The étale Tits process of Jordan algebras revisited

Holger P. Petersson
Fachbereich Mathematik
FernUniversität - Gesamthochschule in Hagen
D-58084 Hagen
Germany
e-mail: Holger.Petersson@FernUni-Hagen.de
and
Maneesh Thakur
Harish-Chandra Research Institute
Chhatnag Road, Jhusi
Allahabad - 211019
India
e-mail: maneesh@mri.ernet.in

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0. Introduction

The étale Tits process, which was called the toral Tits process by Petersson-Racine [12], may be viewed as a Jordan-theoretical method to construct associative algebras with involution. More specifically, starting from a cubic étale algebra E and a quadratic étale algebra L, both over an arbitrary base field, as well as from invertible elements $u \in E,\ b \in L$ having the same norms, the étale Tits process produces an absolutely simple Jordan algebra $J = J(E, L, u, b)$ of degree 3 and dimension 9, which, by structure theory, must be the symmetric elements of a central simple associative algebra of degree 3 with involution of the second kind. In addition, E identifies canonically with a subalgebra of J.

Our first objective in the present paper is to prove the converse of this result: Given a cubic étale algebra E and a central simple associative algebra (B, τ) of degree 3 with involution of the second kind, every isomorphic embedding ι from E to $J = H(B, \tau)$, the Jordan algebra of τ-symmetric elements in B, will be shown in the extension theorem below to allow data L, u, b as above such that ι extends to an isomorphism from the étale Tits process $J(E, L, u, b)$ onto J. Over fields of characteristic different from two and three, this result is due to Knus, Merkurjev, Rost and Tignol [4, (39.5), (6)]. Their proof rests on [4, (36.38)] and hence on certain generic choices which seem to require an ample supply of invertible elements, this being automatic only if the base field is big enough and hence, in particular, if J is a division algebra. Under the very last restriction, the extension theorem also derives quite easily from the classification theory of the Tits process [14, Theorem 3.1 (ii)]. On the other hand, as we shall see below, a unified proof working in full generality turns out to be surprisingly delicate.

The idea is to mimic McCrimmon's classical approach [7] to the enumeration of Albert algebras over arbitrary base fields.

The extension theorem will be applied in two different ways. The first application (3.2) is concerned with cubic étale algebras E, E' giving rise to étale Tits processes J, J', respectively, and spells out necessary and sufficient conditions for an isomorphism $E' \xrightarrow{\sim} E$ to be extendable to an isomorphism $J' \xrightarrow{\sim} J$. The second application (4.2) is concerned with attaching invariants to (isomorphic) embeddings from E to $J = H(B, \tau)$, where E is a cubic étale algebra and (B, τ) is a central simple associative algebra of degree 3 with involution of the second kind. These invariants, called *norm classes*, are analogues of the invariants that under the same name were attached by Albert-Jacobson [2, §9] to "ordered basic sets of idempotents in exceptional simple Jordan algebras". In analogy to [2, Theorem 9] we show that two embeddings from E to J have the same norm class if and only if they are *equivalent*, i.e., can be transformed into one another by an automorphism of J. Thus the presence of distinct norm classes my be viewed as an obstruction to the validity of the Skolem-Noether Theorem.

1. The étale Tits process and the extension theorem

Throughout this paper we fix an arbitrary base field k. All algebras considered in the sequel (as well as subalgebras and homomorphisms thereof) are supposed to be unital. We write $A^\times$ for the set of invertible elements in an algebra A whenever this makes sense. The bilinearization of a quadratic map Q will be denoted by $Q(x, y) = Q(x + y) - Q(x) - Q(y)$.

1.1 Cubic norm structures. Following McCrimmon [6] and Petersson-Racine [13], a *cubic norm structure* over k is a quadruple $(V, N, \sharp, 1)$ consisting of a vector space V over k, a cubic form $N : V \longrightarrow k$ (the *norm*), a quadratic map $\sharp : V \longrightarrow V,\ x \longmapsto x^\sharp$, (the *adjoint*) and a distinguished element $1 \in V$ (the *base point*) such that the relations

$$x^{\sharp\sharp} = N(x)x \qquad \text{("adjoint identity")}, \tag{1}$$

$N(1) = 1,\ T(x^\sharp, y) = (DN)(x)y,\ 1^\sharp = 1,\ 1 \times y = T(y)1 - y$ hold under all scalar extensions, where $T = -(D^2 \log N)(1)$: $V \times V \longrightarrow k$ is the associated trace form, $x \times y = (x+y)^\sharp - x^\sharp - y^\sharp$ is the bilinearization of the adjoint and $T(y) = T(1, y)$. Then the U-operator $U_x y = T(x, y)x - x^\sharp \times y$ and the base point 1 give V the structure of a unital quadratic Jordan algebra denoted by $J(V, N, \sharp, 1)$. From [6, (15), (21) and p. 501] we recall the equations

$$T(x \times y, z) = T(x, y \times z), \tag{2}$$

$$x^\sharp = x^2 - T(x)x + T(x^\sharp)1, \tag{3}$$

$$N(x \times y) + N(x)N(y) = T(x^\sharp, y)T(x, y^\sharp). \tag{4}$$

From [6, Theorem 2] we know that $x \in J$ is invertible if and only if $N(x) \neq 0$, in which case

$$x^{-1} = N(x)^{-1}x^\sharp. \tag{5}$$

1.2 Étale algebras. Quadratic étale k-algebras are classified by $H^1(k, \mathbb{Z}/2\mathbb{Z})$. The element of $H^1(k, \mathbb{Z}/2\mathbb{Z})$ corresponding to a quadratic étale algebra L over k will be denoted by $\delta(L/k)$; it is basically the ordinary discriminant if k has characteristic not two. Conversely, we write $k\{\delta\}$ for the quadratic étale k-algebra corresponding to $\delta \in H^1(k, \mathbb{Z}/2\mathbb{Z})$.

Let E be a cubic étale k-algebra. Then E carries a natural Jordan algebra structure (with U-operator $U_x y = x^2 y$) which, in fact, agrees with $J(V, N, \sharp, 1)$ as in 1.1, where V is the underlying vector space, N is the norm, $\sharp$ is the adjoint (i.e., the numerator of the inversion map), and 1 is the unit of E. By abuse of notation, we do not distinguish carefully between E as a cubic étale k-algebra and its induced Jordan algebra structure. The element of $H^1(k, \mathbb{Z}/2\mathbb{Z})$ corresponding to the discriminant algebra of E [4, §18] will be denoted by $\delta(E/k)$.

1.3 The étale Tits process. Referring to [13] for details, we briefly recall the main ingredients of the étale Tits process, called the toral Tits process in [12]. Let L, E be étale k-algebras of dimension 2,3, respectively. We write $N_E, \sharp, T_E$ both for the norm, adjoint, trace, respectively, of E and for their natural extensions to the cubic étale L-algebra $E \otimes L$, unadorned tensor products always being taken over k. Analogous conventions apply to the norm N_L, the trace T_L and the nontrivial k-automorphism σ of $L, E \otimes L$ this time being viewed as a quadratic étale E-algebra. Notice that E identifies naturally as $E \otimes 1$ with $H(E \otimes L, \sigma)$, the fixed points of $E \otimes L$ under the involution σ.

Now let $u \in E,\ b \in L$ be invertible elements having the same norms, so $N_E(u) = N_L(b) = b\sigma(b) \neq 0$. Extending $N_E, \sharp, 1$ as given on E/k and $E \otimes L/L$ to the vector space $V = E \times (E \otimes L)$ over k according to the rules

$$N\big((v, x)\big) = N_E(v) + bN_E(x) + \sigma(b)\sigma\big(N_E(x)\big) - T_E\big(v, xu\sigma(x)\big), \tag{6}$$

$$(v, x)^\sharp = \big(v^\sharp - xu\sigma(x),\ \sigma(b)\sigma(x)^\sharp u^{-1} - vx\big), \tag{7}$$

$$1 = (1, 0) \tag{8}$$

for $v \in E,\ x \in E \otimes L$, we obtain a cubic norm structure $(V, N, \sharp, 1)$ as in 1.1 whose associated trace form is given by

$$T\big((v, x), (w, y)\big) = T_E(v, w) + T_E\big(xu, \sigma(y)\big) + T_E\big(yu, \sigma(x)\big) \tag{9}$$

for $v, w \in E,\ x, y \in E \otimes L$. The ensuing Jordan algebra will be denoted by $J = J(E, L, u, b) = J(V, N, \sharp, 1)$; it is said to arise from E, L, u, b by the *étale Tits process*. Following [14, Corollary 4.3], J is an absolutely simple Jordan algebra of degree 3 and dimension 9, hence by structure theory (cf., e.g., [8, (15.7)] and [16, Theorem 1]) must be the symmetric elements of a central simple associative algebra of degree 3 with involution of the second kind. In particular, $N = N_J$ is the generic norm and $T = T_J$ is the generic trace of J. Clearly, E identifies with a subalgebra of J through the first factor. The relation

$$v \times (0, x) = (0, -vx) \tag{10}$$

follows immediately by linearizing (7). Since the trace form of J is nondegenerate (by (9)), we deduce from [6, p. 507] that a linear map between étale Tits processes preserving norms and units is necessarily an isomorphism. By [13, Theorem 5.2], J is a division algebra (i.e., all nonzero elements in J are invertible) if and only if $b \notin N_E((E \otimes L)^\times)$.

The following result has been proved in [12]; see also [4, (39.5) (5)] for char $k \neq 2, 3$.

1.4 Theorem. ([12, Theorem 1]) *Let L, E be étale k-algebras of dimension $2, 3$, respectively, and suppose $u \in E,\ b \in L$ are invertible elements satisfying $N_E(u) = N_L(b)$. If (B, τ) is a central simple associative algebra of degree 3 over k with involution of the second kind such that the étale Tits process $J(E, L, u, b)$ becomes isomorphic to the Jordan algebra over k of τ-symmetric elements in B, then the centre of B as a quadratic étale k-algebras corresponds to the element*

$$\delta(E/k) + \delta(L/k) \in H^1(k, \mathbb{Z}/2\mathbb{Z}). \qquad \square$$

1.5 The étale first Tits construction. Let E be a cubic étale k-algebra and $\alpha \in k^\times$. Following [13, Theorem 3.5], we may form the *étale first Tits construction* $J = J(E, \alpha) = J(V, N, \sharp, 1)$, where $N_E, \sharp, 1$ as given on E extend to the vector space $V = E \times E \times E$ over k according to the rules

$$N\big((v_0, v_1, v_2)\big) = N_E(v_0) + \alpha N_E(v_1) + \alpha^{-1} N_E(v_2) - T_E(v_0 v_1 v_2), \tag{11}$$

$$(v_0, v_1, v_2)^\sharp = (v_0^\sharp - v_1 v_2, \alpha^{-1} v_2^\sharp - v_0 v_1, \alpha v_1^\sharp - v_2 v_0), \tag{12}$$

$$1 = (1, 0, 0)$$

for $v_0, v_1, v_2 \in E$. The associated trace form of J is given by

$$T(x, y) = T_E(v_0, w_0) + T_E(v_1, w_2) + T_E(v_2, w_1) \tag{13}$$

for $x = (v_0, v_1, v_2),\ y = (w_0, w_1, w_2) \in J$. Clearly, E identifies with a subalgebra of $J(E, \alpha)$ through the zeroeth factor. The relations

$$v_0 \times (0, v_1, 0) = (0, -v_0 v_1, 0),\ \ v_0 \times (0, 0, v_2) = (0, 0, -v_2 v_0) \tag{14}$$

are well known and follow easily from linearizing (12). Finally, we conclude from [13, Theorem 3.5 and Proposition 3.8[1]] that there are natural identifications of étale Tits processes $J(E, L, u, b)$ as in 1.3, with $L \cong k \times k$ split, and étale first Tits constructions $J(E, \alpha)$ as above which respect the identifications of E in $J(E, L, u, b), J(E, \alpha)$, respectively. More specifically we have

$$J(E, \alpha) = J(E, k \times k, (1, 1), (\alpha, \alpha^{-1})).$$

In particular, $J(E, \alpha)$ is a division algebra if and only if $\alpha \notin N_E(E^\times)$.

We are now ready to state the first main result of the paper.

[1] In the proof there is a misprint: μ^{-1} should be replaced by u^{-1}.

1.6 Extension Theorem. *Let E be a cubic étale k-algebra, (B, τ) a central simple associative algebra of degree 3 with involution of the second kind over k and suppose ι is an isomorphic embedding from E to $J = H(B, \tau)$, the Jordan algebra over k of τ-symmetric elements in B. Writing K for the centre of B and L for the quadratic étale k-algebra corresponding to the element*

$$\delta(K/k) + \delta(E/k) \in H^1(k, \mathbb{Z}/2\mathbb{Z}), \tag{15}$$

there are invertible elements $u \in E,\ b \in L$ satisfying $N_E(u) = N_L(b)$ such that ι extends to an isomorphism from the étale Tits process $J(E, L, u, b)$ onto J.

1.7 Remarks. The proof of the extension theorem will occupy the next section. The specific choice of L is forced upon us by 1.4. We may clearly assume that $E \subset J$ is a subalgebra and ι is the inclusion.

1.8 Special cases. a) If k is an infinite (or a sufficiently big finite) field of characteristic not 2 or 3, the extension theorem follows from [4, (39.5), (5), (6) and (36.38)].
b) More specially, if J is a division algebra, the extension theorem is also implied by the general theory of the Tits process. Indeed, [14, Theorem 3.1 (ii)] shows that some Tits process J' starting from E is still a subalgebra of J. Hence J' is a Jordan division algebra of degree 3 and dimension 6 or 9 (see 2.1 below). But in dimension 6, such creatures do not exist since, otherwise, we would obtain a central associative division algebra of degree 3 with involution of the *first* kind, which is impossible ([4, (3.1)] and [1, V Theorem 17]). Hence $J' = J$, and the extension theorem follows.

2. Proof of the extension theorem

Before starting with the proof, we require a few additional technicalities most of which are well known.

2.1 Springer forms. Let $J = J(V, N, \sharp, 1)$ be the Jordan algebra of a cubic norm structure over k as in 1.1. Given a cubic étale subalgebra $E \subset J$, we obtain an orthogonal decomposition $J = E \oplus E^{\perp}$ relative to T, and the assignment

$$(v, x) \longmapsto v.x := -v \times x \qquad (v \in E, x \in E^{\perp}) \tag{16}$$

endows $E^{\perp}$ with the structure of a left E-module [11, Proposition 2.1 a)]. Furthermore, for $x \in E^{\perp}$ we are allowed to write

$$x^{\sharp} = -q_E(x) + r_E(x) \qquad \big(q_E(x) \in E,\ r_E(x) \in E^{\perp}\big), \tag{17}$$

and the map $q_E : E^{\perp} \longrightarrow E$ is a quadratic form over E [11, Proposition 2.1 b)], called the *Springer form of E in J*. Notice that our definition of q_E differs from the one in [11], [14] by a sign, bringing us back to the original normalization due to Springer, cf. Springer-Veldkamp [17, (6.5)]. Thanks to (2), (16) we have

$$T(v.x, y) = T(x, v.y) = -T(v, x \times y) \tag{18}$$

for all $v \in E,\ x, y \in E^\perp$. Furthermore, (4) implies

$$N(v.x) = N_E(v)N(x). \tag{19}$$

Finally, by [14, Lemma 3.3, (ii): d), e), h) and (vi)],

$$r_E(v.x) = v^\sharp.r_E(x), \tag{20}$$
$$q_E\big(x, r_E(x)\big) = N(x)1, \tag{21}$$
$$r_E\big(r_E(x)\big) = N(x)x - q_E(x).r_E(x), \tag{22}$$
$$r_E\big(x, v.r_E(x)\big) = \big(v \times q_E(x)\big).x. \tag{23}$$

2.2 Lemma. *Notations being as in* 2.1, *we obtain*

$$r_E\big(v.x + w.r_E(x)\big) = \Big(N(x)w^\sharp + [w \times \big(vq_E(x)\big)]\Big).x + \big(v^\sharp - w^\sharp q_E(x)\big).r_E(x)$$

for all $v, w \in E,\ x \in E^\perp$.

Proof. For $v = 1$, the assertion follows by expanding the left-hand side and applying (20), (22), (23). In general, we may extend scalars if necessary and invoke Zariski density to assume $v \in E^\times$. But then the special case $v = 1$ yields

$$\begin{aligned} r_E\big(v.x + w.r_E(x)\big) &= r_E\big(v.[x + (v^{-1}w).r_E(x)]\big) \\ &= v^\sharp.r_E\big(x + (v^{-1}w).r_E(x)\big) && \text{(by (20))} \\ &= v^\sharp.[\big(N(x)(v^{-1}w)^\sharp + (v^{-1}w) \times q_E(x)\big).x \\ &\quad + \big(1 - (v^{-1}w)^\sharp q_E(x)\big).r_E(x)]. \end{aligned}$$

Here we apply the identity $(st)^\sharp = s^\sharp t^\sharp$ and its linearization $(st_1) \times (st_2) = s^\sharp(t_1 \times t_2)$, both valid in E, to arrive at the desired conclusion. $\square$

2.3 Associates of quadratic forms with base point. Let $q : V \longrightarrow k$ be a quadratic form over k which is nonsingular in the sense that its induced symmetric bilinear form is nondegenerate, and suppose $e \in V$ is a base point for q, so $q(e) = 1$. Denote by $k^\bullet$ the multiplicative group $k^\times$ for char $k \neq 2$ and the additive group k for char $k = 2$. Given $\delta \in H^1(k, \mathbb{Z}/2\mathbb{Z})$, choose any representative $d \in k^\bullet$ of δ to define $q_\delta : V \longrightarrow k$ by $q_\delta(x) = dq(x) + \frac{1-d}{4}q(e,x)^2$ (char $k \neq 2$), $q_\delta(x) = q(x) + dq(e,x)^2$(char $k = 2$). Then q_δ is a nonsingular quadratic form with base point e whose isometry class neither depends on d nor on e. This follows from Witt's Theorem and the proof of [11, Proposition 3.1], although for char $k \neq 2$, q_δ as defined here differs from the corresponding notion in [11] by a factor d. We call q_δ the δ-*associate of* q. This concept applies in particular to the norm of a quadratic étale algebra. From [15, 2.9] we recall

$$(N_{k\{\delta\}})_{\delta'} \cong N_{k\{\delta+\delta'\}} \tag{24}$$

for $\delta, \delta' \in H^1(k, \mathbb{Z}/2\mathbb{Z})$.

2.4 We are now ready to prove the extension theorem 1.6, making the adjustments described in 1.7, *and begin by reducing to the case that L is split.* So let us assume for the time being that this case has been settled and suppose L/k is a separable quadratic field extension, with nontrivial k-automorphism σ. Extending scalars from k to L, we then conclude from 1.5 that, for some $b \in L^\times$, the inclusion $E \otimes L \subset J \otimes L$ extends to an isomorphism φ over L from the étale first Tits construction $J(E \otimes L, b)$ onto $J \otimes L$. Using φ, the natural action of σ on $J \otimes L$ through the second factor will be transferred to $J(E \otimes L, b)$, making $\varphi : J(E \otimes L, b) \longrightarrow J \otimes L$ into a σ-equivariant map. Adapting the notations of 1.5 to the present set-up, we first observe that, since $E \otimes L \subset J(E \otimes L, b)$ is stable under σ, so is its orthogonal complement

$$(E \otimes L)^\perp = \{(0, w_1, w_2) \mid w_1, w_2 \in E \otimes L\} \subset J(E \otimes L, b)$$

relative to T (cf. (13)). Hence

$$\sigma\big((0,1,0)\big) = (0, u_1, u_2), \ \sigma\big((0,0,1)\big) = (0, v_1, v_2) \tag{25}$$

for some $u_1, u_2, v_1, v_2 \in E \otimes L$. Applying the norm to the first equation of (25) and using (11), we obtain

$$\sigma(b) = bN_E(u_1) + b^{-1}N_E(u_2). \tag{26}$$

Similarly, applying the adjoint to the first equation of (25) and using (12), we obtain

$$u_1u_2 = 0, \ v_1 = N_L(b)^{-1}u_2^\sharp, \ v_2 = b\sigma(b)^{-1}u_1^\sharp. \tag{27}$$

Finally, applying σ to the relation $\sigma\big((0,1,0)\big) = -u_1 \times (0,1,0) - u_2 \times (0,0,1)$ (by (25), (14)) and using (25), (27), we obtain

$$u_1\sigma(u_1) + N_L(b)^{-1}\sigma(u_2)u_2^\sharp = 1, \ \sigma(u_1)u_2 + b\sigma(b)^{-1}u_1^\sharp\sigma(u_2) = 0. \tag{28}$$

Next we show that $u_2 \in E \otimes L$ is invertible. Arguing indirectly, let us assume $N_E(u_2) = 0$. Then (26) reduces to $\sigma(b) = bN_E(u_1)$, forcing $u_1 \in (E \otimes L)^\times$. Therefore (27) implies

$$u_2 = v_1 = 0, \ v_2 = u_1^{-1}. \tag{29}$$

Moreover, $N_L(u_1) = 1$ by (28), so Hilbert's Theorem 90 yields an element $y \in (E \otimes L)^\times$ satisfying

$$u_1 = y\sigma(y)^{-1}. \tag{30}$$

Now (26), (29), (30) imply that $\alpha := N_E(y)b \in L^\times$ remains fixed under σ and hence belongs to k. Furthermore, the map

$$\psi : J(E, \alpha) \otimes L \longrightarrow J(E \otimes L, b)$$

sending (w_0, w_1, w_2) to $(w_0, w_1y, y^{-1}w_2)$ is an isomorphism over L. Letting σ act on $J(E, a) \otimes L$ through the second factor, we deduce from

$$\begin{aligned}
\sigma\left(\psi\big((w_0, w_1, w_2)\big)\right) &= \sigma\big((w_0, w_1y, y^{-1}w_2)\big) \\
&= \sigma\big(w_0 - w_1y \times (0,1,0) - y^{-1}w_2 \times (0,0,1)\big) && \text{(by (14))} \\
&= \sigma(w_0) - \sigma(w_1)\sigma(y) \times (0, u_1, 0) - \sigma(y)^{-1}\sigma(w_2) \times (0, 0, u_1^{-1}) \\
& && \text{(by (25), (29))} \\
&= \big(\sigma(w_0), \sigma(w_1)y, y^{-1}\sigma(w_2)\big) && \text{(by (14), (30))} \\
&= \psi\left(\sigma\big((w_0, w_1, w_2)\big)\right)
\end{aligned}$$

for all $w_0, w_1, w_2 \in E \otimes L$ that ψ is σ-equivariant. Hence so is $\varphi \circ \psi : J(E, \alpha) \otimes L \longrightarrow J \otimes L$, and restricting to the fixed points of σ gives an isomorphism $J(E, a) \xrightarrow{\sim} J$. From 1.4, 1.5 and (15) we therefore conclude $\delta(L/k) = 0$, so L is split. This contradiction shows that $u_2 \in E \otimes L$ is indeed invertible. Hence (27), (26) collapse to

$$u_1 = v_2 = 0,\ N_E(u_2) = N_L(b),\ v_1 = u_2^{-1}. \tag{31}$$

Now (28) implies $1 = N_L(b)^{-1}\sigma(u_2)u_2^\sharp = \sigma(u_2)u_2^{-1}$, so $u := u_2$ belongs to E and satisfies $N_E(u) = N_L(b)$. Furthermore, by (25), (31), $\sigma\big((0,1,0)\big) = (0,0,u), \sigma\big((0,0,1)\big) = (0, u^{-1}, 0)$. From this and (14) one easily concludes

$$\sigma\big((w_0, w_1, w_2)\big) = \big(\sigma(w_0), \sigma(w_2)u^{-1}, u\sigma(w_1)\big)$$

for $w_0, w_1, w_2 \in E \otimes L$. Identifying $E \subset E \otimes L, J \subset J \otimes L$ canonically and arguing as in the proof of [7, Theorem 9], it now follows that the assignment $(w, x) \longmapsto \varphi\big((w, x, u\sigma(x)\big)$ for $w \in E, x \in E \otimes L$ gives an isomorphism from the étale Tits process $J(E, L, u, b)$ onto J extending the inclusion $E \subset J$.

2.5 In order to complete the proof, the reduction carried out in 2.4 allows us to assume that $L \cong k \times k$ is split, which, by (15), implies

$$\delta := \delta(K/k) = \delta(E/k) \in H^1(k, \mathbb{Z}/2\mathbb{Z}). \tag{32}$$

In view of 1.5, we must show that, for some $\alpha \in k^\times$, the inclusion $E \subset J$ extends to an isomorphism from the étale first Tits construction $J(E, \alpha)$ onto J. To do so, we employ the following specializations of results obtained in [11].

2.6 Proposition. ([11, Proposition 2.2]) *Notations being as in* 1.6, *there exists a nonzero element $\alpha \in k$ such that the inclusion $E \subset J$ extends to an isomorphism from the étale first Tits construction $J(E, \alpha)$ onto J if and only if some $x \in E^\perp \cap J^\times$ satisfies $q_E(x) = 0$.* □

2.7 Theorem. ([11, Theorem 3.2]) *Notations being as in* 1.6 *and setting $\delta := \delta(E/k)$, the Springer form of E in J is similar to $(N_{K\otimes E/E})_\delta$ or $(N_K)_\delta \otimes E$ according as E is a field or not.* □

2.8 Returning to 2.5, we now argue indirectly and assume that *no* $\alpha \in k^\times$ allows an isomorphism $J(E, \alpha) \xrightarrow{\sim} J$ extending the inclusion $E \subset J$. Then 2.6 implies

$$q_E(x) \neq 0 \ \text{ for all } \ x \in E^\perp \cap J^\times. \tag{33}$$

Moreover, it follows from 2.7, (24) and (32) *that the binary quadratic space* $(E^\perp, q_E)$ *over E is the hyperbolic plane.* Accordingly, we choose a hyperbolic pair (x_1, x_2) of $(E^\perp, q_E)$, so $E^\perp$ is a free E-module of rank 2 with basis (x_1, x_2) and

$$q_E(x_1) = q_E(x_2) = 0,\ q_E(x_1, x_2) = 1. \tag{34}$$

In particular, x_1, x_2 cannot be invertible in J (by (33)), and (21) implies

$$q_E(x_i, r_E(x_i)) = N(x_i)1 = 0 \qquad (i = 1, 2). \tag{35}$$

Furthermore, (17), (34) imply $x_i^\sharp = r_E(x_i) = u_i.x_i + v_i.x_{3-i}$ for some $u_i, v_i \in E$, forcing $v_i = 0$ by (34), (35). Hence

$$x_i^\sharp = r_E(x_i) = u_i.x_i \qquad (i = 1, 2). \tag{36}$$

We now claim

$$(v.x_i)^\sharp = r_E(v.x_i) = u_i v^\sharp.x_i \qquad (v \in E, i = 1, 2), \tag{37}$$

$$N_E(u_i) = 0 \qquad (i = 1, 2), \tag{38}$$

$$N(v_1.x_1 + v_2.x_2) = T_E(u_1 v_1^\sharp v_2 + u_2 v_1 v_2^\sharp) \qquad (v_i \in E, i = 1, 2). \tag{39}$$

The first relation of (37) follows from $q_E(v.x_i) = v^2 q_E(x_i) = 0$, the second one from (20), (36). Hence $0 = N(x_i)x_i$ (by (35)) $= x_i^{\sharp\sharp}$ (by (1)) $= (u_i.x_i)^\sharp$ (by (36)) $= (u_i u_i^\sharp).x_i$ (by (37)) $= N_E(u_i)x_i$, giving (38). This allows us to expand the left-hand side of (39):

$$\begin{aligned} N(v_1.x_1 + v_2.x_2) &= N(v_1.x_1) + T\big((v_1.x_1)^\sharp, v_2.x_2\big) + T\big(v_1.x_1, (v_2.x_2)^\sharp\big) + N(v_2.x_2) \\ &= N_E(v_1)N(x_1) + T(u_1 v_1^\sharp.x_1, v_2.x_2) + T(v_1.x_1, u_2 v_2^\sharp.x_2) \\ &\quad + N_E(v_2)N(x_2) && \text{(by (19), (37))} \\ &= -T(u_1 v_1^\sharp v_2 + u_2 v_1 v_2^\sharp, x_1 \times x_2) && \text{(by (18), (35))} \\ &= T_E\big(u_1 v_1^\sharp v_2 + u_2 v_1 v_2^\sharp, q_E(x_1, x_2)\big) && \text{(by (17)),} \end{aligned}$$

and (39) follows from (34).

Special Case. $E \cong k \times k \times k$ splits.
Then $\delta = 0$, and $K \cong k \times k$ splits as well (by (32)). Hence $J \cong M_3(k)^+$, and we may assume that E sits diagonally in J under this identification. But then

$$y := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in E^\perp \cap J^\times$$

satisfies $y^\sharp \in E^\perp$, forcing $q_E(y) = 0$ and contradicting (33).

Returning to the general case, let us assume $u_1 = u_2 = 0$. Then (39) shows that N vanishes on $E^\perp$ even after extending scalars to the algebraic closure. On the other hand, this extension brings us back to special case, where we have just produced invertible elements belonging to $E^\perp$. This is a contradiction, so by symmetry we may assume

$$u_1 \neq 0. \tag{40}$$

Combining with (38), we conclude that E is not a field, forcing $E \cong k \times K$ by (32). Hence we can find $\alpha_i \in k,\ a_i \in K$ satisfying $u_i = (\alpha_i, a_i)$ $(i = 1, 2)$. Now recall from [13, Example 2.2] that the adjoint of E is given by

$$(\beta, b)^\sharp = (N_K(b), \beta\tau(b)) \qquad (\beta \in k,\ b \in K) \tag{41}$$

and consider the idempotent $c = (1, 0) \in E$. From (34) we conclude $q_E(c.x_1 + (1 - c).x_2) = 0$, and the relations $c^\sharp = 0,\ (1 - c)^\sharp = c$ (by (41)) combine with (39) to imply

$N(c.x_1 + (1-c).x_2) = \alpha_2$. Hence $\alpha_2 = 0$ by (33). Interchanging x_1, x_2, we obtain $\alpha_1 = 0$ as well, so

$$u_i = (0, a_i) \qquad (i = 1, 2). \tag{42}$$

For $v_i \in E$, we next expand the expression $r_E(v_1.x_1 + v_2.r_E(x_1))$ in two different ways. On the one hand,

$$\begin{aligned} r_E\big(v_1.x_1 + v_2.r_E(x_1)\big) &= v_1^\sharp.r_E(x_1) && \text{(by 2.2, (34), (35))} \\ &= u_1 v_1^\sharp.x_1 && \text{(by (36))}, \end{aligned}$$

on the other

$$\begin{aligned} r_E\big(v_1.x_1 + v_2.r_E(x_1)\big) &= r_E\big((v_1 + u_1 v_2).x_1\big) && \text{(by (36))} \\ &= u_1(v_1 + u_1 v_2)^\sharp.x_1 && \text{(by (37))} \\ &= \big(u_1 v_1^\sharp + u_1(v_1 \times u_1 v_2)\big).x_1 \end{aligned}$$

since $u_1(u_1 v_2)^\sharp = u_1 u_1^\sharp v_2^\sharp = 0$ by (38). Hence $u_1(v_1 \times u_1 v_2) = 0$. Using (41) to compute this explicitly for $v_1 = c = (1, 0)$, $v_2 = 1 - c = (0, 1)$, we obtain $N_K(a_1) = 0$. But $a_1 \neq 0$ by (40), (42), so K is not a field, and we are back again to the special case, a contradiction. □

3. An Isomorphism criterion

3.1 The general set-up. In this section we fix two étale Tits process algebras $J = J(E, L, u, b)$, $J' = J(E', L', u', b')$ as in 1.3, where E, E' and L, L' are cubic and quadratic étale algebras, respectively, over our base field k and $u \in E$, $u' \in E'$, $b \in L$, $b' \in L'$ are invertible elements satisfying $N_E(u) = N_L(b)$, $N_{E'}(u') = N_{L'}(b')$. We denote by σ, σ' the nontrivial k-automorphism of L, L', respectively. All conventions of 1.3 remain in force. Letting $\varphi : E' \xrightarrow{\sim} E$ be any isomorphism, we will be concerned with the question as to when φ can be extended to an isomorphism from J' onto J. A partial answer to this question may be found in the following result.

3.2 Theorem. *Notations being as in* 3.1, *let* $\eta : J' \longrightarrow J$ *be an arbitrary map. Then the following statements are equivalent.*

(i) *η is an isomorphism extending φ.*

(ii) *There exist an isomorphism $\psi : L' \xrightarrow{\sim} L$ and an invertible element $y \in E \otimes L$ such that*

$$\varphi(u') = N_L(y)u, \;\; \psi(b') = N_E(y)b \tag{43}$$

and

$$\eta\big((v', x')\big) = \big(\varphi(v'),\; y(\varphi \otimes \psi)(x')\big) \tag{44}$$

for all $v' \in E'$, $x' \in E' \otimes L'$.

3.3 Proof of 3.2, (ii) $\Longrightarrow$ (i). This is the easy part. Since η obviously preserves units, it suffices to show that it preserves norms as well. But this follows directly from (6), (43), (44) and the fact that the isomorphism $\varphi \otimes \psi : E' \otimes L' \longrightarrow E \otimes L$ satisfies the relations

$$N_E \circ (\varphi \otimes \psi) = \psi \circ N_{E'}, \ \sigma \circ (\varphi \otimes \psi) = (\varphi \otimes \psi) \circ \sigma',$$
$$T_E \circ \big((\varphi \otimes \psi) \times (\varphi \otimes \psi)\big) = \psi \circ T_{E'}$$

on $E' \otimes L', (E' \otimes L') \times (E' \otimes L')$, respectively.
The converse implication (i) $\Longrightarrow$ (ii) requires a bit more work. First we have to deal with a side-issue.

3.4 Quadratic étale algebras over rings. Let R be a commutative associative ring of scalars. Although we are mostly concerned here with algebras over fields, the following digression to algebras *over* R cannot be avoided. So let A be a quadratic étale R-algebra with norm n, trace t and standard involution σ (cf. Knus [5, I (1.3), especially (1.3.6)]. Observe for the constant group scheme $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}$ (cf. Waterhouse [18, 2.3] that $(\mathbb{Z}/2\mathbb{Z})(R)$ identifies canonically with the group of idempotents in R under the composition $(c, c') \longmapsto c + c' - 2cc'$ [5, III (4.1)]. Also, for $c \in (\mathbb{Z}/2\mathbb{Z})(R)$, the map

$$\psi_c : A \longrightarrow A, \ x \longmapsto \psi_c(x) = cx + (1-c)\sigma(x),$$

is an automorphism of A [5, III (4.1.2)].

3.5 Proposition. ([5, III (4.1.2)]) *Notations being as in* 3.4, *the assignment* $c \longmapsto \psi_c$ *determines an isomorphism from* $(\mathbb{Z}/2\mathbb{Z})(R)$ *onto the automorphism group of* A. $\square$

3.6 Lemma. ([5, III (4.1.1)]) *Notations being as in* 3.4, *suppose* R *is a local ring. Then* A *has a basis* $(1, \theta)$ *over* R *such that* $\theta^2 - \theta + r1 = 0$ *for some* $r \in R$ *satisfying* $1 - 4r \in R^\times$. $\square$

3.7 Lemma. *Notations being as in* 3.4, *let* $\psi : A \longrightarrow A$ *be an* R*-linear map preserving* n *and* 1. *Then* ψ *is an automorphism of* A.

Proof. Since the question is local on R, we may assume that R is a local ring. Then A is generated by a single element (3.6), so it suffices to show that ψ preserves squares. But this follows immediately from $n \circ \psi = n, \ \psi(1) = 1, \ t \circ \psi = t$ and the equation $x^2 - t(x)x + n(x)1 = 0$ for all $x \in A$ [5, I (1.3.3)]. $\square$

3.8 Proof of 3.2, (i) $\Longrightarrow$ (ii). We proceed in several steps.
1^0. Let (B, τ), (B', τ') be central simple associative algebras of degree 3 with involution of the second kind satisfying $J \cong H(B, \tau), \ J' \cong H(B', \tau')$. Since (B', τ') is perfect (cf. Jacobson [3, Theorem 5]), η induces a unique isomorphism $(B', \tau') \xrightarrow{\sim} (B, \tau)$ of algebras with involution. In particular, the centres of B, B' are isomorphic, forcing $\delta(L/k) = \delta(L'/k)$ by 1.4. Hence there exists an isomorphism $\psi_0 : L' \xrightarrow{\sim} L$. The étale Tits process being functorial in the obvious sense, we now obtain an isomorphism

$$\eta_0 := \varphi \times (\varphi \otimes \psi_0) : J' \xrightarrow{\sim} J(E, L, \varphi(u'), \psi_0(b')),$$

and after replacing η by $\eta \circ \eta_0^{-1}$, we may assume $E' = E,\ \varphi = \mathbf{1}_E, L' = L$.
2^0. By 1^0, η induces the identity on E and hence matches the orthogonal complements of E relative to the trace forms of J, J', repectively, which both identify canonically with $E \otimes L$ through the second factor (by (9)). We therefore obtain a k-linear bijection $\psi : E \otimes L \longrightarrow E \otimes L$ satisfying

$$\eta\big((v, x)\big) = \big(v, \psi(x)\big) \qquad (v \in E,\ x \in E \otimes L). \tag{45}$$

Using this an expanding $\eta\big(v \times (0, x)\big) = v \times \eta\big((0, x)\big)$ by means of (10), we conclude that ψ is in fact E-linear. Similarly, expanding $\eta\big((0, x)^\sharp\big) = \eta\big((0, x)\big)^\sharp$ by means of (7) yields

$$N_L\big(\psi(x)\big)u = N_L(x)u', \tag{46}$$

$$\psi\left(\sigma(b')\sigma(x)^\sharp {u'}^{-1}\right) = \sigma(b)\sigma\big(\psi(x)\big)^\sharp u^{-1} \tag{47}$$

for all $x \in E \otimes L$.
3^0. We put $y := \psi(1 \otimes 1) \in E \otimes L$ and specialize $x = 1 \otimes 1$ in (46) to conclude

$$u' = N_L(y)u. \tag{48}$$

In particular, y is invertible. Also, by 3.3, the assignment $(v, x) \longmapsto (v, y^{-1}x)$ gives an isomorphism

$$\eta_1 : J \longrightarrow J(E, L, u_1, b_1)$$

where $u_1 = u', b_1 = N_E(y)b$.
4^0. Using 3^0 and replacing η by $\eta_1 \circ \eta$ if necessary, we may assume $\psi(1 \otimes 1) = 1 \otimes 1$. Then (48) reduces to

$$u' = u, \tag{49}$$

and (46) yields $N_L \circ \psi = N_L$. Hence, by 3.7, ψ is an automorphism of the quadratic étale E-algebra $E \otimes L$. (Notice that E need not be a field, so we do require the generality provided for by 3.4.) Now 3.5 yields an idempotent $e \in E$ satisfying

$$\psi(x) = ex + (1 - e)\sigma(x) \qquad (x \in E \otimes L). \tag{50}$$

On the other hand, since ψ is E-linear, we obtain

$$\begin{aligned} {u'}^{-1}\psi\big(1 \otimes \sigma(b')\big) &= \psi\left({u'}^{-1} \otimes \sigma(b')\right) \\ &= \psi\left(\sigma(b')\sigma(1 \otimes 1)^\sharp {u'}^{-1}\right) \\ &= \sigma(b)\sigma\big(\psi(1 \otimes 1)\big)^\sharp u^{-1} && \text{(by (47))} \\ &= u^{-1}\big(1 \otimes \sigma(b)\big), \end{aligned}$$

and (49) implies

$$\begin{aligned} \sigma(1 \otimes b) = 1 \otimes \sigma(b) &= \psi(1 \otimes \sigma(b')) \\ &= e(1 \otimes \sigma(b')) + (1 - e)(1 \otimes b') && \text{(by (50))} \\ &= \sigma\big(e \otimes b' + (1 - e) \otimes \sigma(b')\big). \end{aligned}$$

Summing up, we have

$$1 \otimes b = e \otimes b' + (1-e) \otimes \sigma(b'). \tag{51}$$

5^0. By (50), (51), the proof will be complete once we have shown $e = 0, 1$. We argue indirectly and assume $e \neq 0, 1$. Then (41) shows either $e^\sharp = 0$ or $(1-e)^\sharp = 0$. Since the assignment $(v, x) \longmapsto \big(v, \sigma(x)\big)$ determines an automorphism η_2 of J by 3.3, we may replace η by $\eta_2 \circ \eta$ if necessary to obtain

$$e^\sharp = 0,\ (1-e)^\sharp = e. \tag{52}$$

Since $e \otimes 1, (1-e) \otimes 1 \in E \otimes L$ are free over L by assumption, the relation

$$\begin{aligned} b'(e \otimes 1) + \sigma(b')\big((1-e) \otimes 1\big) &= e \otimes b' + (1-e) \otimes \sigma(b') \\ &= 1 \otimes b && \text{(by (51))} \\ &= b(e \otimes 1) + b\big((1-e) \otimes 1\big) \end{aligned}$$

implies

$$b = b' \in k. \tag{53}$$

Given $a \in L$, we now to expand, using (52),

$$\begin{aligned} &N_E\big(e \otimes a + (1-e) \otimes \sigma(a)\big) = N_E\Big(a(e \otimes 1) + \sigma(a)\big((1-e) \otimes 1)\big)\Big) \\ &= a^3 N_E(e) + a^2\sigma(a)T_E(e^\sharp, 1-e) + a\sigma(a)^2 T_E\big(e, (1-e)^\sharp\big) + \sigma(a)^3 N_E(1-e) \\ &= a\sigma(a)^2 \end{aligned}$$

to obtain

$$N_E\big(e \otimes a + (1-e) \otimes \sigma(a)\big) = N_L(a)\sigma(a). \tag{54}$$

Next observe that η, being an isomorphism, preserves norms and hence satisfies $N_J \circ \eta = N_{J'}$. Applying both sides to $(0, 1 \otimes a)$, and using (45), (50), (6), (53), (54) we deduce

$$\begin{aligned} N_J \circ \eta\big((0, 1 \otimes a)\big) &= N_J\Big(\big(0, e \otimes a + (1-e) \otimes \sigma(a)\big)\Big) \\ &= b\Big[N_E\big(e \otimes a + (1-e) \otimes \sigma(a)\big) + \sigma\Big(N_E\big(e \otimes a + (1-e) \otimes \sigma(a)\big)\Big)\Big] \\ &= T_L(a)N_L(a)b \end{aligned}$$

on the one hand,

$$\begin{aligned} N_{J'}\big((0, 1 \otimes a)\big) &= b\Big[N_E(1 \otimes a) + \sigma\big(N_E(1 \otimes a)\big)\Big] \\ &= T_L(a^3)b \\ &= \big(T_L(a)^3 - 3T_L(a)N_L(a)\big)b \end{aligned}$$

on the other. Comparing we conclude $T_L(a)^3 = 4T_L(a)N_L(a)$ for all $a \in L$. But 3.6 yields an element $\theta \in L$ satisfying $T_L(\theta) = 1,\ 1 - 4N_L(\theta) \neq 0$, a contradiction. This completes the proof of 3.2. $\square$

4. Isomorphic embeddings

4.1 Norm classes. In this section, we fix a central simple associative algebra (B, τ) of degree 3 with involution of the second kind over k and a cubic étale k-algebra E. As in 1.6, we write K for the centre of B, $J = H(B, \tau)$ for the Jordan algebra of τ-symmetric elements in B and L for the quadratic étale k-algebra corresponding to the element $\delta(E/k) + \delta(K/k) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. Given an isomorphic embedding $\iota : E \longrightarrow J$, the extension theorem 1.6 yields invertible elements $u \in E,\ b \in L$ satisfying $N_E(u) = N_L(b)$ such that ι extends to an isomorphism form $J(E, L, u, b)$ onto J. Applying 3.2 to $\varphi = \mathbf{1}_E$, it follows that

$$[\iota] := u \bmod N_L((E \otimes L)^\times) \in E^\times / N_L((E \otimes L)^\times)$$

is independent of all choices made. We call $[\iota]$ the *norm class* of ι. Our main objective in this section is to prove the following result.

4.2 Theorem. *Given isomorphic embeddings $\iota, \iota' : E \longrightarrow J$, the following statements are equivalent.*

(i) *ι and ι' have the same norm class*, i.e., $[\iota] = [\iota']$.

(ii) *ι and ι' are equivalent, i.e., there exists an automorphism φ of J moving ι to ι', so $\varphi \circ \iota = \iota'$.*

Since the norm class by 4.1 is well defined, (ii) obviously implies (i). To establish the converse, we argue similarly to the proof of the Skolem-Noether Theorem for ninedimensional subalgebras of Albert algebras due to Parimala-Sridharan-Thakur [9], see [4, (40.15)]. We begin by examining the étale first Tits construction 1.5.

4.3 Proposition. *For $\alpha, \alpha' \in k^\times$ the following statements are equivalent.*

(i) *$J(E, \alpha)$ and $J(E, \alpha')$ are isomorphic.*

(ii) *$J(E, \alpha)$ and $J(E, \alpha')$ are isotopic.*

(iii) *$\alpha \equiv \alpha'^{\varepsilon} \bmod N_E(E^\times)$ for some $\varepsilon \in \{\pm 1\}$.*

(iv) *The identity of E can be extended to an isomorphism from $J(E, \alpha)$ to $J(E, \alpha')$.*

Proof. The implications (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) are clear.
(iii) $\Longrightarrow$ (iv). We have $\alpha = \alpha'^{\varepsilon} N_E(v)$ for some $\varepsilon \in \{\pm 1\}$ and $v \in E^\times$. Hence, if $\varepsilon = 1$, the assignment $(w_0, w_1, w_2) \longmapsto (w_0, w_1 v, v^{-1} w_2)$ gives an isomorphism of the desired kind. On the other hand, if $\varepsilon = -1$, the assignment $(w_0, w_1, w_2) \longmapsto (w_0, w_2 v^{-1}, v w_1)$ gives an isomorphism of the desired kind (since E is commutative).
(ii) $\Longrightarrow$ (iii). We distinguish the following cases.

Case 1. $\delta(E/k) = 0$.
Then E is either a cyclic cubic field extension or it splits. Furthermore, 1.4 yields central simple associative algebras A, A' of degree 3 over k satisfing $J(E, \alpha) \cong A^+$, $J(E, \alpha') \cong A'^+$, and (ii) allows us to assume $A \cong A'$. Let $\Psi : A' \xrightarrow{\sim} A$ be an isomorphism. Then we may either apply the classical Skolem-Theorem to $E, \Psi(E) \subset A$ if E is a field or

use standard facts about complete orthogonal systems of primitive idempotents in full matrix algebras otherwise to conclude (iv). Using the indentifications

$$J(E,\alpha) = J(E, k\times k, (1,1), (\alpha,\alpha^{-1})), J(E,\alpha') = J(E, k\times k, (1,1), (\alpha', \alpha'^{-1}))$$

guaranteed by 1.5, we may therefore apply 3.2 to $\varphi = \mathbf{1}_E$ and obtain (iii).

Case 2. $\delta(E/k) \neq 0$.
Changing scalars from k to $M = k\{\delta(E/k)\}$ brings us back to Case 1, so some $\varepsilon \in \{\pm 1\}$ has $\alpha\alpha'^{\varepsilon} \in N_E((E\otimes M)^\times)$, forcing the norm of $J(E, \alpha\alpha'^{\varepsilon})$ to become isotropic over M (1.5). Hence it must have been isotropic to begin with [17, 4.2.11], and (iii) follows. □

4.4 Remark. For char $k \neq 2, 3$, the preceding result can also be derived by using the cohomological invariants attached to associative algebras of degree 3 [4, §30.C]. Indeed, given E, α as in 4.3, E for simplicity assumed to be a field, write $J(E,\alpha) = H(B_\alpha, \tau_\alpha)$ for some central simple associative algebra (B_α, τ_α) of degree 3 with involution of the second kind. Then the centre of B_α agrees with the discriminant algebra of E (1.4), so $f_1(B_\alpha, \tau_\alpha) \in H^1(k, \mathbb{Z}/2\mathbb{Z})$ is independent of α. Furthermore, τ_α is distinguished [4, (39.5)(3)], forcing $f_3(B_\alpha, \tau_\alpha) = 0$. Finally, if K is a field and ρ denotes a fixed nontrivial K-automorphism of $E \otimes K$, we obtain $B_\alpha \cong (E, \rho, \alpha)$ or $B_\alpha \cong (E, \rho, \alpha^{-1})$ as cyclic K-algebras of degree 3, and $g_2(B_\alpha, \tau_\alpha)$ is uniquely determined by the Brauer class of B_α; on the other hand, if $K \cong k\times k$ splits, E is cyclic, so fixing a nontrivial k-automorphism ρ of E, we obtain $B_\alpha \cong A_\alpha \times A_\alpha^{\mathrm{op}}$ where $A_\alpha \cong (E,\rho,\alpha)$ or $A_\alpha \cong (E, \rho, \alpha^{-1})$, and $g_2(B_\alpha, \tau_\alpha)$ is uniquely determined by the Brauer class of A_α. Summing up, 4.3 now follows from the fact that central simple associative algebras of degree 3 with involutions of the second kind are classified by their invariants f_1, f_3, g_2 [4, (30.21)]. □

The following technicality is a variant of [4, (40.13)] over fields of arbitrary characteristic. We keep the notational conventions of 1.3.

4.5 Lemma. *Let L, E be étale k-algebras of dimension $2, 3$ respectively. Given $y \in E\otimes L$ such that $c := N_E(y) \in L$ satisfies $N_L(c) = 1$, there exists an element $y' \in E\otimes L$ satisfying $N_E(y') = c, N_L(y') = 1$.*

Proof. By Hilbert's Theorem 90, we have $c = d\sigma(d)^{-1}$ for some $d \in L^\times$. If E is not a field, N_E is surjective, so some $z \in E\otimes L$ has $N_E(z) = d$, and $y' = z\sigma(z)^{-1}$ does the job. Hence we may assume that E is a field. Choose $\theta \in L - k,\ \kappa \in L^\times$ satisfying $\theta + \sigma(\theta) = 1, \sigma(\kappa) = -\kappa$. Then the map $F : E\times k \longrightarrow k$ defined by

$$F((x,\xi)) := \kappa\Big(\sigma(d)N_E(x+\xi\theta 1) - dN_E(x + \xi(1-\theta)1)\Big)$$

for $x \in E,\ \xi \in k$ is a cubic form. We now distinguish the following cases.

Case 1. $L \cong k\times k$ splits.
Then $E\otimes L = E\times E,\ y = (y_1, y_2), y_i \in E^\times (i = 1, 2)$ and $c = (\gamma, \gamma^{-1}), \gamma = N_E(y_1)$, so the lemma follows by setting $y' = (y_1, y_1^{-1})$. We also claim that F is isotropic. To see this, we may assume $d = (\gamma, 1)$ and have $\theta = (\alpha, 1-\alpha)$ for some $\alpha \in k,\ 2\alpha \neq 1$. Since $y_1 = 1$ implies $d = (1,1)$, hence $F((1,0)) = 0$, we may assume $y_1 \neq 0$. Then a direct computation shows that $x := (y_1 - 1)^{-1}(\alpha 1 + (\alpha - 1)y_1) \in E$ satisfies $F((x,1)) = 0$, proving our claim.

Case 2. L is a field.
Since by Case 1, the cubic form F becomes isotropic after changing scalars from k to L, it must have been so all along [17, 4.2.11]. Hence there exists a nonzero element $(x, \xi) \in E \times k$ satisfying

$$\sigma(d)N_E(x + \xi\theta 1) = dN_E(x + \xi(1 - \theta)1).$$

This implies $x + \xi\theta 1 \in (E \otimes L)^\times$, and the lemma follows by setting

$$y' := (x + \xi\theta 1)\sigma(x + \xi\theta 1)^{-1}. \qquad \square$$

4.6 Remark. A similar argument will give a proof of [4, (40.13)] in all characteristics, see [10].

4.7 Proof of 4.2, (i) $\Longrightarrow$ (ii). Keeping the notations of 4.1, we apply 1.6 to obtain invertible elements $u, u' \in E, b, b' \in L$ satisfying $N_E(u) = N_L(b), N_E(u') = N_L(b')$ such that ι, ι' extend to isomorphisms

$$\eta : J_1 := J(E, L, u, b) \xrightarrow{\sim} J,\ \eta' : J_1' = J(E, L, u', b') \xrightarrow{\sim} J,$$

respectively; in particular, $J_1 \cong J_1'$ under $\eta'^{-1} \circ \eta$. We now distinguish the following cases.

Case 1. $L \cong k \times k$ splits.
Then we may identfy $J_1 = J(E, \alpha), J_1' = J(E, \alpha')$ for some $\alpha, \alpha' \in k^\times$, and 4.3 yields an isomorphism $\phi : J_1 \xrightarrow{\sim} J_1'$ which is the identity on E. Hence $\varphi := \eta' \circ \phi \circ \eta^{-1} \in \mathrm{Aut}(J)$ satisfies $\varphi \circ \iota = \iota'$, forcing ι, ι' to be equivalent.

Case 2. L is a field.
Changing scalars from k to L, $J_1 \otimes L \cong J(E \otimes L, b)$ and $J_1' \otimes L \cong J(E \otimes L, b')$ [13, 3.5, 3.8] continue to be isomorphic, so 4.3 yields $b = b'^{\varepsilon} N_E(z)$ for some $\varepsilon \in \{\pm 1\},\ z \in (E \otimes L)^\times$. Applying 3.2 for $E = E',\ L = L',\ \varphi = \mathbf{1}_E,\ \psi = \sigma, u = u'^{-1},\ b = b'^{-1}, y = u'$ we may in fact assume $\varepsilon = 1$, so $b = b' N_E(z)$. Applying 3.2 again, this time for $E = E', L = L', \varphi = \mathbf{1}_E, \psi = \mathbf{1}_L, y = z^{-1}$ we even reduce to the case $z = 1$, so $b = b'$. Now we use the fact that ι and ι' have the same norm class, so $u' = N_L(y)u$ for some invertible element $y \in E \otimes L$. This implies $N_L(c) = 1$ for $c = N_E(y) \in L$, and by 4.5 we obtain an element $y' \in E \otimes L$ such that $N_E(y') = N_E(y),\ N_L(y') = 1$. Hence the assignment $(v, x) \longmapsto (v, y'^{-1} yx)$ gives an isomorphism $\phi : J_1 \xrightarrow{\sim} J_1'$ which is the identity on E (3.2), so $\varphi := \eta' \circ \phi \circ \eta^{-1} \in \mathrm{Aut}(J)$ moves ι to ι', and the proof is complete. $\square$

4.8 Remark. If K is isomorphic to the discriminant algebra of E, then $L \cong k \times k$ must be split, forcing the norm class of group $E^\times / N_L((E \otimes L)^\times)$ to be trivial. Hence, thanks to 4.2, any two embeddings from E to J are equivalent. According to the discussion of Case 1 in the preceding proof, this statement amounts to exactly the same as 4.3. $\square$

4.9 Example. We now specialize 4.1 to the case that $E = k \times k \times k$ is split. Giving an isomorphic embedding $\iota : E \longrightarrow J$ amounts to the same as giving a complete orthogonal system (e_1, e_2, e_3) of absolutely primitive idempotents in J. This in turn leads to a matrix $g = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_3) \in \mathrm{GL}_3(k)$ and an identification $J = H_3(K, g)$, the

Jordan algebra g-hermitian 3-by-3 matrices x having entries in K (so $x = g^{-1}x^*g, x^*$ the conjugate thranspose of x), which matches e_1, e_2, e_3 with the diagonal idempotents in $H_3(K,g)$. We may clearly assume $\det g = 1$, so $u = (\gamma_1, \gamma_2, \gamma_3) \in E$ has norm 1. It now follows easily that the linear bijection

$$\phi : J(E, K, u, 1) \longrightarrow J = H_3(K, g)$$

defined by

$$\phi((v, y)) := \begin{pmatrix} \alpha_1 & \gamma_3\gamma_2 a_3 & \gamma_2\gamma_3\sigma(a_2) \\ \gamma_3\gamma_1\sigma(a_3) & \alpha_2 & \gamma_1\gamma_3 a_1 \\ \gamma_2\gamma_1 a_2 & \gamma_1\gamma_2\sigma(a_1) & \alpha_3 \end{pmatrix}$$

for $v = (\alpha_1, \alpha_2, \alpha_3) \in E,\ y = (a_1, a_2, a_3) \in E \otimes K = K \times K \times K$, preserves norms and units, hence is an isomorphism extending ι. Making the obvious identifications and writing $[\alpha]$ for the coset of $\alpha \in k^\times$ in $k^\times/N_K(K^\times)$, we conclude

$$[\iota] = ([\gamma_1], [\gamma_2], [\gamma_3]) = ([\gamma_2^{-1}\gamma_3], [\gamma_3^{-1}\gamma_1], [\gamma_1^{-1}\gamma_2]).$$

Hence the norm classes of isomorphic embeddings from E to ninedimensional absolutely simple Jordan algebras of degree 3 are exactly analogous to the norm classes of complete orthogonal systems of absolutely primitive idempotents in Albert algebras [2].

4.10 Example. We close the paper by constructing isomorphic embeddings that have distinct norm classes and hence are not equivalent (4.2). Let k_0 be a field, for simplicity assumed to be of characteristic not two, and K_0/k_0 a quadratic field extension. Given an indeterminate X, we put $k = k_0(X),\ K = K_0(X)$,

$$g = \mathrm{diag}(1, X, X^{-1}),\ g' = \mathrm{diag}((X+1)^{-1}, X, X^{-1}(X+1)),$$

and $J = H_3(K, g),\ J' = H_3(K, g')$. Since $X + 1$ is represented by the 2-fold Pfister form $N_K \perp < X > N_K$, we conclude that

$$N_K \perp < X^{-1} > N_K \perp < X > N_K$$

and

$$< X + 1 > N_K \perp < X^{-1} > N_K \perp < (X+1)^{-1}X > N_K$$

are isometric, forcing J and J' to have isometric trace forms [6, p. 502]. Hence they are isomorphic [4, (19.6)]. Now let $\phi : J' \xrightarrow{\sim} J$ be any isomorphism, put $E = k \times k \times k$ and denote by ι (resp. ι') the isomorphic embedding $E \longrightarrow J$ corresponding to the diagonal idempotents in J (resp. the image under ϕ of the diagonal idempotents in J'). Then 4.9 implies

$$[\iota] = ([1], [X], [X]),\ [\iota'] = ([X+1], [X], [X(X+1)]).$$

In particular, $[\iota] \neq [\iota']$ since, otherwise, we would obtain $X + 1 \in N_K(K^\times)$, which is easily seen to be impossible (for example, K is unramified at X). A similar argument shows $X \not\equiv (X+1) \bmod N_K(K^\times)$, so no embedding $E \longrightarrow J$ arising from ι by composing it with an automorphism of E can be equivalent to ι'.
It would be interesting to have an example of this kind where E or even J as in 4.1 are division algebras. $\square$

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