$:

(N1) $$\rho^\varepsilon(x) = \infty \Longleftrightarrow x = 0.$$

(N2) $$\rho^\varepsilon(x + x') \geq \min(\rho^\varepsilon(x), \rho^\varepsilon(x')).$$

(N3) $$\rho^\varepsilon(Q(x)y) \geq 2\rho^\varepsilon(x) + \rho^{-\varepsilon}(y).$$

(N4) $$\rho^\varepsilon(ax) = \lambda(a) + \rho^\varepsilon(x).$$

See Bruhat-Tits [2, 1.1] for a related concept. Given a norm ρ of V with values in $\triangle$ as above, $\mathfrak{O}(V,\rho)$, where

$$\mathfrak{O}(V,\rho)^\varepsilon = \{x \in V^\varepsilon : \rho^\varepsilon(x) \geq 0\},$$

is an $\mathfrak{o}(K,\lambda)$-subpair of V, called the *norm pair of* (V,ρ). Notice that $\mathfrak{P}(V,\rho)$, where

$$\mathfrak{P}(V,\rho)^\varepsilon = \{x \in V^\varepsilon : \rho^\varepsilon(x) > 0\},$$

need not be an ideal in $\mathfrak{O}(V,\rho)$. If it is, however, we call it the *norm ideal* and $\kappa(V,\rho) = \mathfrak{O}(V,\rho)/\mathfrak{P}(V,\rho)$ the *residue class pair of* (V,ρ).

4.2 Example. Notations being as in 4.1, let λ be discrete (so we may assume $\Gamma = \mathbb{Z}$) and $\mathfrak{O}$ be an $\mathfrak{o}$-order [19, §4 **4.**] in a finitedimensional Jordan pair V oder K. Since $\mathfrak{O}^\varepsilon \subset V^\varepsilon$ is a full $\mathfrak{o}$-lattice, we obtain a filtration

$$\ldots \subset \mathfrak{p}^{m+1}\mathfrak{O} \subset \mathfrak{p}^m\mathfrak{O} \subset \ldots \subset \mathfrak{O} \subset \ldots \subset \mathfrak{p}^{-m}\mathfrak{O} \subset \mathfrak{p}^{-m-1}\mathfrak{O} \subset \ldots$$

for $m \in \mathbb{N}$ such that

$$\bigcap_{m\in\mathbb{Z}} \mathfrak{p}^m\mathfrak{O} = 0, \ \bigcup_{m\in\mathbb{Z}} \mathfrak{p}^m\mathfrak{O} = V$$

This filtration induces maps $\rho^\varepsilon : V^\varepsilon \longrightarrow \mathbb{Z}_\infty$ by setting

$$\rho^\varepsilon(x) = \sup\ \{m \in \mathbb{Z} : x \in \mathfrak{p}^m\mathfrak{O}\ \} \qquad (x \in V^\varepsilon),$$

and it is readily checked that $\rho = (\rho^+,\ \rho^-)$ is a norm of V with values in $\mathbb{Z}$ satisfying $\mathfrak{O}(V,\rho) = \mathfrak{O}$, $\mathfrak{P}(V,\rho) = \mathfrak{p}\mathfrak{O}$ and $\kappa(V,\rho) = \mathfrak{O} \otimes_{\mathfrak{o}} \kappa$. Moreover, ρ satisfies the triple product inequality (cf. 2.15)

(TPI) $$\rho^\varepsilon(\{xyz\}) \geq \rho^\varepsilon(x) + \rho^{-\varepsilon}(y) + \rho^\varepsilon(z)$$

for all $x, z \in V^\varepsilon,\ y \in V^{-\varepsilon}$.

We are now prepared to state the most difficult result of the paper, whose proof will occupy the better part of this section. As a matter of terminology, a cog in a Jordan pair over a field is said to be *r-finitedimensional* ($r \in \mathbb{Z},\ r > 0$) if all its Peirce components are r-finitedimensional Jordan pairs in the sense of 2.10.

4.3 Norm Theorem. *Let (K, Γ, λ) be a Henselian field, V a Jordan pair over K and G a 2-finitedimensional covering standard division grid of V. Write (R, R_1) for the 3-graded root system associated with G and denote by μ_α, for each $\alpha \in R_1$, the unique extension of λ to a valuation of $V_\alpha(G)$ with values in $\triangle = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfying $\mu_\alpha^\varepsilon(c_\alpha^\varepsilon) = 0$ (2.13). Then*
a) *$\rho = (\rho^+,\ \rho^-)$, with $\rho^\varepsilon : V^\varepsilon \longrightarrow \triangle_\infty$ given by*

$$\rho^\varepsilon(\sum_{\alpha \in R_1} x_\alpha) = \inf\nolimits_{\alpha \in R_1} \mu_\alpha^\varepsilon(x_\alpha) \qquad (x_\alpha \in V_\alpha(G)^\varepsilon,\ \alpha \in R_1),$$

is a λ-norm of V.
b) *If every connected component of G contains more than one element, ρ satisfies the triple product inequality*

$$\text{(TPI)} \qquad \rho^\varepsilon(\{xyz\}) \geq \rho^\varepsilon(x) + \rho^{-\varepsilon}(y) + \rho^\varepsilon(z)$$

for all $x, z \in V^\varepsilon,\ y \in V^{-\varepsilon}$.

Comments. If G is any grid in any Jordan pair V, there exists a standard grid G_1 in V *associated with* G [13, 3.8], i.e., some bijection $\varphi : G \longrightarrow G_1$ (necessarily unique) satisfies $\varphi(c) \approx c$ for all $c \in G$. In particular, G and G_1 have isomorphic 3-graded root systems [13, 3.4] and the same Peirce components. Hence, if G is a covering (resp. a division) grid, so is G_1. We also recall from [14, 2.8] that if V is nondegenerate, it admits a finite covering division grid if and only if it is nondegenerate and has dcc on inner ideals. In particular, if V as in 4.3 is finitedimensional and semi-simple, covering standard division grids exist.

Before turning to the proof of the Norm Theorem, we establish an easy corollary.

4.4 Corollary. *Notations being as in 4.3, $\mathfrak{P} = \mathfrak{P}(V, G) = \mathfrak{P}(V, \rho)$ is an ideal in $\mathfrak{O} = \mathfrak{O}(V, G) = \mathfrak{O}(V, \rho)$; more precisely, we have*

$$\text{(4.4.1)} \qquad G \subset \mathfrak{O} = \sum_{\alpha \in R_1} \mathfrak{O}(V_\alpha(G), \mu_\alpha),\ \mathfrak{P} = \sum_{\alpha \in R_1} \mathfrak{P}(V_\alpha(G), \mu_\alpha)$$

and $\mathfrak{P} = \mathrm{Rad}\ \mathfrak{O}$. Furthermore, the natural map from $\mathfrak{O}$ to $\mathfrak{O}/\mathfrak{P}$ maps G as an abstract closed cog isomorphically onto a covering standard division grid G' of $\mathfrak{O}/\mathfrak{P}$. In particular, $\mathfrak{O}/\mathfrak{P}$ is simple if and only if V is simple.

Proof. By 4.3, $\mathfrak{P}$ is an ideal in $\mathfrak{O}$ and (4.4.1) holds. Each $\mu_\alpha,\ \alpha \in R_1$, being a separated valuation of $V_\alpha = V_\alpha(G)$, forcing $\mathfrak{P}(V_\alpha, \mu_\alpha) = \mathrm{Rad}\ \mathfrak{O}(V_\alpha, \mu_\alpha)$ by 2.3e), we deduce $\mathfrak{P} = \mathrm{Rad}\ \mathfrak{O}$ from (4.4.1) and 3.11. The assertion that G canonically induces G' as indicated is obvious and by 3.13 implies the rest.

□

4.5 We now turn to the proof of the Norm Theorem 4.3, which consists of two parts, the first (and shorter) one being purely algebraic in nature. For the time being, we therefore let V be an arbitrary Jordan pair over any commutative associative ring of scalars. An idempotent $c \in V$ is said to be *invertible* if $V = V_2(c)$.

4.6 Lemma. *Suppose $x \in V^+,\ y \in V^-$ are invertible in V and put $v = x^{-1} - y$. Then (x, v) is quasi-invertible in V, with quasi-inverse $x^v = y^{-1}$. Furthermore,*

$$\beta(x, v) = (B(x, v),\ B(-v,\ y^{-1})).$$

Proof. This follows immediately from [7, 2.12, 3.2 and JP35]. □

4.7 Lemma. *Let c be an idempotent in V and put $U = V_2(c),\ W = V_1(c)$.*
a) *For all $x \in U^{\varepsilon\times},\ y \in U^{-\varepsilon}$, we have*

$$D(Q(x)y, x^{-1}) = D(x, y).$$

b) *For all $x \in U^{\varepsilon},\ y \in U^{-\varepsilon},\ y' \in U^{-\varepsilon\times}$, we have*

$$D(Q(x)y, y') = D(x, y')D(y'^{-1}, y)D(x, y') \text{ on } W^{\varepsilon}$$

and

$$D(y', Q(x)y) = D(y', x)D(y, y'^{-1})D(y', x) \text{ on } W^{-\varepsilon}.$$

c) *For all $x \in U^{\varepsilon\times},\ y \in U^{-\varepsilon\times}$, the linear map $D(x, y)$ is bijective on W^{ε} with inverse $D(y^{-1}, x^{-1})$ on W^{ε}.*
d) *Let $x \in U^{+\times},\ y \in U^{-\times}$ and put $v = x^{-1} - y \in U^-$* (cf. 4.6). *Then*

$$\beta(x, v) = (D(x, y), D(x^{-1}, y^{-1})) \text{ on } W;$$

in particular, the right-hand side gives an automorphism of W*.*

Proof. Throughout we may assume $\varepsilon = +$. Observe in a) (resp. b)) that (x, x^{-1}) (resp. $(y'^{-1},\ y')$) is an idempotent of V associated with c [12, I.2.3].
a) Linearizing [7, JP2] yields

$$\begin{aligned} D(Q(x)y, x^{-1}) + D(x, y) &= D(Q(x)y, x^{-1}) + D(Q(x)x^{-1}, y) \\ &= D(x, \{yxx^{-1}\}) = 2D(x, y), \end{aligned}$$

as claimed.
b) We may assume $y' = c^-$ and conclude, using [7, 8.1 (2)],

$$D(Q(x)y, c^-) = D(x, c^-)D(Q(c^+)y, c^-)D(x, c^-)$$

on W^ε. Applying a), the first relation of b) follows, which in turn yields the second:

$$\begin{aligned} D(y', Q(x)y) &= D(Q(y')Q(x)y, y'^{-1}) && \text{(by a))} \\ &= D(Q(Q(y')x)Q(y'^{-1})y, y'^{-1}) \\ &= D(Q(y')x, y'^{-1})D(y', Q(y'^{-1})y)D(Q(y')x, y'^{-1}) \\ &= D(y', x)D(y, y'^{-1})D(y', x) \end{aligned}$$

on $W^{-\varepsilon}$.
c), d) By 4.6, (x, v) is quasi-invertible in U, hence in V [7, 3.2], and $B(x, v) : V^+ \longrightarrow V^+$ is bijective. For $z \in W^+$ we obtain

$$B(x, v)z = z - \{x, x^{-1} - y, z\} + Q(x)Q(v)z = \{xyz\}$$

by the Peirce rules, so $B(x, v) = D(x, y)$ on W^+, giving the first part of c). From a similar computation we deduce $B(-v, y^{-1}) = D(x^{-1}, y^{-1})$ on W^-, giving d) by 4.6. Finally, [7, JP35] and 4.6 yield $B(x, v)^{-1} = B(x^v, -v) = B(y^{-1}, -v)$, which is easily seen to agree with $D(y^{-1}, x^{-1})$ on W^+ and so gives the second part of c) as well. □

4.8 Lemma. *Let $c \in V$ be an idempotent and $U, U' \subset V_2(c)$, $W, W' \subset V_1(c)$ be subpairs satisfying*

$$c \in U,\ \{UU'W\} \subset W',\ \{U'UW'\} \subset W.$$

Suppose $x \in U^{\varepsilon\times}$, $y' \in U'^{-\varepsilon\times}$ make $D(x, y') : W^{\varepsilon} \longrightarrow W'^{\varepsilon}$ injective. Then

$$D(y'^{-1}, x^{-1})D(x, y') = \mathbf{1} \text{ on } W^{\varepsilon}.$$

Proof. By 3.2a), x is invertible in $V_2(c)$. Hence 4.7b) yields

$$D(x, y') = D(x, Q(y')y'^{-1}) = D(x, y')D(y'^{-1}, x^{-1})D(x, y')$$

on W^{ε}, and the assertion follows. □

4.9 Lemma. *Let $G \subset V$ be a covering grid with associated 3-graded root system (R, R_1), write $V_\alpha = V_\alpha(G)$ for $\alpha \in R_1$ and suppose $\alpha, \beta, \gamma \in R_1$ are distinct such that $\{V_\alpha V_\beta V_\gamma\} \neq 0$. Then one of the following configurations is fulfilled.*

(4.9.1) $$c_\alpha \top c_\beta \top c_\gamma \perp c_\alpha.$$

(4.9.2) $$c_\alpha \top c_\beta \top c_\gamma \top c_\alpha.$$

(4.9.3) $$c_\alpha \dashv c_\beta \top c_\gamma \perp c_\alpha.$$

(4.9.4) $$c_\alpha \top c_\beta \vdash c_\gamma \perp c_\alpha.$$

(4.9.5) $$c_\alpha \dashv c_\beta \vdash c_\gamma \perp c_\alpha.$$

(4.9.6) $$c_\alpha \vdash c_\beta \dashv c_\gamma \top c_\alpha.$$

Moreover, $\alpha - \beta + \gamma \in R_1$ and $\{c_\alpha c_\beta c_\gamma\} \in V_{\alpha-\beta+\gamma}$ is a nontrivial idempotent unless we are in configuration (4.9.2).

Proof. By [12, I.3.6b)], $c_\alpha, c_\beta, c_\gamma$ fall into one of the cases 2, 8, 9, 13, 21, 24 - 27 of [12, I.3.5]. The final statement of the lemma being a consequence of [12, I.3.6c)], it therefore suffices to show that the configurations 24 - 26 are impossible. Arguing indirectly, we may invoke [12, I.3.9] to assume $c_\alpha \dashv c_\beta \vdash c_\gamma \top c_\alpha$. Thanks to the No-Tower-Lemma [12, I.3.4] we then have $c_\alpha, c_\gamma \in G^{(1)}$, forcing them to bei rigid collinear by 3.9 and [12, I (1.32)]. But this implies $\{V_\alpha V_\beta V_\gamma\} = 0$ by [12, I.3.9c)], a contradiction. □

4.10 Returning now to the arithmetic setting, we let (K, Γ, λ) be a Henselian field with valuation ring $\mathfrak{o} = \mathfrak{o}(K, \lambda)$, valuation ideal $\mathfrak{p} = \mathfrak{p}(K, \lambda)$ and residue class field $\kappa = \kappa(K, \lambda)$. For the time being, we let V be an arbitrary Jordan pair over K. If V is division, a valuation of V taking values in $\triangle = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ and extending λ will be called a λ-*valuation* for short.

4.11 Proposition. *Let $c \in V$ be an idempotent and $U \subset V_2(c)$, $W \subset V_1(c)$ be 2-finitedimensional division subpairs satisfying $c \in U$ and $\{UUW\} \subset W$. Then*

$$\nu^\varepsilon(\{xyz\}) = \mu^\varepsilon(x) + \mu^{-\varepsilon}(y) + \nu^\varepsilon(z) \tag{4.11.1}$$

for all λ-valuations μ of U, ν of W and for all $x \in U^\varepsilon$, $y \in U^{-\varepsilon}$, $z \in W^\varepsilon$.

Proof. Since (4.11.1) is invariant under translates of valuations (cf. 2.4), it suffices to show that, given a λ-valuation ν of W, there exists a λ-valuation μ of U satisfying (4.11.1). To do so, observe $U^\times \subset V_2(c)^\times$ by 3.2a), fix $v \in U^{-\times}$ and let $x \in U^{+\times}$. Then $(D(x, v), D(x^{-1}, v^{-1}))$ determines an automorphism of W (4.7d)), so 2.12 produces an element $\mu^+(x) \in \triangle$ satisfying

$$\nu^+(\{xvz\}) = \mu^+(x) + \nu^+(z) \qquad (z \in W^+),$$

$$\nu^-(\{x^{-1}v^{-1}z'\}) = -\mu^+(x) + \nu^-(z') \qquad (z' \in W^-).$$

We now put $v^+ = v^{-1}, v^- = v, \mu^+(0) = \mu^-(0) = \infty,\ \mu^-(x') = -\mu^+(x'^{-1})$ for $x' \in U^{-\times}$, and thus obtain maps $\mu^\varepsilon : U^\varepsilon \longrightarrow \triangle_\infty$ satisfying

$$\nu^\varepsilon(\{xv^{-\varepsilon}z\}) = \mu^\varepsilon(x) + \nu^\varepsilon(z) \qquad (x \in U^\varepsilon,\ z \in W^\varepsilon), \tag{1}$$

$$\nu^{-\varepsilon}(\{x^{-1}v^\varepsilon z'\}) = -\mu^\varepsilon(x) + \nu^{-\varepsilon}(z') \qquad (x \in U^{\varepsilon\times},\ z' \in W^{-\varepsilon}). \tag{2}$$

We claim that (2) implies

$$\nu^\varepsilon(\{v^\varepsilon yz\}) = \mu^{-\varepsilon}(y) + \nu^\varepsilon(z) \qquad (y \in U^{-\varepsilon},\ z \in W^\varepsilon). \tag{3}$$

To see this, we may assume $y \neq 0$ and conclude

$$\nu^\varepsilon(z) = \nu^\varepsilon(\{y^{-1}v^{-\varepsilon}\{v^\varepsilon yz\}) \qquad \text{(by 4.7c))}$$

$$= -\mu^{-\varepsilon}(y) + \nu^{\varepsilon}(\{v^{\varepsilon}yz\}),$$

as desired. Using 4.7b) and (1), (3) above, it is now readily checked that $\mu = (\mu^+, \mu^-)$ is a λ-valuation of U; observe, however, that $\mu = \mu_v$ depends on v. In order to prove (4.11.1), we may assume $y \neq 0$ and have

$$\nu^{\varepsilon}(\{xyz\}) = \mu_y^{\varepsilon}(x) + \nu^{\varepsilon}(z) \qquad (x \in U^{\varepsilon},\ z \in W^{\varepsilon}) \tag{4}$$

by (1) (applied to V^{op} for $\varepsilon = -$). Hence 2.11 implies $\mu_y = \mu + \delta$ for some $\delta \in \triangle$, and putting $x = y^{-1}$ in (4) gives $\varepsilon\delta = -\mu^{\varepsilon}(y^{-1}) = \mu^{-\varepsilon}(y)$ by 2.3a). Returning to (4) yields (4.11.1). $\square$

4.12 Proposition. *Let $F = (e_1, e_2)$ be an orthogonal system of idempotents in V. Suppose for $i, j = 1, 2$ that $U_i \subset V_{ii}(F)$, $W_j \subset V_{12}(F)$ are 2-finitedimensional division subpairs satisfying $e_i \in U_i$ as well as the relations*

$$\{U_1 W_j U_2\} \subset W_{3-j}.$$

If $f_j \in W_j$ $(j = 1, 2)$ are nontrivial idempotents, then so are $\{e_1 f_j e_2\} \in W_{3-j}$, and

$$\begin{aligned}(4.12.1)\qquad & \nu_{3-j}^{\varepsilon}(\{x_1 y x_2\}) + \nu_{3-j}^{-\varepsilon}(\{e_1^{-\varepsilon} f_j^{\varepsilon} e_2^{\varepsilon}\}) = \\ & \mu_1^{\varepsilon}(x_1) + \nu_j^{-\varepsilon}(y) + \mu_2^{\varepsilon}(x_2) + \mu_1^{-\varepsilon}(e_1) + \nu_j^{\varepsilon}(f_j) + \mu_2^{-\varepsilon}(e_2)\end{aligned}$$

for all λ-valuations μ_i of U_i, ν_j of W_j and all $x_i \in U_i^{\varepsilon}$, $y \in W_j^{-\varepsilon}$. Furthermore,

$$\nu_1^{\varepsilon}(f_1^{\varepsilon}) + \nu_2^{\varepsilon}(\{e_1^{\varepsilon} f_1^{-\varepsilon} e_2^{\varepsilon}\}) = \nu_2^{\varepsilon}(f_2^{\varepsilon}) + \nu_1^{\varepsilon}(\{e_1^{\varepsilon} f_2^{-\varepsilon} e_2^{\varepsilon}\}). \tag{4.12.2}$$

Proof. We have $U_i^{\varepsilon\times} \subset V_{ii}(F)^{\varepsilon\times}$ by 3.2a). Given $u_i^+ \in U_i^{+\times}$ with inverse $u_i^- = (u_i^+)^{-1} \in U_i^{-\times}$, this is easily seen to imply that

$$\eta = (Q(u_1^+ + u_2^+),\ Q(u_1^- + u_2^-)) : W_j^{\mathrm{op}} \xrightarrow{\sim} W_{3-j} \qquad (j = 1, 2) \tag{1}$$

is an isomorphism with inverse

$$\eta^{-1} = (Q(u_1^- + u_2^-),\ Q(u_1^+ + u_2^+)) : W_{3-j} \xrightarrow{\sim} W_j^{\mathrm{op}} \tag{2}$$

(notice $\eta^+(y) = \{u_1^+ y\ u_2^+\}$ for $y \in W_j^-$ etc.). In particular, specializing u_i^+ to e_i^+ yields an isomorphism $W_j^{\mathrm{op}} \longrightarrow W_{3-j}$ sending f_j^{op} to $\{e_1 f_j e_2\}$, which therefore is a nontrivial idempotent in W_{3-j}. Hence, if some λ-valuations $\mu_i,\ \nu_j\ (i,j=1,2)$ satisfy (4.12.1,2), all do (2.14). It therefore suffices to prove that, given λ - valuations μ_2 of $U_2,\ \nu_j$ of $W_j\ (j=1,2)$, there exists a λ-valuation μ_1 of U_1 satisfying

$$\nu_{3-j}^{\varepsilon}(\{x_1 y x_2\}) = \mu_1^{\varepsilon}(x_1) + \nu_j^{-\varepsilon}(y) + \mu_2^{\varepsilon}(x_2) \tag{3}$$

for all $x_i \in U_i^{\varepsilon},\ y \in W_j^{-\varepsilon}\ (i,j=1,2)$. Indeed, (3) easily implies (4.12.1,2) by using the specializations $x_i = e_i^{\varepsilon},\ y = f_j^{-\varepsilon}$ and 2.3a). Now fix $u_2^+ \in U_2^{+\times}$ with inverse $u_2^- \in U_2^{-\times}$ as above. Given $x_1^+ \in U_1^{+\times}$ with inverse $x_1^- \in U_1^{-\times}$, the isomorphism η of (1) (with $j=1$ and x_1^{ε} replacing u_1^{ε}) together with 2.12 yields an element $\mu_1^+(x_1^+) \in \triangle$ satisfying $\nu_2 \circ \eta = \nu_1^{\mathrm{op}} + \mu_1^+(x_1^+) + \mu_2^+(u_2^+)$, i.e.,

$$\nu_2^+(\{x_1^+ y u_2^+\}) = \mu_1^+(x_1^+) + \nu_1^-(y) + \mu_2^+(u_2^+) \qquad (y \in W_1^-),$$

$$\nu_2^-(\{x_1^- y u_2^-\}) = -\mu_1^+(x_1^+) + \nu_1^+(y) - \mu_2^+(u_2^+) \qquad (y \in W_1^+).$$

Setting $\mu_1^+(0) = \mu_1^-(0) = \infty$ and $\mu_1^-(z_1^-) = -\mu_1^+((z_1^-)^{-1})$ for $z_1^- \in U_1^{-\times}$, we thus obtain mappings $\mu_1^{\varepsilon} : U_1^{\varepsilon} \longrightarrow \triangle_\infty$ such that

$$\nu_2^{\varepsilon}(\{x_1^{\varepsilon} y u_2^{\varepsilon}\}) = \mu_1^{\varepsilon}(x_1^{\varepsilon}) + \nu_1^{-\varepsilon}(y) + \mu_2^{\varepsilon}(u_2^{\varepsilon}) \qquad (x_1^{\varepsilon} \in U_1^{\varepsilon},\ y \in W_1^{-\varepsilon}).$$

Assuming here $x_1^{\varepsilon} \neq 0$ and replacing y by $\{x_1^{-\varepsilon} y\ u_2^{-\varepsilon}\}$ $(x_1^{-\varepsilon} = (x_1^{\varepsilon})^{-1},\ y \in W_2^{\varepsilon})$, we obtain the same equation with $-\varepsilon$ instead of ε and the indices $j=1,2$ interchanged. Hence

$$\nu_{3-j}^{\varepsilon}(\{x_1^{\varepsilon} y u_2^{\varepsilon}\}) = \mu_1^{\varepsilon}(x_1^{\varepsilon}) + \nu_j^{-\varepsilon}(y) + \mu_2^{\varepsilon}(u_2^{\varepsilon}) \qquad (x_1^{\varepsilon} \in U_1^{\varepsilon},\ y \in W_j^{-\varepsilon}) \tag{4}$$

for $j=1,2$. Using (4) and the fundamental formula, it is now straightforward to check that μ_1 is a λ-valuation of U_1. This allows us to prove (3) for the valuations thus constructed. Indeed, we may assume $x_1^{\varepsilon} = x_1 \neq 0$ and, putting $x_1^{-\varepsilon} = x_1^{-1}$, apply the previous considerations with e_1, e_2 interchanged to obtain a λ-valuation μ_2' of U_2 satisfying

$$\nu_{3-j}^{\varepsilon}(\{x_1^{\varepsilon} y x_2^{\varepsilon}\}) = \mu_1^{\varepsilon}(x_1^{\varepsilon}) + \nu_j^{-\varepsilon}(y) + \mu_2'^{\varepsilon}(x_2^{\varepsilon}) \qquad (x_2^{\varepsilon} \in U_2^{\varepsilon},\ y \in W_j^{-\varepsilon}) \tag{5}$$

for $j=1,2$. By 2.11, $\mu_2' = \mu_2 + \delta$ for some $\delta \in \Delta$. Specializing x_2^{ε} to u_2^{ε} in (5) and comparing the result with (4), we conclude $\delta = 0$, and (3) follows. $\square$

4.13 Proposition. *In the situation of* 4.12, *assume* $W_1 = W_2 = W$, $f_1 = f_2 = f$ *and let* μ_i *(resp.* ν*) be arbitrary* λ*-valuations of* U_i *(resp.* W*)* $(i = 1, 2)$.
a) *If the relations*

$$Q(W)U_i \subset U_{3-i},\ \{U_iU_iW\} \subset W \tag{4.13.1}$$

hold for $i = 1, 2$, *then*

$$\begin{aligned}(4.13.2)\qquad & \mu^{\varepsilon}_{3-i}(Q(y)x) + \mu^{-\varepsilon}_{3-i}(e^{-\varepsilon}_{3-i}) = 2\nu^{\varepsilon}(y) + \\ & \qquad \mu^{-\varepsilon}_i(x) + \nu^{-\varepsilon}(f^{-\varepsilon}) + \nu^{-\varepsilon}(\{e^{-\varepsilon}_1 f^{\varepsilon} e^{-\varepsilon}_2\}) + \mu^{\varepsilon}_i(e^{\varepsilon}_i) \\ (4.13.3)\qquad & \mu^{\varepsilon}_{3-i}(\{yxz\}) + \mu^{-\varepsilon}_{3-i}(e^{-\varepsilon}_{3-i}) \geq \nu^{\varepsilon}(y) + \\ & \qquad \mu^{-\varepsilon}_i(x) + \nu^{\varepsilon}(z) + \nu^{-\varepsilon}(f^{-\varepsilon}) + \nu^{-\varepsilon}(\{e^{-\varepsilon}_1 f^{\varepsilon} e^{\varepsilon}_2\}) + \mu^{\varepsilon}_i(e^{\varepsilon}_i)\end{aligned}$$

for all $y, z \in W^{\varepsilon}$, $x \in U^{-\varepsilon}_i$.
b) *If the relations* (4.13.1) *and*

$$\{WWU_i\} \subset U_i \tag{4.13.4}$$

hold for $i = 1, 2$, *then*

$$\mu^{\varepsilon}_i(\{yzx\}) \geq \nu^{\varepsilon}(y) + \nu^{-\varepsilon}(z) + \mu^{\varepsilon}_i(x) \tag{4.13.5}$$

for all $y \in W^{\varepsilon}, z \in W^{-\varepsilon}, x \in U^{\varepsilon}_i$.

Proof. By the usual argument involving (2.14), it suffices to *exhibit* λ-valuations μ_i, ν as above satisfying the asserted conditions. For this purpose, we may assume $\mu^{\varepsilon}_i(e^{\varepsilon}_i) = \nu^{\varepsilon}(f^{\varepsilon}) = 0$ for $i = 1, 2$ by (2.13).
a) Setting $x_i = x$ and choosing $x_{3-i} \in U^{-\varepsilon}_{3-i}$, [7, JP1] yields

$$\{Q(y)x_i, x_{3-i}, y\} = Q(y)\{x_1yx_2\},$$

and we obtain

$$\begin{aligned}&\mu^{\varepsilon}_{3-i}(Q(y)x_i) + \mu^{-\varepsilon}_{3-i}(x_{3-i}) + \nu^{\varepsilon}(y) \\ &\qquad = \nu^{\varepsilon}(\{Q(y)x_i, x_{3-i}, y\}) && \text{(by 4.11)} \\ &\qquad = \nu^{\varepsilon}(Q(y)\{x_1yx_2\}) = 2\nu^{\varepsilon}(y) + \nu^{-\varepsilon}(\{x_1yx_2\})\end{aligned}$$

$$= 2\nu^\varepsilon(y) + \mu_i^{-\varepsilon}(x) + \nu^\varepsilon(y) + \mu_{3-i}^\varepsilon(x_{3-i}) -$$
$$\nu^\varepsilon(\{e_1^\varepsilon f^{-\varepsilon} e_2^\varepsilon\}) \qquad \text{(by (4.12.1))},$$

hence (4.13.2). In order to deduce (4.13.3), we compute

$$\begin{aligned}\{\{yx_iz\}x_{3-i}z\} &= Q(z, \{yx_iz\})x_{3-i} \\ &= D(y, x_i)Q(z)x_{3-i} + Q(z)D(x_i, y)x_{3-i} \qquad \text{(by [7, JP12])},\end{aligned}$$

which implies

$$\{\{yx_iz\}x_{3-i}z\} = \{y, x_i, Q(z)x_{3-i}\} + Q(z)\{x_1yx_2\}.$$

Hence

$$\begin{aligned}&\mu_{3-i}^\varepsilon(\{yx_iz\}) + \mu_{3-i}^{-\varepsilon}(x_{3-i}) + \nu^\varepsilon(z) \\ &= \nu^\varepsilon(\{\{yx_iz\}x_{3-i}z\} \qquad \text{(by 4.11)} \\ &= \nu^\varepsilon\big(\{y, x_i, Q(z)x_{3-i}\} + Q(z)\{x_1yx_2\}\big) \\ &\geq \min\left(\nu^\varepsilon(\{y, x_i, Q(z)x_{3-i}\}), \nu^\varepsilon(Q(z)\{x_1yx_2\})\right) \\ &= \min\big(\nu^\varepsilon(y) + \mu_i^{-\varepsilon}(x_i) + \mu_i^\varepsilon(Q(z)x_{3-i}), 2\nu^\varepsilon(z) + \nu^{-\varepsilon}(\{x_1yx_2\})\big) \qquad \text{(by 4.11)} \\ &= \min\big(\nu^\varepsilon(y) + \mu_i^{-\varepsilon}(x_i) + 2\nu^\varepsilon(z) + \mu_{3-i}^{-\varepsilon}(x_{3-i}) + \nu^{-\varepsilon}(\{e_1^{-\varepsilon} f^\varepsilon e_2^{-\varepsilon}\}), \\ &\qquad 2\nu^\varepsilon(z) + \mu_1^{-\varepsilon}(x_1) + \nu^\varepsilon(y) + \mu_2^{-\varepsilon}(x_2) \\ &\qquad - \nu^\varepsilon(\{e_1^\varepsilon f^{-\varepsilon} e_2^\varepsilon\})\big) \qquad \text{(by (4.13.2), (4.12.1))} \\ &= \nu^\varepsilon(y) + \mu_i^{-\varepsilon}(x) + \nu^\varepsilon(z) + \nu^{-\varepsilon}(\{e_1^{-\varepsilon} f^\varepsilon e_2^{-\varepsilon}\}) + \mu_{3-i}^{-\varepsilon}(x_{3-i}) + \nu^\varepsilon(z),\end{aligned}$$

which yields (4.13.3).
b) Putting $x_i = x$ and choosing $x_{3-i} \in U_{3-i}^\varepsilon$, we use [7, JP7] to compute

$$\begin{aligned}\{\{yzx_i\}zx_{3-i}\} &= D(\{yzx_i\}, z)x_{3-i} \\ &= D(x_i, Q(z)y)x_{3-i} + D(y, Q(z)x_i)x_{3-i} \\ &= \{x_1, Q(z)y, x_2\} + \{y, Q(z)x_i, x_{3-i}\}.\end{aligned}$$

Hence

$$
\begin{aligned}
&\mu_i^\varepsilon(\{yzx_i\}) + \nu^{-\varepsilon}(z) + \mu_{3-i}^\varepsilon(x_{3-i}) \\
&\qquad = \nu^\varepsilon(\{\{yzx_i\}zx_{3-i}\}) + \nu^{-\varepsilon}(\{e_1^{-\varepsilon} f^\varepsilon e_2^{-\varepsilon}\}) && \text{(by (4.12.1))} \\
&\qquad = \nu^\varepsilon(\{x_1, Q(z)y, x_2\} + \{y, Q(z)x_i, x_{3-i}\}) + \nu^{-\varepsilon}(\{e_1^{-\varepsilon} f^\varepsilon e_2^{\varepsilon}\}) \\
&\qquad \geq \min\bigl(\nu^\varepsilon(\{x_1, Q(z)y, x_2\}) + \nu^{-\varepsilon}(\{e_1^{-\varepsilon} f^\varepsilon e_2^{-\varepsilon}\}), \\
&\qquad\qquad \nu^\varepsilon(\{y, Q(z)x_i, x_{3-i}\} + \nu^{-\varepsilon}(\{e_1^{-\varepsilon} f^\varepsilon e_2^{-\varepsilon}\})\bigr) \\
&\qquad = \min\bigl(\mu_i^\varepsilon(x_i) + 2\nu^{-\varepsilon}(z) + \nu^\varepsilon(y) + \mu_{3-i}^\varepsilon(x_{3-i}), \\
&\qquad\qquad \nu^\varepsilon(y) + \mu_{3-i}^{-\varepsilon}(Q(z)x_i) + \mu_{3-i}^\varepsilon(x_{3-i}) + \nu^{-\varepsilon}(\{e_1^{-\varepsilon} f^\varepsilon e_2^{-\varepsilon}\})\bigr) \\
&\qquad && \text{(by (4.12.1), (4.11))} \\
&\qquad = \mu_i^\varepsilon(x_i) + 2\nu^{-\varepsilon}(z) + \nu^\varepsilon(y) + \mu_{3-i}^\varepsilon(x_{3-i}) && \text{(by 4.13.2),}
\end{aligned}
$$

and this is (4.13.5). ☐

4.14 Our next aim is to show that ρ as defined in 4.3 is a λ-norm of V. Conditions (N1,2,4) of 4.1 being trivially fulfilled, it remains to prove (N3). To this end, we put $V_\alpha = V_\alpha(G)$ for $\alpha \in R_1$, consider the Peirce decomposition of x, y relative to G and conclude that it suffices to establish the relations

$$\mu_{2\alpha-\beta}^\varepsilon(Q(x)y) \geq 2\mu_\alpha^\varepsilon(x) + \mu_\beta^{-\varepsilon}(y) \tag{4.14.1}$$

for all $\alpha, \beta \in R_1,\ x \in V_\alpha^\varepsilon,\ y \in V_\beta^{-\varepsilon}$ provided $2\alpha - \beta \in R_1$, and

$$\mu_{\alpha-\beta+\gamma}^\varepsilon(\{xyz\}) \geq \mu_\alpha^\varepsilon(x) + \mu_\beta^{-\varepsilon}(y) + \mu_\gamma^\varepsilon(z) \tag{4.14.2}$$

for all $\alpha, \beta, \gamma \in R_1$, $x \in V_\alpha^\varepsilon,\ y \in V_\beta^{-\varepsilon},\ z \in V_\gamma^\varepsilon$ provided $\alpha - \beta + \gamma \in R_1$, and α, β, γ are not all equal.

4.15 We wish to derive (4.14.1,2), or, more precisely, modified versions of these formulae in the spirit of 4.12, 4.13, under the weaker assumption that G is just a 2-finitedimensional covering division grid of V; so, until further notice, *we no longer demand that G be standard.* As before, we write (R, R_1) for the 3-graded root system associated with G and let μ_α, for $\alpha \in R_1$, be *any* λ-valutation of $V_\alpha = V_\alpha(G)$.

4.16 Proposition. *Notations being as in* 4.15, *suppose* $\alpha, \beta \in R_1$ *satisfy* $c_\alpha \vdash c_\beta$. *Then* $\gamma = 2\alpha - \beta \in R_1$ *and* $(c_\alpha, c_\beta, c_\gamma)$ *is a root triangle; more precisely,*

$$c_\beta \dashv c_\alpha \vdash c_\gamma \perp c_\beta,\ c_\gamma \approx Q(c_\alpha)c_\beta. \tag{4.16.1}$$

Furthermore, the relations

$$\begin{aligned}
&\mu_\alpha^\varepsilon(\{uvw\}) + \mu_\alpha^{-\varepsilon}(c_\alpha^{-\varepsilon}) = && (4.16.2)\\
&\mu_\gamma^\varepsilon(u) + \mu_\alpha^{-\varepsilon}(v) + \mu_\beta^\varepsilon(w) + \mu_\gamma^{-\varepsilon}(Q(c_\alpha^{-\varepsilon})c_\beta^\varepsilon) + \\
&\mu_\alpha^\varepsilon(c_\alpha^\varepsilon) + \mu_\beta^{-\varepsilon}(c_\beta^\varepsilon),\\
&\mu_\gamma^\varepsilon(Q(x)y) + \mu_\gamma^{-\varepsilon}(Q(c_\alpha^{-\varepsilon})c_\beta^\varepsilon) = && (4.16.3)\\
&2\mu_\alpha^\varepsilon(x) + \mu_\beta^{-\varepsilon}(y) + 2\mu_\alpha^{-\varepsilon}(c_\alpha^{-\varepsilon}) + \mu_\beta^\varepsilon(c_\beta^\varepsilon),\\
&\mu_\gamma^\varepsilon(\{xyz\}) + \mu_\gamma^{-\varepsilon}(Q(c_\alpha^{-\varepsilon})c_\beta^\varepsilon) \geq && (4.16.4)\\
&\mu_\alpha^\varepsilon(x) + \mu_\beta^{-\varepsilon}(y) + \mu_\alpha^\varepsilon(z) + 2\mu_\alpha^{-\varepsilon}(c_\alpha^{-\varepsilon}) + \mu_\beta^\varepsilon(c_\beta^\varepsilon),\\
&\mu_\beta^\varepsilon(\{xvy\}) \geq \mu_\alpha^\varepsilon(x) + \mu_\alpha^{-\varepsilon}(v) + \mu_\beta^\varepsilon(y) && (4.16.5)
\end{aligned}$$

hold for all $u \in V_\gamma^\varepsilon,\ v \in V_\alpha^{-\varepsilon},\ w \in V_\beta^\varepsilon,\ x, z \in V_\alpha^\varepsilon,\ y \in V_\beta^{-\varepsilon}$.

Proof. [12, I.3.6a)] implies $\gamma \in R_1$. From [12, I.2.5] combined with [12, I.3.3] we conclude (4.16.1) and that $(c_\alpha, c_\beta, Q(c_\alpha)c_\beta)$ is a triangle; in particular, $\{c_\beta, c_\alpha, Q(c_\alpha)c_\beta\} = c_\alpha$. Hence the hypotheses of 4.12, 4.13 are fulfilled by setting $e_1 = c_\beta, e_2 = Q(c_\alpha)c_\beta, U_1 = V_\beta, U_2 = V_\gamma,\ W = W_1 = W_2 = V_\alpha,\ f = f_1 = f_2 = c_\alpha,\ \mu_1 = \mu_\beta,\ \mu_2 = \mu_\gamma,\ \nu = \nu_1 = \nu_2 = \mu_\alpha$, and (4.16.2-5) follow. □

4.17 Proposition. *Notations being as in* 4.15, *suppose* $\alpha, \beta \in R_1$ *satisfy* $c_\alpha \top c_\beta$ *or* $c_\alpha \dashv c_\beta$. *Then*

$$\mu_\beta^\varepsilon(\{xyz\}) = \mu_\alpha^\varepsilon(x) + \mu_\alpha^{-\varepsilon}(y) + \mu_\beta^\varepsilon(z)$$

for all $x \in V_\alpha^\varepsilon, y \in V_\alpha^{-\varepsilon}, z \in V_\beta^\varepsilon$.

Proof. Put $c = c_\alpha,\ U = V_\alpha,\ W = V_\beta,\ \mu = \mu_\alpha,\ \nu = \mu_\beta$ in 4.11 and apply (4.11.1). $\square$

4.18 Using 4.16 we can derive (4.14.1). To do so, we may assume $c_\alpha \vdash c_\beta$ [12, I.3.6a)]. But then $(c_\alpha, c_\beta, c_\gamma), \gamma = 2\alpha - \beta$, is a root triangle (4.16), hence a triangle since G is standard (3.6.(SG1)); in particular $Q(c_\alpha)c_\beta = c_\gamma$, and the assertion follows from (4.16.3). Similarly, (4.16.4) implies (4.14.2) for $\alpha = \gamma \neq \beta$. On the other hand, if $\alpha = \beta \neq \gamma$ and without loss $\{V_\alpha V_\alpha V_\gamma\} \neq 0$, then $c_\alpha \top c_\gamma$ or $c_\alpha \dashv c_\gamma$ or $c_\alpha \vdash c_\gamma$. In the first two cases, (4.14.2) follows from 4.17. In the third, we may apply 4.16 with the roles of β, γ interchanged to deduce (4.14.2) from (4.16.5).
To complete the proof of (4.14.2), we may therefore assume from now on *that* $\alpha, \beta, \gamma \in R_1$ *are distinct.* In order to establish (4.14.2) under this additional hypothesis, we will apply 4.9 to assume without loss that $c_\alpha, c_\beta, c_\gamma$ satisfy one of the configuration (4.9.1-6), which will now be treated separately under the more general circumstances described in 4.15.

4.19 Proposition. *Notations being as in* 4.15, *suppose* $\alpha, \beta, \gamma \in R_1$ *are distinct and one of the following configurations holds.*
(i) $c_\alpha \top c_\beta \top c_\gamma \perp c_\alpha$,
(ii) $c_\alpha \dashv c_\beta \top c_\gamma \perp c_\alpha$,
(iii) $c_\alpha \top c_\beta \vdash c_\gamma \perp c_\alpha$,
(iv) $c_\alpha \dashv c_\beta \vdash c_\gamma \perp c_\alpha$,
(v) $c_\alpha \vdash c_\beta \dashv c_\gamma \top c_\alpha$.
Then $\alpha - \beta + \gamma \in R_1$, *and*

$$\mu_{\alpha-\beta+\gamma}^\varepsilon(\{xyz\}) + \mu_{\alpha-\beta+\gamma}^{-\varepsilon}(\{c_\alpha^{-\varepsilon} c_\beta^\varepsilon c_\gamma^{-\varepsilon}\}) = \mu_\alpha^\varepsilon(x) + \mu_\beta^{-\varepsilon}(y) + \mu_\gamma^\varepsilon(z) + \mu_\alpha^{-\varepsilon}(c_\alpha^{-\varepsilon}) + \mu_\beta^\varepsilon(c_\beta^\varepsilon) + \mu_\gamma^{-\varepsilon}(c_\gamma^{-\varepsilon}) \tag{4.19.1}$$

for all $x \in V_\alpha^\varepsilon,\ y \in V_\beta^{-\varepsilon},\ z \in V_\gamma^\varepsilon$.

Proof. By 4.9, $\alpha - \beta + \gamma \in R_1$, and $\{c_\alpha c_\beta c_\gamma\}$ is a nontrivial idempotent. Suppose first that (i) - (iv) hold. Then (4.19.1) follows from (4.12.1) by specializing $e_1 = c_\alpha,\ e_2 = c_\gamma,\ U_1 = V_\alpha,\ U_2 = V_\gamma,\ W_1 = V_\beta,\ W_2 = V_{\alpha-\beta+\gamma}, f_1 =$

$c_\beta, f_2 = \{c_\alpha c_\beta c_\gamma\},\ \mu_1 = \mu_\alpha,\ \mu_2 = \mu_\gamma,\ \nu_1 = \mu_\beta,\ \nu_2 = \mu_{\alpha-\beta+\gamma}$. The case (v) is more troublesome. Observing symmetry in α, γ and setting $c = \{c_\alpha c_\beta c_\gamma\}$, we conclude from [12, I.2.7b)] that $(c_\beta, c_\gamma, c, c_\alpha)$ is a diamond; in particular, the defining equations [12, I (2.11,12)] yield

$$c_\beta \dashv c_\gamma \top c \perp c_\beta,\ \{c_\beta c_\gamma c\} = c_\alpha. \tag{1}$$

Observe now that replacing $c_{\alpha-\beta+\gamma}$ by c converts G into a 2-finitedimensional covering division grid $G_1 \approx G$ and $c_\beta, c_\gamma, c \in G_1$ satisfy configuration (ii) above. Hence (4.19.1) applies and, when combined with (1), gives

$$\begin{aligned}\mu_\alpha^{-\varepsilon}(\{yzw\}) + \mu_\alpha^\varepsilon(c_\alpha^\varepsilon) = \mu_\beta^{-\varepsilon}(y) + \mu_\gamma^\varepsilon(z) + \\ \mu_{\alpha-\beta+\gamma}^{-\varepsilon}(w) + \mu_\beta^\varepsilon(c_\beta^\varepsilon) + \mu_\gamma^{-\varepsilon}(c_\gamma^{-\varepsilon}) + \mu_{\alpha-\beta+\gamma}^\varepsilon(c^\varepsilon)\end{aligned} \tag{2}$$

for $y \in V_\beta^{-\varepsilon},\ z \in V_\gamma^\varepsilon,\ w \in V_{\alpha-\beta+\gamma}^\varepsilon$. In order to prove (4.19.1) for α, β, γ (rather than $\beta, \gamma, \alpha - \beta + \gamma$), we may of course assume that x, y, z are invertible in their respective Peirce components, with inverses x^{-1}, y^{-1}, z^{-1}, respectively. Hence they give rise canonically to nontrivial idempotents $e_\alpha \in V_\alpha, e_\beta \in V_\beta, e_\gamma \in V_\gamma$ satisfying (v), so $\{zyx\} = \{xyz\} \neq 0$ by [12, I.3.6c)]. This amounts to $D(z, y) : V_\alpha^\varepsilon \longrightarrow V_{\alpha-\beta+\gamma}^\varepsilon$ being injective and allows us to apply 4.8. We conclude

$$\begin{aligned}&\mu_{\alpha-\beta+\gamma}^\varepsilon(\{xyz\}) = \mu_{\alpha-\beta+\gamma}^\varepsilon(\{zyx\}) \\ &= \mu_\alpha^\varepsilon(\{y^{-1}z^{-1}\{zyx\}\}) + \mu_\alpha^{-\varepsilon}(c_\alpha^{-\varepsilon}) - \mu_\beta^\varepsilon(y^{-1}) - \mu_\gamma^{-\varepsilon}(z^{-1}) \\ &\quad -\mu_\beta^{-\varepsilon}(c_\beta^{-\varepsilon}) - \mu_\gamma^\varepsilon(c_\gamma^\varepsilon) - \mu_{\alpha-\beta+\gamma}^{-\varepsilon}(c^{-\varepsilon}) && \text{(by (2))} \\ &= \mu_\alpha^\varepsilon(x) + \mu_\beta^{-\varepsilon}(y) + \mu_\gamma^\varepsilon(z) + \mu_\alpha^{-\varepsilon}(c_\alpha^{-\varepsilon}) + \mu_\beta^\varepsilon(c_\beta^\varepsilon) + \mu_\gamma^{-\varepsilon}(c_\gamma^{-\varepsilon}) \\ &\quad -\mu_{\alpha-\beta+\gamma}^{-\varepsilon}(c^{-\varepsilon}) && \text{(by 2.3a)),}\end{aligned}$$

and (4.19.1) follows. □

4.20 Proposition. *Notations being as in* 4.15, *suppose* $\alpha, \beta, \gamma \in R_1$ *are distinct such that* $c_\alpha \top c_\beta \top c_\gamma \top c_\alpha$ *and* $V_{\alpha-\beta+\gamma} \neq 0$. *Then* $\alpha - \beta + \gamma \in R_1, \{c_\beta c_\alpha c_{\beta-\alpha+\gamma}\}$ *is a nontrivial idempotent and*

(4.20.1) $\mu^{\varepsilon}_{\alpha-\beta+\gamma}(\{xyz\}) + \mu^{-\varepsilon}_{\alpha-\beta+\gamma}(c^{-\varepsilon}_{\alpha-\beta+\gamma}) \geq$

$$\mu^{\varepsilon}_{\alpha}(x) + \mu^{-\varepsilon}_{\beta}(y) + \mu^{\varepsilon}_{\gamma}(z) + \mu^{-\varepsilon}_{\alpha}(c^{-\varepsilon}_{\alpha}) + \mu^{\varepsilon}_{\beta}(c^{\varepsilon}_{\beta}) + \mu^{-\varepsilon}_{\gamma}(\{c^{-\varepsilon}_{\beta} c^{\varepsilon}_{\alpha} c^{-\varepsilon}_{\alpha-\beta+\gamma}\})$$

for all $x \in V^{\varepsilon}_{\alpha}, y \in V^{-\varepsilon}_{\beta}, z \in V^{\varepsilon}_{\gamma}$.

Proof. The relation $\alpha - \beta + \gamma \in R_1$ being obvious, it follows easily that $c_{\alpha-\beta+\gamma}$ fits into the configurations

(1) $$c_\beta \top c_\alpha \vdash c_{\alpha-\beta+\gamma} \perp c_\beta,\ c_\gamma \vdash c_{\alpha-\beta+\gamma}.$$

Hence $\{c_\beta c_\alpha c_{\alpha-\beta+\gamma}\} \in V_\gamma,\ \{c_\beta c_\gamma c_{\alpha-\beta+\gamma}\} \in V_\alpha$ are nontrivial idempotents [12, I.3.6c)]. Setting $e_1 = c_\beta, e_2 = c_{\alpha-\beta+\gamma},\ U_1 = V_\beta,\ U_2 = V_{\alpha-\beta+\gamma},\ W_1 = V_\alpha, W_2 = V_\gamma,\ f_1 = c_\alpha,\ f_2 = c_\gamma, \mu_1 = \mu_\beta,\ \mu_2 = \mu_{\alpha-\beta+\gamma},\ \nu_1 = \mu_\alpha,\ \nu_2 = \mu_\gamma$ in 4.12, we conclude

(2) $\mu^{-\varepsilon}_{\alpha}(\{yzw\}) + \mu^{\varepsilon}_{\alpha}(\{c^{\varepsilon}_{\beta} c^{-\varepsilon}_{\gamma} c^{\varepsilon}_{\alpha-\beta+\gamma}\}) =$

$$\mu^{-\varepsilon}_{\beta}(y) + \mu^{\varepsilon}_{\gamma}(z) + \mu^{-\varepsilon}_{\alpha-\beta+\gamma}(w) + \mu^{\varepsilon}_{\beta}(c^{\varepsilon}_{\beta}) + \mu^{-\varepsilon}_{\gamma}(c^{-\varepsilon}_{\gamma}) + \mu^{\varepsilon}_{\alpha-\beta+\gamma}(c^{\varepsilon}_{\alpha-\beta+\gamma}),$$

(3) $\mu^{-\varepsilon}_{\gamma}(\{yxw\}) + \mu^{\varepsilon}_{\gamma}(\{c^{\varepsilon}_{\beta} c^{-\varepsilon}_{\alpha} c^{\varepsilon}_{\alpha-\beta+\gamma}\}) =$

$$\mu^{-\varepsilon}_{\beta}(y) + \mu^{\varepsilon}_{\alpha}(x) + \mu^{-\varepsilon}_{\alpha-\beta+\gamma}(w) + \mu^{\varepsilon}_{\beta}(c^{\varepsilon}_{\beta}) + \mu^{-\varepsilon}_{\alpha}(c^{-\varepsilon}_{\alpha}) + \mu^{\varepsilon}_{\alpha-\beta+\gamma}(c^{\varepsilon}_{\alpha-\beta+\gamma}),$$

(4) $\mu^{-\varepsilon}_{\alpha}(c^{-\varepsilon}_{\alpha}) + \mu^{-\varepsilon}_{\gamma}(\{c^{-\varepsilon}_{\beta} c^{\varepsilon}_{\alpha} c^{-\varepsilon}_{\alpha-\beta+\gamma}) = \mu^{-\varepsilon}_{\gamma}(c^{-\varepsilon}_{\gamma}) + \mu^{-\varepsilon}_{\alpha}(\{c^{-\varepsilon}_{\beta} c^{\varepsilon}_{\gamma} c^{-\varepsilon}_{\alpha-\beta+\gamma}\})$

from (4.12.1,2) for $w \in V^{-\varepsilon}_{\alpha-\beta+\gamma}$ and x, y, z as above. Next we claim

(5) $$\{\{xyz\}wz\} = \{x\{yzw\}z\} + Q(z)\{yxw\}.$$

To prove this, we compute

$$\begin{aligned}\{\{xyz\}wz\} &= Q(z, \{xyz\})w \\ &= D(x,y)Q(z)w + Q(z)D(y,x)w \qquad \text{(by [7, JP12])} \\ &= \{x, y, Q(z)w\} + Q(z)\{yxw\}\end{aligned}$$

and

$$\{x, y, Q(z)w\} = D(Q(z)w, y)x$$

$$= D(z, \{wzy\})x - D(Q(z)y, w)x \qquad \text{(by [7, JP8])}$$
$$= \{x\{yzw\}z\}$$

since $Q(z)y = 0$ by [12, I.3.6a)]. Combining we arrive at (5). From 4.17 we deduce

$$\mu_\gamma^\varepsilon(\{xx'z\}) = \mu_\alpha^\varepsilon(x) + \mu_\alpha^{-\varepsilon}(x') + \mu_\gamma^\varepsilon(z), \tag{6}$$
$$\mu_\gamma^\varepsilon(\{w'wz\}) = \mu_{\alpha-\beta+\gamma}^\varepsilon(w') + \mu_{\alpha-\beta+\gamma}^{-\varepsilon}(w) + \mu_\gamma^\varepsilon(z) \tag{7}$$

for $x' \in V_\alpha^{-\varepsilon}, w' \in V_{\alpha-\beta+\gamma}^\varepsilon$. Hence

$$\begin{aligned}
&\mu_{\alpha-\beta+\gamma}^\varepsilon(\{xyz\}) + \mu_{\alpha-\beta+\gamma}^{-\varepsilon}(w) + \mu_\gamma^\varepsilon(z) \\
&= \mu_\gamma^\varepsilon(\{\{xyz\}w\}z\}) && \text{(by (7))} \\
&= \mu_\gamma^\varepsilon(\{x\{yzw\}z\} + Q(z)\{yxw\}) && \text{(by (5))} \\
&\geq \min\big(\mu_\gamma^\varepsilon(\{x\{yzw\}z\}), \mu_\gamma^\varepsilon(Q(z)\{yxw\})\big) \\
&= \mu_\alpha^\varepsilon(x) + \mu_\beta^{-\varepsilon}(y) + 2\mu_\gamma^\varepsilon(z) + \mu_{\alpha-\beta+\gamma}^{-\varepsilon}(w) + \mu_\beta^\varepsilon(c_\beta^\varepsilon) + \mu_\alpha^{-\varepsilon}(c_\alpha^{-\varepsilon}) \\
&\quad + \mu_{\alpha-\beta+\gamma}^\varepsilon(c_{\alpha-\beta+\gamma}^\varepsilon) + \mu_\gamma^{-\varepsilon}(\{c_\beta^{-\varepsilon}c_\alpha^\varepsilon c_{\alpha-\beta+\gamma}^{-\varepsilon}\}) && \text{(by (6), (2)-(4))},
\end{aligned}$$

which implies (4.20.1). □

4.21 We can now finisch the proof of (4.14.2), where we may assume that $\alpha, \beta, \gamma \in R_1$ we distinct and $c_\alpha, c_\beta, c_\gamma$ satisfy one of the configurations (4.9.1-6) (cf. 4.18). Then, since G is standard, we apply [13, 3.5.(2'.a)] to derive (4.14.2) from 4.19, 4.20. Summing up, ρ as defined in 4.3 is therefore a λ-norm of V.

4.22 To complete the proof of the Norm Theorem 4.3, it remains to show that ρ satisfies the triple product inequality (TPI) if every connected component of G contains more than one element. For this purpose, we may assume that G itself is connected (3.10). In view of (4.14.2), 4.21, it evidently suffices to establish (TPI) for each $\mu_\alpha, \alpha \in R_1$. To this end, let $x, z \in V_\alpha^\varepsilon, y \in$

$V_\alpha^{-\varepsilon}$. Since G is connected, there exists a $\beta \in R_1$, $\beta \neq \alpha$, such that c_α and c_β are not orthogonal, leaving us with the following cases.

Case 1 $c_\beta \vdash c_\alpha$.
Then $\gamma = 2\beta - \alpha \in R_1$ and $(c_\beta, c_\alpha, c_\gamma)$ ist a root triangle (4.16), hence a triangle since G is standard (3.6(SG1)), giving $Q(c_\beta)c_\gamma = c_\alpha$ by (3.5.2′). Furthermore, $Q(c_\beta^\varepsilon) : V_\gamma^{-\varepsilon} \longrightarrow V_\alpha^\varepsilon$ is bijective, so $z = Q(c_\beta^\varepsilon)w$ for some $w \in V_\gamma^{-\varepsilon}$. This implies

$$\begin{aligned}\{xyz\} &= D(x,y)Q(c_\beta^\varepsilon)w \\ &= Q(c_\beta^\varepsilon, \{xyc_\beta^\varepsilon\})w - Q(c_\beta^\varepsilon)D(y,x)w && \text{(by [7, JP12])} \\ &= \{c_\beta^\varepsilon w\{xyc_\beta^\varepsilon\}\} - Q(c_\beta^\varepsilon)\{yxw\},\end{aligned}$$

hence

$$\begin{aligned}\mu_\alpha^\varepsilon(\{xyz\}) &\geq \min\bigl(\mu_\alpha^\varepsilon(\{c_\beta^\varepsilon w\{xyc_\beta^\varepsilon\}\}), \mu_\alpha^\varepsilon(Q(c_\beta^\varepsilon)\{yxw\})\bigr) \\ &\geq 2\mu_\beta^\varepsilon(c_\beta^\varepsilon) + \mu_\gamma^{-\varepsilon}(w) + \mu_\alpha^\varepsilon(x) + \mu_\alpha^{-\varepsilon}(y) && \text{(by (4.16.3,4), 4.17)} \\ &= \mu_\alpha^\varepsilon(x) + \mu_\alpha^{-\varepsilon}(y) + \mu_\alpha^\varepsilon(Q(c_\beta^\varepsilon)(w)) && \text{(by (4.16.3))} \\ &= \mu_\alpha^\varepsilon(x) + \mu_\alpha^{-\varepsilon}(y) + \mu_\alpha^\varepsilon(z).\end{aligned}$$

Case 2 $c_\alpha \top c_\beta$.
Then we consider the McCrimmon-Meyberg automorphism $\varphi = \beta(c_\alpha^+ + c_\beta^+, c_\alpha^- + c_\beta^-)$ of V (cf. [12, I.1.13]), which has order 2, interchanges c_α, c_β and permutes the Peirce spaces of G [12, I.3.12]. More specifically,

$$(1) \qquad \varphi^\varepsilon(x) = \{c_\beta^\varepsilon c_\alpha^{-\varepsilon} x\} \qquad (x \in V_\alpha^\varepsilon).$$

Writing $\varphi_{\alpha\beta}$ for the isomorphism $V_\alpha \longrightarrow V_\beta$ induced by φ, $\mu_\beta \circ \varphi_{\alpha\beta}$ agrees with μ_α up to a translate (2.12), which must be trivial since $\varphi_{\alpha\beta}(c_\alpha) = c_\beta$. Hence $\mu_\beta \circ \varphi_{\alpha\beta} = \mu_\alpha$. Putting $x' = \varphi^\varepsilon(x) \in V_\beta^\varepsilon$, we therefore obtain

$$\begin{aligned}(2) \qquad \mu_\alpha^\varepsilon(x) + \mu_\alpha^{-\varepsilon}(y) + \mu_\alpha^\varepsilon(z) &= \mu_\beta^\varepsilon(x') + \mu_\alpha^{-\varepsilon}(y) + \mu_\alpha^\varepsilon(z) \\ &= \mu_\beta^\varepsilon(\{x'yz\}) && \text{(by 4.17)} \\ &= \mu_\beta^\varepsilon(\{\{c_\beta^\varepsilon c_\alpha^{-\varepsilon} x\}yz\}) && \text{(by (1))}\end{aligned}$$

$$= \mu^\varepsilon_\beta(\{c^\varepsilon_\beta c^{-\varepsilon}_\alpha\{xyz\}\} - \{xy\{c^\varepsilon_\beta c^{-\varepsilon}_\alpha z\}\} + \{x\{c^{-\varepsilon}_\alpha c^\varepsilon_\beta y\}z\}\}) \qquad \text{(by [7, JP14])}$$

$$= \mu^\varepsilon_\beta(\{c^\varepsilon_\beta c^{-\varepsilon}_\alpha\{xyz\} - \{xy\{c^\varepsilon_\beta c^{-\varepsilon}_\alpha z\}\})$$

since $\{c^{-\varepsilon}_\alpha c^\varepsilon_\beta y\} = 0$ by the Peirce rules. On the other hand, we have

$$\begin{aligned}
&(3) \qquad & \mu^\varepsilon_\beta(\{c^\varepsilon_\beta c^{-\varepsilon}_\alpha\{xyz\}\}) &= \mu^\varepsilon_\beta(\varphi^\varepsilon_{\alpha\beta}(\{xyz\})) && \text{(by (1))} \\
& & &= \mu^\varepsilon_\alpha(\{xyz\}), \\
&(4) \qquad & \mu^\varepsilon_\beta(\{xy\{c^\varepsilon_\beta c^{-\varepsilon}_\alpha z\}\}) &= \mu^\varepsilon_\alpha(x) + \mu^{-\varepsilon}_\alpha(y) + \mu^\varepsilon_\beta(\varphi^\varepsilon_{\alpha\beta}(z)) && \text{(by 4.17)} \\
& & &= \mu^\varepsilon_\alpha(x) + \mu^{-\varepsilon}_\alpha(y) + \mu^\varepsilon_\alpha(z).
\end{aligned}$$

Hence the assumption

$$\mu^\varepsilon_\alpha(\{xyz\}) < \mu^\varepsilon_\alpha(x) + \mu^{-\varepsilon}_\alpha(y) + \mu^\varepsilon(z)$$

combined with (2) - (4) would imply the contradiction

$$\mu^\varepsilon_\alpha(\{xyz\}) = \mu^\varepsilon_\alpha(x) + \mu^{-\varepsilon}_\alpha(y) + \mu^\varepsilon_\alpha(z).$$

Case 3 $\quad c_\alpha \vdash c_\beta$.
Setting $\gamma = 2\alpha - \beta$, we have $Q(c_\alpha)c_\beta = c_\gamma$, which implies

$$\begin{aligned}
2\mu^\varepsilon_\alpha(\{xyz\}) &= \mu^\varepsilon_\alpha(Q(\{xyz\})c^{-\varepsilon}_\beta) && \text{(by (4.16.3))} \\
&= \mu^\varepsilon_\alpha\big(Q(x)(Q(y)Q(z)c^{-\varepsilon}_\beta + Q(z)Q(y)Q(x)c^{-\varepsilon}_\beta + \\
&\quad \{x, Q(y)\{xc^{-\varepsilon}_\beta z\}, z\} - \{Q(x)y, c^{-\varepsilon}_\beta, Q(z)y\}\big) \\
& && \text{(by [7, JP20])} \\
&\geq 2(\mu^\varepsilon_\alpha(x) + \mu^{-\varepsilon}_\alpha(y) + \mu^\varepsilon_\alpha(z)) && \text{(by (4.16.3,4)),}
\end{aligned}$$

as claimed.

This completes the proof of the Norm Theorem 4.3. $\square$

4.23 The triple product inequality revisited. Strangely enough, the Norm Theorem 4.3 combined with structure theory implies the triple product inequality (2.15) for 2-finitedimensional Jordan division pairs over Henselian fields. We merely sketsch the details. Let (K, Γ, λ) be a Henselian field and V a 2-finitedimensional Jordan division pair over K. Since (TPI) is translation invariant, one easily reduces to the case $V = (J, J)$ where J is a finitedimensional Jordan division algebra on two generators over K. But then, by structure theory, there exist a finitedimensional simple *nondivision* Jordan pair W over K and a covering standard division grid G of W such that V identifies with a Peirce component of W relative to G. Hence (TPI) follows from 4.3b).

5. Saturated Orders.

In this section, the norm theorem will be applied to an arithmetic setting. We fix a Dedekind domain $\mathfrak{o}$ with quotient field K and a finitedimensional Jordan pair V over K. Recall from [19] that an *order in* V is an $\mathfrak{o}$-subpair $\mathfrak{O} \subset V$ such that $\mathfrak{O}^\varepsilon \subset V^\varepsilon$ is a full $\mathfrak{o}$-lattice, i.e., a finitely generated $\mathfrak{o}$-module spanning V^ε as a vector space over K; maximal orders are defined in the obvious way. We say an order in V is *saturated* if it contains a covering division grid of V. By 3.11 or [14, Lemma 2.4], for saturated orders to exist it is necessary that V be semi-simple.

5.1 Until further notice, the preceding set-up will be specified to the case of a local field K, its complete, discrete and surjective valuation $\lambda : K \longrightarrow \mathbb{Z}_\infty$ being understood. $\mathfrak{o} = \mathfrak{o}(K, \lambda)$ is a discrete valuation ring containing $\mathfrak{p} = \mathfrak{p}(K, \lambda)$ as its unique nonzero (principal) prime ideal; we put $\kappa = \mathfrak{o}/\mathfrak{p}$.

5.2 Proposition. *Let V be a finitedimensional Jordan division pair over the local field K. Then every order is contained in a maximal order. $\mathfrak{O} \subset V$ is a maximal order if and only if there exists a separated valuation $\mu : V \longrightarrow \mathbb{Q}_\infty$ extending λ and satisfying $\mathfrak{O} = \mathfrak{O}(V, \mu)$; in particular, maximal orders of V are saturated.*

Proof. We may assume $V = (J, J)$ for some finitedimensional Jordan division algebra J over K. By 2.8, the separated valuations $V \longrightarrow \mathbb{Q}_\infty$ are exactly the objects of the form $\mu = (\nu^{(y)}, \nu^{(y^{-1})})$, where $\nu : J \longrightarrow \mathbb{Q}_\infty$ is a valuation of J and $y \in J^\times$; also μ extends λ if and only if ν does. Hence the asserted description of maximal orders follows from [19, Proposition 6], whose proof also shows that every order is contained in a maximal one. The final statement now easily derives from 2.3a),e) and the fact that the covering division grids of V have the form $\{(x, x^{-1})\}, x \in J^\times$.

5.3 Theorem. *Let V be a finitedimensional Jordan pair over the local field K. Then every saturated order is contained in a saturated maximal order. $\mathfrak{O} \subset V$ is a saturated maximal order if and only if there exists a covering standard division grid G in V satisfying $\mathfrak{O} = \mathfrak{O}(V, G)$.*

Proof. Suppose first that $G \subset V$ is a covering standard division grid satisfying $\mathfrak{O} = \mathfrak{O}(V, G)$. Write (R, R_1) for the corresponding 3-graded root system and put $V_\alpha = V_\alpha(G)$ for $\alpha \in R_1$. Then $\mathfrak{O}$ is obviously a saturated order, and G is a covering standard grid of every order $\mathfrak{O}' \subset V$ containing $\mathfrak{O}$, with Peirce components $\mathfrak{O}'_\alpha = \mathfrak{O}' \cap V_\alpha \supset \mathfrak{O} \cap V_\alpha = \mathfrak{O}(V_\alpha, \mu_\alpha)$ ((4.4.1)) for $\alpha \in R_1$. Here 5.2 gives equality, and the covering property of G implies $\mathfrak{O}' = \mathfrak{O}$, i.e., $\mathfrak{O}$ is a maximal order of V. Conversely, let $\mathfrak{O}$ be any saturated order in V, so $G' \subset \mathfrak{O}$ for some covering division grid G' of V. Hence $\mathfrak{O}$ contains a covering standard division grid G of V associated with G' [13, 3.8]. Adopting the previous notation, we obtain $c_\alpha \in \mathfrak{O}_\alpha = \mathfrak{O} \cap V_\alpha \subset \mathfrak{O}(V_\alpha, \mu'_\alpha)$ ($\alpha \in R_1$) where $\mu'_\alpha : V_\alpha \longrightarrow \mathbb{Q}_\infty$ is a separated valuation extending λ (5.2). This implies $\mu'^{\varepsilon}_\alpha(c^\varepsilon_\alpha) = 0$ by (2.1.2) and 2.3a), forcing $\mu'_\alpha = \mu_\alpha$ in the sense of 4.3. Thus $\mathfrak{O} \subset \mathfrak{O}(V, G)$, which completes the proof. □

5.4 Corollary. *In finitedimensional semi-simple Jordan pairs over a local field, saturated maximal orders exist.* □

Note. Even if the property of being saturated is ignored, maximal orders were previously known to exist only for finitedimensional Jordan pairs with nonsingular generic trace forms [19, Proposition 3]. These Jordan pairs are separable but not conversely. Hence it seems worth pointing out that the argument used in the proof of 7.12 below leads to the following conclusion: *Every finitedimensional separable Jordan pair over the quotient field of a Dedekind domain contains maximal orders.*

5.5 Example. Notations being as in 5.3, we show that *a maximal order in V need not be saturated.* Let C be composition algebra without zero divisors over K having ramification order 1 [18], and let $r > 1$ be an integer with $r \leq 3$ unless C is associative. Write V for the Jordan pair belonging to the Jordan algebra $H_r(C)$ of r-by-r hermitian matrices with entries in C and scalars down the diagonal. We assume that κ contains at least r elements and pick a prime element $\pi \in \mathfrak{o}$. Working with the customary hermitian matrix units [5], we claim that the partially twisted diagonal idempotents

$$c_j = (e_{jj}, e_{jj}), c_r = (\pi e_{rr}, \pi^{-1} e_{rr}) \qquad (1 \leq j < r)$$

all belong to a single maximal order $\mathfrak{O} \subset V$. Indeed, combining a Vandermonde argument with the fact that integral pairs always imbed in an order [19, Satz 2], we first find an order $\mathfrak{O}' \subset V$ containing every c_j for $1 \leq j \leq r$. But then, the generic trace of V being nonsingular, $\mathfrak{O}'$ can be enlarged to a maximal order $\mathfrak{O}$ of V [19, Proposition 3]. We claim that $\mathfrak{O}$ is not saturated. Otherwise, 5.3 yields a covering standard division grid $G \subset V$ satisfying $\mathfrak{O} = \mathfrak{O}(V, G)$. Applying [14, Lemma 1.5b)] and its proof, we find a frame $F = \{d_1, \ldots, d_r\}$ of $\mathfrak{O}$ which is entirely contained in $G^{(1)}$. But $F_0 = \{c_1, \ldots, c_r\}$ is a frame of $\mathfrak{O}$ as well, and any two frames of $\mathfrak{O}$ are associated up to conjugation by elementary automorphisms of $\mathfrak{O}$ [9, Corollary 2 of Theorem 2]. Hence we may assume $F \approx F_0$. In particular, relabeling the d's if necessary, there are $a_1, a_r \in \mathfrak{o}^\times$ satisfying

$$d_1 = (a_1 e_{11}, a_1^{-1} e_{11}),\ d_r = (a_r \pi e_{rr}, a_r^{-1} \pi^{-1} e_{rr}).$$

Up to conjugation by elementary automorphisms of V, G is associated with the grid of standard hermitian matrix units [14, 1.9]. Hence G is connected and satisfies $G^{(2)} \neq \emptyset$. In particular, d_1, d_r imbed into a root triangle $(d_0; d_1, d_r)$ of G [12, I.4.7b)], which is actually a triangle since G is standard (3.6.(SG1)). From

$$d_0 \in V_1(d_1) \cap V_1(d_r) = V_1(e_{11}) \cap V_1(e_{rr})$$

we conclude $d_0^+ = u[1r]$ for some $u \in C^\times$, and the relation $Q(d_0)d_1 = d_r$ ((3.5.2)) gives $n(u) = a_1 a_r \pi$, n being the norm of C. But this is a contradiction since C has ramification order 1. $\square$

5.6 Weak separability. Let k be a commutative associative ring of scalars. Following Loos [8], a Jordan paar W over k whose underlying k-modules W^ε are finitely generated projective is said to be *separable* if, for all $\mathfrak{p} \in \operatorname{Spec} k$, the finitedimensional Jordan pairs $W(\mathfrak{p}) = W \otimes_k \kappa(\mathfrak{p})$ over $\kappa(\mathfrak{p})$ (the quotient field of $k/\mathfrak{p}$) are separable in the classical sense. If these Jordan pairs are merely required to be semi-simple, W will be called *weakly separable.* The importance of this notion derives from the fact that it leads to a class of orders which are automatically saturated. This is an unpublished result due to Neher which will be reproduced here with his kind permission. We begin with a few elementary preliminaries.

5.7 Lemma. *Let W be a Jordan pair over k whose underlying k-modules are finitely generated. Then* $(\operatorname{Rad} k)W \subset \operatorname{Rad} W$.

Proof. For $a \in \operatorname{Rad} k$ and $\alpha \in \operatorname{End}_k(M)$, M a finitely generated k-module, $\mathbf{1}_M - a\alpha$ is surjective by Nakayama's Lemma, hence bijective by [28, 38.15]. For all $x \in W^\varepsilon, y \in W^{-\varepsilon}$,

$$B(ax, y) = \mathbf{1} - a(D(x, y) - aQ(x)Q(y))$$

in therefore bijective as well, forcing (ax, y) to be quasi-invertible and $ax \in (\operatorname{Rad} W)^\varepsilon$. $\square$

5.8 Proposition. *$\mathfrak{o}$ being a discrete valuation ring with maximal ideal $\mathfrak{p}$ and quotient field K, let V be a finite-dimensional Jordan pair over K.*
a) *An order $\mathfrak{O} \subset V$ is weakly separable if and only if V is semi-simple and* $\operatorname{Rad} \mathfrak{O} = \mathfrak{p}\mathfrak{O}$.
b) *Let $\mathfrak{O} \subset V$ be a weakly separable order. An idempotent $c \in \mathfrak{O}$ is a division idempotent of V if and only if $\check{c} = c \otimes 1$ is a division idempotent of $\check{\mathfrak{O}} = \mathfrak{O}(\mathfrak{p})$.*

Proof. a) By definition, $\mathfrak{O}$ is weakly separable iff $V = \mathfrak{O}\ ((0))$ and $\check{\mathfrak{O}} \cong \mathfrak{O}/\mathfrak{p}\mathfrak{O}$ are semi-simple. Now a) follows from 5.7.

b) $\mathfrak{O}_2(c) = \mathfrak{O} \cap V_2(c)$ is an order in $V_2(c)$ satisfying $\mathfrak{O}_2(c)\check{} = \check{\mathfrak{O}}_2(\check{c})$ since Peirce components are compatible with base change. By a) combined with [7, 5.8], $\mathfrak{O}_2(c)$ is weakly separable, and we are reduced to the case $V = V_2(c)$. Up to nonzero scalar factors, all nonzero elements $x \in V^\varepsilon$ belong to $\mathfrak{O}^\varepsilon - \mathfrak{p}\mathfrak{O}^\varepsilon$. Clearly, $\alpha = Q(x)Q(c^{-\varepsilon}) : \mathfrak{O}^\varepsilon \longrightarrow \mathfrak{O}^\varepsilon$ is bijective iff $\check{\alpha} = \alpha(\mathfrak{p})$ is, so V is a division pair iff $\check{\mathfrak{O}}$ is. □

Remark. 5.8 b) was derived in [19, Satz 4b)] for the more special case of a local field and selfdual orders.

5.9 Proposition. (Neher). *Notations being as in* 5.8, *suppose* $\mathfrak{O} \subset V$ *is an* $\mathfrak{o}$*-subpair and* $E \subset \mathfrak{O}$ *is a cog. Then* $\mathfrak{O}$ *is a (weakly) separable order in* V *if and only if* $\mathfrak{O}_I(E)$*, for every* $I \in \mathbb{Z}^E$*, is a (weakly) separable order in* $V_I(E)$*.*

Proof. Setting $V_I = V_I(E), \mathfrak{O}_I = \mathfrak{O}_I(E)$ $(I \in \mathbb{Z}^E)$, the decompositions

$$V = \sum V_I, \quad \text{Rad } V = \sum \text{ Rad } V_I,$$
$$\mathfrak{O} = \sum \mathfrak{O}_I, \quad \text{Rad } \mathfrak{O} = \sum \text{ Rad } \mathfrak{O}_I, \ \mathfrak{p}\mathfrak{O} = \sum \mathfrak{p}\mathfrak{O}_I,$$

summation always being taken over all of $\mathbb{Z}^E$, hold either trivially or by virtue of [12, I.6.1]; the assertion follows from 5.8a). □

5.10 Lemma. (Neher). *Let* E *be a cog in a Jordan pair* W *such that no Peirce component* $W_I = W_I(E)$*,* $I \in \text{supp } E$*, has zero divisors. If* $J \in \mathbb{Z}^E$ *and* $d \in W_J$ *is a nontrivial idempotent, then*
a) *every* $c \in E$ *belongs to a Peirce component of* d*;*
b) $W_J \cap E = \emptyset$ *implies that* $E \cup \{d\}$ *is a cog in* W*.*

Note. This generalizes a result of McCrimmon [12, I.1.12]

Proof. a) We have $c \in W_I$ for some $I \in \text{supp } E$. Since d belongs to a Peirce component of c, the idempotents c, d are compatible [12, I.1.8], forcing

$$c^\varepsilon = c_2^\varepsilon + c_1^\varepsilon + c_0^\varepsilon, \ c_i^\varepsilon \in W_2(c)^\varepsilon \cap W_i(d)^\varepsilon$$

for $i = 0, 1, 2$ (loc. cit.). In fact, the relations

$$c_2^\varepsilon = Q(d^\varepsilon)Q(d^{-\varepsilon})c^\varepsilon \in Q(W_J^\varepsilon)Q(W_J^{-\varepsilon})W_I^\varepsilon \subset W_I^\varepsilon \qquad \text{(by (3.3.2))},$$

$$c_1^\varepsilon = \{d^\varepsilon d^{-\varepsilon} c^\varepsilon\} - 2c_2^\varepsilon \in W_I^\varepsilon \qquad \text{(loc. cit.)}$$

show, more specifically,

$$c_i \in W_I \cap V_i(d) \qquad (i = 0, 1, 2).$$

Since W_I has no zero divisors, we now deduce from $Q(c_2^\varepsilon)c_1^{-\varepsilon} = Q(c_0^\varepsilon)(c_2^{-\varepsilon} + c_1^{-\varepsilon}) = 0$ that either $c_0 \neq 0$, forcing $c_1 = c_2 = 0$ (i.e., $c \in V_0(d)$), or $c_0 = 0$, forcing $c_2 = 0$ (i.e., $c \in V_1(d)$) or $c_1 = 0$ (i.e., $c \in V_2(d)$).
b) By hypothesis, d and any $c \in E$ belong to distinct Peirce components of E, so they are not associated [12, I.3.3]. Hence $E \cup \{d\}$ is a cog. □

5.11 Theorem. (Neher). *Let V be a finitedimensional Jordan pair over the local field K. Then every weakly separable order in V is saturated.*

Proof. Let $\mathfrak{O} \subset V$ be a weakly separable order. By 5.8a), V is semi-simple. We write $\mathcal{E}$ for the totality of division cogs of V that are contained in $\mathfrak{O}$. Then $\mathcal{E}$ is not empty ([19, Proposition 5] combined with 5.8b)), and the cardinality of each member is bounded by the dimension of V. Letting $G \in \mathcal{E}$ be maximal with respect to inclusion, it suffices to show that G covers V (3.7). Otherwise, setting $V_I = V_I(G)$, $\mathfrak{O}_I = \mathfrak{O}_I(G)$ for $I \in \mathbb{Z}^G$, some $J \in \mathbb{Z}^G$ would satisfy $\mathfrak{O}_J \neq 0$ and $V_J \cap G = \emptyset$. Since $\mathfrak{O}_J$ is a weakly separable order in V_J (5.9), combining [19, Proposition 5] with 5.8b) again produces a division idempotent d of V_J contained in $\mathfrak{O}_J$. But then, by 5.10b), $G \cup \{d\} \subset \mathfrak{O}$ is a cog in V which strictly exceeds G and is division since its Peirce components are contained in those of G, a contradiction. □

5.12 Proposition *Let V be a finitedimensional Jordan pair over the local field K. Then every weakly separable order in V is maximal.*

Note. This is a variant of a result of Loos [8, Proposition 3].

Proof. Let $\mathfrak{O} \subset V$ be a weakly separable order. Then V is semisimple. In fact, as we proceed to show now, we may assume that V is simple. To do so, we write $V = V^{(1)} \oplus \ldots \oplus V^{(n)}$ as a direct sum of simple ideals $V^{(j)} \subset V$ $(1 \leq j \leq n)$. Then $\mathfrak{O}^{(j)} = \mathfrak{O} \cap V^{(j)}$ is an ideal in $\mathfrak{O}$, and the short exact sequence

$$0 \longrightarrow \mathfrak{O}^{(j)\varepsilon} \longrightarrow \mathfrak{O}^{\varepsilon} \longrightarrow \mathfrak{O}^{\varepsilon}/\mathfrak{O}^{(j)\varepsilon} \longrightarrow 0$$

of $\mathfrak{o}$-modules splits since $\mathfrak{O}^{\varepsilon}/\mathfrak{O}^{(j)\varepsilon}$ is easily seen to be torsion-free, hence free, as an $\mathfrak{o}$-module. Therefore $\mathfrak{O}^{(j)}(\mathfrak{p})$ canonically imbeds into $\mathfrak{O}(\mathfrak{p})$ as an ideal and, since $\mathfrak{O}(\mathfrak{p})$ is semi-simple, so is $\mathfrak{O}^{(j)}(\mathfrak{p})$. It follows that $\mathfrak{O}^{(j)} \subset V^{(j)}$ is a weakly separable, hence maximal order. If $\mathfrak{O}' \subset V$ is any order containing $\mathfrak{O}$, then $\mathfrak{O}^{(j)}$, sitting inside the natural projection of $\mathfrak{O}'$ to $V^{(j)}$, agrees with that projection, which implies $\mathfrak{O}' \subset \mathfrak{O}^{(1)} \oplus \ldots \oplus \mathfrak{O}^{(n)} \subset \mathfrak{O} \subset \mathfrak{O}'$, and $\mathfrak{O}$ is indeed maximal. We may therefore assume from now on that V is simple. Let F' be a frame of $\mathfrak{O}(\mathfrak{p})$. Thanks to [19, Proposition 5] and 5.8b), F' lifts to a frame F in V belonging to $\mathfrak{O}$. Since V is simple, the off-diagonal Peirce components of F are nontrivial. This property being inherited by $F' = F(\mathfrak{p})$, the semisimple Jordan pair $\mathfrak{O}(\mathfrak{p})$ must in fact be simple. Now the proof of [8, Proposition 3] carries over and shows that $\mathfrak{O}$ is a maximal order in V. $\square$

5.13 Remark. a) In connection with the above reduction to the simple case, the following trivial observation is sometimes useful: If $V^{(1)}, \ldots V^{(n)}$ are finitedimensional Jordan pairs over $K = \mathrm{Quot}\ \mathfrak{o}$, $\mathfrak{o}$ a Dedekind domain, then $\mathfrak{O} \subset V = V^{(1)} \oplus \ldots \oplus V^{(n)}$ is a maximal order if and only if $\mathfrak{O} = \mathfrak{O}^{(1)} \oplus \ldots \oplus \mathfrak{O}^{(n)}$ with maximal orders $\mathfrak{O}^{(j)} \subset V^{(j)}$, $1 \leq j \leq n$. The argument of [24], p. 12, to prove this for Jordan algebras rather than pairs carries over verbatim.

b) In the proof of 5.12 we have shown that, if $\mathfrak{O}$ is a weakly separable order in a finitedimensional *simple* Jordan pair over a local field, then $\mathfrak{O}(\mathfrak{p})$ is simple as well.

6. The anisotropic part of a Jordan pair.

Without recourse to structure theory, we define in this section the anisotropic part of a nondegenerate simple Artinian Jordan pair V, generalizing uniformly classical notions in Witt's theory of quadratic forms and the

Wedderburn-Artin theory of associative algebras; connections with algebraic groups will be discussed elsewhere. Our approach relies on results due to Loos [9] and Loos-Neher [10], which we shall briefly recall for convenience. Until further notice, we let V be an arbitrary Jordan pair over k, a commutative associative ring of scalars.

6.1 Elementary automorphisms. Let e be an idempotent of V. For all $x \in V_1(e)^+$, $y \in V_1(e)^-$, the pairs $(e^+, y), (x, e^-)$ are quasi-invertible [7, 5.7], forcing $\beta(e^+, y), \beta(x, e^-)$ to be inner automorphisms of V. Following Loos [9], the subgroup of $\mathrm{Inn}(V)$ generated by these elements as x, y vary over $V_1(e)^+, V_1(e)^-$, respectively, is called the group of *e-elementary automorphisms* of V and will be denoted here by $\mathrm{Elt}_e(V)$. The *elementary automorphism group* of V, written as $\mathrm{Elt}(V)$, is then defined to be the subgroup of $\mathrm{Inn}(V)$ generated by all the $\mathrm{Elt}_e(V), e \in V$ an arbitrary idempotent. The elements of $\mathrm{Elt}(V)$ are called *elementary automorphisms*. It is a trivial but crucial observation that elementary automorphisms of a subpair always extend to elementary automorphisms of V.

6.2 Fact. [9, Corollary 3 of Theorem 2]. *Suppose V contains a frame and is connected* (i.e., any two orthogonal local idempotents in V are connected in the usual sense). *Then any two ordered sets of orthogonal local idempotents in V having the same cardinality are conjugate up to association under* $\mathrm{Elt}(V)$. $\square$

6.3 Capacity and length. The *capacity of V*, which we denote here by $\mathrm{cap}\, V$, is defined as the infimum of cardinalities of all finite sets of orthogonal division idempotents in V whose common Peirce-0-component vanishes. If V is nondegenerate of finite capacity, all frames $F \subset V$ are strong (i.e., consist of division idempotents and satisfy $V_{00}(F) = 0$) and have cardinality $\mathrm{cap}\, V$ [9, Theorem 3]. Following Loos-Neher [10, 4.6], we define the *length of V^ε*, denoted by $l(V^\varepsilon)$, as the supremum of cardinalities of all properly ascending finite chains of nonzero inner ideals in V^ε. If V is nondegenerate, we have $l(V^+) = l(V^-)$ [10, 4.8], which we write as $l(V)$ and call the *length of V*. A Jordan pair is said to be *Artinian* if it satisfies the dcc on all inner ideals. The property of being (nondegenerate resp. simple) Artinian is trivially inherited by Peirce-i-spaces of idempotents for $i = 0, 2$ [10, 10.2]. A nondegenerate Jordan pair is Artinian if and only if it has finite length [10, 5.2], in which case it has finite capacity as well [7, §10].

6.4 Fact. [10, 4.1, 4.7]. *Suppose V is simple nondegenerate and $e \in V$ is a division idempotent. Then $V_1(e)$ is nondegenerate, and V is Artinian if and only if $V_1(e)$ is Artinian. In this case,*

$$l(V_1(e)) = l(V) - 1. \qquad \square$$

6.5 Fact. [10, 5.10]. *Suppose V is simple nondegenerate and $e \in V$ is a division idempotent. Then there are the following mutually exclusive possibilities for $V' = V_1(e)$.*
(0) $V' = 0$.
(1) $\operatorname{cap} V' = 1$, *and every nonzero idempotent of V' is rigidly collinear to e.*
(2) $\operatorname{cap} V' = 1$, *and every nonzero idempotent of V' governs e.*
(3) $\operatorname{cap} V' = 2$, *and V' is the direct sum of two simple ideals of capacity* 1.
(4) $\operatorname{cap} V' = 2$, *and V' is simple.*
A division idempotent of V' remains a division idempotent in V and is rigidly collinear to e except in case (2). *Also,* $\operatorname{cap} V = 1$ *in cases* $(0), (1)$, *and* $\operatorname{cap} V \geq 2$ *in the other cases.* $\square$

6.6 Anisotropic subpairs. From now on we assume that V is nondegenerate, simple and Artinian. We proceed to define by induction on the length $l = l(V)$ a certain collection of subpairs of V, written as $\mathfrak{An}(V)$, in the following way. For $l = 1$, i.e., if V is a division pair [10, 4.12(i)], we put $\mathfrak{An}(V) = \{V\}$. For $l > 1$, we pick a division idempotent $e \in V$ and consider the following cases. If $V_1(e)$ is not simple, we define $\mathfrak{An}(V)$ as the totality of all subpairs $V_2(c) \subset V$ where c varies over the division idempotents of V. On the other hand, if $V_1(e)$ is simple, necessarily of length $l - 1$ (6.4), so is $V_1(c)$ for every division idempotent $c \in V$ (6.2), and we define $\mathfrak{An}(V)$ as the union of all $\mathfrak{An}(V_1(c))$ where c again varies over the division idempotents of V. In what follows, it will be important to view the elements of $\mathfrak{An}(V)$ not just as Jordan pairs in their own right but, more accurately, in their capacity as subpairs of V; see 6.16 below for examples.

Note. My original definition of $\mathfrak{An}(V)$ proceeded by induction on the dimension, performing the induction step by looking at off-diagonal Peirce components relative to a frame rather than the subpairs $V_1(e)$ above, and thus was confined to the finitedimensional case. The idea of using the length instead and thus extending to the Artinian case is due to Neher.

The following statement is an immediate consequence of 6.1, 6.2 and the definition.

6.7 Proposition. *Let V be a nondegenerate simple Artinian Jordan pair.*
a) *Every element of $\mathfrak{An}(V)$ is a division subpair of V.*
b) $\mathfrak{An}(V)$ *is stable under isomorphisms: If $\varphi : V \xrightarrow{\sim} W$ is an isomorphisms of Jordan pairs, then $\varphi(U) \in \mathfrak{An}(W)$ for all $U \in \mathfrak{An}(V)$.*
c) *Any two members of $\mathfrak{An}(V)$ are conjugate under the elementary automorphisms group of V.* □

6.8 The anisotropic part. In view of 6.7, we are allowed to call any element of $\mathfrak{An}(V)$ *the anistropic part of V* and to denote it by V_{an}. Thus V_{an} is a division subpair of V, uniquely determind up to conjugation by elementary automorphisms. Given a division (resp. nondegenerate simple Artinian) subpair W of V, we write $V_{\mathrm{an}} \hat{=} W$ (resp. $V_{\mathrm{an}} \hat{=} W_{\mathrm{an}}$) for $W \in \mathfrak{An}(V)$ (resp. $\mathfrak{An}(W) \subset \mathfrak{An}(V)$).

6.9 Proposition. *Let V be a nondegenerate simple Artinian Jordan pair of capacity 1 and e a division idempotent in V. Then $V_{\mathrm{an}} \hat{=} V_2(e)$.*

Proof. By induction on $l = l(V)$. For $l = 1$ there is nothing to prove. For $l > 1$, we may assume that $V' = V_1(e)$ is simple. Then we are in case (1) of 6.5, so a division idempotent d of V' continues to be one of V and satisfies $V_2(d) = V_2'(d)$. Hence, by the induction hypothesis, $V_{\mathrm{an}} \hat{=} V'_{\mathrm{an}} \hat{=} V_2(d)$, and 6.2, 6.7b) imply $V_{\mathrm{an}} \hat{=} V_2(e)$. □

6.10 Lemma. *Let V be a nondegenerate simple Jordan pair with dcc on principal inner ideals and $F = (e_1, e_2)$ a connected orthogonal system of idempotents in V. Then $V' = V_1(e_1)$ contains a maximal idempotent f satisfying $V_{12}(F) = V_2'(f)$; in particular, $V_{12}(F)$ has the same capacity as V' and is simple if and only if V' is simple.*

Proof. Put $V_{ij} = V_{ij}(F)$ for $i, j \in \mathbb{Z}$ and choose an element $f^+ \in V_{12}^+$ which is invertible in $V_2(e_1 + e_2)$, with inverse $f^- \in V_{12}^-$. Then $f = (f^+, f^-)$ is an idempotent of V satisfying $f \approx e_1 + e_2$ [12, I.2.3]. Hence the Peirce rules give $V' = V_{12} + V_{01}, V_2'(f) = V_{12}, V_1'(f) = V_{01}$. In particular, f is a maximal idempotent of V', and since V' inherits nondegeneracy as well as principal dcc from V [10, 4.1], the assertion follows from [7, 10.14a),10.17]. □

6.11 Proposition. *Let V be a nondegenerate simple Artinian Jordan pair and $F = (e_1, e_2)$ an orthogonal system of division idempotents in V. If $V_{12}(F)$ is not simple, then $V_{\mathrm{an}} \hat{=} V_2(e_1)$.* If $V_{12}(F)$ *is simple, then* $V_{\mathrm{an}} \hat{=} V_{12}(F)_{\mathrm{an}}$.

Proof. By induction on $l = l(V)$. For $l = 1$ the statement is empty. For $l > 1$, we choose f as in 6.10, allowing us to assume that $W = V_{12}(F)$ and $V' = V_1(e_1)$ are both simple. If W is division pair, then V' has capacity 1 (6.10), which implies $V_{\mathrm{an}} \hat{=} V'_{\mathrm{an}} \hat{=} V_2'(f)$ (by 6.9) $= W \hat{=} W_{\mathrm{an}}$. Hence we may assume that W is not a division pair, forcing it and V' to have capacity 2 (6.5). Let $F' = (e_1', e_2')$ be a frame in W, hence in V'. Since we are not in case (2) of 6.5, F' is an orthogonal system of division idempotents in V whose Peirce-2-components in V' and V are the same; also $V_{12}'(F') = W_{12}(F') = W_1(e_1')$. If $V_{12}'(F')$ is not simple, the induction hypothesis implies $V_{\mathrm{an}} \hat{=} V'_{\mathrm{an}} \hat{=} V_2'(e_1') \hat{=} V_2(e_1')$, hence $V_{\mathrm{an}} \hat{=} V_2(e_1)$. Now suppose that $V_{12}'(F') = W_{12}(F')$ is simple. Since $W = V_2'(f)$ satisfies $l(W) \le l(V')$ (by [7, 10.2]) $= l(V) - 1$ (by 6.4), we may apply the induction hypothesis twice to conclude

$$V_{\mathrm{an}} \hat{=} V'_{\mathrm{an}} \hat{=} V_{12}'(F')_{\mathrm{an}} = W_{12}(F')_{\mathrm{an}} \hat{=} W_{\mathrm{an}}.$$ □

6.12 Corollary. *Let V be a nondegenerate simple Artinian Jordan pair, c an idempotent in V and $i \in \{0, 2\}$. If $V_i(c)$ has capacity > 1, then $V_{\mathrm{an}} \hat{=} V_i(c)_{\mathrm{an}}$.* □

6.13 Corollary. *Let V be a nondegenerate simple Artinian Jordan Pair and e a maximal idempotent in V. Then $V_{\mathrm{an}} \hat{=} V_2(e)_{\mathrm{an}}$.*

Proof. If V has capacity 1, e is a division idempotent, and the assertion follows from 6.9. If V has capacity > 1, so has $V_2(e)$, and the assertion follows from 6.12. $\square$

6.14 Remark. Let V be a nondegenerate simple Artinian Jordan pair and c an idempotent in V. The connection expressed in 6.12 between the anisotropic part of V and that of its Peirce-i-component $(i = 0, 2)$ relative to c can be extended to the case $i = 1$. We state the result but omit the proof. If c is not maximal, then $V_1(c)$ is simple if and only if $V_1(e)$ is simple, e being any division idempotent of V; in this case, $V_{\mathrm{an}} \hat{=} V_1(c)_{\mathrm{an}}$. If $V_1(c)$ is not simple, then $V_1(c) = W^{(1)} \oplus W^{(2)}$ with simple ideals $W^{(j)} \subset V_1(c)$ satisfying $V_{\mathrm{an}} \hat{=} (W^{(j)})_{\mathrm{an}}$ for $j = 1, 2$. Finally, for c maximal, $V_1(c)$ is either simple or zero; in the former case, $V_{\mathrm{an}} \hat{=} V_1(c)_{\mathrm{an}}$.

6.15 Examples. We briefly describe up to isomorphism the anisotropic part for the standard examples of nondegenerate simple Artinian Jordan pairs V (cf. [7, 12.12, 12.13]). The following statements are easily deduced from the definition, 6.9 and 6.11.

(I) $V = (M_{pq}(D), M_{pq}(D^{\mathrm{op}}))^J$ where D is an associative division algebra and $p, q \geq 1$ are integers. Then $V_{\mathrm{an}} \cong (D, D^{\mathrm{op}})^J$.

(II) $V = (A_n(K), A_n(K))$ where K is an extension field of k and $n \geq 4$. Then $V_{\mathrm{an}} \cong (K, K)^J$.

(III) $V = (H_n(D, D_0), H_n(D, D_0))$ where D is an associative division algebra with involution and D_0 is an ample subspace of the symmetric elements. Then $V_{\mathrm{an}} \cong (D, D)^J$ for $n \geq 2$ and $V_{\mathrm{an}} \cong (D_0, D_0)$ for $n = 1$.

(IV) $V = \mathrm{Jord}(q)$, the Jordan pair of a nondegenerate quadratic form q, for simplicity assumed to be finitedimensional, over an extension field K of k. If q_{an} denotes the anisotropic part of q, then $V_{\mathrm{an}} \cong \mathrm{Jord}(q_{\mathrm{an}})$ unless q is hyperbolic, in which case $V_{\mathrm{an}} \cong (K, K)$.

(V), (VI) $V = (M_{12}(C), M_{12}(C^{\mathrm{op}}))$ or $V = (H_3(C), H_3(C))$ where C is an octonion algebra over an extension field K of k. Then $V_{\mathrm{an}} \cong (C, C)^J$ if C is a division algebra and $V_{\mathrm{an}} \cong (K, K)^J$ if C is split. $\square$

6.16 Remark. Let k be a field, $n \in \mathbb{Z}, n \geq 2$ and $V = (\mathrm{Sym}_n(k), \mathrm{Sym}_n(k))$ the Jordan pair of symmetric n-by-n matrices over k. Then 6.11 yields $V_{\mathrm{an}} \hat{=} (k1[12], k1[12])$, hence $V_{\mathrm{an}} \cong (k,k)^J \cong (ke_{11}, ke_{11})$. However, *we do not have* $V_{\mathrm{an}} \hat{=} (ke_{11}, ke_{11})$ since this would imply by 6.7c) that the idempotents $c = (e_{11}, e_{11})$ and $d = (1[12], 1[12])$ up to association are conjugate by elementary automorphisms, which is impossible because d governs c. We are thus lead to conclude that, in dealing with the anisotropic part of a Jordan pair, its subpair structure plays a very significant role. $\square$

6.17 The anisotropic part and grids. In dealing with the anisotropic part of Jordan pair, we have so far avoided the theory of grids. For our subsequent applications, however, it will be vitally important to bring the two together. Recall from [10, 5.2] that a Jordan pair is nondegenerate Artinian if and only if it can be covered by a finite division grid. With this in mind we can now prove the following central result.

6.18 Theorem. *Let V be a nondegenerate simple Artinian Jordan pair and $G \subset V$ a finite covering division grid. Write (R, R_1) for the 3-graded root system associated with G.*
a) *If the roots of R_1 all have the same length, then $V_{\mathrm{an}} \hat{=} V_\alpha(G)$ for every $\alpha \in R_1$.*
b) *If the roots of R_1 have (two) different lengths, then $V_{\mathrm{an}} \hat{=} V_\alpha(G)$ for every short root $\alpha \in R_1$, and $V_1(e)$ is simple for every division idempotent $e \in V$.*

Proof. Setting $V_\alpha = V_\alpha(G)$ for $\alpha \in R_1$, suppose α is a long root (this being automatic in case a) by convention). Then c_α is a division idempotent satisfying $V_2(c_\alpha) = V_\alpha$ (3.9), and $V' = V_1(c_\alpha)$ has

$$G' = G \cap V' = \{c_\beta;\ \beta \in R_1,\ \alpha \top \beta \ \text{ or } \ \alpha \dashv \beta\}$$

as a finite covering division grid whose Peirce components agree with certain Peirce components of V relative to G [14, Lemma 1.5 and its proof]. Hence (R', R'_1), the 3-graded root system corresponding to G', may be viewed as a subsystem of (R, R_1) in such a way that

(1) $$R'_1 = \{\beta \in R_1 : \alpha \top \beta \ \text{ or } \ \alpha \dashv \beta\}.$$

This being so, wo now proceed by induction on $l = l(V)$. For $l = 1$ there is nothing to prove. For $l > 1$ we pick α as above and first treat case a), allowing us to assume that V' is simple. Because R'_1 satisfies a) as well, the induction hypothesis implies $V_{\mathrm{an}} \hat{=} V'_{\mathrm{an}} \hat{=} V'_\beta \hat{=} V_\beta$ for every $\beta \in R'_1$. But since $\alpha \top \beta, V_\alpha$ and V_β are conjugate under $\mathrm{Aut}(V)$ [12, I.3.12], forcing $V_{\mathrm{an}} \hat{=} V_\alpha$ by 6.7b). We are left with case b). Given a short root $\beta \in R_1$, we conclude $\beta \vdash \alpha$ for some $\alpha \in R_1$, to which our preliminary remark applies. Assuming $c_\beta \perp c_\gamma$ for some $\gamma \in R'_1$ would yield $c_\alpha \in V_2(c_\beta) \subset V_0(c_\gamma)$ and hence $c_\alpha \perp c_\gamma$, which is impossible by (1). This contradiction shows that G' is connected. Therefore V' is simple (3.13), proving the second part of b) (6.2). Suppose now that R'_1 satisfies a). Then $V'_2(c_\beta) = V_\beta$ by 3.9, forcing c_β to be a division idempotent in V' governing c_α. Therefore V' has capacity 1 (6.5) and, using 6.9, we conclude from the induction hypothesis $V_{\mathrm{an}} \hat{=} V'_{\mathrm{an}} \hat{=} V'_2(c_\beta) = V_\beta$. Finally suppose R'_1 satisfies b). Then β must be a short root of R'_1 as well since, otherwise, $\gamma \vdash \beta$ for some $\gamma \in R'_1$ [12, I.4.5], contradicting the no-tower-lemma [12, I.3.4]. Now the induction hypothesis implies $V_{\mathrm{an}} \hat{=} V'_{\mathrm{an}} \hat{=} V_\beta$.

□

6.19 Split versus reduced Jordan pairs. Let k be a field and V a nondegenerate simple Artinian Jordan pair over k. Following [21], we say that V is *reduced* if $V_2(c)$ has dimension 1 for every (equivalently (by 6.2): for some) division idempotent $c \in V$. On the other hand, we say that V is *split* if V_{an} has dimension 1; see 6.22 below for the connection with Neher's notion of splitness [15]. Observing 3.9, the first of the following two propositions becomes obvious.

6.20 Proposition. *Let V be a nondegenerate simple Artinian Jordan pair over the field k and (R, R_1) the 3-graded root system associated with a finite covering division grid $G \subset V$.*
a) *If the roots of R_1 all have the same length, then V is reduced if and only if $V_\alpha(G)$ has dimension 1 for every $\alpha \in R_1$.*
b) *If the roots of R_1 have (two) different lengths, then V is reduced if and only if $V_\alpha(G)$ has dimension 1 for every long root $\alpha \in R_1$.* □

6.21 Proposition. *Let V be a nondegenerate simple Artinian Jordan pair over the field k.*
a) *If V is split, then V is reduced.*
b) *For V to be separable it is necessary and sufficient that V_{an} be separable.*

Proof. Up to a point, we treat a), b) simultaneously and let (R, R_1) be the 3-graded root system associated with a finite covering division grid $G \subset V$. By 3.11, 6.18, 6.20 we may assume that the roots of R_1 have different lengths. Letting α be any long root of R_1, there exists a short root $\beta \in R_1$ governing α [12, I.4.5]. At this stage we need the following standard fact:

(1) The natural action of V_α^ε on V_β^ε via $(x, z) \mapsto \{xyz\}$, $y = c_\alpha^{-\varepsilon}$, gives an injective algebra homomorphism from $(V_\alpha)_y^\varepsilon$ to $\mathrm{End}_k(V_\beta^\varepsilon)^J$.

a) We know that V_β^ε has dimension 1 (6.18) and must show the same for V_α^ε (6.20). But this follows immediately from (1).
b) Let k'/k be an arbitrary field extension, put $V' = V \otimes_k k'$, $V'_\gamma = V_\gamma \otimes_k$ for $\gamma \in R_1$ and identify $V \subset V'$ canonically. We know that V'_β is semi-simple (6.18) and must show the same for V'_α (3.11). Let $x' \in (\mathrm{Rad}\ V'_\alpha)^\varepsilon \subset (\mathrm{Rad}\ V')^\varepsilon$ and assume $x' \neq 0$. Then $\{x'yc_\beta^\varepsilon\}$ belongs to $(\mathrm{Rad}\ V'_\beta)^\varepsilon$ and hence must zero. Writing $x' = \sum a'_i x_i, 0 \neq x_i \in V_\alpha, a'_i \in k'$ linearly independent over k, we conclude $\sum a'_i \{x_i y c_\beta^\varepsilon\} = 0$ and $\{x_i y c_\beta^\varepsilon\} \neq 0$ for all i, according to the standard fact (1) above. This is a contradiction and completes the proof. $\square$

6.22 Remark. Let V be a a nondegenerate split simple Artinian Jordan pair over a field. By 6.18, 6.20, 6.21a), the Peirce components of V relative to a finite covering division grid all have dimension 1, showing that our notion of splitness is compatible with the one introduced by Neher [15] in another context.

We conclude this section with an auxiliary result that relates the concept of anisotropic part to the Norm Theorem 4.3.

6.23 Proposition. *Let (K, Γ, λ) be a Henselian field, V a finite-dimensional simple Jordan pair over K and G a covering standard division grid of V. Then the anisotropic part of the simple finitedimensional Jordan pair $\mathfrak{O}(V, G)/\mathfrak{P}(V, G)$ over the residue class field $\kappa(K, \lambda)$ is isomorphic* *to* $\kappa(V_{\mathrm{an}}, \mu)$ *for some separated valuation* $\mu : V_{\mathrm{an}} \longrightarrow \Delta_\infty$ $(\Delta = \Gamma \otimes_{\mathbb{Z}} \mathbb{Q})$ extending λ.

Proof. The image G' of G in $\mathfrak{O}' = \mathfrak{O}(V,G)/\mathfrak{P}(V,G)$ is a covering standard division grid whose 3-graded root system canonically identifies with that of G (4.4). Also, in the terminology of 4.3, 4.4, $\mathfrak{O}'_\alpha(G') = \kappa(V_\alpha(G), \mu_\alpha)$ is finitedimensional over $\kappa(K, \lambda)$ for each $\alpha \in R_1$. Hence so is $\mathfrak{O}'$. The remaining assertion now follows from 6.18. $\square$

7. Ramification.

We now return to the arithmetic setting of Section 5 to discuss the following question: When does a finitedimensional, say: simple, Jordan pair over a local field contain a (weakly) separable order. We will answer this question purely in terms of the anisotropic part and its ramification properties. We also present a global version of this result that works over arbitrary nonsingular schemes of dimension 1 (e.g., algebraic curves). For the time being, we let K be a local field as in 5.1. All algebras, pairs etc. over K are tacitly assumed to be finite-dimensional.

7.1 (Weakly) unramified Jordan division algebras. We slightly modify the terminology of [17], bringing it into line with the established usage. A Jordan division algebra J over K is said to be *weakly unramified* if it has ramification order 1 in the sense of [17, §5 **3**.]. Writing $\nu : J \longrightarrow \mathbb{Q}_\infty$ for the unique valuation of J extending λ (1.7), this means that $\mathfrak{P}(J, \nu) = \mathfrak{p}\mathfrak{O}(J, \nu)$. If, in addition, the residue class algebra $\kappa(J, \nu)$ is separable over κ, J is said to be *unramified.*

7.2 Lemma. *If* J *is a (weakly) unramified Jordan division algebra over* K*, so is every isotope of* J*.*

Proof. The ramification order does not change when passing to an isotope [17, Satz 6.3]. Hence the property of J being weakly unramified is inherited by $J^{(y)}, y \in J^\times$. Also, $y \equiv y'$ mod $K^\times$ for some $y' \in \mathfrak{O}(J, \nu)^\times$ (ν being as in

7.1), whence $\kappa(J^{(y)}) \cong \kappa(J)^{(z)}$ (where z stands for the canonical image of y' in $\kappa(J)$) is separable over κ if $\kappa(J)$ is. 7.2 follows. $\square$

7.3 Proposition. *Let V be a Jordan division pair over K. Then the following statements are equivalent.*
(i) *V contains weakly separable orders.*
(ii) *All maximal orders in V are weakly separable.*
(iii) *There exists a separated valuation $\mu : V \longrightarrow \mathbb{Q}_\infty$ extending λ such that $\mathfrak{P}(V, \mu) = \mathfrak{p}\mathfrak{O}(V, \mu)$.*
(iv) *$\mathfrak{P}(V, \mu) = \mathfrak{p}\mathfrak{O}(V, \mu)$ for all separated valuations $\mu : V \longrightarrow \mathbb{Q}_\infty$ extending λ.*
(v) *$V \cong (J, J)$ where J is a weakly unramified Jordan division algebra over K.*

Proof. Weakly separable orders are maximal (5.12), and the maximal orders of V have precisely the form $\mathfrak{O}(V, \mu)$ where μ varies over the separated valuations of V extending λ (5.2). Hence 5.8a) shows the equivalence of (i) and (iii), (ii) and (iv), respectively. The implication (iv) $\Rightarrow$ (iii) being obvious, it remains to establish (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv).
(iii) $\Rightarrow$ (v). We may assume $V = (J, J)$ for some Jordan division algebra J over K. Then 2.8 yields an element $y \in J^\times$ such that $J^{(y)}$ is weakly unramified. Hence so is J by 7.2.
(v) $\Rightarrow$ (iv). We may identify $V = (J, J)$ as above. Given a separated valuation $\mu : V \longrightarrow \mathbb{Q}_\infty$ extending λ, we conclude from 2.8 that $\mu^+ = \nu^{(y)}, \mu^- = \nu^{(y^{-1})}$ for some $y \in J^\times$. Since $J^{(y)}$ continues to be weakly unramified (7.2), (iv) follows. $\square$

An analogous criterion for the existence of separable orders can be derived in exactly the same manner; we record the result without proof. Notice that a Jordan pair over K admits separable orders only if it is itself separable.

7.4 Proposition. *Let V be a separable Jordan division pair over K. Then the following statements are equivalent.*
(i) *V contains separable orders.*
(ii) *All maximal orders in V are separable.*

(iii) *There exists a separated valuation* $\mu : V \longrightarrow \mathbb{Q}_\infty$ *extending* λ *such that* $\mathfrak{O}(V,\mu)$ *is a separable order in* V.
(iv) $\mathfrak{O}(V,\mu)$ *is a separable order in* V *for every separated valuation* $\mu : V \longrightarrow \mathbb{Q}_\infty$ *extending* λ.
(v) $V \cong (J,J)$ *where* J *is an unramified Jordan division algebra over* K. $\square$

7.5 (Weakly) unramified Jordan pairs. Observing 6.21b), a separable simple (resp. simple) Jordan pair over K is said to the *unramified* (resp. *weakly unramified*) if its anisotropic part satisfies the equivalent conditions of 7.4 (resp. 7.3). An arbitrary Jordan pair over K is said to be *unramified* (resp. *weakly unramified*) if it is separable (resp. semi-simple) and all its simple summands are unramified (resp. weakly unramified).

7.6 Theorem. *Let* V *be a Jordan pair over* K. *Then the following statements are equivalent.*
(i) V *contains separable (resp. weakly separable) orders.*
(ii) V *is separable (resp. semi-simple), and all saturated maximal orders in* V *are separable (resp. weakly separable).*
(iii) *There exists a covering standard division grid* $G \subset V$ *such that* $\mathfrak{O}(V,G)$ *is a separable (resp. weakly separable) order in* V.
(iv) V *is separable (resp. semi-simple) and* $\mathfrak{O}(V,G)$ *is a separable (resp. weakly separable) order in* V *for every covering standard division grid* $G \subset V$.
(v) V *is unramified (resp. weakly unramified).*

Proof. Just as in the beginning of the proof to 7.3 one shows, appealing to 5.3 rather than 5.2 and to 5.11, that it suffices to establish the implications (iii) $\Rightarrow$ (v) $\Rightarrow$ (iv). These in turn will be implied by the following proposition.

7.7 Proposition. *Let* V *be a Jordan pair over* K *and* G *a covering standard division grid of* V. *Writing* (R, R_1) *for the* 3*-graded root system associated with* G, *the following statements are equivalent.*
(i) $\mathfrak{O}(V,G)$ *is a separable (resp. weakly separable) order in* V.
(ii) $V_\alpha(G)$ *is unramified (resp. weakly unramified) for every* $\alpha \in R_1$.
(iii) V *is unramified (resp. weakly unramified).*

Proof. As in 4.3, let $\mu_\alpha, \alpha \in R_1$, be the unique valuation of $V_\alpha = V_\alpha(G)$ extending λ and satisfying $\mu_\alpha^\varepsilon(c_\alpha^\varepsilon) = 0$; put $\mathfrak{O}_\alpha = \mathfrak{O}(V_\alpha, \mu_\alpha)$, $\mathfrak{P}_\alpha = \mathfrak{P}(V_\alpha, \mu_\alpha)$.
(i) $\Rightarrow$ (ii). Suppose $\mathfrak{O} = \mathfrak{O}(V, G)$ is weakly separable. Then Rad $\mathfrak{O} = \mathfrak{p}\mathfrak{O}$ implies Rad $\mathfrak{O}_\alpha = \mathfrak{p}\mathfrak{O}_\alpha$ for each $\alpha \in R_1$ by 3.11 and 4.4, so $\mathfrak{O}_\alpha$ is weakly separable and V_α is weakly unramified. Also, $\mathfrak{O}_\alpha(\mathfrak{p})$ is a Peirce component of the Jordan pair $\mathfrak{O}(\mathfrak{p})$ over κ relative to the covering standard division grid $G(\mathfrak{p})$. Hence if $\mathfrak{O}$ is separable, so ist $\mathfrak{O}_\alpha$, forcing V_α to be unramified (7.4).
(ii) $\Rightarrow$ (iii). Follows from 6.18.
(iii) $\Rightarrow$ (i). Suppose first V is weakly unramified. We may assume that V is simple and, by 6.18, that the roots of R_1 have different lengths, the short roots posing no problem whatsever. Let $\alpha \in R_1$ be a long root. Then [12, I.4.5] yields a $\beta \in R_1$ satisfying $c_\alpha \dashv c_\beta$, and from 4.17 we conclude

$$\mu_\beta^\varepsilon(\{x c_\alpha^{-\varepsilon} c_\beta^\varepsilon\}) = \mu_\alpha^\varepsilon(x) \tag{1}$$

for all $x \in V_\alpha^\varepsilon$. Now let π be a prime element of $\mathfrak{o}$. Given $x \in \mathfrak{P}_\alpha^\varepsilon$, we obtain $\{x c_\alpha^{-\varepsilon} c_\beta^\varepsilon\} = \mathfrak{P}_\beta^\varepsilon$ (by (1)) $= \mathfrak{p}\mathfrak{O}_\beta^\varepsilon$ (since $V_\beta \mathrel{\hat{=}} V_{\mathrm{an}}$ is weakly unramified). Replacing x by $\pi^{-1}x$ in (1) now implies $x \in \mathfrak{p}\mathfrak{O}_\alpha^\varepsilon$, hence that each $\mathfrak{O}_\alpha, \alpha \in R_1$, is weakly separable. This property trivially extends to $\mathfrak{O}$. If V is unramified, so is its anisotropic part, and combining 6.21b) with 6.23 we see that $\mathfrak{O}$ is separable. $\square$

7.8 Corollary. *Let V be an unramified Jordan pair over K. Then every weakly separable order in V is separable.*

Proof. By Neher's Theorem 5.11, weakly separable orders are saturated, and by 5.12 they are maximal. Hence 7.6 applies. $\square$

7.9 Corollary. *Let V be a split simple Jordan pair over K. Then V is unramified, and any two separable orders in V are conjugate under the automorphism group of V.*

Proof. Since $V_{\mathrm{an}} \cong (K, K)^J$ is trivially unramified, so is V. Let $\mathfrak{O}, \mathfrak{O}'$ be separable orders in V. Then there exist covering standard division grids G, G' in V such that $\mathfrak{O} = \mathfrak{O}(V, G), \mathfrak{O}' = \mathfrak{O}(V, G')$ (5.11, 5.3). Since V splits, [13, 3.10] (cf. also [14, p. 467]) yields an automorphism φ of V satisfying $\varphi(G) = \pm G'$. Hence $\varphi(\mathfrak{O}) = \mathfrak{O}'$. $\square$

7.10 Remark. Example 5.5 shows that a maximal order in a (weakly) unramified Jordan pair over K need not be saturated, so in 7.6(ii) the restriction to *saturated* maximal orders is essential.

7.11 Regular integral schemes of dimension 1. Let X be a scheme. The definition of (weak) separability (5.6) being local in nature, it makes sense for Jordan pairs over X: A Jordan pair $\mathcal{V}$ over X whose underlying $\mathcal{O}_X$-modules are locally free of finite rank is said to be *separable* (resp. *weakly separable*) at $p \in X$ if $\mathcal{V}(p) = \mathcal{V} \otimes \kappa(p)$ is separable (resp. semi-simple) over $\kappa(p)$. If this is so for all $p \in X$, we say that $\mathcal{V}$ is *separable* (resp. *weakly separable*).

Now suppose X is integral regular of dimension 1 and let K be the function field of X. For $p \in X$, the local ring $\mathfrak{o}_p = \mathcal{O}_{p,X}$ of X at p is a discrete valuation ring with quotient field K, valuation ideal $\mathfrak{m}_p = \mathfrak{m}_{p,X}$ and residue class field $\kappa(p)$; we write $\lambda_p : K \to \mathbb{Z}_\infty$ for the corresponding discrete valuation. The process of completion leads to a local field $\hat{K}_p = (\hat{K}_p, \hat{\lambda}_p)$, with valuation ring $\hat{\mathfrak{o}}_p$, valuation ideal $\hat{\mathfrak{m}}_p$ and residue class field $\kappa(p)$. Let V be a finitedimensional Jordan pair over K. If $\hat{V}_p = V \otimes_K \hat{K}_p$ is (weakly) unramified over $\hat{K}_p$, we say that V is (*weakly*) *unramified at* p.

7.12 Theorem. *Let X be a regular integral scheme of dimension* 1 *and V a finitedimensional Jordan pair over the function field K of X. Then the following statements are equivalent.*
(i) *V is unramified at each point of X.*
(ii) *V extends to a separable Jordan pair spread over all of X, i.e., there exists a separable Jordan pair over X whose generic fibre is isomorphic to V.*

Proof. (ii) $\Rightarrow$ (i). Let $\mathfrak{O}$ be a separable Jordan pair over X whose generic fibre identifies with V. For $p \in X$, $\hat{\mathfrak{O}}_p = \mathfrak{O}_p \otimes_{\mathfrak{o}_p} \hat{\mathfrak{o}}$ is a separable order in $\hat{V}_p$, whence V must be unramified at p (7.6).
(i) $\Rightarrow$ (ii). Letting (e_i^ε) be a basis of V^ε over K and putting $\mathfrak{O}'^\varepsilon = \sum \mathcal{O}_X e_i^\varepsilon$, $\mathfrak{O}' = (\mathfrak{O}'^+, \mathfrak{O}'^-)$ will become a Jordan pair over some dense open subscheme $U \subset X$ and, by shrinking U even further if necessary, we may assume that $\mathfrak{O}'$ is separable over U [8, Theorem 1]. For $p \in X - U, \hat{V}_p$ is

unramified over $\hat{K}_p$, so 7.6 produces a separable order $\mathfrak{O}''_p$ in $\hat{V}_p$. By [25, (4.22)], the separable Jordan pairs $\mathfrak{O}'$ over U and $\mathfrak{O}''_p = \hat{\mathfrak{O}}''_p \cap V$ over each $p \in X - U$ glue to give a separable Jordan pair over all of X whose generic fibre brings us back to V. $\square$

7.13 Example. Assume for simplicity that in 7.12 all fields $\kappa(p), p \in X$, have characteristic not two, let $q : M \longrightarrow K$ be a nondegenerate quadratic form and V the corresponding Jordan pair. By 7.2 and [17, §7 **4**.] V is unramified at $p \in X$ if and only if $\partial_p q$, the second residue class form of q in the sense of Springer, vanishes in the Witt ring of $\kappa(p)$. On the other hand, for V to extend to a separable Jordan pair over all of X it is necessary and sufficient that there exist a selfdual $\mathcal{O}_X$-lattice in M relative to q. Thus, in the totally nondyadic situation, 7.12 may be regarded as a generalization of [11, IV Theorem 3.1].

7.14 Remark. 7.12 applies in particular to (complete smooth algebraic) curves over a field k. For example, if X is such a curve of genus zero, results due to Van den Bergh-Van Geel [30] and Van Geel [31] for associative algebras and quadratic forms, to Harder [4] for algebraic groups and to the author [22] for composition algebras suggest the following question: Suppose a finitedimensional Jordan pair V over the function field K of X is unramified everywhere. Does this imply that V is extended from k, i.e., that there exists a Jordan pair V_0 over k satisfying $V \cong V_0 \otimes_k K$? We will take up this question in another paper.

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