On the Invariants mod 2 of Albert Algebras

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Introduction

The existence of a new invariant for central simple exceptional Jordan algebras (nowadays called *Albert algebras*) over an arbitrary base field k of characteristic not 2 or 3, following a suggestion of Serre [Se], has recently been established by Rost [Ro1]. In fact, Serre (loc. cit.) has raised the question as to whether this new invariant, called the *invariant mod* 3, together with the ordinary trace form already employed in the work of Springer [Sp], classifies Albert algebras over k.

The invariant mod 3 belongs to $H^3(k, Z/3Z)$ and so ties up nicely with results and techniques from Galois cohomology. Serre (loc. cit.) has also supplied an analogous cohomological characterization for the trace form by means of two decomposable elements in $H^3(k, Z/2Z)$ and $H^5(k, Z/2Z)$, respectively, called the *invariants mod* 2, provided the underlying Albert algebra is reduced. It is of course desirable to define the invariants mod 2 also in the case of an Albert division algebra. In order to accomplish this, one must deal with the following question, raised by Serre in a private communication to the second author of this paper: If k'/k is a cubic field extension reducing a given Albert algebra $\mathcal{J}$ over k (so that $\mathcal{J} \otimes_k k'$ is reduced and hence isomorphic to the algebra of twisted 3-by-3 hermitian matrices with entries in an octonion algebra C' over k'), does it follow that C' *is uniquely extended from* k, i.e., that there exists a unique octonion algebra C over k satisfying $C \otimes_k k' \cong C'$?

As Serre himself has pointed out in another communication to the second author, this question has an affirmative answer thanks to a descent property for Pfister forms established by Rost [Ro2]. This approach, however, does not reveal any, even remotely explicit, description of the octonion algebra C in terms of the Albert algebra $\mathcal{J}$.

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The purpose of the present paper is twofold. First we answer Serre's original question in a somewhat more general form (1.8) by using only elementary properties of the generalized second Tits construction (1.4). Our main concern, however, will be to derive an explicit description, valid in all characteristics, if not of C itself but at least of its norm form. The description presented here (4.2) depends on the constituents used to build up $\mathcal{J}$ by means of the generalized second Tits construction and requires certain elementary manipulations of quadratic forms, expanding earlier ones developed in [PR1] and revolving around the notion of the pseudo-discriminant (section 2). In addition, certain results on separable commutative-associative algebras of dimension 3 have to be separated from the rest of the argument in section 3. The proof of 4.2 will then be carried out in section 5. Ignoring certain technical adjustments necessary to include characteristics 2 and 3, the idea is to exhibit, in any reduced Albert algebra realized by means of the generalized second Tits construction, a complete orthogonal system of three absolutely primitive idempotents and to compute the quadratic trace (1.1) on the corresponding Peirce-(2,3)-component.

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1. Albert Algebras and their Coordinate Algebras

1.1 Albert algebras are best treated in the more general set-up of Jordan algebras arising from cubic forms with adjoint and base point [M1]. For the convenience of the reader we recall the definition.

Let k be an arbitrary field, remaining fixed throughout this paper. By a *cubic form with adjoint and base point* we mean a triple $(N, \#, 1)$ consisting of a cubic form N on a finite-dimensional vector space V over k, a quadratic mapping $x \mapsto x^\#$ from V to V and a distinguished element $1 \in V$ such that the following identities hold under all scalar extensions.

$$
\begin{aligned}
&(1) & x^{\#\#} &= N(x)x, && (\text{“adjoint identity”})\\
& & N(1) &= 1, && (\text{“base point identity”})\\
&(2) & T(x^{\#}, y) &= (DN)(x)y, &&\\
&(3) & 1^{\#} &= 1, &&\\
&(4) & 1 \times y &= T(y)1 - y; &&
\end{aligned}
$$

here $T = -(D^2 \log N)(1)$ is the *associated trace form* (“D” indicating the total derivative of a rational map), $\times$ is the bilinearization of $\#$ and $T(y) = T(1, y)$. Then the operation

$$(5) \qquad U_x y = T(x, y)x - x^{\#} \times y$$

together with the base point 1 gives V the structure of a unital quadratic Jordan algebra over k, written as $\mathcal{J} = \mathcal{J}(N, \#, 1)$ [M1, Theorem 1]. In addition to the norm $N = N_{\mathcal{J}}$ and the trace $T = T_{\mathcal{J}}$, extensive use will be made of the *quadratic trace* $S = S_{\mathcal{J}}$ of $\mathcal{J}$, i.e., the quadratic form defined by

$$(6) \qquad S(x) = T(x^{\#}).$$

Its bilinearization satisfies

$$(7) \qquad S(x, y) = T(x)T(y) - T(x, y) = T(1 \times x, y)$$

by [M1, (16)] and (4). From [M1] we finally recall the identities

$$
\begin{aligned}
&(8) & T(1) &= 3\\
&(9) & x^{\#} &= x^2 - T(x)x + S(x)1,\\
&(10) & x \times x^{\#} &= [T(x^{\#})T(x) - N(x)]1 - T(x^{\#})x - T(x)x^{\#},\\
&(11) & T(x \times y, z) &= T(x, y \times z),\\
&(12) & x^{\#} \times (x \times y) &= N(x)y + T(x^{\#}, y)x,\\
&(13) & (x \times y)^{\#} &= T(x^{\#}, y)y + T(x, y^{\#})x - x^{\#} \times y^{\#}.
\end{aligned}
$$

1.2 Examples Let R be a finite-dimensional unital associative k-algebra and suppose R has degree 3. Then R^+, the associated Jordan algebra (with cubic operation $U_x y = xyx$ and identity element $1_{R^+} = 1_R$) agrees with $\mathcal{J}(N, \#, 1)$ where $1 = 1_R$, N is the reduced norm of R (i. e., the exact denominator of the inversion map, normalized by $N(1) = 1$) and # is the adjoint (i.e., the corresponding numerator). Notice that this implies

$$xx^{\#} = N(x)1 = x^{\#}x. \tag{14}$$

1.3 Returning to 1.1, let us assume that $u \in \mathcal{J}$ is invertible (i.e., satisfies $N(u) \neq 0$). Then the u-isotope $\mathcal{J}^{(u)}$ of $\mathcal{J}$ again arises from a cubic form with adjoint and basepoint [M1, Theorem 2]:

$$\mathcal{J}^{(u)} = \mathcal{J}(N^{(u)},\ \#^{(u)},\ 1^{(u)}),$$

where $N^{(u)} = N(u)N$, $\#^{(u)} = N(u)U_u^{-1} \circ \#$ and $1^{(u)} = u^{-1} = N(u)^{-1}u^{\#}$. Writing $T_{\mathcal{J}}^{(u)}$ for the trace and $S_{\mathcal{J}}^{(u)}$ for the quadratic trace of $\mathcal{J}^{(u)}$, we have

$$T_{\mathcal{J}}^{(u)}(x,y) = T_{\mathcal{J}}(U_u x, y), \tag{15}$$

$$T_{\mathcal{J}}^{(u)}(x) = T_{\mathcal{J}}(u, x), \tag{16}$$

$$S_{\mathcal{J}}^{(u)}(x) = T_{\mathcal{J}}(u^{\#}, x^{\#}) \tag{17}$$

for all $x, y \in \mathcal{J}$. Indeed, (15) may be found in [M1, p. 500] and easily implies the rest.

1.4 By [PR4, Theorem 3.1], all Albert algebras over k may be obtained from a slight generalization of the second Tits construction [M1, Theorem 7], which we now proceed to describe [PR3, Theorem 3.4]. Let $(B, *)$ be a central simple associative algebra of degree 3 with involution of the second kind over k, write K for the center of B (which is either a separable quadratic field extension of k or isomorphic to $k \oplus k$), N for its (reduced) norm, T for its (reduced) trace, # for its adjoint and put

$$A = \mathrm{H}(B, *) = \{a \in B : a^* = a\}$$

(which is a k-subalgebra of B^+). Suppose further that we are given invertible elements $u \in A$, $\beta \in K$ satisfying $N(u) = \beta\beta^*$. Extending $N, \#, 1 = 1_B = 1_A$ as given on B and A to the $k-$vector space

$$V = A \oplus B$$

according to the rules

$$N((a,b)) = N(a) + \beta N(b) + \beta^* N(b)^* - T(a, bub^*),$$

(18) $$(a,b)^\# = (a^\# - bub^*, \beta^* b^{*\#} u^{-1} - ab),$$

(19) $$1 = (1_A, 0)$$

for $a \in A, b \in B$, we obtain a cubic form $(N, \#, 1)$ with adjoint and base point over k. The ensuing Jordan algebra structure will be written as $\mathcal{J} = \mathcal{J}(B, *, u, \beta)$ and is in fact an Albert algebra. The associated trace form is given by

(20) $$T(x,y) = T(a,c) + T(bu, d^*) + T(du, b^*)$$

for $x = (a,b),\ y = (c,d) \in \mathcal{J}$, which implies

(21) $$T(x) = T(x,(1,0)) = T(a).$$

For future reference, we also record the bilinearization of the adjoint:

(22) $$x \times y = (a \times c - bud^* - dub^*, \beta^*(b^* \times d^*)u^{-1} - ad - cb).$$

Finally, $\mathcal{J}$ is a division algebra if and only if β is not a norm of B [PR3, Theorem 5.2]. The following result combines [PR3, Propositions 3.7 and 3.9].

1.5 Proposition *Let $\mathcal{J} = \mathcal{J}(B, *, u, \beta)$ be an Albert algebra over k, realized by means of the gereralized second Tits construction as in* 1.4.
a) *Given an invertible element $w \in B$ the rule $(a,b) \mapsto (a, bw)$ determines an isomorphism from $\mathcal{J}$ onto $\mathcal{J}(B, *, wuw^*, N(w)\beta)$.*
b) *Given an invertible element $y \in A = \mathrm{H}(B, *)$ and writing $*^{(y)}$ for the involution $b \mapsto yb^*y^{-1}$* of B, *the rule $(a,b) \mapsto (y^{-1}a, b)$ determines an isomorphism from $\mathcal{J}(B, *^{(y)}, uy^\#, N(y)\beta)$ onto the y-isotope $\mathcal{J}^{(y)}$.*

1.6 An Albert algebra $\mathcal{J}$ over k is either a division algebra or it is reduced. In the latter case, there exists a unique octonion algebra C over k and a diagonal matrix $g \in \mathrm{GL}_3(k)$ such that $\mathcal{J} \cong \mathrm{H}_3(C, g)$, the Jordan algebra of 3-by-3 matrices x which have diagonal entries in k, off-diagonal entries in C and are *g-twisted hermitian*, i.e., satisfy $x = g^{-1t}\overline{x}g$, $^-$ being the canonical involution of C. We call C the *coordinate algebra* and its norm the *coordinate norm* of $\mathcal{J}$. The Jordan algebra structure of $\mathrm{H}_3(C, g)$ is again derived from a cubic form $(N, \#, 1)$ with adjoint and base point [M1, Theorem 3]. Noting that an arbitrary element of $\mathrm{H}_3(C, g)$ has the form

$$x = \sum_{i=1}^{3} \alpha_i e_{ii} + \sum_{i=1}^{3} a_i[jl], \quad \alpha_i \in k, a_i \in C, \tag{23}$$

where (ijl) are cyclic permutations of (123) and $a[ij] = g_j a e_{ij} + g_i \bar{a} e_{ji}$ in terms of the matrix units e_{ij}, we set, n and t being the norm and trace, respectively, of C,

$$\begin{aligned} N(x) &= \alpha_1\alpha_2\alpha_3 - \sum_{i=1}^{3} g_j g_l \alpha_i n(a_i) + g_1 g_2 g_3 t(a_1 a_2 a_3), \\ x^{\#} &= \sum_{i=1}^{3}\{\alpha_j\alpha_l - g_j g_l n(a_i)\}e_{ii} + \sum_{i=1}^{3}\{g_i \overline{a_j a_l} - \alpha_i a_i\}[jl], \\ 1 &= e_{11} + e_{22} + e_{33}. \end{aligned}$$

The associated trace form is given by

$$T(x,y) = \sum_{i=1}^{3}\{\alpha_i\beta_i + g_j g_l n(a_i, b_i)\},$$

$y \in \mathrm{H}_3(C,g)$ being represented in a form analogous to (23). For the quadratic trace, this implies the relation

$$S(x) = \sum_{i=1}^{3}\{\alpha_j\alpha_l - g_j g_l n(a_i)\}. \tag{24}$$

1.7 Let A be an absolutely simple Jordan algebra of degree 3 and dimension 9 over k. By structure theory, there exists a central simple associative algebra $(B,*)$ of degree 3 over k with involution of the second kind, unique up to isomorphism, satisfying $A \cong \mathrm{H}(B,*)$. Then we define the *octonion algebra of* A, written as Oct A, to be the coordinate algebra (1.6) of the reduced Albert algebra $\mathcal{J}(B,*,1,1)$ (1.4). The relationship of this notion to an earlier construction of Faulkner's [F2] will be taken up in another paper. Calling a field extension k'/k a *reducing field* of an Albert algebra $\mathcal{J}$ over k if the extended algebra $\mathcal{J} \otimes_k k'$ over k' is reduced, we can now prove the following result.

1.8 Theorem (Serre, Rost) *Let $\mathcal{J}$ be an Albert algebra over k. Then there exists an octonion algebra C over k, unique up to isomorphism, such that, given any reducing field k'/k of $\mathcal{J}$, $C \otimes_k k'$ is the coordinate algebra of $\mathcal{J} \otimes_k k'$. Moreover, realizing $\mathcal{J} = \mathcal{J}(B,*,u,\beta)$ by means of the generalized second Tits construction as in* 1.4 *and setting $A = \mathrm{H}(B,*)$, C is isomorphic to the octonion algebra of $A^{(u)}$.*

Proof We may assume that $\mathcal{J} = \mathcal{J}(B,*,u,\beta)$ is reduced, and must then show it has the coordinate algebra Oct $A^{(u)}$. Writing $\beta = N(t)$ for some invertible element $t \in B$ (1.4) and $w = t^{-1}$, $\mathcal{J}$ and $\mathcal{J}(B, *, wuw^*, 1)$ are isomorphic by 1.5 a), as are $A^{(u)}$ and $A^{(v)}$, $v = wuw^*$, since the map $a \mapsto waw^*$ belongs to the structure group of A. Hence we may assume $\beta = 1$. But then, by the Albert-Jacobson-Faulkner Theorem [F1, Theorem 1.8], $\mathcal{J}$ and $\mathcal{J}^{(u)} \cong \mathcal{J}(B, *(u), 1, 1)$ (1.5 b)) have the same coordinate algebra, which by definition agrees with Oct $A^{(u)}$.

1.9 According to 1.8, we are allowed to talk about the *coordinate algebra* and the *coordinate norm* of an Albert algebra $\mathcal{J}$ even if $\mathcal{J}$ is not reduced. Using this terminology, we will be concerned in the rest of the paper with an explicit description of the coordinate norm of $\mathcal{J}$ realized by means of the generalized second Tits construction as in 1.4. In order to understand this description, a few preparations are required that will be taken up in the next section.

2. Associates of quadratic forms and the pseudo-discriminant

2.1 In this section, we modify and expand certain notions introduced in [PR1]. Writing $k^\bullet$ for the multiplicative group $k^\times = k - \{0\}$ if char $k \neq 2$ and for the additive group k otherwise, we define $\wp : k^\bullet \longrightarrow k^\bullet$ by $\wp(\alpha) = \alpha^2$ in the first case and by $\wp(\alpha) = \alpha + \alpha^2$ in the second ($\alpha \in k^\bullet$). In contrast to [PR1], the group $\Gamma(k) = k^\bullet/\wp(k^\bullet)$ will be written *additively* even if the characteristic is not two. Notice that $\Gamma(k)$ canonically identifies with $H^1(k, Z/2Z)$ and also with the group of isomorphism classes of separable quadratic k-algebras in the sense of Knus [K, III §4].

2.2 Let V be a (finite-dimensional) vector space over k and $q : V \longrightarrow k$ a quadratic form. As in (7), the symmetric bilinear form induced by q will also be written as q, so

$$q(u, v) = q(u + v) - q(u) - q(v)$$

for $u, v \in V$. If this symmetric bilinear form is nondegenerate in the usual sense, we call q *nonsingular*. By contrast, q is said to be *nondegenerate* if $q(u) = q(u, v) = 0$ for all $v \in V$ implies $u = 0$. These two notions agree for char $k \neq 2$ but are distinct in general. For example, nonsingularity of q in characteristic two forces V to have even dimension. By Witt cancellation [Sc, (7.9.2)], the following classical result of Springer, being well known for char $k \neq 2$ [Sc, (2.5.3), (2.5.4)], easily extends to arbitrary characteristics as follows.

2.3 Springer's Theorem *Let k'/k be a finite field extension of* odd *degree.*
a) *If q is a quadratic form over k such that $q \otimes_k k'$ is isotropic, then q is isotropic.*
b) *If q_1, q_2 are nonsingular quadratic forms over k and q_2' is a nonsingular quadratic form over k' such that*

$$q \otimes_k k' \cong (q_1 \otimes_k k') \perp q_2'$$

then there exists a nonsingular quadratic form q_2 over k, uniquely determined up to isometry, satisfying

$$q_2' \cong q_2 \otimes_k k' \ , \ q \cong q_1 \perp q_2.$$

In particular, $q \otimes_k k' \cong q_1 \otimes_k k'$ implies $q \cong q_1$.

2.4 Let $q : V \longrightarrow k$ be a nonsingular quadratic form representing 1 and $e \in V$ be a *basepoint* of q, so $q(e) = 1$. Fixing $d \in k^\bullet$, we define a quadratic form $q_d : V \longrightarrow k$ by

$$q_d(v) = dq(v) + \tfrac{1-d}{4} q(e,v)^2 \qquad (v \in V)$$

for char $k \neq 2$ and by

$$q_d(v) \;=\; q(v) + dq(e,v)^2 \qquad (v \in V) \tag{25}$$

for char $k = 2$. Writing q^0 for the restriction of q to the orthogonal complement of e, we conclude

$$q_d \;\cong\; <1> \;\perp dq^0 \qquad (\text{char } k \neq 2), \tag{26}$$

whereas, for char $k = 2$, q and q_d induce the same symmetric bilinear form on V. Hence, in all characteristics, q_d is a nonsingular quadratic form representing 1 (in fact, $q_d(e) = 1$). Finally, by Witt's Theorem [Sc,(7.9.1)], q_d up to isometry does not depend on the base point e.

2.5 Remark For char $k \neq 2$, q_d as defined in 2.4 differs from q_d as defined in [PR1, 3.1] by a factor d. If one is interested only in *similarity* of quadratic forms (as happens, for example, in [PR1, 3.2]), this difference may therefore be safely ignored.

2.6 Arguing as in the proof of [PR1,3.1b)], it now follows easily that, given a nonsingular quadratic form q representing 1 and $d \in k^\bullet$, the quadratic form q_d up to isometry only depends on the class of d modulo $\wp(k^\bullet)$. Hence, for any $\delta \in \Gamma(k)$, the notion q_δ is unambiguous and refers to a nonsingular quadratic form representing 1, called the *δ-associate of q*. Obviously, $q_0 \cong q$, i. e., the 0-associate of q is isometric to q. The following lemma is an immediate consequence of the definitions.

2.7 Lemma *Let q be a nonsingular quadratic form representing 1 and $\delta \in \Gamma(k)$.*
a) *For all $\delta' \in \Gamma(k)$ we have $(q_\delta)_{\delta'} \cong q_{\delta+\delta'}$.*
b) *If q' is a strongly nondegenerate quadratic form, then*

$$(q \perp q')_\delta \cong q_\delta \perp \delta' q',$$

where $\delta' = \delta$ for char $k \neq 2$ *and $\delta' = 1$ for* char $k = 2$.

2.8 By a *torus of rank n* we mean as in [PR1] a separable commutative-associative algebra E/k of dimension n, which we usually identify with its corresponding Jordan algebra E^+. If E has rank 2 or 3 it comes along with an important invariant attached, written as $\delta(E/k) \in \Gamma(k)$ and called its *pseudo-discriminant* because it is the usual discriminant in characteristic not two. For the convenience of the reader, we recall the definition in characteristic two.

So assume char $k = 2$, let K/k be a torus of rank 2, with norm n and trace t, and write $K = k[\theta]$ for some $\theta \in K$ having $t(\theta) = 1$. Then K has discriminant 1 and $\delta(K/k)$ is the class of $n(\theta)$ in $\Gamma(k)$ (easily seen to be independent of the choice of θ). On the other hand, if E/k is a torus of rank 3, one of the following cases occurs.

(i) E/k is not a field, so $E = k \oplus K$ for some torus K/k of rank 2. Then we set $\delta(E/k) = \delta(K/k)$.
(ii) E/k is a cyclic field extension. Then we set $\delta(E/k) = 0$.
(iii) E/k is a noncyclic field extension, so its Galois closure contains a unique torus K/k of rank 2. Then we set $\delta(E/k) = \delta(K/k)$.

In arbitrary characteristic, given any $\delta \in \Gamma(k)$, we write $k\{\delta\}$ for the unique torus of rank 2 over k having pseudo-discriminant δ (2.1). Clearly, $k\{\delta\}$ is a composition algebra, so it makes sense to talk about its norm and its trace, written as $n_{k\{\delta\}}$ and $t_{k\{\delta\}}$, respectively, the former being a nonsingular binary quadratic form representing 1. The following lemma is an adaptation of [PR2, Lemma 2] to the present set-up.

2.9 Lemma *Let $\delta, \delta' \in \Gamma(k)$. Then*

$$(n_{k\{\delta\}})_{\delta'} \cong n_{k\{\delta+\delta'\}}.$$

3. Tori of rank three

3.1 Let E/k be a torus of rank 3 containing zero divisors. Then $E = k \oplus C$, where C is a torus of rank 2 over k, necessarily of the form $C = k\{\delta\}$ for some $\delta \in \Gamma(k)$; in fact $\delta = \delta(E/k)$ (2.8). Write $n, ^-, t$ for the norm, canonical involution, trace, respectively, of C and $N, \#, T, S$ for the norm, adjoint, trace, quadratic trace, respectively, of E. For $a, b \in k,\ u, v \in C$, [PR2, Example 2.2] implies

$$\begin{aligned} N\big((a,u)\big) &= an(u), \\ (a,u)^{\#} &= (n(u), a\bar{u}), \\ T\big((a,u),(b,v)\big) &= ab + t(uv), \end{aligned} \tag{27}$$

forcing

$$\begin{aligned} T\big((a,u)\big) &= a + t(u), \\ S\big((a,u)\big) &= n(u) + at(u). \end{aligned} \tag{28}$$

Hence E^0, the space of trace zero elements in E, is the same as C under the identification $u = (-t(u), u)$, and S^0, the restriction of S to E^0, becomes the binary quadratic form $S^0 : C \longrightarrow k$ given by

$$S^0(u) = n(u) - t(u)^2. \tag{29}$$

Now choose $\theta \in C$ satisfying $\theta + \bar{\theta} = t(\theta) = 1$ and $\theta - \bar{\theta} \in C^{\times}$. Then it is easily checked that

$$d = 1 - 4n(\theta) \tag{30}$$

is the discriminant of C/k and E/k. Also, setting

$$h_+ = (1, 0),\ h_- = (-n(\theta), \theta),\ w = (-d, 1 - 2\theta), \tag{31}$$

an equally straightforward computation, involving (28) and its linearization, shows that

$$S(w) = -d \tag{32}$$

and (h_+, h_-) is a hyperbolic pair of the quadratic space (E, S) satisfying

(33) $$S(h_\pm, w) = 0.$$

We are now in a position to establish the following results.

3.2 Proposition *Assume* char $k = 2$, *let* E *be a torus of rank* 3 *over* k *and write* S *for the quadratic trace of* E. *Then* E *has discriminant* 1, *the radical of the symmetric bilinear form induced by* S *is* $k1$ *and* S^0, *the restriction of* S *to the space of trace zero elements in* E, *is given by*

$$S^0 \cong n_{k\{\delta(E/k)+1\}}.$$

Proof Because of (7), the radical of the bilinear form induced by S agrees with kernel of the map $x \mapsto 1 \times x = T(x)1 - x$ (by (4)), hence with $k1$ since we are in characteristic 2. Therefore, as $T(1) = 3 \neq 0$ by (8), S^0 is nonsingular, so by Springer's Theorem 2.3 we may assume that E has zero divisors. But then $d = 1$ by (30), and (25), (29) as well as 2.9 imply

$$S^0 \cong \left(n_{k\{\delta(E/k)\}}\right)_1 \cong n_{k\{\delta(E/k)+1\}},$$

as desired.

In what follows, we write $\mathbf{h}$ for the hyperbolic plane in the theory of quadratic forms.

3.3 Proposition *Let* E *be a torus of rank* 3 *and discriminant* d *over* k *and let* S *be its quadratic trace. Then*

$$S \cong \mathbf{h} \perp <-d>.$$

Proof If E has zero divisors this follows immediately from (32), (33), so let us assume that E is a field. If char $k \neq 2$ we are done by Springer's Theorem 2.3, so let us assume char $k = 2$. Because of 3.2, the radical of the bilinear form induced by S agrees with $k1$. Also, $S(1) = 1 = -d$ by (3), (6) and (30). Finally, S becoming isotropic after extending scalars from k to E, it must have been so all along ((2.3a)). Hence there exists a hyperbolic plane in the quadratic space (E, S), which is automatically orthogonal to 1, and the assertion follows.

3.4 The pseudo-discriminant of a torus of rank 3 is an important invariant since, for example, it characterizes Galois extensions. On the other hand, the original definition reproduced in 2.8 is not particularly explicit. To get around this deficiency in characteristic not two, we can always fall back on the ordinary discriminant, whose determination is quite straightforward. That such an explicit characterization exists also in characteristic two is the content of our next theorem.

3.5 Theorem *Assume* char $k = 2$ *and let* E/k *be a torus of rank* 3, *with trace* T *and quadratic trace* S. *Then, given any hyperbolic pair* (h_+, h_-) *of the quadratic space* (E, S) (cf. 3.3), *the pseudo-discriminant of* E/k *is given by*

$$\delta(E/k) = T(h_+, h_-) \bmod \wp(k^\bullet).$$

Proof Our first aim is to show that the class of $T(h_+, h_-)$ in $\Gamma(k)$ does not depend on the choice of (h_+, h_-). So let (h'_+, h'_-) be another hyperbolic pair of (E, S) and assume first that $\{h_+, h_-\} \cap \{h'_+, h'_-\} \neq \emptyset$, hence without loss of generality $h'_+ = h_+$. Notice that $1, h_+, h_-$ is a basis of E over k and that $k1$ is the radical of the symmetric bilinear form induced by S (3.2). Choosing scalars $\alpha, \beta_+, \beta_- \in k$ such that $h'_- = \alpha 1 + \beta_+ h_+ + \beta_- h_-$, the relations $S(h_+, h'_-) = 1, S(h'_-) = 0$ imply $\beta_- = 1, \beta_+ = \alpha^2$, forcing $h'_- = \alpha 1 + \alpha^2 h_+ + h_-$. On the other hand, $0 = S(h_+, h_+) = T(h_+)^2 - T(h_+, h_+)$ (by (7)), and we obtain $T(h_+, h_+) = T(h_+)^2$, hence

$$\begin{aligned} T(h'_+, h'_-) &= T(h_+, \alpha 1 + \alpha^2 h_+ + h_-) \\ &= T(h_+, h_-) + \alpha T(h_+) + \alpha^2 T(h_+)^2 \\ &\equiv T(h_+, h_-) \bmod \wp(k^\bullet), \end{aligned}$$

as claimed. We are left with the case $\{h_+, h_-\} \cap \{h'_+, h'_-\} = \emptyset$. This time we write $h'_+ = \alpha 1 + \beta_+ h_+ + \beta_- h_-$ with $\alpha, \beta_\pm \in k$. If $\alpha = 0$, $S(h'_+) = 0$ implies $\beta_+\beta_- = 0$, so without loss $\beta_- = 0, \beta_+ = 1$ (since T is invariant under the action of $k^\times$ on hyperbolic pairs of (E, S) given by $\alpha.(h_+, h_-) = (\alpha h_+, \alpha^{-1} h_-)$), forcing $h'_+ = h_+$, a contradiction. If $\alpha \neq 0$ we may assume $\alpha = 1$, and $S(h'_+) = 0$ implies $\beta_+\beta_- = 1$. Arguing as before, we may even assume $\beta_+ = \beta_- = 1$. Then (h_+, h'_+) is a hyperbolic pair of (E, S), so applying the previous case twice establishes our first aim. In order to complete the proof of the theorem, we note that, given any odd degree field extension k'/k, the natural map $\Gamma(k) \longrightarrow \Gamma(k')$ is injective, so extending scalars from k to E if necessary we may assume that E has zero divisors. Then the invariance property just established allows us to further assume $h_+ = (1, 0), h_- = (-n(\theta), \theta)$ as in (31), which implies $T(h_+, h_-) = n(\theta)$ (by (27)) and finishes the proof by 2.8.

3.6 Corollary *In the situation of* 3.5, *let* a *be a* primitive element of E, *so* $E = k[a]$ *is generated by* a *as a unital commutative-associative* k*-algebra. Then* $T(a)S(a) + N(a) \neq 0$ *and*

$$\delta(E/k) = 1 + \frac{(T(a)^2 + S(a))^3}{(T(a)S(a) + N(a))^2} \bmod \wp(k^\bullet)$$

Proof Replacing a by $T(a)1 + a$ if necessary we may assume $T(a) = 0$. Then we must first show $N(a) \neq 0$. Otherwise, the relations $S(a, 1) = 0$ (by 3.2), $S(a, a) = 2S(a) = 0$, $S(a, a^\#) = T(a, a^\#)$ (by (7)) $= 3N(a)$ (by (2) and Euler's Differential

Equation) = 0 would imply $a \in k1$ (again by 3.2), hence $a = 0$, a contradiction. Thus $N(a) \neq 0$ and, by (1), $h_+ = a^\#$ is an isotropic vector of (E, S), extending to the hyperbolic pair (h_+, h_-) where $h_- = N(a)^{-1}a + N(a)^{-2}S(a)h_+$. Now

$$\begin{aligned} T(h_+, h_-) &= N(a)^{-1}T(a^\#, a) + N(a)^{-2}S(a)T(a^\#, a^\#) \\ &= 1 + N(a)^{-2}S(a)^3 \end{aligned}$$

since $0 = 2S(a^\#) = S(a^\#, a^\#) = S(a)^2 - T(a^\#, a^\#)$ (by (7)). Comparing with 3.5 completes the proof.

For char $k \neq 2$, there is a more technical result, in a sense complementary to 3.5, which reads as follows.

3.7 Lemma *Assume* char $k \neq 2$ *and let E/k be a torus of rank* 3, *with trace T and quadratic trace S. Then there exists a hyperbolic pair (h_+, h_-) of the quadratic space (E, S)* (cf. 3.3) *satisfying*

$$T(h_+, h_-) = -\frac{1}{4} \quad \textit{and} \quad 1 = 1_E \in kh_+ + kh_-.$$

Proof We first assume char $k = 3$. Then $S(1) = T(1) = 3 = 0$, so 1 extends to a hyperbolic pair (h_+, h_-) of (E, S), and we obtain

$$\begin{aligned} T(h_+, h_-) &= -(T(h_+)T(h_-) - T(h_+, h_-)) \\ &= -S(h_+, h_-) && \text{(by (7))} \\ &= -1 = -\frac{1}{4}, \end{aligned}$$

as claimed. Now let char $k \neq 3$ and put $S' = \frac{1}{3}S$. Then $S' \cong \mathbf{h} \perp \; < -3d >$ by 3.3, d being the discriminant of E/k. By Witt's Theorem, we can find a hyperbolic pair (h'_+, h'_-) of (E, S') satisfying $h'_+ + h'_- = 1$, so $(h_+, h_-) = (h'_+, \frac{1}{3}h'_-)$ is a hyperbolic pair of (E, S) satisfying $h_+ + 3h_- = 1$. We further contend

(34) $$T(h_-) = \tfrac{1}{2}, \; T(h_-, h_-) = \tfrac{1}{4}.$$

Indeed, this follows from $T(h_-)^2 - T(h_-, h_-) = S(h_-, h_-)$ (by (7)) $= 0$ and

$$\begin{aligned} 1 &= S(h_+, h_-) = T(1 - 3h_-)T(h_-) - T(1 - 3h_-, h_-) \\ &= 3T(h_-) - 3T(h_-)^2 - T(h_-) + 3T(h_-, h_-) \\ &= 2T(h_-). \end{aligned}$$

Equation (34) implies

$$\begin{aligned} T(h_+, h_-) &= T(1 - 3h_-, h_-) = T(h_-) - 3T(h_-, h_-) \\ &= \frac{1}{2} - \frac{3}{4} = -\frac{1}{4}, \end{aligned}$$

as claimed.

4. The coordinate norm of a generalized second Tits construction

4.1 Returning to 1.9, we can now state the main result of the paper.

4.2 Main Theorem *Let $\mathcal{J} = \mathcal{J}(B, *, u, \beta)$ be an Albert algebra over k, realized by means of the generalized second Tits construction as in* 1.4, *and write n for the coordinate norm of $\mathcal{J}$* (cf. 1.9). *Then the following holds.*
(i)

$$S_A^{(u)} \cong < -1 > \perp n_\delta,$$

where $\delta = \delta(K/k)$ *for* char $k \neq 2$ *and* $\delta = \delta(K/k) + \delta(E/k)$ *for* char $k = 2$, $E \subset A^{(u)}$ *being any maximal torus.*
(ii)

$$n \cong n_{k\{\delta(K/k)+\delta(E/k)\}} \perp (d\, S_A^{(u)}|_{E^\perp}),$$

where $E \subset A^{(u)}$ *is any maximal torus,* $E^\perp \subset A^{(u)}$ *stands for the orthogonal complement of* E *relative to* $T_A^{(u)}$ *and* d *is the discriminant of* K/k.
(iii)

$$S_A^{(u)0} \cong n_{\delta(K/k)+1}$$

for char $k = 2$, where $S_A^{(u)0}$ *is the restriction of* $S_A^{(u)}$ *to the space of trace zero elements in* $A^{(u)}$ *(i.e., to the kernel of* $T_A^{(u)}$*).*

4.3 The proof of 4.2 will be postponed to the next section. For the time being, we settle with the following explanatory comments.

To begin with, 4.2 is nontrivial even when $\mathcal{J}$ is reduced; in fact, the reduced case lies at the heart of the matter.

Secondly, without going into the details, it is worth recording that our proof of 4.2 with only quite minor changes works equally well for a ninedimensional absolutely simple Jordan algebra $\mathcal{J}$ of degree 3 (rather than an Albert algebra) realized by means of the toral Tits process as defined in [PR2]. Then we necessarily have $E^\perp = 0$ in 4.2 (ii), forcing $n_{k\{\delta(K/k)+\delta(E/k)\}}$ to be the coordinate norm of $\mathcal{J}$ and thus leading to a new proof of [PR2, Theorems 1,2].

Thirdly, for char $k \neq 2$, the isometry 4.2 (i) determines n uniquely; in fact, we have $S_A^{(u)} \cong \mathbf{h} \perp d\,n^0$ by 2.4, forcing

$$d\,S_A^{(u)} \cong \mathbf{h} \perp n^0 \cong < -1 > \perp n$$

and thus recovering [PR5, Theorem 2] also in characteristic 3. For char $k = 2$, in addition to 4.2 (i) being awkwardly dependent on the choice of a maximal torus $E \subset A^{(u)}$, all this is no longer true since $S_A^{(u)}$ induces a *degenerate* symmetric bilinear form, whose radical in fact agrees with $k1^{(u)}$ (cf. 1.3), and $S_A^{(u)}(1^{(u)}) = 1 = -1$. By contrast, 4.2 (ii), though still dependent on E as above, describes the coordinate norm of $\mathcal{J}$ as explicitly as one could possibly wish. It should be noted here that maximal tori in Albert algebras need not be isomorphic (in fact, they may have distinct pseudo-discriminants), and if they are they need not be conjugate under the automorphism group [AJ, Theorem 9]. Yet, even though the orthogonal constituents of

$$n_{k\{\delta(K/k)+\delta(E/k)\}} \perp (d\,S_A^{(u)}|_{E^\perp}),$$

vary with E, the quadratic form as a whole by 4.2 (ii) only depends on the isotopy class of $\mathcal{J}$.

Finally, 4.2 (iii), supplementing 4.2 (i) in as much as it determines n uniquely in characteristic 2, is a peculiarity (reminiscent of 3.2) which doesn't stand much chance of being extendable to arbitrary characteristics. For example, if char $k = 3$, $S_A^{(u)0}$, being degenerate, can never be isometric to an associate of n.

4.4 In 4.2, the various descriptions of n in terms of the parameters realizing $\mathcal{J}$ by means of the generalized second Tits construction are strongly interdependent. In fact, we shall establish the following set of implications:

$$\text{(35)} \qquad \text{(i)} \Longleftarrow \text{(ii)} \Longrightarrow \text{(iii)},$$

$$\text{(36)} \qquad \text{(i)} \Longrightarrow \text{(ii)} \qquad (\text{char } k \neq 2).$$

To see this, we let $E \subset A^{(u)}$ be a maximal torus and note first that, because of (7), $A^{(u)} = E \perp E^\perp$ is an orthogonal splitting relative to $S_A^{(u)}$. Hence

$$\text{(37)} \qquad \begin{aligned} S_A^{(u)} &\cong S_E \perp (S_A^{(u)}\,|_{E^\perp}) \\ &\cong \mathbf{h} \perp < -d' > \perp (S_A^{(u)}\,|_{E^\perp}) \qquad \text{(by 3.3)}, \end{aligned}$$

d' being the discriminant of E/k. For char $k \neq 2$, this implies

$$(38)\qquad \begin{aligned} S_A^{(u)} &\cong\ <-1> \perp\ <1,-d'> \perp (S_A^{(u)}\,|_{E^\perp}) \\ &\cong\ <-1> \perp n_{k\{\delta(E/k)\}} \perp (S_A^{(u)}\,|_{E^\perp}) \\ &\cong\ <-1> \perp \Big(n_{k\{\delta(K/k)+\delta(E/k)\}} \perp (d\,S_A^{(u)}\,|_{E^\perp})\Big)_\delta \end{aligned}$$

by 2.7, 2.9, whereas, for char $k=2$, we have $d'=1$ by 3.2, forcing

$$(39)\qquad \begin{aligned} S_A^{(u)} &\cong\ <-1> \perp n_{k\{0\}} \perp \Big(S_A^{(u)}\,|_{E^\perp}\Big) \\ &\cong\ <-1> \perp \Big(n_{k\{\delta(K/k)+\delta(E/k)\}} \perp (d\,S_A^{(u)}\,|_{E^\perp})\Big)_\delta . \end{aligned}$$

Now one reads off (36) and the left-hand implication in (35) by consulting (38), (39). It remains to establish (ii) $\Longrightarrow$ (iii) in characteristic 2. Starting from the orthogonal splitting $A^{(u)0} = \ker T_A^{(u)} = E^0 \perp E^\perp$ relative to $S_A^{(u)}$, we conclude

$$\begin{aligned} S_A^{(u)0} &\cong S_E^0 \perp (S_A^{(u)}|_{E^\perp}) \\ &\cong n_{k\{\delta(E/k)+1\}} \perp (S_A^{(u)}|_{E^\perp}) && \text{(by 3.2)} \\ &\cong (n_{k\{\delta(K/k)+\delta(E/k)\}} \perp (d\,S_A^{(u)}\,|_{E^\perp}))_{\delta(K/k)+1} \end{aligned}$$

again by 2.7, 2.9, and the assertion follows; actually, we also get (iii) $\Longrightarrow$ (ii).

5. Proof of the Main Theorem

5.1 In order to establish 4.2, we proceed in several steps. The *first step* consists in showing *that it suffices to deduce* 4.2 (ii) *in the special case* $u = y^\#$, $\beta = N(y)$, *for some invertible element* $y \in A = \mathrm{H}(B,*)$ *satisfying* $T_A(y) = 1$. Indeed, by 1.8 we may assume $u = \beta = 1$. Then the assertion follows by picking y as above and passing to the y-isotope $\mathcal{J}^{(y)} \cong \mathcal{J}(B, *^{(y)}, y^\#, N(y))$ (1.5 b)) which, in view of [F1, Theorem 1.8], leaves the coordinate norm unaffected.

5.2 According to 5.1, we assume from now on that $\mathcal{J}$ agrees with $\mathcal{J}(B,*,u,\beta)$ as in 1.4, where $u = y^\#$, $\beta = N(y)$ for some invertible element $y \in A$ satisfying $T(y) = 1$. Then

$$e = (y, 1) \in \mathcal{J}$$

by (21), (18), (14) satisfies $T(e) = T(y) = 1$ as well as

$$e^\# = (y^\# - u,\ \beta^* u^{-1} - y) = 0.$$

Hence e *is an absolutely primitive idempotent in* $\mathcal{J}$, so, writing $\mathcal{J}_i = \mathcal{J}_i(e)$

$(i = 0, 1, 2)$ for the Peirce–i–component of $\mathcal{J}$ relative to e in the labelling of Loos [L], we have $\mathcal{J}_2 = ke$.

(*Remark* By passing from an arbitrary generalized second Tits construction $\mathcal{J}(B, *, u, \beta)$ to an Albert algebra of the form $\mathcal{J}(B, *, y^{\#}, N(y))$ as above, we have thus explicitly produced an absolutely primitive idempotent. For char $k \neq 3$, this is much easier to accomplish since we are always allowed to assume $u = \beta = 1$ (1.8) and then obtain the absolutely primitive idempotent $e' = \frac{1}{3}(1, 1)$ [PR5].)

The *second step* now consists in computing $\mathcal{J}_0 = \mathcal{J}_0(e)$. To this end, we require the following auxiliary result.

5.3 Lemma *Let $\mathcal{J}$ be a reduced Albert algebra over k, $e \in \mathcal{J}$ an absolutely primitive idempotent (so $e^{\#} = 0$, $T(e) = 1$) and $f = 1 - e$. Then, setting $\mathcal{J}_0 = \mathcal{J}_0(e)$, we have*:

a) $f^{\#} = e$.

b) $x^{\#} = S(x)e$ *for all* $x \in \mathcal{J}_0$.

c) *Writing* S_0 *for* $S = S_{\mathcal{J}}$ *restricted to* $\mathcal{J}_0$, (S_0, f) *is a quadratic form with base point whose associated Jordan algebra agrees with* $\mathcal{J}_0$ *viewed as a subalgebra of* $\mathcal{J}$: $\mathcal{J}_0 = \mathcal{J}(S_0, f)$.

d) $\mathcal{J}_0 = \{x \in \mathcal{J} : e \times x = T(x)f - x\}$.

Proof a) - c) are well known and due to Faulkner [F1, pp. 16 - 17], see also [Ra, pp. 96 - 97].

d) First of all, suppose $x \in \mathcal{J}$ satisfies $T(e, x) = 0$. Then

$$\begin{aligned} U_f x &= T(f, x)f - f^{\#} \times x && \big(\text{by } (5)\big) \\ &= T(e + f, x)f - e \times x && \big(\text{by a)}\big) \\ &= T(x)f - e \times x, \end{aligned}$$

and we conclude that $T(e, x) = 0$ implies (40) x $\in \mathcal{J}_0 \Longleftrightarrow U_f x = x \Longleftrightarrow e \times x = T(x)f - x$. Now let $x \in \mathcal{J}_0$. Then $0 = U_e x = T(e, x)e - e^{\#} \times x = T(e, x)e$ yields $T(e, x) = 0$, hence $e \times x = T(x)f - x$ by (40). Conversely, suppose $x \in \mathcal{J}$ satisfies $e \times x = T(x)f - x$. Then

$$\begin{aligned} T(x) - T(e,x) &= T(e)T(x) - T(e,x) \\ &= T(e \times x) && \big(\text{by (7) and (6) linearized}\big) \\ &= T(x)T(f) - T(x) \\ &= T(x) && \big(\text{since } T(f) = 2\big), \end{aligned}$$

so $T(e,x) = 0$. This implies $x \in \mathcal{J}_0$, again by (40).

5.4 Returning to 5.2, recall from 1.4 that K is the center of B and pick an element $\theta \in K$ satisfying (41) $\theta + \theta^* = 1,\ \theta - \theta^* \in K^\times$. *Then we claim* (42) $\mathrm{J}_0 = k(y^\#, -\theta y) \oplus \{(y \times a, -a); a \in A\}$. Since the sum on the right-hand side is direct (assuming $(y^\#, -\theta y)) = (y \times a, -a)$ for some $a \in A$ would imply $y^\# = 2\theta y^\#$, hence $2\theta = 1$, contradicting (41)), so the right-hand side has the correct dimension (ten), it suffices to show that it is contained in the left. To see this, we first note

$$f = 1 - e = (1 - y, -1)$$

and then linearize $bb^\# = N(b)1 = b^\# b$ ($b \in B$, cf. (14)) to obtain (43) $\mathrm{d}b^\# + b(b \times d) = T(b^\#, d)1 = (b \times d)b + b^\# d$ for $b, d \in B$ $\big(\text{by (2)}\big)$. Now put $x = (y^\#, -\theta y)$. Observing $T(x) = T(y^\#)$ $\big(\text{by (21)}\big)$, we must show by 5.3 d) that both components of

$$e \times x - T(x)f + x = (y,1) \times (y^\#, -\theta y) - T(y^\#)(1 - y, -1) + (y^\#, -\theta y)$$

are zero. Computing the first yields

$$\begin{aligned} & y \times y^\# + y^\# \theta^* y + \theta y y^\# - T(y^\#)(1-y) + y^\# \quad \big(\text{by (22)}\big) = \\ & [T(y^\#) - N(y)]1 - T(y^\#)y - y^\# + N(y)1 - T(y^\#)(1-y) + y^\# \\ & \quad \big(\text{by (10), (14), (41)}\big) = 0, \end{aligned}$$

whereas computing the second yields

$$\begin{aligned} & N(y)\big(1 \times (-\theta^* y)\big)y^{\#-1} + \theta y^2 - y^\# + T(y^\#)1 - \theta y && \big(\text{by (22)}\big) && = \\ & -\theta^* y + \theta^* y^2 + \theta y^2 - y^\# + T(y^\#)1 - \theta y && \big(\text{by (4), (14)}\big) && = \\ & -y^\# + y^2 - T(y)y + S(y)1 && (\text{by (41)}) && = \ 0 \ \big(\text{by (9)}\big). \end{aligned}$$

Hence $x \in \mathcal{J}_0$. Now let $a \in A$ and put $x = (y \times a, -a)$. Then $T(x) = T(y \times a)$, and again by 5.3 d) we must show that both components of

$$e \times x - T(x)f + x = (y,1) \times (y \times a, -a) - T(y \times a)(1 - y, -1) + (y \times a, -a)$$

are zero. Computing the first yields

$$
\begin{aligned}
&y \times (y \times a) + y^{\#}a + ay^{\#} - T(y \times a)(1-y) + y \times a && \big(\text{by } (22)\big) &=\\
&y(y \times a) + (y \times a)y - y \times a - T(y \times a)y + T\big(y \times (y \times a)\big)1 + \\
&y^{\#}a + ay^{\#} - T(y \times a)(1-y) + y \times a && \big(\text{by (9) linearized}\big) &=\\
&T(y^{\#}, a)1 + T(y^{\#}, a)1 + T(y \times a)1 - T(y, y \times a)1 - T(y \times a)1 && (\text{by } (43)) &=\\
&2T(y^{\#}, a)1 - T(y \times y, a)1 && (\text{by } (11)) &= 0,
\end{aligned}
$$

whereas computing the second yields

$$
\begin{aligned}
&N(y)\big(1 \times (-a)\big)y^{\#-1} + ya - y \times a + T(y \times a)1 - a && \big(\text{by } (22)\big) &=\\
&-T(a)y + ay + ya - y \times a + T(y \times a)1 - T(y)a && \big(\text{by } (4)\big) &=\\
&0 \ \big(\text{by (9) linearized}\big)\ .
\end{aligned}
$$

Hence $x \in \mathcal{J}_0$, and the proof of (42) is complete.

5.5 To simplify notation, we put $u = y^{\#}$ (as in 5.2), (44) u' = N(y)$^{-1}(y^{\#}, -\theta y)$, $a' = (y \times a, -a)$ $(a \in A)$, $A' = \{a'; a \in A\}$, whence (42) reads (45) J$_0 = ku' \oplus A'$. Also, we put $A_1 = A^{(u)}$ and write $T_1 = T_A^{(u)}$ for the trace, $S_1 = S_A^{(u)}$ for the quadratic trace and $c_1 = 1^{(u)}$ for the unit element of A_1. The *third step* now consists in computing the quadratic form $S_0' = N(y)S_0$ where S_0 denotes the restriction of the quadratic trace $S = S_{\mathcal{J}}$ of $\mathcal{J}$ to $\mathcal{J}_0$. Writing n_K for the norm of the torus K/k of rank 2, we claim *that the following relations hold for all $a \in A$*:

(46) $$S_0'(u') = 1 - n_K(\theta),$$

(47) $$S_0'(u', a') = T_1(a),$$

(48) $$S_0'(a') = S_1(a).$$

Indeed,

$$\begin{aligned}
S_0'(u') &= N(y)^{-1}T\big((y^\#,-\theta y)^\#\big) && \big(\text{by (44), (6)}\big)\\
&= N(y)^{-1}T\left(y^{\#\#}-\theta y y^\#\theta^* y\right) && \big(\text{by (18), (21)}\big)\\
&= N(y)^{-1}T\big(N(y)y-N(y)n_K(\theta)y\big) && \big(\text{by (1), (14)}\big)\\
&= 1-n_K(\theta) && \big(\text{since } T(y)=1\big),\\
S_0'(u',a') &= T\left((y^\#,-\theta y)\times(y\times a,-a)\right) && \big(\text{by (44), (6)}\big)\\
&= T\left(y^\#\times(y\times a)-\theta y y^\# a - a y^\#\theta^* y\right) && \big(\text{by (22), (21)}\big)\\
&= T\left(N(y)a+T(y^\#,a)y-N(y)a\right) && \big(\text{by (11), (14)}\big)\\
&= T(y^\#,a)=T_1(a) && \big(\text{by (16)}\big),
\end{aligned}$$

$$\begin{aligned}
S_0'(a') &= N(y)T\left((y\times a,-a)^\#\right)\\
&= N(y)T\left((y\times a)^\# - a y^\# a\right) && \big(\text{by (18), (21)}\big)\\
&= N(y)T\big(T(y^\#,a)a+T(y,a^\#)y-y^\#\times a^\#-U_a y^\#\big) && \big(\text{by (13), (14)}\big)\\
&= T(N(y)y,a^\#) && \big(\text{by (5)}\big)\\
&= T(u^\#,a^\#) && \big(\text{by (1)}\big)\\
&= S_1(a) && \big(\text{by (17)}\big),
\end{aligned}$$

and the proof of (46) – (48) is complete.

5.6 The standard procedure to determine the coordinate norm n of a reduced Albert algebra is to pick a complete orthogonal system c_1, c_2, c_3 of absolutely primitive idempotents and to restrict the quadratic trace to the corresponding Peirce–(2,3)–component. Indeed, this yields a quadratic form S_{23} which, by (21), is similar to n, forcing $\gamma S_{23} \cong n$ for any $\gamma \in k^\times$ represented by S_{23}. This procedure requires a refinement in order to become applicable in our situation. The refinement we have in mind is spelled out in the following lemma.

5.7 Lemma *Let $\mathcal{J}$ be a reduced Albert algebra over $k, e \in \mathcal{J}$ an absolutely primitive idempotent and S_0 the restriction of $S = S_{\mathcal{J}}$ to $\mathcal{J}_0 = \mathcal{J}_0(e)$. Let $\gamma \in k^\times$, (h^+, h^-) be a hyperbolic pair relative to $S_0' = \gamma S_0$, $H^\perp$ the orthogonal complement of $H = kh^+ + kh^-$ in $(\mathcal{J}_0, S_0')$ and $S_0^\perp$ the restriction of S_0' to $H^\perp$. Then $\gamma_0\, S_0^\perp$ is isometric to the coordinate norm of $\mathcal{J}$ for any $\gamma_0 \in k^\times$ repesented by $S_0^\perp$.*

Proof Setting $f = 1 - e$, we have $\mathcal{J}_0 = \mathcal{J}(S_0, f)$ by 5.3 c); also, $(h^+, \gamma h^-)$ is a hyperbolic pair relative to S_0, and $H^\perp$ agrees with the orthogonal complement of H in $(\mathcal{J}_0, S_0)$. Now write v for the inverse of $h^+ + \gamma h^-$ in $\mathcal{J}_0$ and $\hat{\mathcal{J}}_0$ for the

v-isotope of $\mathcal{J}_0$. Then (c_2, c_3), where $c_2 = h^+, c_3 = \gamma h^-$, is a complete orthogonal system of absolutely primitive idempotents in $\hat{\mathcal{J}}_0$ whose corresponding Peirce–(1,2)–component agrees with $H^\perp$. Setting $w = e + v$, [PR3, Example 2.2] applies to $ke \oplus \mathcal{J}_0$ and yields (49) $\quad N(w) = S_0(v) = 1, \ w^\# = e + \overline{v} = e + h^+ + \gamma h^-$, where $^-$ refers to the canonical involution of $\mathcal{J}_0 = \mathcal{J}(S_0, f)$. Now (c_1, c_2, c_3), where $c_1 = e$, becomes a complete orthogonal system of absolutely primitive idempotents in $\hat{\mathcal{J}} = \mathcal{J}^{(w)}$. Hence, adopting the terminology of 5.6 and putting $\hat{S} = S_{\hat{\mathcal{J}}}, \hat{S}_{23}$ is similar to the coordinate norm of $\hat{\mathcal{J}}$, hence to that of $\mathcal{J}$. On the other hand, the Peirce–(2,3)–component of $\hat{\mathcal{J}}$ relative to (c_1, c_2, c_3) agrees with $H^\perp$, and for all $x \in H^\perp$ we obtain

$$
\begin{aligned}
\hat{S}(x) &= T(w^\#, x^\#) && \big(\text{by } (17)\big) \\
&= T(e + h^+ + \gamma h^-, S(x)e) && \big(\text{by } (49),\ 5.3\ \text{b})\big) \\
&= S(x),
\end{aligned}
$$

since the Peirce decomposition is always an orthogonal splitting relative to the generic trace. Hence $\hat{S}_{23}$ and $S_0^\perp$ are similar quadratic forms, which completes the proof.

5.8 We now return to 5.5 and let $E \subset A_1$ be any maximal torus. By 3.3, 3.7, E contains a hyperbolic pair (h_+, h_-) relative to S_1 satisfying (50) $\quad c_1 \in kh_+ + kh_-$ and $T_1(h_+, h_-) = -\frac{1}{4}$ if char $k \neq 2$, which, via (44), translates into a hyperbolic pair $(h^+, h^-) = (h'_+, h'_-)$, belonging to A', of the quadratic space $(\mathcal{J}_0, S'_0)$. Writing V for the orthogonal complement of $kh_+ + kh_-$ in (A_1, S_1) and adopting the terminology of 5.5 with $\gamma = N(y)$, the *fourth step* consists in computing $H^\perp$ and $S_0^\perp$: *Setting*

$$v' = u' - T_1(h_+)h^- - T_1(h_-)h^+ \in \mathcal{J}_0$$

and $V' = \{a'; a \in V\} = H^\perp \cap A'$, *we claim*

$$
\begin{aligned}
&(51) \qquad & H^\perp &= kv' \oplus V' \\
&(52) \qquad & S_0^\perp(v') &= -n_K(\theta) - T_1(h_+, h_-), \\
&(53) \qquad & S_0^\perp(v', a') &= T_1(a), \\
&(54) \qquad & S_0^\perp(a') &= S_1(a)
\end{aligned}
$$

for all $a \in V$. Indeed, for $\sigma = \pm$ we get

$$\begin{aligned} S_0'(v', h^\sigma) &= S_0'(u' - T_1(h_\sigma)h^{-\sigma} - T_1(h_{-\sigma})h^\sigma, h^\sigma) \\ &= S_0'(u', h^\sigma) - T_1(h_\sigma) = 0 \qquad \big(\text{by } (47)\big), \end{aligned}$$

showing $v' \in H^\perp$, hence (51). Furthermore,

$$\begin{aligned} S_0'(v') &= S_0'(u' - T_1(h_+)h^- - T_1(h_-)h^+) \\ &= S_0'(u') - T_1(h_+)S_0'(u', h^-) - T_1(h_-)S_0'(u', h^+) + T_1(h_+)T_1(h_-) \\ &= 1 - n_K(\theta) - T_1(h_+)T_1(h_-) \qquad \big(\text{by } (46), (47)\big) \\ &= -n_K(\theta) + S_1(h_+, h_-) - T_1(h_+)T_1(h_-) \\ &= -n_K(\theta) - T_1(h_+, h_-) \qquad \big(\text{by } (7)\big), \end{aligned}$$

yielding (52), whereas (53) follows immediately from (47) and the fact that, for each $a \in V$, a' is perpendicular to $h^\pm$ relative to S_0'. Finally, (54) is simply a restatement of (48).

5.9 We can now complete the proof of 4.2 (ii) and first treat the case char $k \neq 2$. Then (52), in view of (50), (30), reduces to

$$S_0^\perp(v') = -n_K(\theta) + \frac{1}{4} = d.$$

Also, for $a \in V$,

$$\begin{aligned} 2S_0^\perp(v', a') &= 2T_1(a) \qquad \big(\text{by } (53)\big) \\ &= T_1(c_1 \times^{(u)} a) \qquad \big(\text{by } (4) \text{ applied to } A_1 = A^{(u)}\big), \\ &= S_1(c_1, a) = 0 \qquad \big(\text{by } (50)\big). \end{aligned}$$

Hence (51) is an orthogonal splitting relative to $S_0^\perp$, giving rise to the decomposition $S_0^\perp \cong \langle d \rangle \perp (S_1|_V)$. In particular, $S_0^\perp$ represents d, so 5.7 implies

$$n \cong d\, S_0^\perp \cong \langle 1 \rangle \perp (d\, S_1|_V),$$

which in turn yields

$$\langle -1 \rangle \perp n_\delta \cong \mathbf{h} \perp (S_1|_V) \cong S_1,$$

and we have established 4.2 (i), hence 4.2 (ii) as well, by (36).

5.10 We are left with the case char $k = 2$. Since $S_0^\perp(c_1') = S_1(c_1) = 1$, we may identify $n = S_0^\perp$ by 5.7. Furthermore, setting $\varepsilon = \delta(K/k) + \delta(E/k)$, 2.8 and 3.5

yield

$$\varepsilon = (n_K(\theta) + T_1(h_+, h_-)) \bmod \wp(k^\bullet).$$

Setting $w' = v' + c_1'$, this implies

$$\begin{aligned}
n_\varepsilon(v') &= n(v') + \varepsilon n(c_1', v')^2 && \big(\text{by } (25)\big)\\
&= \varepsilon + \varepsilon T_1(c_1)^2 && \big(\text{by } (52), (53)\big)\\
&= 0,\\
n_\varepsilon(w') &= n_\varepsilon(v') + n(c_1', v') + n_\varepsilon(c_1') = 1 + 1 = 0,\\
n_\varepsilon(v', w') &= n(v', v' + c_1') = 1.
\end{aligned}$$

Hence $\mathbf{h} = kv' + kw'$ is a hyperbolic plane relative to n_ε, and one easily checks that the corresponding orthogonal complement is given by

$$\mathbf{h}^\perp = \{a'; a \in E^\perp\}.$$

This yields $n_\varepsilon \cong \mathbf{h} \perp (S_1|_{E^\perp})$, hence

$$\begin{aligned}
n &\cong \Big(n_{k\{0\}} \perp (S_1|_{E^\perp})\Big)_\varepsilon && \big(\text{by } 2.7\text{a})\big)\\
&\cong n_{k\{\varepsilon\}} \perp (d\, S_1|_{E^\perp}) && \big(\text{by } 2.7 \text{ a}), 2.9\big),
\end{aligned}$$

and the proof of 4.2. (ii) is complete.

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