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Title

INEQUALITIES RELATED TO HEINZ MEAN AND SOME TRACE INEQUALITIES FOR MATRICES

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Abstract

Many questions were asked by mathematicians after the well-known Heinz mean inequality. In this thesis we discuss a question posed by Bourin about a related inequality to the Heinz mean. We couldn't prove or disprove his inequality but Alakhrass found few inequalities related to his question. Hayajneh and Kittaneh proved that Bourin's inequality holds true in the case of the trace norm. Moreover, they were able to prove that the norm of the real and imaginary part of the left hand side of Bourin's inequality is less or equal to the right hand side of his inequality. This gives a partial answer to his question. The work of Alakhrass, Hayajneh and Kittaneh concerning inequalities related to Bourin's question will be presented in this thesis. Furthermore, the Cauchy-Schwarz inequality will be discussed. We will present one of the refinements of this inequality along with new type of inequalities related to Cauchy-Schwarz as done by Burqan. Finally, two trace inequalities related to Lieb and Thirring trace inequality, proven by Ando and Hiai for nonnegative real numbers satisfying some conditions were disproved by two counterexamples found by Plevnik. We present Plevnik's work and his generalization of a trace inequality, and we suggest a small improvement.

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Contents

Chapter 1

Introduction

The well-known Heinz mean inequality for matrices says that for positive semidefinite matrices A and B in $M_n(\mathbb{C})$, for $p, q > 0$ and for every unitarily invariant norm $\|.\|$, we have

$$\|A^pB^q + A^qB^p\| \leq \left\|A^{p+q} + B^{p+q}\right\|. \tag{1.1}$$

A related inequality to the above Heinz mean inequality, is

$$\|A^pB^q + B^pA^q\| \leq \left\|A^{p+q} + B^{p+q}\right\|. \tag{1.2}$$

In his work on subadditivity of concave functions of positive semidefinite matrices, Jean-Christophe Bourin asked in [8] whether the unitarily invariant norm inequality (1.2) still holds true for any positive semidefinite matrices A, B and for any $p, q > 0$.
Hayajneh and Kittaneh in [14] has answered Bourin's question and proved it true for $p = 1, 2, 3$ and $q = 1$ in the case of the Schatten 2-norm.
In the following chapters we will give more general results and Bourin's question will be answered for some special cases.
Replacing A and B by $A^{\frac{1}{p+q}}$ and $B^{\frac{1}{p+q}}$ respectively in (1.1) and (1.2) we get:

$$\left\|A^tB^{1-t} + A^{1-t}B^t\right\| \leq \|A + B\| \tag{1.3}$$

and

$$\left\|A^tB^{1-t} + B^tA^{1-t}\right\| \leq \|A + B\|, \quad \text{where } t = \frac{p}{p+q}. \tag{1.4}$$

In order to solve Bourin's question positively, Hayajneh and Kittaneh have conjectured that:

$$\|A^pB^q + B^pA^q\| \leq \|A^pB^q + A^qB^p\|. \tag{1.5}$$

Again, replacing A and B by $A^{\frac{1}{p+q}}$ and $B^{\frac{1}{p+q}}$ respectively in (1.5) we get

$$\left\|A^tB^{1-t} + B^tA^{1-t}\right\| \leq \left\|A^tB^{1-t} + A^{1-t}B^t\right\|, \quad \text{where } t = \frac{p}{p+q}. \tag{1.6}$$

In the case of the Hilbert-Schmidt norm, we see that the inequality (1.6) is equivalent to saying

$$Re\, tr(A^tB^{1-t}A^{1-t}B^t) \leq tr(AB). \tag{1.7}$$

We can see this by using the definition $\|A\|_2 = (tr(A^*A))^{\frac{1}{2}}$, and the cycle permutation of the trace.

$$\begin{aligned}
tr(A^tB^{1-t}A^{1-t}B^t) + tr(B^tA^{1-t}B^{1-t}A^t) &\leq 2tr(AB) \\
tr(A^tB^{1-t}A^{1-t}B^t) + tr((A^tB^{1-t}A^{1-t}B^t)^*) &\leq 2tr(AB) \\
2\,Re\;\; tr(A^tB^{1-t}A^{1-t}B^t) &\leq 2tr(AB) \qquad \text{since } \; tr(A^t) = tr(A) \\
Re\, tr(A^tB^{1-t}A^{1-t}B^t) &\leq tr(AB).
\end{aligned}$$

In [5] Bhatia has proved (1.7) for $t \in [\frac{1}{4}, \frac{3}{4}]$ in the case of the Hilbert Schmidt norm (or 2-Schatten norm), which is an extension of the result of Hayajneh and Kittaneh [14].
Recently, Bottazzi *et al.* [9] proved the inequality (1.7) actually holds whenever t belongs to a vertical strip in a complex plane containing the interval $[\frac{1}{4}, \frac{3}{4}]$. Moreover, they have found a counterexample which shows that the inequality (1.6) does not hold for all $t \in [0, 1]$ if $\|.\|$ is the operator norm.

Until now, inequality (1.4) is not proved yet. But after the preliminaries in chapter 2, the following inequalities, weaker than (1.4) but more general than previous results, will be proved in chapter 3. These inequalities are the work of Mouhammad Al Akhrass in [1].

1. For positive semidefinite matrices A, B and $t \in [0,1]$ and for any unitarily invariant norm:
$$\left\|A^t B^{1-t} + B^t A^{1-t}\right\| \leq 2^{\left|\frac{1}{2}-t\right|} \|A+B\|. \tag{1.8}$$
2. If A and B are Hermitian matrices and $p, q \geq 0$ such that $\frac{1}{p}+\frac{1}{q}=1$ then,
$$\|A^p B^q + B^p A^q\| \leq 2^{\left|\frac{1}{2}-\frac{1}{p}\right|} \left\| |A|^{p+q} + |B|^{p+q} \right\| \tag{1.9}$$
for any unitarily invariant norm $\|.\|$.

In the next chapter, we give a proof of the following inequality, done by Hayajneh and Kittaneh in [15], and that holds for any unitarily invariant norm.
$$\left\|A^t X B^{1-t} + B^t X^\star A^{1-t}\right\| \leq \|AX\| + \|XB\| \tag{1.10}$$
where $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite and $t \in [0,1]$.
As an application of inequality (1.10) it is shown that:
$$\left\|A^t B^{1-t} + B^t A^{1-t}\right\|_1 \leq \|A+B\|_1 \quad \text{for } t \in [0,1]. \tag{1.11}$$
This gives an affirmative answer to Bourin's question (inequality 1.4) in the case of the trace norm. However, the question remains open for other unitarily invariant norms like the Schatten p-norms.
Moreover, Hayajneh and Kittaneh proved in the same article, the following two norm inequalities, weaker than the inequality (1.4) for any unitarily invariant norm:
$$\left\|Re(A^t B^{1-t} + B^t A^{1-t})\right\| \leq \|A+B\| \tag{1.12}$$
and
$$\left\|Im(A^t B^{1-t} + B^t A^{1-t})\right\| \leq \|A+B\| \tag{1.13}$$
where $A, B \in M_n(\mathbb{C})$ are positive semidefinite and $t \in [0,1]$.
These inequalities give a partial answer to Bourin's question by inserting the real and imaginary parts in the left-hand side of the inequality (1.4).

In chapter 5, a different, yet important inequality will be discussed. It is the Cauchy-Schwarz norm inequality for operators which asserts that
$$\left\| \left|A^{\frac{1}{2}} X B^{\frac{1}{2}}\right|^r \right\|^2 \leq \| |AX|^r \| \,.\, \| |XB|^r \|$$
for any real number $r > 0$ and every unitarily invariant norm $\|.\|$, where A, B and X are operators on a complex separable Hilbert space such that A and B are positive.
Aliaa Abed Al-Jawwad Burqan in [10], was able to derive several refinements of Cauchy-Schwarz norm inequality for operators. We will present one of them in chapter 5.
Furthermore, after proving the convexity of the function $f(\nu) = \| |A^\nu X B^\nu|^r \| \,.\, \left\| \left|A^{1-\nu} X B^{1-\nu}\right|^r \right\|$ on the interval $[0,1]$, Burqan found new types of inequalities close to Cauchy-Schwarz norm inequality.
She proved that for any $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive, for $0 \leq \nu \leq 1$, $r > 0$, and for every unitarily invariant norm, we have
$$\left\| \left|A^{1/2} X B^{1/2}\right|^r \right\|^2 \leq \| |A^\nu X B^\nu|^r \| \,.\, \left\| \left|A^{1-\nu} X B^{1-\nu}\right|^r \right\| \leq \| |X|^r \| \,.\, \| |AXB|^r \| \tag{1.14}$$
and then got related inequalities from special cases of (1.14).

Finally, the last section contains a new type of inequalities; the trace inequalities.
The famous inequality due to Lieb and Thirring reads as
$$tr\, AB \ldots AB = tr(AB)^n \leq tr\, A^n B^n$$
for all $A, B \geq 0$ and natural number n.
It has been of interest in the literature whether the trace of some other word in A and B in which each of the

letters occurs n times can be bounded with $tr\, A^n B^n$ from above and $tr\,(AB)^n$ from below, see e.g.[13]. But in these estimates we need to consider either a real part or an absolute value of the trace of a word, as the latter needs not to be neither positive [17] nor real ([13] Remark 2.7). Generally, Ando *et al.* considered in [2] the inequality of the form

$$Re\; tr\, A^{p_1} B^{q_1} \dots A^{p_k} B^{q_k} \leq tr\, A^{p_1+\dots+p_k} B^{q_1+\dots+q_k} \tag{1.15}$$

where $k \geq 2$ is a natural number and $p_1, q_1, \dots, p_k, q_k$ are real numbers. We notice that inequality (1.7) is a special case of (1.15). In [3], Ando and Hiai were studying related inequalities of the form

$$\|A^{p_1} B^{q_1} \dots A^{p_k} B^{q_k}\| \leq \left\|A^{p_1+\dots+p_k} B^{q_1+\dots+q_k}\right\| \tag{1.16}$$

for any unitarily invariant norm.
They showed that inequalities (1.15) and (1.16) are valid whenever $p_1, q_1, \dots, p_k, q_k$ are nonnegative and satisfy some additional restrictions. They pointed out the problems were subtle for general positive real numbers $p_1, q_1, \dots, p_k, q_k$ and they did not have any counterexamples. It is natural to ask what is the complete range of validity of these inequalities.
Plevnik in [25] found a counterexample to (1.15) and a counterexample to (1.16) which shows that both inequalities are not valid for all nonnegative numbers $p_1, q_1, \dots, p_k, q_k$.

As mentioned before, in the papers [13] and [14] some inequalities of the type (1.15) were studied. For example, for all $A, B \geq 0$ and $p, q \geq 0$ we have

$$tr\left(AB^{\frac{p+q}{2}}\right)^2 \leq tr\; AB^p AB^q \leq tr\; A^2 B^{p+q}. \tag{1.17}$$

Plevnik in [25] gave a generalization to Inequality (1.17). In the last section, we will state his counterexamples and generalization along with a small improvement I did to this generalization.

Chapter 2

Preliminaries

2.1 Linear algebra preliminaries

Let $M_n(\mathbb{C})$ denotes the space of square matrices of order n with entries in $\mathbb{C}$.
In what follows we give some basic definitions, notations and properties of the elements of this space that will be useful in the upcoming chapters.

Definition 2.1.1. Let $\lambda \in \mathbb{C}$.

- If there exist $x \in \mathbb{C}^n \backslash \{0\}$ such that $Ax = \lambda x$, then we say that λ is an eigenvalue of A, and x is an eigenvector of A associated with λ.

  Remark 2.1.1. If A is a triangular matrix (A triangular matrix has the property that either all of its entries below the main diagonal are 0 or all of its entries above the main diagonal are 0), the eigenvalues of A are the entries on the main diagonal of A.

  Throughout this book, we'll use the following ordering for the eigenvalues of the matrix $A \in M_n(\mathbb{C})$:

  $$|\lambda_1(A)| \geq \ldots \geq |\lambda_n(A)|.$$

- The multi-set of all the eigenvalues of A is called the spectrum of A, and it is denoted by $Sp(A)$:

  $$Sp(A) = \{\lambda \in \mathbb{C}; \lambda \text{ is an eigenvalue of } A\}.$$

- Let $A = (a_{ij}) \in M_n(\mathbb{C})$, the trace of A, denoted by $tr(A)$, is defined as:

  $$tr(A) = \sum_{i=1}^{n} a_{ii}.$$

- Let $A = (a_{ij}) \in M_n(\mathbb{C})$, the determinant of A, denoted by $det(A)$, is defined as :

  $$det(A) = \sum_{(i_1 i_2 \ldots i_n)} \pm a_{i_1} a_{i_2} \ldots a_{i_n}$$

  Where the terms are summed over all permutations $(i_1 i_2 \cdots i_n)$, and the sign is + if the permutation is even, otherwise it is -.

Proposition 2.1.1. For all $A, B \in M_n(\mathbb{C})$ we have $Sp(AB) = Sp(BA)$.

Proof. See Zhang's book [28] page 51 Theorem 2.8. □

Proposition 2.1.2. If A is a square matrix with eigenvalues λ_i with $i = 1, \ldots, n$ then we have,

$$tr(A) = \sum_{i=1}^{n} \lambda_i$$

$$det(A) = \prod_{i=1}^{n} \lambda_i.$$

Proof. See Proposition C.3.7 in [26]. □

Next we give some properties of the trace of matrices that are easy to prove.

Properties 2.1.1. Let $A, B \in M_n(\mathbb{C})$, and $a \in \mathbb{C}$, then:

1. $tr\,(A+B) = tr\,(A) + tr\,(B)$.
2. $tr\,(a\,A) = a\,tr\,(A)$.
3. $tr\,(AB) = tr\,(BA)$.
4. $tr(ABCD) = tr(BCDA) = tr(CDAB) = tr(DABC)$. (Cycle permutation of a trace)

Having a matrix A in $M_n(\mathbb{C})$, we can define three related matrices.

Definition 2.1.2. Let $A = (a_{ij}) \in M_n(\mathbb{C})$

- The conjugate matrix of A is given by
$$\overline{A} = (\overline{a_{ij}})\,.$$
- The transpose of A, denoted by A^T, is the matrix
$$A^T = (a_{ji})\,.$$
- The adjoint of A, denoted by $A^\star$, is the matrix given by
$$A^\star = \overline{A}^T.$$

Next we give some properties of the transpose and adjoint matrices of A.

Properties 2.1.2. Let $A, B \in M_n(\mathbb{C})$, then:

1. $\left(A^T\right)^T = A$.
2. $(A+B)^T = A^T + B^T$.
3. $(A\,B)^T = B^T\,A^T$
4. $Sp\left(A^T\right) = Sp\,(A)\,.$
5. $tr\left(A^T\right) = tr\,(A)\,.$
6. $\det\left(A^T\right) = \det\,(A)\,.$

Properties 2.1.3. Let $A, B \in M_n(\mathbb{C})$ and $a \in \mathbb{C}$, then:

1. $(A^\star)^\star = A$.
2. $(a\,A)^\star = \overline{a}\,A^\star$.
3. $(A+B)^\star = A^\star + B^\star$.
4. $(A\,B)^\star = B^\star A^\star$.
5. $Sp\,(A^\star) = \overline{Sp(A)}$, where $\overline{Sp(A)} = \{\overline{\lambda}\,;\lambda \in Sp(A)\}$.
6. $tr\,(A^\star) = \overline{tr\,(A)}$.
7. $det\,(A^\star) = \overline{\det\,(A)}$.

In the following we are going to define the Hermitian inner product and give some of its important properties and propositions.

Definition 2.1.3. (Hermitian inner product)

- The Hermitian inner product of two vectors $x, y \in \mathbb{C}^n$ is given by

$$\langle x, y \rangle = \sum_{i=1}^{n} \overline{y_i} x_i = y^\star x.$$

- The Euclidean norm of $x \in \mathbb{C}^n$ is given by

$$\|x\|^2 = \langle x, x \rangle = \sum_{i=1}^{n} \overline{x_i} x_i,$$

  where x_i is denoted the i^{th} components of x, for all $i \in \{1, \ldots, n\}$.

The Hermitian inner product satisfies the following properties.

Property 2.1.1. Let $x, y, z \in C^n$, and $a \in \mathbb{C}$.

1. $\langle y, x \rangle = \overline{\langle x, y \rangle}$
2. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
3. $\langle a\, x, y \rangle = a \langle x, y \rangle$
4. $\langle x, x \rangle \geq 0$ and if $\langle x, x \rangle = 0 \Rightarrow x = 0$

Theorem 2.1.1. (Cauchy-Schwarz inequaliy)

For any $x, y \in \mathbb{C}^n$, we have

$$\begin{aligned} |\langle x, y \rangle|^2 &\leq \langle x, x \rangle \langle y, y \rangle \\ \text{i.e.} \quad |\langle x, y \rangle| &\leq \|x\| \, \|y\| \,. \end{aligned}$$

Proof. See Theorem 1.8 page 22 in [28].

□

Theorem 2.1.2 (Hölder Inequality).

Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n$ or $\mathbb{C}^n$, and $p, q \in \mathbb{R}_*^+$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We have

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}$$

Proof. See [23].

□

Remark 2.1.2. For any $x, y \in \mathbb{C}^n$ and any $A \in M_n(\mathbb{C})$ we have

$$y^\star A x = \langle Ax, y \rangle = \langle x, A^\star y \rangle = \overline{\langle y, Ax \rangle}.$$

2.2 Hermitian matrices

We will start this section by defining some special matrices and giving few related remarks that will be often used in this book.

Definition 2.2.1. Let $A \in M_n(\mathbb{C})$.

1. If $A = A^T$ then A is called **Symmetric**.
2. If $A = A^\star$ then A is called **Hermitian**.
3. If $AA^\star = A^\star A = I_n$ then A is called **Unitary**, where I_n is the identity matrix of order n.
4. If $AA^\star = A^\star A$ then A is called **Normal**.
5. If $AA^T = I_n$ then A is called **Orthogonal**, where I_n is the identity matrix of order n.

Remarks 2.2.1.

1. Every unitary matrix $A \in M_n(\mathbb{C})$ is invertible and $A^{-1} = A^\star$.
2. For every two unitary matrices A and B in $M_n(\mathbb{C})$, AB and BA are also unitary matrices.
3. Every Hermitian matrix $A \in M_n(\mathbb{C})$ is normal. But not every normal matrix is Hermitian. Below is a counterexample.

   Let $A = \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix}$ and $A^\star = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}$. We have $AA^\star = A^\star A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ but $A \neq A^\star$.
4. The sum of two Hermitian matrices is Hermitian, whereas the product of two Hermitian matrices is Hermitian if and only if these two matrices commute as the following example shows:

   Let $A = \begin{pmatrix} i & 0 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -i \\ 1 & 0 \end{pmatrix}$. We have $AB = \begin{pmatrix} 0 & 1 \\ 0 & -i \end{pmatrix}$ but $(AB)^\star = \begin{pmatrix} 0 & 0 \\ 1 & i \end{pmatrix}$.

   $AB \neq (AB)^\star$ so AB is not Hermitian. (We notice that A and B don't commute).

Here are some properties about Hermitian matrices

Property 2.2.1. Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix then:

1. $Sp(A) \subset \mathbb{R}$.
2. $A + A^\star$, $A\,A^\star$ and $A^\star A$ are all Hermitian matrices for.

Proof. 1. Let $\lambda \in Sp(A)$, and x a nonzero eigenvector associated to λ, we have,

$$\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda \langle x, x\rangle = \lambda \|x\|^2$$
$$\langle Ax, x\rangle = \langle x, A^\star x\rangle = \overline{\lambda} \langle x, x\rangle = \overline{\lambda} \|x\|^2 .$$

Thus $\lambda = \overline{\lambda}$ so $\lambda \in \mathbb{R}$ and $Sp(A) \subset \mathbb{R}$.

□

Definition 2.2.2. (Cartesian decomposition)

For a matrix $A \in M_n(\mathbb{C})$ we define the two Hermitian matrices H and K:

$$\begin{aligned} H &= \frac{A + A^\star}{2} \quad (\text{ real part of A }) \\ K &= \frac{A - A^\star}{2i} \quad (\text{ imaginary part of A }) \end{aligned} \tag{2.1}$$

The Cartesian decomposition of A is $A = H + iK$.

Next we give two very important theorems in Linear Algebra.

Theorem 2.2.1 (Schur decomposition theorem).

If $A \in M_n(\mathbb{C})$ then there exist $U \in M_n(\mathbb{C})$ unitary such that $U^\star AU = \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & \mathbf{0} & \ddots & \\ & & & \lambda_n \end{pmatrix}$ is a upper-triangular matrix where λ_i are the eigenvalues of A.

Proof. See Theorem 3.3 page 64 in [23]. □

Corollary 2.2.1.1 (Spectral Decomposition theorem).

$A \in M_n(\mathbb{C})$ is normal if and only if there exists $U \in M_n(\mathbb{C})$ unitary such that $U^\star AU = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & \mathbf{0} & \ddots & \\ & & & \lambda_n \end{pmatrix}$.

Proof. The proof can be found in [24]. But since we are concerned by the Hermitian matrices, we will give a proof of this theorem when A is Hermitian.
Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix.

By the Schur decomposition theorem, there exists $U \in M_n(\mathbb{C})$ unitary such that $U^\star AU = \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & \mathbf{0} & \ddots & \\ & & & \lambda_n \end{pmatrix}$.

Since A is Hermitian, we have $(U^\star AU)^\star = U^\star A^\star U = U^\star AU$,

thus $\begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & * & \ddots & \\ & & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & \mathbf{0} & \ddots & \\ & & & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & \mathbf{0} & \ddots & \\ & & & \lambda_n \end{pmatrix}$. □

2.3 Positive definite and positive semidefinite matrices

In this section we give a definition of positive definite and semidefinite matrices with some related proposition and remarks.

Definition 2.3.1. (Positive definite, Positive semidefinite matrices)

- A matrix $A \in M_n(\mathbb{C})$ is positive definite (p.d) if for any $x \in \mathbb{C}^n \backslash \{0\}$, $\langle Ax, x\rangle > 0$,. We use the notation $A > 0$.
- A matrix $A \in M_n(\mathbb{C})$ is positive semidefinite (p.s.d) if for any $x \in \mathbb{C}^n$, $\langle Ax, x\rangle \geq 0$. We use the notation $A \geq 0$.

Remarks 2.3.1. Let $A \in M_n(\mathbb{C})$

1. If A is positive semidefinite then $Sp(A) \subset \mathbb{R}^+$.
2. If A is positive definite then $Sp(A) \subset \mathbb{R}^*_+$.

Proof. A is positive semidefinite so A is Hermitian, then $Sp(A) \subseteq \mathbb{R}$.
Let $\lambda \in Sp(A)$, then there exists $x \in \mathbb{C}^n \backslash \{0\}$ such that $Ax = \lambda x$.

$$\langle Ax, x\rangle = \langle \lambda x, x\rangle = \lambda \|x\|_2^2 \geq 0$$
$$\Rightarrow \lambda \geq 0 \quad \text{since } \|x\|_2^2 > 0.$$

The same holds for 2. □

Proposition 2.3.1.

1. Every positive semidefinite matrix is Hermitian. The converse is not generally correct. A Hermitian matrix is positive semidefinite if its eigenvalues are positive.
2. For any $A \in M_n(\mathbb{C})$, $A^\star A$ is positive semidefinite.

Proof. 1.

- We will start by proving that every p.s.d matrix is Hermitian. Let $A = (a_{ij})$ be a p.s.d matrix and $A^\star = (\overline{a}_{ji})$.
Our goal is to prove that $a_{sk} = \overline{a}_{ks}\ \forall s, k$.

$$\text{Let} \quad x = (x_{i1}) = \begin{cases} x_{s1} = 1 \\ x_{k1} = \alpha \\ x_{i1} = 0 \quad \forall i \neq s \text{ and } i \neq k \end{cases} \qquad \text{where } \alpha \in \mathbb{C}$$

$$Ax = \left(\sum_{j=1}^{n} a_{ij} x_j \right)_{1 \leq i \leq n} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{pmatrix} = \begin{pmatrix} a_{1s} + \alpha\, a_{1k} \\ \vdots \\ a_{ss} + \alpha\, a_{sk} \\ \vdots \\ a_{ks} + \alpha\, a_{kk} \\ \vdots \\ a_{ns} + \alpha\, a_{nk} \end{pmatrix}$$

$$\begin{aligned} &\Rightarrow \langle Ax, x \rangle = (a_{ss} + \alpha\, a_{sk})\overline{1} + (a_{ks} + \alpha\, a_{kk})\overline{\alpha} \geq 0 \\ &\Rightarrow a_{ss} + |\alpha|^2 a_{kk} + \alpha\, a_{sk} + \overline{\alpha}\, a_{ks} \geq 0 \end{aligned}$$

If $\alpha = 0$, then $a_{ss} \geq 0, \quad \forall\, 1 \leq s \leq n$

Let $a_{sk} = a + ib \in \mathbb{C}$ *and* $a_{ks} = c + id \in \mathbb{C}$

If $\alpha = 1$, then

$$\begin{aligned} &a_{ss} + a_{kk} + (a + ib) + (c + id) \geq 0 \quad \text{with } a_{ss} + a_{kk} \in \mathbb{R} \\ &\Rightarrow\ b + d = 0 \\ &\Rightarrow\ b = -d \end{aligned}$$

If $\alpha = i$, then

$$\begin{aligned} &a_{ss} + akk + i(a + ib) - i(c + id) \geq 0 \\ &\Rightarrow a - c = 0 \\ &\Rightarrow a = c \end{aligned}$$

Hence $a_{sk} = a + ib = c - id = \overline{a}_{ks} \quad \forall\ 1 \leq k, s \leq n$.
Therefore $A = A^\star$ and A is Hermitian.

- This is a counter example that shows why the converse is not generally correct.
Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, A is Hermitian since $A = A^\star$ but A is not positive semidefinite since it has a negative eigenvalue.

- Finally we will prove that if the eigenvalues of a Hermitian matrix are positive, then the matrix is p.s.d.
Let A be a Hermitian matrix and $x \in \mathbb{C}^n$. We have to prove that $\langle Ax, x \rangle \geq 0$.
Since A is Hermitian, by the spectral decomposition, there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that

$$U^\star A U = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix}.$$

The columns of U form an orthonormal basis of $\mathbb{C}^n$.
$U = [U_1\ U_2\ \ldots U_n]$ with $\|U_i\|_2 = 1$ and $\langle U_i, U_j \rangle = 0$ for all $i \neq j$.

Now $x \in \mathbb{C}^n$, $x = \sum_{i=1}^n \alpha_i u_i$ with $\alpha_i \in \mathbb{C}$

$$\begin{aligned}
Ax &= \sum_{i=1}^n \alpha_i A u_i = \sum_{i=1}^n \alpha_i \lambda_i u_i \qquad (Au_i = \lambda_i u_i) \\
\langle Ax, x \rangle &= \left(\sum_{i=1}^n \alpha_i \lambda_i u_i \, , \sum_{j=1}^n \alpha_j u_j \right) \\
&= \sum_{i=1}^n \alpha_i \overline{\alpha_i} \lambda_i \|u_i\|_2^2 \\
&= \sum_{i=1}^n \lambda_i |\alpha_i|^2 \geq 0 \quad \text{since } \lambda_i \geq 0 \quad \forall\, 1 \leq i \leq n.
\end{aligned}$$

Hence $\langle Ax, x \rangle \geq 0$ for all $x \in \mathbb{C}^n$. Thus A is p.s.d.

2. We have

$$\begin{aligned}
\langle A^\star A x, x \rangle &= \left\langle Ax, (A^*)^\star x \right\rangle \\
&= \langle Ax, Ax \rangle \\
&= ||Ax||^2 \geq 0
\end{aligned}$$

for all $x \in \mathbb{C}^n$, so A is positive semidefinite. □

Theorem 2.3.1 (Square root).

For any positive semidefinite matrix $A \in M_n(\mathbb{C})$, there exists a unique positive semidefinite matrix $B \in M_n(\mathbb{C})$ such that

$$B^2 = A.$$

B is called the square root of the matrix A and is denoted by $\sqrt{A}$, or $A^{\frac{1}{2}}$.

Proof. See [28], Theorem 6.4 page 162. □

2.4 Singular value decomposition

In this section, we will give the definition of singular values of a matrix, and some related and basic theorems.
Since $A^\star A$ is positive semidefinite then by the previous remark, $Sp(A^\star A) \subseteq \mathbb{R}^+$.
Thus the following definition is possible.

Definition 2.4.1.

- The singular values of $A \in M_n(\mathbb{C})$ are the square roots of the eigenvalues of $A^\star A$,

$$s_i(A) = \sqrt{\lambda_i(A^\star A)} = \sqrt{\lambda_i(AA^\star)}.$$

  Throughout this book, we'll use the following ordering for the singular values of the matrix $A \in M_n(\mathbb{C})$:

$$s_1(A) \geq \ldots \geq s_n(A).$$

- A non-negative real number μ is a singular value for A if and only if there exist unit-length vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{C}^n$ such that

$$A\vec{v} = \mu\vec{u} \text{ and } A^\star\vec{u} = \mu\vec{v}.$$

  The vectors $\vec{u}$ and $\vec{v}$ are called left-singular and right-singular vectors for μ, respectively.

Remark 2.4.1. If A is Hermitian then,

- The left-singular vector is equal to the right-singular vector and $A\vec{u} = \mu\vec{u}$.

- $s_i(A) = \lambda_i(A)$ because $s_i(A) = \lambda_i^{1/2}(A^\star A) = \lambda_i^{1/2}(A^2) = \lambda_i(A)$.

Theorem 2.4.1. (Singular value decomposition theorem: SVD)

For any $A \in M_{n,m}(\mathbb{C})$, there exist two unitary matrix $U \in M_n(\mathbb{C})$ and $V \in M_m(\mathbb{C})$, such that

$$UAV = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with } S = \begin{pmatrix} s_1(A) & & 0 \\ & \ddots & \\ 0 & & s_n(A) \end{pmatrix}.$$

Proof. See Proposition C.5.1 in [26] or Theorem 3.5 page 66 in [28] □

Corollary 2.4.1.1. (Polar decomposition)

For any square matrix $A \in M_n(\mathbb{C})$, there exist a unique positive semidefinite matrix $P \in M_n(\mathbb{C})$, and there exist U unitary $\in M_n(\mathbb{C})$ such that

$$A = UP$$

where P is denoted by $|A|$ and is given by

$$|A| = (A^\star A)^{\frac{1}{2}}.$$

Moreover if A is invertible than U is unique.

Proof. See [27]. □

Theorem 2.4.2 (Courant-Fisher or Min-max theorem).

Let $A = A^\star \in M_n(\mathbb{C})$ and $\lambda_1(A) \geq \cdots \geq \lambda_k(A) \geq \cdots \geq \lambda_n(A)$ then:

$$\lambda_k(A) = \max_{dimM=k} \; \underset{\substack{x\in M \\ \|x\|_2=1}}{min} < Ax, x > \tag{2.2}$$

$$\lambda_{n-k+1}(A) = \min_{dimM=n-k+1} \; \underset{\substack{x\in M \\ \|x\|_2=1}}{max} < Ax, x > . \tag{2.3}$$

The Min-max principle for singular values

$$\begin{aligned} \lambda_k(A^\star A) &= \max_{dimM=k} \; \underset{\substack{x\in M \\ \|x\|_2=1}}{min} < A^\star Ax, x > \\ &= \max_{dimM=k} \; \underset{\substack{x\in M \\ \|x\|_2=1}}{min} < Ax, Ax > \\ \Rightarrow s_j(A) &= \max_{dimM=k} \; \underset{\substack{x\in M \\ \|x\|_2=1}}{min} \|Ax\| . \end{aligned} \tag{2.4}$$

Proof. See Theorem 7.7 page 220 in [28]. □

2.5 Matrix norms

In the following section we will define the matrix norm, present some examples of it and then we'll give the definition of some special norms with their properties.

Definition 2.5.1. A function $\|.\| : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \longrightarrow \mathbb{R}^+$ is said to be a norm on $M_n(\mathbb{C})$ if for any $A, B \in M_n(\mathbb{C})$, and $\alpha \in \mathbb{C}$, it satisfies the following conditions :

1. $\|\alpha A\| = |\alpha| \, \|A\|$.
2. $\|A + B\| \leq \|A\| + \|B\|$.
3. $\|A\| \geq 0$, and $\|A\| = 0$ if and only if $A = 0$.

A norm $\|.\|$ over $M_n(\mathbb{C})$ that verifies $\|A\,B\| \leq \|A\| \, \|B\|$ for all $A, B \in M_n(\mathbb{C})$ is called a submultiplicative matrix norm.

Example 2.5.1. For $A = (a_{ij}) \in M_n(\mathbb{C})$, we give the following examples of norm over the space $M_n(\mathbb{C})$.

1. **Sum matrix norm:**
$$\|A\|_{sum} = \sum_{i,j} |a_{ij}| \, .$$
2. **Max element norm:**
$$\|A\|_{max} = \max_{i,j} |a_{ij}| \, .$$
3. **Frobenius matrix norm** or **Schur norm** or **Hilbert-schmidt norm:**
$$\|A\|_F = \|A\|_S = \|A\|_2 = \sqrt{tr\,(AA^\star)} = \sqrt{\sum_{i,j} |a_{ij}|^2}.$$
4. **Trace norm:**
$$\|A\|_1 = tr|A| = tr(A^\star A)^{\frac{1}{2}}.$$
5. **p-Schatten norm**:
$$\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A) \right)^{\frac{1}{p}}, \qquad \forall\, p \geq 1 \;\; (p \in \mathbb{R}).$$
6. **Ky Fan k-norm**:
$$\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A) \, , \qquad \forall 1 \leq k \leq n.$$
7. **Operator norm**(Spectral norm)**:**
   - $\|A\|_{op} = \sqrt{\rho(A^\star A)} = s_1(A), \qquad$ with $\rho(X) = \max|\lambda|$ for $\lambda \in \sigma(X)$.
   - $\|A\|_{op} = \sup\,\{\|Ah\| \; ; \; \|h\| \leq 1\}$.
   - $\|A\|_{op} = \sup\,\{\dfrac{\|Ah\|}{\|h\|} \; ; \; h \in \mathbb{C}, \; \|h\| \neq 0\}$.
   - $\|A\|_{op} = \inf\,\{c > 0 \; : \; \forall\, h \in \mathbb{C} \; : \; \|Ah\| \leq c\,\|h\|\}$.

   These definitions are all equivalent and the operator norm is a submultiplicative matrix norm.

Proposition 2.5.1. Let $A \in M_n(\mathbb{C})$, the operator norm of A satisfies the following:
$$\|A\|^2 = \|A^\star\|^2 = \|A^\star A\| = \|AA^\star\| \, .$$

Proof. Let $h \in \mathbb{C}$ such that $\|h\| \leq 1$. Then:
$$\begin{aligned}
\|Ah\|^2 &= \langle Ah, Ah \rangle \\
&= \langle A^\star Ah, h \rangle \\
&\leq \|A^\star Ah\| \, \|h\| && \text{(By Cauchy-Schwarz Inequality)} \\
&\leq \|A^\star A\| \, \|h\|^2 && \text{(By Submultiplicativity of Operator Norm)} \\
&\leq \|A^\star A\| && \text{(By assumption on h)} \\
&\leq \|A^\star\| \, \|A\| && \text{(By submultiplicativity of Operator Norm)}
\end{aligned}$$

Therefore, $\|Ah\|^2 \leq \|A^\star\| \, \|A\|$.
By the second definition of the operator norm, it follows that $\|A\|^2 \leq \|A^\star A\| \leq \|A^\star\| \, \|A\|$.
That is, $\|A\| \leq \|A^\star\|$. By substituting $A^\star$ for A, and using $(A^\star)^\star = A$, the reverse inequality is obtained.
Hence $\|A\|^2 = \|A^\star A\| = \|A^\star\|^2$.

□

Remark 2.5.1. We notice that

- $\|A\|_{(1)} = \|A\|_{op} \; \forall A \in M_n(\mathbb{C})$.
- The Hilbert-Schmidt norm is a particular case of the p-Schatten norm for $p = 2$.
- The trace norm is a special case of the p-Schatten norm for $p = 1$.

Definition 2.5.2. (Unitarily invariant norm)
A norm $\|.\|$ on $M_n(\mathbb{C})$ is called unitarily invariant norm if for all $A \in M_n(\mathbb{C})$ and all U, V unitary matrices in $M_n(\mathbb{C})$, it satisfies

$$\|UAV\| = \|A\|.$$

Example 2.5.2. $\|A\|_{op}$, $\|A\|_p$, $\|A\|_F$, $\|A\|_{(k)}$ and the trace norm are all examples of unitarily invariant norms.

Proposition 2.5.2. Let $\|.\|$ be a unitarily invariant norm over $M_n(\mathbb{C})$, then for any $A, B \in M_n(\mathbb{C})$, we have

1. $\|A\| = \|A^\star\|$.
2. $\|AA^\star\| = \|A^\star A\|$.
3. $\| \, |A| \, \| = \| \, |A^\star| \, \|$.
4. $\| \, |A| \, |A^\star| \, \| = \left\|A^2\right\|$.

2.6 Heinz mean

Since our main interest in this book is the Heinz mean inequality, we will define the Heinz mean of nonnegative real numbers after defining the arithmetic and geometric means and present their fundamental properties.

Definition 2.6.1 (The arithmetic and geometric means).
Let $a_1, \dots, a_n$ be nonnegative real numbers.

- The arithmetic mean is given by

$$A\left(a_1, \dots, a_n\right) = \frac{1}{n}\sum_{i=1}^{n} a_i = \frac{a_1 + \cdots + a_n}{n}.$$

- The geometric mean is given by

$$G\left(a_1, \dots, a_n\right) = \left(a_1 . \dots . a_n\right)^{\frac{1}{n}}.$$

Proposition 2.6.1 (The arithmetic-geometric mean inequality).

Let $a_1, \dots, a_n$ be nonnegative real numbers. We always have

$$G\left(a_1, \dots, a_n\right) \leq A\left(a_1, \dots, a_n\right) \quad \text{i.e} \quad \left(a_1 . \dots . a_n\right)^{\frac{1}{n}} \leq \frac{a_1 + \cdots + a_n}{n}.$$

Proof. Let $\alpha = \dfrac{a_1 + \cdots + a_n}{n}$. We always have that $e^{x-1} \geq x$ so

$$\begin{cases} e^{\frac{a_1}{\alpha}-1} \geq \dfrac{a_1}{\alpha} \\ \quad \vdots \\ e^{\frac{a_n}{\alpha}-1} \geq \dfrac{a_n}{\alpha} \end{cases} \Rightarrow \; e^{\frac{a_1}{\alpha}-1} \dots e^{\frac{a_n}{\alpha}-1} \geq \frac{a_1 \dots a_n}{\alpha^n} \; \Rightarrow \; e^{\frac{a_1+\cdots+a_n}{\alpha}-n} \geq \frac{a_1 \dots a_n}{\alpha^n} \; \Rightarrow \; 1 \geq \frac{a_1 \dots a_n}{\alpha^n}.$$

The result follows by replacing α by it's value. □

Definition 2.6.2 (**The Heinz mean**).
Let a and b be two nonnegative real numbers and $0 \le t \le 1$.
We define their Heinz mean as follows

$$H_t(a,b) = \frac{a^{1-t}b^t + a^t b^{1-t}}{2}.$$

These means have been studied a lot in the literature and their fundamental property is that they interpolate geometric and arithmetic mean in the following sense:

Property 2.6.1. For any a and b nonnegative real numbers and for $0 \le t \le 1$, we have:

$$H_{\frac{1}{2}}(a,b) = \sqrt{ab} \le H_t(a,b) = \frac{a^{1-t}b^t + a^t b^{1-t}}{2} \le \frac{a+b}{2} = H_0(a,b) = H_1(a,b). \tag{2.5}$$

Proof. We derive $H_t(a,b)$ with respect to t.
After a small calculation we'll get $H_t'(a,b) = (\ln a - \ln b)(a^t b^{1-t} - a^{1-t}b^t)$.

$$\begin{aligned} H_t'(a,b) = 0 &\Rightarrow a^t b^{1-t} = a^{1-t}b^t \\ &\Rightarrow t = 1-t \\ &\Rightarrow t = \frac{1}{2}. \end{aligned}$$

So $H_t(a,b)$ attains it's minimum at $t = \dfrac{1}{2}$ and it maximum at $t = 0$ and $t = 1$.
We have $H_0(a,b) = H_1(a,b) = \dfrac{a+b}{2}$ and $H_{\frac{1}{2}}(a,b) = \sqrt{ab}$.

□

In accordance with the definition for nonnegative real numbers, for $A, B \ge 0$ and $0 \le t \le 1$, we set $H_t(A,B) = \dfrac{A^{1-t}B^t + A^t B^{1-t}}{2}$.

In the introduction we noticed the use of the matrix A^p. One can easily define A^p when p is a positive integer. But what will be the definition of A^p when p is a positive real number?
At the end of this chapter, we give a remark that explains why A^p is defined when A is Hermitian and $p \ge 0$.

Remark 2.6.1. Let A be a positive semidefinite matrix and $p \ge 0$.
By Schur's decomposition theorem, there exists U unitary such that

$$U^*AU = \begin{pmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{pmatrix}.$$

Thus $\quad (U^*)^p A^p U^p = \begin{pmatrix} \lambda_1^p & & & \mathbf{0} \\ & \lambda_2^p & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^p \end{pmatrix} \Rightarrow A^p = V \begin{pmatrix} \lambda_1^p & & & \mathbf{0} \\ & \lambda_2^p & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n^p \end{pmatrix} V^*$

with $V = U^p$ unitary since U is unitary (λ_i^p exists since $\lambda_i \in \mathbb{R}$).

Chapter 3

Inequalities related to Heinz mean

During his work on the Heinz mean inequality and his search for a solution of Bourin's question in [1], Mohammad Alakhrass was able to prove an inequality that hods true for positive matrices A and B and for every unitarily invariant norm when $t \in [0,1]$,

$$\left\|A^t B^{1-t} + B^t A^{1-t}\right\| \leq 2^{\left|\frac{1}{2}-t\right|} \|A+B\|.$$

Although it is a weaker inequality than the one he desired to prove (inequality 1.4) but it is more general than previous results. In this chapter we present his proof along with some related inequalities.

3.1 Basic definitions and theorems

Notation 1. If X and Y are Hermitian matrices, then we write $X \leq Y$ if $Y - X \geq 0$.

Theorem 3.1.1. (Weyl's Monoticity Theorem:)
Let X and Y be two Hermitian matrices in $M_n(\mathbb{C})$.

$$\text{If} \quad X \leq Y \quad \text{then} \quad \lambda_j(X) \leq \lambda_j(Y) \quad \text{for } j = 1, 2, \ldots, n.$$

Proof. It is proven in [6] Theorem III.2.1, that for A and B Hermitian matrices in $M_n(\mathbb{C})$ we have:

$$\lambda_j^{\downarrow}(A+B) \leq \lambda_i^{\downarrow}(A) + \lambda_{j-i+1}^{\downarrow}(B) \quad \text{for} \quad i \leq j$$

and

$$\lambda_j^{\downarrow}(A+B) \geq \lambda_i^{\downarrow}(A) + \lambda_{j-i+n}^{\downarrow}(B) \quad \text{for} \quad i \geq j.$$

Here $\lambda_1^{\downarrow}(A), \ldots, \lambda_n^{\downarrow}(A)$ are the eigenvalues of A rearranged in a decreasing order.
By putting $i = j$ we get:

$$\lambda_j^{\downarrow}(A) + \lambda_n^{\downarrow}(B) \leq \lambda_j^{\downarrow}(A+B) \leq \lambda_j^{\downarrow}(A) + \lambda_1^{\downarrow}(B). \tag{3.1}$$

If H is a positive matrix then by (3.1) we have: $\lambda_j^{\downarrow}(A+H) \geq \lambda_j^{\downarrow}(A) + \lambda_n^{\downarrow}(H)$. But all the eigenvalues of H are nonnegative, therefore $\lambda_j^{\downarrow}(A+H) \geq \lambda_j^{\downarrow}(A)$.

□

Proposition 3.1.1. The inequalities in Theorem 3.1.1 are equivalent to the following:

$$\text{There exists a unitary matrix U such that } X \leq U^{\star} Y U. \tag{3.2}$$

Proof. We will start by proving the necessary condition.
Let X and Y be Hermitian matrices such that $X \leq Y$ so we have:

$$\lambda_j(X) \leq \lambda_j(Y) \Rightarrow \begin{pmatrix} \lambda_1(X) & & & \mathbf{0} \\ & \lambda_2(X) & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n(X) \end{pmatrix} \leq \begin{pmatrix} \lambda_1(Y) & & & \mathbf{0} \\ & \lambda_2(Y) & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n(Y) \end{pmatrix}.$$

Thus there exists U and V unitary matrices such that

$$\begin{aligned} & U^\star XU \leq V^\star YV \\ \Rightarrow \quad & X \leq UV^\star YVU^\star \\ \Rightarrow \quad & X \leq W^\star YW \qquad \text{with } W = VU^\star \text{ and } W \text{ unitary.} \end{aligned}$$

To prove the sufficient condition, let's suppose that there exists a unitary matrix U such that such that

$$X \leq U^\star YU.$$

By Weyl's Monotonicity theorem we have:

$$\lambda_j(X) \leq \lambda_j(U^\star YU) \quad \text{for all } j.$$

But $\lambda_j(U^\star YU) = \lambda_j(YU^\star U) = \lambda_j(Y)$ by Proposition 2.1.1.
Therefore $\lambda_j(X) \leq \lambda_j(Y) \quad \forall j = 1, 2, \ldots, n$.

□

Definition 3.1.1.
Let $x = (x_1, \ldots, x_2)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ with $x_1 \geq \cdots \geq x_n \geq 0$ and $y_1 \geq \cdots \geq y_n \geq 0$.

- The vector x is said to be weakly majorized by y, denoted by $x \prec_w y$, if and only if

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j \quad k = 1, 2, \ldots, n.$$

- The vector x is said to be weakly log-majorized by y, denoted by $x \prec_{wlog} y$, if and only if

$$\prod_{j=1}^{k} x_j \leq \prod_{j=1}^{k} y_j \quad k = 1, 2, \ldots, n.$$

- The vector x is said to be log-magorized by y, denoted $x \prec_{log} y$, if and only if,

$$x \prec_{wlog} y \quad \text{and} \quad \prod_{j=1}^{n} x_j = \prod_{j=1}^{n} y_j.$$

The weak log-majorization implies the weak majorization. (See [16] Proposition 1.3)

Theorem 3.1.2. For any $X, Y \in M_n$ we have,

$$s^r(XY) \prec_w s^r(X)s^r(Y) \quad \forall\, r > 0$$

Proof. See [6] page 94.

□

Theorem 3.1.3 (Fan dominance theorem).
For every unitarily invariant norm,

$$s(X) \prec_w s(Y) \quad \text{if and only if} \quad \|X\| \leq \|Y\| \tag{3.3}$$

We refer the reader for the Bhatia's book [6] page 93 for more details for the theory of this topic.

Definition 3.1.2.

- A real-valued function $f(t)$ defined on a real interval Ω is said to be operator monotone if

$$A \leq B \;\Rightarrow\; f(A) \leq f(B)$$

for all Hermitian matrices A, B of all orders whose eigenvalues are contained in Ω.

- A function f is called operator concave if for all Hermitian matrices X and Y,

$$\frac{f(X)+f(Y)}{2} \leq f\left(\frac{X+Y}{2}\right).$$

- A function f is called operator convex if for all Hermitian matrices X and Y,

$$f\left(\frac{X+Y}{2}\right) \leq \frac{f(X)+f(Y)}{2}.$$

Theorem 3.1.4 (Löwner-Heinz inequality).
If $A \geq B \geq 0$ and $0 \leq r \leq 1$ then

$$A^r \geq B^r.$$

Proof. See [29] Theorem 1.1 .

□

Proposition 3.1.2. Let f be a nonnegative continuous function on $[0, \infty)$, we have

f is operator monotone if and only if it is operator concave.

Proof. See Theorem $V.2.5$ in [2].

□

Theorem 3.1.5. If $f : [0, \infty) \to [0, \infty)$ is operator concave, then for any unitarily invariant norm,

$$\|f(X+Y)\| \leq \|f(X)+f(Y)\|. \tag{3.4}$$

By Theorem 3.1.3, this is equivalent to

$$s(f(X+Y)) \prec_w s(f(X)+f(Y)). \tag{3.5}$$

For the proof we will need the following lemma.

Lemma 3.1.6. Let $A, B \geq 0$, and let u_j be the orthonormal eigenvectors of $A+B$ corresponding to $\lambda_j(A+B)$ with $j = 1, 2, \ldots, n$. Then we have:

$$\sum_{j=1}^{k} < \{A(A+I)^{-1} + B(B+I)^{-1}\}u_j, u_j > \geq \sum_{j=1}^{k} < (A+B)(A+B+I)^{-1}u_j, u_{j>}.$$

The proof of the Lemma can be found in [4].

Proof. By Theorem 1.3 in [29] we have that if f is an non-negative operator monotone function on $[0, \infty)$, then there exists μ positive measure on $[0, \infty)$ such that

$$f(t) = \alpha + \beta t + \int_0^\infty \frac{st}{s+t} d\mu(s) \qquad \text{for } \alpha \in \mathbb{R} \text{ and } \beta \geq 0$$

Thus for any $A \geq 0$ and any vector u,

$$\begin{aligned}
< f(A)u, u > &= \alpha < u, u > + \beta < Au, u > + \int_0^\infty s < A(A+sI)^{-1}u, u > d\mu(s) \\
< f(A)u, u > &\leq \int_0^\infty s < A(A+sI)^{-1}u, u > d\mu(s) \\
\sum_{j=1}^{k} < f(A+B)u_j, u_j > &\leq \sum_{j=1}^{k} \int_0^\infty s < (A+B)(A+B+sI)^{-1}u_j, u_j > d\mu(s) \\
&\leq \sum_{j=1}^{k} < \{\int_0^\infty sA(A+sI)^{-1}d\mu(s) + \int_0^\infty sB(B+sI)^{-1}d\mu(s)\}u_j, u_j > \text{ (By Lemma 3.1.6)} \\
&\leq \sum_{j=1}^{k} < (f(A)+f(B))u_j, u_j > .
\end{aligned} \tag{3.6}$$

Since $f(t)$ is non-decreasing, for $j = 1, 2, \ldots, n$, a unit eigenvector u_j of A corresponding to $\lambda_j(A)$ becomes a unit eigenvector of $f(A)$ corresponding to $\lambda_j(f(A)) = f(\lambda_j(A))$, so that by the definition of Ky Fan norms

$$\begin{aligned}
\|f(A)\|_{(k)} &= \sum_{i=1}^{k} s_j(f(A)) = \sum_{i=1}^{k} \lambda_j^{\frac{1}{2}}\left((f(A))^{\star}.f(A)\right) \\
&= \sum_{i=1}^{k} \lambda_j^{\frac{1}{2}}(f(A))^2 \quad \text{since } f(A) = (f(A))^{\star} \\
&= \sum_{i=1}^{k} \lambda_j(f(A)) \\
&= \sum_{j=1}^{k} < f(A)u_j, u_j > \qquad k = 1, 2, \ldots, n.
\end{aligned} \tag{3.7}$$

By applying (3.7) to $A + B$ in place of A we have

$$\|f(A+B)\|_{(k)} = \sum_{j=1}^{k} < f(A+B)u_j, u_j > .$$

On the other hand, the Ky Fan Maximum principle says that if $A \in M_n$ is Hermitian then for $1 \leq k \leq n$

$$\sum_{j=1}^{k} \lambda_j(A) = \max \sum_{j=1}^{k} < Ax_j, x_j > \tag{3.8}$$

where the maximum is taken over all orthonormal k-tuples $(x_1, ..., x_k) \in \mathbb{C}^n$.
Thus

$$\|f(A) + f(B)\|_{(k)} = \sum_{j=1}^{k} s_j(f(A) + f(B)) \geq \sum_{j=1}^{k} \lambda_j(f(A) + f(B)).$$

Because

$$\sum_{j=1}^{k} |\lambda_j|^p \leq \sum_{j=1}^{k} s_j^p \quad \forall\, p \geq 0. \tag{3.9}$$

On the other hand

$$\begin{aligned}
\sum_{j=1}^{k} \lambda_j(f(A) + f(B)) &\geq \sum_{j=1}^{k} < (f(A) + f(B))x_j, x_j > \quad \text{by (3.8)} \\
&\geq \sum_{j=1}^{k} < f(A+B)x_j, x_j > \quad \text{by (3.6)} \\
&= \|f(A+B)\|_{(k)} .
\end{aligned}$$

Thus $\|f(A+B)\|_{(k)} \leq \|f(A) + f(B)\|_{(k)}$ and by the Fan Dominance principle (Theorem 3.1.3), we obtain $\|f(A+B)\| \leq \|f(A) + f(B)\|$ for any unitarily invariant norm. □

3.2 Inequalities related to Heinz mean

We start this section by three lemmas.

Lemma 3.2.1. Let A, B be Hermitian in M_n and let $r \geq 0$.

If $X = \begin{pmatrix} A^r & B^r \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} A^r & 0 \\ B^r & 0 \end{pmatrix}$ then $s_j(X) = s_j(Y) = \lambda_j^{1/2}(A^{2r} + B^{2r})$ for $j = 1, 2, \ldots, n$.

Proof. Since $s_j(X) = \lambda_j^{1/2}(X^\star X) = \lambda_j^{1/2}(XX^\star)$ we have:

$$\begin{aligned} s_j(X) &= \lambda_j^{1/2}\begin{pmatrix} A^r & B^r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (A^r)^\star & 0 \\ (B^r)^\star & 0 \end{pmatrix} \\ &= \lambda_j^{1/2}\begin{pmatrix} A^r & B^r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^r & 0 \\ B^r & 0 \end{pmatrix} \\ &= \lambda_j^{1/2}\begin{pmatrix} A^{2r}+B^{2r} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \lambda_j^{1/2}(A^{2r}+B^{2r}). \end{aligned}$$

For $1 \le j \le n$ with the convention $s_j(X) = 0$ for $j > n$. Similar computation implies $s_j(Y) = (A^{2r} + B^{2r})$. □

This can be extended for normal matrices, and we'll have $s_j(X) = s_j(Y) = \lambda_j^{1/2}(|A|^{2r} + |B|^{2r})$ for $j = 1, 2, \ldots, n$. (See [1]).

Lemma 3.2.2. If $X \ge 0$, $Y \ge 0$ in M_n and $r \in [0, 1]$ then

$$\lambda^r(X+Y) \prec_w \lambda(X^r+Y^r).$$

Equivalently,

$$\sum_{j=1}^{k}\lambda_j^r(X+Y) \le \sum_{j=1}^{k}\lambda_j(X^r+Y^r). \tag{3.10}$$

Proof. Note that $f(t) = t^r$ is an operator monotone for $0 \le r \le 1$ by Theorem 3.1.4.

Thus by applying (3.4) we get, for $0 \le r \le 1$,

$$\begin{aligned} \|(X+Y)^r\| \le \|X^r+Y^r\| &\Leftrightarrow s((X+Y)^r) \prec_w s(X^r+Y^r) \quad \text{(by Theorem 3.1.3)} \\ &\Leftrightarrow \sum_{j=1}^{k} s_j((X+Y)^r) \le \sum_{j=1}^{k} s_j(X^r+Y^r). \end{aligned} \tag{3.11}$$

Thus by using inequality (3.9) we get:

$$\begin{aligned} \sum_{j=1}^{k}|\lambda_j((X+Y)^r)| &\le \sum_{j=1}^{k} s_j((X+Y)^r) \qquad (p=1) \\ \sum_{j=1}^{k}\lambda_j^r(X+Y) &\le \sum_{j=1}^{k} s_j((X+Y)^r) \\ \sum_{j=1}^{k}\lambda_j^r(X+Y) &\le \sum_{j=1}^{k} s_j(X^r+Y^r) \quad \text{(by 3.11)} \end{aligned}$$

The lemma follows by noticing that

$$\sum_{j=1}^{k} s_j(X^r+Y^r) = \sum_{j=1}^{k}\lambda_j(X^r+Y^r) \quad \text{as } X^r+Y^r \text{ is Hermitian.}$$

□

Now we use the definition of operator concave to prove the following result.

Lemma 3.2.3. If $X \ge 0$, $Y \ge 0$ are two matrices in M_n and $r \ge 1$, then for $1 \le j \le n$ we have:

$$\lambda_j^r(X+Y) \le 2^{r-1}\lambda_j(X^r+Y^r) \tag{3.12}$$

Hence,

$$\lambda^r(X+Y) \prec_w 2^{r-1}\lambda(X^r+Y^r) \tag{3.13}$$

Proof. Consider the function $f(t) = t^r$ which is an operator monotone for $0 \le r \le 1$ by Theorem 3.1.4. So $g(x) = x^{1/r}$ is an operator monotone since $\frac{1}{r} \le 1$. Thus $g(x)$ is an operator concave by (Proposition 3.1.2).

So we get:

$$\frac{X+Y}{2} = \frac{g(X^r)+g(Y^r)}{2} \le g\left(\frac{X^r+Y^r}{2}\right) = \left(\frac{X^r+Y^r}{2}\right)^{1/r}.$$

Hence, by Weyl's Monoticity Theorem and for all $j = 1, 2, \ldots, n$ we have,

$$\begin{aligned}
\lambda_j\left(\frac{X+Y}{2}\right) \le \lambda_j\left(\frac{X^r+Y^r}{2}\right)^{1/r} \quad &\Rightarrow \frac{1}{2}\lambda_j(X+Y) \le \lambda_j^{1/r}\left(\frac{X^r+Y^r}{2}\right) \\
&\Rightarrow \frac{1}{2^r}\lambda_j^r(X+Y) \le \frac{1}{2}\lambda_j(X^r+Y^r) \\
&\Rightarrow \lambda_j^r(X+Y) \le 2^{r-1}\lambda_j(X^r+Y^r) \\
&\Rightarrow \lambda^r(X+Y) \prec_w 2^{r-1}\lambda(X^r+Y^r).
\end{aligned}$$

□

We notice that combining (3.12) and (3.2) implies the following.

Proposition 3.2.1. If $X \ge 0$, $Y \ge 0$ are two matrices in M_n and $r \ge 1$, then there exists a unitary matrix U such that

$$U^\star(X+Y)^rU \le 2^{r-1}(X^r+Y^r).$$

Proof. By the above proof we have,

$$\lambda_j\left(\frac{X+Y}{2}\right)^r \le \lambda_j\left(\frac{X^r+Y^r}{2}\right).$$

Therefore, by (3.2) there exists V unitary such that

$$\left(\frac{X+Y}{2}\right)^r \le V^\star\left(\frac{X^r+Y^r}{2}\right)V$$

$$V(X+Y)^rV^\star \le 2^{r-1}(X^r+Y^r).$$

Let $U = V^\star$ then $U^\star(X+Y)^rU \le 2^{r-1}(X^r+Y^r)$ and U is unitary.

□

Theorem 3.2.4. Let $A \ge 0$, $B \ge 0$ in $M_n(\mathbb{C})$ and $t \in [0,1]$. Then:

$$s(A^tB^{1-t}+B^tA^{1-t}) \prec_w 2^{|\frac{1}{2}-t|}s(A+B). \tag{3.14}$$

Proof. Since the result is obvious when $t = 0, 1$. We only need to prove it for $0 < t < 1$.

Let's first assume $0 < t \le \frac{1}{2}$. Let $X = \begin{pmatrix} A^t & B^t \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} A^{1-t} & 0 \\ B^{1-t} & 0 \end{pmatrix}$.

Using Lemma 3.2.1 and Theorem 3.1.2 we get:

$$\begin{aligned}
s(A^tB^{1-t}+B^tA^{1-t}) &= s(XY) \\
&\prec_w s(X)s(Y) \\
&= \lambda^{1/2}(A^{2t}+B^{2t})\lambda^{1/2}(A^{2(1-t)}+B^{2(1-t)}).
\end{aligned}$$

Then, for $1 \leq k \leq n$, we have

$$\sum_{j=1}^{k} s_j(A^tB^{1-t} + B^tA^{1-t}) \leq \sum_{j=1}^{k} \lambda_j^{1/2}(A^{2t} + B^{2t})\lambda_j^{1/2}(A^{2(1-t)} + B^{2(1-t)}). \tag{3.15}$$

Let $p = \dfrac{1}{t}$ and $q = \dfrac{1}{1-t}$. Then $1 \leq q \leq 2 \leq p$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.
Using Hölder inequality (Theorem 2.1.2) for the above p and q, Inequality (3.15) implies that

$$\sum_{j=1}^{k} s_j(A^tB^{1-t} + B^tA^{1-t}) \leq \left(\sum_{j=1}^{k} \lambda_j^{p/2}(A^{2t} + B^{2t})\right)^{1/p} \left(\sum_{j=1}^{k} \lambda_j^{q/2}(A^{2(1-t)} + B^{2(1-t)})\right)^{1/q}. \tag{3.16}$$

Note that $\dfrac{q}{2} \leq 1$. So by Lemma 3.2.2, we get

$$\sum_{j=1}^{k} \lambda_j^{q/2}(A^{2(1-t)} + B^{2(1-t)}) \leq \sum_{j=1}^{k} \lambda_j(A^{q(1-t)} + B^{q(1-t)}).$$

This implies

$$\begin{aligned}\left(\sum_{j=1}^{k} \lambda_j^{q/2}(A^{2(1-t)} + B^{2(1-t)})\right)^{1/q} &\leq \left(\sum_{j=1}^{k} \lambda_j(A^{q(1-t)} + B^{q(1-t)})\right)^{1/q} \\ &= \left(\sum_{j=1}^{k} \lambda_j(A+B)\right)^{1-t}.\end{aligned} \tag{3.17}$$

On the other hand, $\dfrac{p}{2} \geq 1$. So Lemma 3.2.3 implies that

$$\sum_{j=1}^{k} \lambda_j^{p/2}(A^{2t} + B^{2t}) \leq \frac{2^{p/2}}{2} \sum_{j=1}^{k} \lambda_j(A^{pt} + B^{pt}).$$

Then

$$\begin{aligned}\left(\sum_{j=1}^{k} \lambda_j^{p/2}(A^{2t} + B^{2t})\right)^{1/p} &\leq 2^{\frac{1}{2}-\frac{1}{p}} \left(\sum_{j=1}^{k} \lambda_j(A^{pt} + B^{pt})\right)^{1/p} \\ &= 2^{\frac{1}{2}-t} \left(\sum_{j=1}^{k} \lambda_j(A^{pt} + B^{pt})\right)^{t}.\end{aligned} \tag{3.18}$$

The inequalities (3.16), (3.17) and (3.18) lead to

$$\begin{aligned}\sum_{j=1}^{k} s_j(A^tB^{1-t} + B^tA^{1-t}) &\leq 2^{\frac{1}{2}-t} \left(\sum_{j=1}^{k} \lambda_j(A+B)\right)^{t} \left(\sum_{j=1}^{k} \lambda_j(A+B)\right)^{1-t} \\ &= 2^{\frac{1}{2}-t} \left(\sum_{j=1}^{k} \lambda_j(A+B)\right) \\ &= 2^{\frac{1}{2}-t} \left(\sum_{j=1}^{k} s_j(A+B)\right).\end{aligned}$$

Thus for $0 \leq t \leq \frac{1}{2}$, we have

$$s(A^tB^{1-t} + B^tA^{1-t}) \prec_w 2^{\frac{1}{2}-t} s(A+B). \tag{3.19}$$

Now we suppose $\dfrac{1}{2} \leq t \leq 1$. If we replace t by $1-t$ and interchange A and B in (3.19) we get

$$s(B^{1-t}A^t + A^{1-t}B^t) \prec_w 2^{t-\frac{1}{2}} s(A+B).$$

Then for $\frac{1}{2} \leq t \leq 1$, we have

$$\begin{aligned} s(A^tB^{1-t} + B^tA^{1-t}) &= s((A^tB^{1-t} + B^tA^{1-t})^\star) \\ &= s(B^{1-t}A^t + A^{1-t}B^t) \\ &\prec_w 2^{t-\frac{1}{2}} s(A+B). \end{aligned} \tag{3.20}$$

The result follows by combining (3.19) and (3.20).

□

Corollary 3.2.4.1. Suppose $A \geq 0$, $B \geq 0$ two matrices in M_n and $t \in [0,1]$. Then by the Fan dominance Theorem we have, for any unitarily invariant norm,

$$\left\| A^tB^{1-t} + B^tA^{1-t} \right\| \leq 2^{|\frac{1}{2}-t|} \left\| A+B \right\|.$$

In the following theorem, Alakhrass extended his previous result to Hermitian and normal matrices.

Theorem 3.2.5. Let A and B be Hermitian in M_n. Let $p, q \geq 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$s(A^pB^q + B^pA^q) \prec_w 2^{|\frac{1}{2}-\frac{1}{p}|} s(A^{p+q} + B^{p+q}).$$

Proof. Without loss of generality, we may assume that $p \leq q$. So, $1 \leq p \leq 2 \leq q$.

Let $X = \begin{pmatrix} A^p & B^p \\ 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} A^q & 0 \\ B^q & 0 \end{pmatrix}$. Using Lemma 4.2.1 and Theorem 3.1.2 we get

$$\begin{aligned} s(A^pB^q + B^pA^q) &= s(XY) \\ &\prec_w s(X)s(Y) \\ &= \lambda^{1/2}(A^{2p} + B^{2p})\lambda^{1/2}(A^{2q} + B^{2q}) \\ &= s^{1/2}(A^{2p} + B^{2p})s^{1/2}(A^{2q} + B^{2q}). \end{aligned}$$

since $A^{2p} + B^{2p}$ and $A^{2q} + B^{2q}$ are Hermitian matrices.
Therefore

$$s(A^pB^q + B^pA^q) \prec_w s^{1/2}(A^{2p} + B^{2p})s^{1/2}(A^{2q} + B^{2q}).$$

This implies that, for $0 \leq k \leq n$,

$$\sum_{j=1}^{k} s_j(A^pB^q + B^pA^q) \leq \sum_{j=1}^{k} s_j^{1/2}(A^{2p} + B^{2p})s_j^{1/2}(A^{2q} + B^{2q}).$$

Hence by Hölder inequality (Theorem 2.1.2) we obtain,

$$\sum_{j=1}^{k} s_j(A^pB^q + B^pA^q) \leq \left(\sum_{j=1}^{k} s_j^{q/2}(A^{2p} + B^{2p}) \right)^{1/q} \left(\sum_{j=1}^{k} s_j^{p/2}(A^{2q} + B^{2q}) \right)^{1/p}.$$

Since $\frac{p}{2} \leq 1$, Lemma 3.2.2 implies,

$$\sum_{j=1}^{k} s_j^{p/2}(A^{2q} + B^{2q}) \leq \sum_{j=1}^{k} s_j(A^{pq} + B^{pq}).$$

Also, because $\frac{q}{2} \geq 1$, Lemma 3.2.3 implies that,

$$\sum_{j=1}^{k} s_j^{q/2}(A^{2p} + B^{2p}) \leq \frac{2^{q/2}}{2} \sum_{j=1}^{k} s_j(A^{pq} + B^{pq}).$$

So, for $0 \le k \le n$ we have,

$$\begin{aligned}\sum_{j=1}^{k} s_j(A^pB^q + B^pA^q) &\le \left(\sum_{j=1}^{k} s_j^{q/2}(A^{2p} + B^{2p})\right)^{1/q} \left(\sum_{j=1}^{k} s_j^{p/2}(A^{2q} + B^{2q})\right)^{1/p} \\ &\le \frac{2^{1/2}}{2^{1/q}} \left(\sum_{j=1}^{k} s_j(A^{pq} + B^{pq})\right)^{1/q} \left(\sum_{j=1}^{k} s_j(A^{pq} + B^{pq})\right)^{1/p} \\ &\le 2^{\frac{1}{2}-\frac{1}{q}} \sum_{j=1}^{k} s_j(A^{p+q} + B^{p+q}).\end{aligned}$$

Hence

$$s(A^pB^q + B^pA^q) \prec_w 2^{\left|\frac{1}{2}-\frac{1}{p}\right|} s(A^{p+q} + B^{p+q}).$$

$\square$

Corollary 3.2.5.1. Let A and B be Hermitian in M_n. Let $p, q \ge 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then by the Fan dominance theorem, we have

$$\|A^pB^q + B^pA^q\| \le 2^{\left|\frac{1}{2}-\frac{1}{p}\right|} \left\|A^{p+q} + B^{p+q}\right\|$$

for any unitarily invariant norm.

These results can be extended for normal matrices as follows, (See [1])

$$\|A^pB^q + B^pA^q\| \le 2^{\left|\frac{1}{2}-\frac{1}{p}\right|} \left\| \, |A|^{p+q} + |B|^{p+q} \right\|.$$

By the end of this chapter, Bourin's questions could not be completely solved yet but, by using three important lemmas, Mohammad Alakhrass was able to prove two weaker inequalities that holds true for positive semidefinite matrices and for Hermitian matrices. However, the search for a solution is not over yet, therefore in the next chapter, Bourin's question will be answered affirmatively for the trace norm and two related inequalities will be proved.

Chapter 4

Norm inequalities for positive semidefinite matrices

In one of their latest work (in [15]) , Hayajneh and Kittaneh proved that for positive semidefinite matrices A and B and $X \in M_n(\mathbb{C})$ we have

$$\left\|A^t X B^{1-t} + B^t X^\star A^{1-t}\right\| \leq \|AX\| + \|XB\|$$

where $t \in [0,1]$ and $\|.\|$ is any unitarily invariant norm.
This gives an affirmative answer to Bourin's question regarding subadditivity inequalities in the case of the trace norm. New norm inequalities related to Bourin's question are also presented.

4.1 Basics

Before presenting the main results we will give a small definition and a proposition about the symmetric norm. This will be helpful for the following theorems.

Definition 4.1.1. A norm σ is called a symmetric norm on M_n if for A, B and $C \in M_n(\mathbb{C})$

$$\sigma(ABC) \leq \|A\|\, \sigma(B)\, \|C\|$$

where $\|.\|$ is the operator norm.

Proposition 4.1.1. For any matrices A, B in $M_n(\mathbb{C})$ and for all $j > 0$, we have

$$\begin{aligned} s_j(AB) &\leq \|B\|\, s_j(A) \\ s_j(AB) &\leq \|A\|\, s_j(B), \end{aligned}$$

where $\|.\|$ is the operator norm.

Proof.

$$\begin{aligned} s_j(AB) &= \lambda_j^{1/2}((AB)^\star(AB)) = \lambda_j^{1/2}(B^\star A^\star AB) && (4.1) \\ &= \lambda_j^{1/2}((AB)(AB)^\star) = \lambda_j^{1/2}(ABB^\star A^\star). && (4.2) \end{aligned}$$

For any vector v we have

$$\begin{aligned} (Av, Av) = v^\star A^\star Av &= \|Av\|^2 \\ &\leq \|A\|^2\, \|v\|^2 \\ &\leq \|A\|^2\, v^\star v. \end{aligned}$$

Apply that to $v = Bw$, we get

$$\begin{aligned} w^\star B^\star A^\star ABw &\leq \|A\|^2\ w^\star B^\star Bw \\ \langle B^\star A^\star ABw, w\rangle &\leq \|A\|^2 \langle B^\star Bw, w\rangle\,. \end{aligned}$$

By the Min-max Theorem 2.4.2 and by using (4.1) we get

$$\lambda(B^\star A^\star AB) \leq \|A\|^2 \lambda(B^\star B) \Rightarrow s_j(AB) \leq \|A\| s_j(B).$$

Similarly for any vector v we have

$$\begin{aligned}(B^\star v, B^\star v) = v^\star BB^\star v &= \|B^\star v\|^2 \\ &\leq \|B^\star\|^2 \|v\|^2 \\ &\leq \|B\|^2 v^\star v \quad \text{(By Proposition 2.5.1)}.\end{aligned}$$

Apply that to $v = A^\star w$, we get

$$\begin{aligned} w^\star ABB^\star A^\star w &\leq \|B\|^2 \; w^\star AA^\star w \\ \langle ABB^\star A^\star w, w\rangle &\leq \|B\|^2 \langle AA^\star w, w\rangle .\end{aligned}$$

By the Min-max Theorem 2.4.2 and by using (4.2) we get:

$$\lambda(ABB^\star A^\star) \leq \|B\|^2 \lambda(A^\star A) \Rightarrow s_j(AB) \leq \|B\| s_j(A).$$

□

Proposition 4.1.2. A norm on M_n is symmetric if and only if it is unitarily invariant.

Proof. Starting by the necessary condition, if σ is a symmetric norm then for U, V unitary matrices we have:

$$\sigma(UAV) \leq \sigma(A)$$

On the other hand, $\sigma(A) = \sigma(U^{-1}UAVV^{-1}) \leq \sigma(UAV)$. Thus $\sigma(UAV) = \sigma(A)$ therefore σ is a unitarily invariant norm.
Conversely, let σ be a unitarily invariant norm. By the previous proposition we have,

$$\begin{aligned} s_j(ABC) = s_j(ACB) &\leq \|AC\| s_j(B) \\ &\leq \|A\| \; \|C\| \; s_j(B).\end{aligned}$$

Since σ is a unitarily invariant norm, by the Fan dominance Theorem, this is equivalent to

$$\sigma(ABC) = \|A\| \; \|C\| \; \sigma(B).$$

Therefore σ is a symmetric norm.

□

4.2 Bourin's question for the trace norm

We start with the following lemma that has been proved by Kittaneh in [21].

Lemma 4.2.1. Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite. Then for $t \in [0, 1]$ and for every unitarily invariant norm, we have

$$\left\|A^t X B^{1-t}\right\| \leq \|AX\|^t \|XB\|^{1-t} .$$

Proof. For each $\varepsilon > 0$, let $B_\varepsilon = B + \varepsilon I$ then B_ε is invertible for every $\varepsilon > 0$.
By Theorem 1 in [21] , if A and B are positive matrices in $M_n(\mathbb{C})$, and if X an arbitrary matrix in $M_n(\mathbb{C})$, then for $0 \leq r \leq 1$ we have

$$\|A^r X B^r\| \leq \|X\|^{1-r} \|AXB\|^r .$$

If we apply this inequality to B_ε we get

$$\begin{aligned}\left\|A^t X B_\varepsilon^{1-t}\right\| = \left\|A^t (XB_\varepsilon) B_\varepsilon^{-t}\right\| &\leq \|XB_\varepsilon\|^{1-t} \left\|A(XB_\varepsilon)B_\varepsilon^{-1}\right\|^t \\ &= \|XB_\varepsilon\|^{1-t} \|AX\|^t .\end{aligned}$$

By letting $\varepsilon \to 0$ and by using the continuity argument we get

$$\left\|A^t X B^{1-t}\right\| \leq \|AX\|^t \|XB\|^{1-t} .$$

□

Theorem 4.2.2. Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite. Then for $t \in [0,1]$ and for every unitarily invariant norm we have

$$\left\|A^t X B^{1-t} + B^t X^\star A^{1-t}\right\| \leq \|AX\| + \|XB\|. \tag{4.3}$$

Proof. We have

$$\begin{aligned}
\left\|A^t X B^{1-t} + B^t X^\star A^{1-t}\right\| &\leq \left\|A^t X B^{1-t}\right\| + \left\|B^t X^\star A^{1-t}\right\| \qquad \text{(by the triangle inequality)} \\
&\leq \|AX\|^t \|XB\|^{1-t} + \|BX^\star\|^t \|X^\star A\|^{1-t} \qquad \text{(by Lemma 4.2.1)} \\
&= \|AX\|^t \|XB\|^{1-t} + \|XB\|^t \|AX\|^{1-t} \qquad (\text{since } \|Z\| = \|Z^\star\| \ \ \forall\ Z) \\
&\leq \|AX\| + \|XB\| \quad \text{(by the arithmetic-geometric mean inequality (2.5))}.
\end{aligned}$$

□

Now using this theorem, we have the following corollary.

Corollary 4.2.2.1. Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive semidefinite. Then for $t \in [0,1]$ we have :

$$\left\|A^t X B^{1-t} + B^t X^\star A^{1-t}\right\|_1 \leq \|A+B\|_1 \|X\|$$

where $\|.\|_1$ is the trace norm.

Proof. Since the trace norm is unitarily invariant and therefore symmetric by Proposition 4.1.2, we have $\|AX\|_1 \leq \|A\|_1 \|X\|$ and the same for B. Thus we get

$$\begin{aligned}
\left\|A^t X B^{1-t} + B^t X^\star A^{1-t}\right\|_1 &\leq \|AX\|_1 + \|XB\|_1 \quad \text{(by Theorem 4.2.2)} \\
&\leq \|A\|_1 \|X\| + \|X\| \ \|B\|_1 \\
&= (\|A\|_1 + \|B\|_1) \|X\| \\
&= tr(A+B) \|X\| \\
&= \|A+B\|_1 \|X\|.
\end{aligned}$$

The last equality is true because $A+B$ is p.s.d and if A is p.s.d we have $\|A\|_1 = tr\sqrt{A^\star A} = tr\sqrt{A^2} = tr(A)$. □

Letting $X = I$ in this previous corollary, we have the following result which answers Bourin's question affirmatively for the trace norm.

Corollary 4.2.2.2. Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite. Then for $t \in [0,1]$ we have

$$\left\|A^t B^{1-t} + B^t A^{1-t}\right\|_1 \leq \|A+B\|_1.$$

Now using theorem 4.2.2 again, we have the following remark.

Remark 4.2.1. It follows from Theorem 4.2.2 that for every unitarily invariant norm,

$$\left\|A^t B^{1-t} + B^t A^{1-t}\right\| \leq \|A\| + \|B\|,$$

where $A, B \in M_n(\mathbb{C})$ are positive semidefinite and $t \in [0,1]$.

4.3 Norm Inequalities Related to the Question of Bourin

We start with the following lemma which has been proved by Kittaneh in [22]

Lemma 4.3.1. Let $X, Y \in M_n(\mathbb{C})$ be positive semidefinite, and let $\|.\|$ be the operator norm. Then

$$\|XY - YX\| \leq \frac{1}{2} \|X\| \ \|Y\|.$$

A different version of Heinz inequality (1.3) can be stated as follows.

Lemma 4.3.2. Let $X, Y \in M_n(\mathbb{C})$ be positive semidefinite. Then for $t \in [0,1]$ and for every unitarily invariant norm, we have

$$\left\| A^t B^{1-t} - A^{1-t} B^t \right\| \leq |2t-1| \ \|A-B\| .$$

Proof. See [7]. □

We introduce some notations regarding the following results which are related to Bourin's question (Inequality 1.4). Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite and let $b_t = A^t B^{1-t} + B^t A^{1-t}$ and $h_t = A^t B^{1-t} + A^{1-t} B^t$ for $t \in [0,1]$

Theorem 4.3.3. Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite. Then for $t \in [0,1]$ and for every unitarily invariant norm, we have

$$\|Re\ b_t\| \leq \|A+B\| .$$

Proof. By the Cartesian decomposition (Definition 2.2.2) we have,

$$\begin{aligned}
\|Re\ b_t\| &= \frac{1}{2}\left\| A^t B^{1-t} + B^t A^{1-t} + B^{1-t} A^t + A^{1-t} B^t \right\| \\
&= \frac{1}{2}\left\| (A^t B^{1-t} + A^{1-t} B^t) + (B^{1-t} A^t + B^t A^{1-t}) \right\| \\
&= \frac{1}{2}\left\| h_t + h_t^{\star} \right\| \\
&= \|Re\ h_t\| \\
&\leq \|h_t\| \\
&\leq \|A+B\| \quad \text{(by inequality (1.3))}.
\end{aligned}$$

□

Proposition 4.3.1. Let $X, Y \in M_n(\mathbb{C})$ be Hermitian matrices such that $\pm Y \leq X$. Then $\|Y\| \leq \|X\|$ for every unitarily invariant norm.

Proof. Choose an orthonormal basis $e_1, e_2, \ldots, e_n$ such that $Ye_j = \mu_j e_j$ where $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n|$. Then for $1 \leq k \leq n$ we have,

$$\sum_{j=1}^{k} s_j(Y) = \sum_{j=1}^{k} |\mu_j| = \sum_{j=1}^{k} |\left\langle e_j, \mu_j e_j \right\rangle| = \sum_{j=1}^{k} |\left\langle e_j, Y e_j \right\rangle| \leq \sum_{j=1}^{k} \left\langle e_j, X e_j \right\rangle .$$

By Ky Fan's maximum principle, the sum of the extreme right is bounded by $\sum_{j=1}^{k} s_j(X)$. So the assertion of the proposition follows from the Fan Dominance theorem. □

Theorem 4.3.4. Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite. Then for $t \in [0,1]$ and for every unitarily invariant norm, we have

$$\|Im\ b_t\| \leq \|A+B\| .$$

Proof. By the Cartesian decomposition we have,

$$\begin{aligned}
\|Im\ b_t\| &= \frac{1}{2}\left\| A^t B^{1-t} + B^t A^{1-t} - B^{1-t} A^t - A^{1-t} B^t \right\| \\
&= \frac{1}{2}\left\| A^t B^{1-t} - A^{1-t} B^t - (A^t B^{1-t} - A^{1-t} B^t)^{\star} \right\| \\
&\leq \left\| A^t B^{1-t} - A^{1-t} B^t \right\| \\
&\leq |2t-1| \ \|A-B\| \quad \text{(by Lemma 4.3.2)}.
\end{aligned}$$

Since $\pm(A-B) \leq (A+B)$ then by the previous proposition, $\|A-B\| \leq \|A+B\|$. And since $|2t-1| \leq 1$ for $t \in [0,1]$, it follows that $\|Im\ b_t\| \leq \|A+B\|$. □

In the case of the spectral norm, we have the following estimates for $\|Im\ b_t\|$.

Theorem 4.3.5. Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite. Then for $t \in [0,1]$, we have

$$\|Im\ b_t\| \le \frac{1}{4}(\|A\| + \|B\|).$$

Proof. By the Cartesian decomposition we have,

$$\begin{aligned}
\|Im\ b_t\| &= \frac{1}{2}\left\|A^tB^{1-t} + B^tA^{1-t} - B^{1-t}A^t - A^{1-t}B^t\right\| \\
&\le \frac{1}{2}\left\|A^tB^{1-t} - B^{1-t}A^t\right\| + \left\|B^tA^{1-t} - A^{1-t}B^t\right\| \\
&\le \frac{1}{2}\left(\frac{1}{2}\left\|A^t\right\|\ \left\|B^{1-t}\right\| + \frac{1}{2}\left\|B^t\right\|\ \left\|A^{1-t}\right\|\right) \quad \text{(by Lemma 4.3.1)} \\
&= \frac{1}{4}(\|A\|^t\,\|B\|^{1-t} + \|B\|^t\,\|A\|^{1-t}) \\
&\le \frac{1}{4}(\|A\| + \|B\|) \quad \text{(by inequality (2.5))}.
\end{aligned}$$

□

If $A, B \in M_n(\mathbb{C})$ are positive semidefinite, then $\max(\|A\|, \|B\|) \le \|A + B\|$. Note that this together with Theorem 4.3.5, yields the following result which is better than Theorem 4.3.4 for the spectral norm.

Corollary 4.3.5.1. Let $A, B \in M_n(\mathbb{C})$ be positive semidefinite. Then for $t \in [0,1]$, we have

$$\begin{aligned}
\|Im\ b_t\| &\le \frac{1}{4}(\|A\| + \|B\|) \\
&\le \frac{1}{4}(2\max(\|A\|, \|B\|)) \\
&\le \frac{1}{4}(2\,\|A + B\|) \\
&= \frac{1}{2}\,\|A + B\|.
\end{aligned}$$

The two inequalities proven in Theorems 4.3.4 and 4.3.3 gives a partial answer to Bourin's question by inserting the real and imaginary parts in the left handed side of inequality (1.4). However, after answering Bourin's question in the case of the trace norm, the question remains open for all other unitarily invariant norm. In the next chapter a different, yet important inequality will be discussed. It's the Cauchy-Schwarz inequality.

Chapter 5

Improved Cauchy-Schwarz norm operator and Trace Inequalities

Let A, B and X be operators on a complex separable Hilbert space such that A and B are positive.
The Cauchy-Schwarz norm inequality for operators asserts that

$$\left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 \leq \| \, |AX|^r \| \, . \, \| \, |XB|^r \|$$

for any real number $r > 0$ and every unitarily invariant norm $\|.\|$.
We show, throughout this chapter, one of the improvements of the results of Hiai and Zhan [Linear Algebra Appl. 341 (2002) 151-169]. Besides, new type inequalities close to Cauchy-Schwarz norm inequality will be introduced.

5.1 Introduction

It has been shown by Horn and Mathias [19] that if $A, B \in M_n(\mathbb{C})$ then

$$\| \, |A^\star B|^r \|^2 \leq \|(AA^\star)^r\| \, . \, \|(BB^\star)^r\|$$

for every positive real number r and every unitarily invariant norm.
This inequality can be considered as an operator version of the familiar Cauchy-Schwarz inequality for real numbers.

A stronger version of this inequality, which has been proved by Bhatia and Davis in [7], asserts that

$$\| \, |A^\star X B|^r \|^2 \leq \| \, |AA^\star X|^r \| \, . \, \| \, |XBB^\star|^r \| \tag{5.1}$$

for all $A, B, X \in M_n(\mathbb{C})$, $r > 0$ and for any unitarily invariant norm.
For A and B positive matrices in $M_n(\mathbb{C})$ and X arbitrary, inequality (5.1) will be equivalent to

$$\left\| \left| A^{1/2} X B^{1/2} \right|^r \right\|^2 \leq \| \, |AX|^r \| \, . \, \| \, |XB|^r \| \tag{5.2}$$

since $|A^\star X B| = |AXB| = |A^{1/2}A^{1/2}XB^{1/2}B^{1/2}| = |A^{1/2}YB^{1/2}|$ for $Y = A^{1/2}XB^{1/2}$.
By applying (5.1) to $|A^{1/2}YB^{1/2}|$ we get the result.

Hiai and Zhan [18] proved that if $A, B, X \in M_n(\mathbb{C})$ such that A, B are positive and $r > 0$, then the function $f(\nu) = \| \, |A^\nu X B^{1-\nu}|^r \| \, . \, \| \, |A^{1-\nu} X B^\nu|^r \|$ is convex on $[0, 1]$, attains its minimum at $\nu = \frac{1}{2}$ and it's maximum at $\nu = 0$ and $\nu = 1$. Moreover, $f(\nu) = f(1-\nu)$.
Thus for every unitarily invariant norm, we have the following refinement of Cauchy-Schwarz norm inequality for operators

$$\left\| \left| A^{1/2} X B^{1/2} \right|^r \right\|^2 \leq \left\| \, |A^\nu X B^{1-\nu}|^r \right\| . \left\| \, |A^{1-\nu} X B^\nu|^r \right\| \leq \| \, |AX|^r \| \, . \, \| \, |XB|^r \| \, . \tag{5.3}$$

Using some basic properties of convex functions, Hu [20] obtained the following refinement of the second inequality in (5.3).

$$\left\| \left|A^{\nu}XB^{1-\nu}\right|^{r}\right\| . \left\| \left|A^{1-\nu}XB^{\nu}\right|^{r}\right\| \leq 2\nu_0 \left\| \left|A^{1/2}XB^{1/2}\right|^{r}\right\|^{2} + (1-2\nu_0)\left\| |AX|^{r}\right\| . \left\| |XB|^{r}\right\|$$

where $\nu_0 = \min\{\nu, 1-\nu\}$.

Our main task in this chapter is to derive one of the improvement of Cauchy-Schwarz norm inequality with the help of the well-known Hermite-Hadamard inequality. In addition, Burquan establish new type inequalities close to Cauchy-Schwarz norm inequality for operators.

5.2 Refinement of Cauchy-Schwarz norm inequality for operators via the convexity

Hermite-Hadamard inequality, which includes a basic property of convex function and plays a central role in our investigation, asserts that if g is a convex real valued function on the interval $[a, b]$,then

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(t)dt \leq \frac{g(a)+g(b)}{2}.$$

A recent refinement of the second inequality in Hermite-Hadamard inequality, due to Feng [11], asserts that

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(t)dt \leq \frac{1}{4}\left[g(a) + 2g\left(\frac{a+b}{2}\right) + g(b)\right] \leq \frac{g(a)+g(b)}{2} \tag{5.4}$$

where g is a convex real valued function on $[a, b]$.

In the following lemma, Burqan constructed a refinement of the first inequality in the Hermite-Hadamard inequality ([10]).

Lemma 5.2.1. Let g be a real valued function which is convex on the interval $[0, 1]$. Then

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[g\left(\frac{3a+b}{4}\right) + g\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{b-a}\int_a^b g(t)dt \leq \frac{g(a)+g(b)}{2}.$$

Proof. Using Hermie-Hadamard inequality, we have

$$g\left(\frac{a+b}{2}\right) = g\left[\frac{1}{2}\left(\frac{3a+b}{4}\right) + \frac{1}{2}\left(\frac{a+3b}{4}\right)\right] \leq \frac{1}{2}\left[g\left(\frac{3a+b}{4}\right) + g\left(\frac{a+3b}{4}\right)\right].$$

To prove the second inequality, note that

$$\begin{aligned}
\frac{1}{2}\left[g\left(\frac{3a+b}{4}\right) + g\left(\frac{a+3b}{4}\right)\right] &= \frac{1}{2}\left[g\left(\frac{\frac{a+b}{2}+a}{2}\right) + g\left(\frac{\frac{a+b}{2}+b}{2}\right)\right] \\
&\leq \frac{1}{2}\left[\frac{1}{a-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^{a} g(t)dt + \frac{1}{b-\frac{a+b}{2}}\int_{\frac{a+b}{2}}^{b} g(t)dt\right] \\
&= \frac{1}{2}\left[\frac{2}{b-a}\int_a^{\frac{a+b}{2}} g(t)dt + \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} g(t)dt\right] \\
&= \frac{1}{b-a}\int_a^b g(t)dt.
\end{aligned}$$

This completes the proof. $\square$

In order to derive refinements of Cauchy-Schwarz norm inequality, we apply Lemma 5.2.1 and Inequality (5.4) to the function $f(\nu) = \| \, |A^{\nu} X B^{1-\nu}|^r \| \, . \, \| \, |A^{1-\nu} X B^{\nu}|^r \|$ on the interval $[\mu, 1-\mu]$, when $0 \le \mu < \frac{1}{2}$, and on the interval $[1-\mu, \mu]$, when $\frac{1}{2} < \mu \le 1$.

Theorem 5.2.2. Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive. Then for $0 \le \mu \le 1$, $r > 0$, and for every unitarily invariant norm,

$$\begin{aligned} \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 &\le \left\| \left| A^{\frac{2\mu+1}{4}} X B^{\frac{3-2\mu}{4}} \right|^r \right\| . \left\| \left| A^{\frac{3-2\mu}{4}} X B^{\frac{2\mu+1}{4}} \right|^r \right\| \\ &\le \frac{1}{|1-2\mu|} \left| \int_{\mu}^{1-\mu} \left\| \left| A^{\nu} X B^{1-\nu} \right|^r \right\| . \left\| \left| A^{1-\nu} X B^{\nu} \right|^r \right\| d\nu \right| \\ &\le \frac{1}{2} \left[\left\| \left| A^{\mu} X B^{1-\mu} \right|^r \right\| . \left\| \left| A^{1-\mu} X B^{\mu} \right|^r \right\| + \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 \right] \\ &\le \left\| \left| A^{\mu} X B^{1-\mu} \right|^r \right\| . \left\| \left| A^{1-\mu} X B^{\mu} \right|^r \right\| \\ &\le \| \, |AX|^r \| \; \| \, |XB|^r \| . \end{aligned} \tag{5.5}$$

Proof. First we assume that $0 \le \mu < \frac{1}{2}$. Then it follows by applying Lemma 5.2.1 and the Inequality (5.4) to the function $f(\nu) = \| \, |A^{\nu} X B^{1-\nu}|^r \| \, . \, \| \, |A^{1-\nu} X B^{\nu}|^r \|$ on the interval $[\mu, 1-\mu]$ that

$$\begin{aligned} f\left(\frac{1}{2}\right) &\le \frac{1}{2}\left[f\left(\frac{2\mu+1}{4}\right) + f\left(\frac{3-2\mu}{4}\right) \right] \le \frac{1}{1-2\mu} \int_{\mu}^{1-\mu} f(t)dt \\ &\le \frac{1}{4}\left[f(\mu) + 2f(\frac{1}{2}) + f(1-\mu) \right] \le \frac{1}{2}\left[f(\mu) + f\left(\frac{1}{2}\right) \right] \quad (\text{since } f(\mu) = f(1-\mu)) \\ &\le f(\mu). \end{aligned} \tag{5.6}$$

Thus,

$$\begin{aligned} \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 &\le \left\| \left| A^{\frac{2\mu+1}{4}} X B^{\frac{3-2\mu}{4}} \right|^r \right\| . \left\| \left| A^{\frac{3-2\mu}{4}} X B^{\frac{2\mu+1}{4}} \right|^r \right\| . \\ &\le \frac{1}{|1-2\mu|} \int_{\mu}^{1-\mu} \left\| \left| A^{\nu} X B^{1-\nu} \right|^r \right\| . \left\| \left| A^{1-\nu} X B^{\nu} \right|^r \right\| d\nu \\ &\le \frac{1}{2} \left[\left\| \left| A^{\mu} X B^{1-\mu} \right|^r \right\| . \left\| \left| A^{1-\mu} X B^{\mu} \right|^r \right\| + \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 \right] \\ &\le \left\| \left| A^{\mu} X B^{1-\mu} \right|^r \right\| . \left\| \left| A^{1-\mu} X B^{\mu} \right|^r \right\| . \end{aligned} \tag{5.7}$$

Now we assume that $\frac{1}{2} < \mu \le 1$. Then by applying (5.7) to $1-\mu$, we get

$$\begin{aligned} \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 &\le \left\| \left| A^{\frac{2\mu+1}{4}} X B^{\frac{3-2\mu}{4}} \right|^r \right\| . \left\| \left| A^{\frac{3-2\mu}{4}} X B^{\frac{2\mu+1}{4}} \right|^r \right\| . \\ &\le \frac{1}{|1-2\mu|} \int_{\mu}^{1-\mu} \left\| \left| A^{\nu} X B^{1-\nu} \right|^r \right\| . \left\| \left| A^{1-\nu} X B^{\nu} \right|^r \right\| d\nu \\ &\le \frac{1}{2} \left[\left\| \left| A^{\mu} X B^{1-\mu} \right|^r \right\| . \left\| \left| A^{1-\mu} X B^{\mu} \right|^r \right\| + \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 \right] \\ &\le \left\| \left| A^{\mu} X B^{1-\mu} \right|^r \right\| . \left\| \left| A^{1-\mu} X B^{\mu} \right|^r \right\| . \end{aligned} \tag{5.8}$$

Since

$$\mathop{Lim}_{\mu \to \frac{1}{2}} \left\| \left| A^{\frac{1}{2}} X B^{\frac{1}{2}} \right|^r \right\|^2 = \mathop{Lim}_{\mu \to \frac{1}{2}} \frac{1}{|2\mu - 1|} \left| \int_{1-\mu}^{\mu} \left\| \left| A^{\nu} X B^{1-\nu} \right|^r \right\| . \left\| \left| A^{1-\nu} X B^{\nu} \right|^r \right\| d\nu \right| ,$$

the inequalities in (5.5) follows by combining the inequalities (5.7) and (5.3.3). $\square$

5.3 Inequalities close to Cauchy-Schwarz norm inequality

In this section we present Burqan's work in [10]. Burqan proved that the function $f(\nu) = \| \, |A^\nu X B^\nu|^r \| \, . \, \| \, |A^{1-\nu} X B^{1-\nu}|^r \|$ is convex on the interval $[0,1]$ and used this convexity to obtain some inequalities that are related Cauchy-Schwarz norm inequality.

Theorem 5.3.1. Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive. Then for $r > 0$, and for every unitarily invariant norm, the function $f(\nu) = \| \, |A^\nu X B^\nu|^r \| \, . \, \| \, |A^{1-\nu} X B^{1-\nu}|^r \|$ is convex on $[0,1]$ and attains its minimum at $\nu = \frac{1}{2}$.
Consequently, it is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.

Proof. Without loss of generality, we may assume that $A > 0$ and $B > 0$. Since $f(\nu)$ is continuous and symmetric with respect to $\nu = \frac{1}{2}$, $\left(\text{because } f(\frac{1}{2} + \nu) = f(\frac{1}{2} - \nu)\right)$, all the conclusions will follow after we show that

$$f\left(\frac{\nu - s + \nu + s}{2}\right) = f(\nu) \le \frac{1}{2}[f(\nu - s) + f(\nu + s)] \tag{5.9}$$

for $\nu \pm s \in [0,1]$.
By the inequality (5.2), we have

$$\begin{aligned}
\| \, |A^\nu X B^\nu|^r \| &= \left\| \, \left|A^s (A^{\nu-s} X B^{\nu+s}) B^{-s}\right|^r \right\| \\
&= \left\| \, \left| \left(A^{2s}\right)^{\frac{1}{2}} \left(A^{\nu-s} X B^{\nu+s}\right) \left(B^{-2s}\right)^{\frac{1}{2}} \right|^r \right\| \\
&\le \left[\left\| \, \left|A^{2s} \left(A^{\nu-s} X B^{\nu+s}\right)\right|^r \right\| . \left\| \, \left|\left(A^{\nu-s} X B^{\nu+s}\right) B^{-2s}\right|^r \right\| \right]^{\frac{1}{2}} \\
&= \left[\| \, |A^{\nu+s} X B^{\nu+s}|^r \| \, . \, \| \, |A^{\nu-s} X B^{\nu-s}|^r \| \right]^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\left\| \, \left|A^{1-\nu} X B^{1-\nu}\right|^r \right\| &= \left\| \, \left|A^s (A^{1-\nu-s} X B^{1-\nu+s}) B^{-s}\right|^r \right\| \\
&= \left\| \, \left| \left(A^{2s}\right)^{\frac{1}{2}} \left(A^{1-\nu-s} X B^{1-\nu+s}\right) \left(B^{-2s}\right)^{\frac{1}{2}} \right|^r \right\| \\
&\le \left[\left\| \, \left|A^{2s} \left(A^{1-\nu-s} X B^{1-\nu+s}\right)\right|^r \right\| . \left\| \, \left|\left(A^{1-\nu-s} X B^{1-\nu+s}\right) B^{-2s}\right|^r \right\| \right]^{\frac{1}{2}} \\
&= \left[\left\| \, \left|A^{1-(\nu-s)} X B^{1-(\nu-s)}\right|^r \right\| . \left\| \, \left|A^{1-(\nu+s)} X B^{1-(\nu+s)}\right|^r \right\| \right]^{\frac{1}{2}} .
\end{aligned}$$

Upon multiplication of the above two inequalities, we obtain

$$\| \, |A^\nu X B^\nu|^r \| \, . \left\| \, \left|A^{1-\nu} X B^{1-\nu}\right|^r \right\| \le [f(\nu - s) + f(\nu + s)]^{\frac{1}{2}} . \tag{5.10}$$

Applying the arithmetic-geometric mean inequality (Lemma 2.5) to the right hand side of (5.10) leads to inequality (5.9). This completes the proof.

□

Since the function $f(\nu) = \| \, |A^\nu X B^\nu|^r \| \, . \, \| \, |A^{1-\nu} X B^{1-\nu}|^r \|$ is convex on $[0,1]$, attains its minimum at $\nu = \frac{1}{2}$, and its maximum at $\nu = 0$ and $\nu = 1$, and $f(\nu) = f(1 - \nu)$ for $0 \le \nu \le 1$, we have the following results.

Theorem 5.3.2. Let $A, B, X \in M_n(\mathbb{C})$ such that A and B are positive. Then for $0 \le \nu \le 1$, $r > 0$, and for every unitarily invariant norm,

$$\left\| \, \left|A^{1/2} X B^{1/2}\right|^r \right\|^2 \le \| \, |A^\nu X B^\nu|^r \| \, . \left\| \, \left|A^{1-\nu} X B^{1-\nu}\right|^r \right\| \le \| \, |X|^r \| \, . \, \| \, |AXB|^r \| \, . \tag{5.11}$$

A special case of the inequality (5.11) can be obtained as follows.

Theorem 5.3.3. Let $A, B \in M_n(\mathbb{C})$ such that A and B are positive. Then for $0 \leq \nu \leq 1$, $r > 0$, and for the operator norm,

$$\left\| \left|A^{1/2}B^{1/2}\right|^r \right\|^2 \leq \| \left|A^\nu B^\nu\right|^r \| . \left\| \left|A^{1-\nu}B^{1-\nu}\right|^r \right\| \leq \| \left|AB\right|^r \| . \tag{5.12}$$

In particular,

$$\left\|A^{1/2}B^{1/2}\right\|^2 \leq \|A^\nu B^\nu\| . \left\|A^{1-\nu}B^{1-\nu}\right\| \leq \|AB\| .$$

Remark 5.3.1. The operator norm inequality $\left\|A^{1/2}B^{1/2}\right\| \leq \|AB\|^{1/2}$ for $A, B \leq 0$ mentioned above is equivalent to the Löwner-Heinz inequality (Theorem 3.1.4), Heinz-Kato inequality and moreover,

$$\|A^r B^r\| \leq \|AB\|^r \text{ for } r \in [0,1]$$

Proof. See [12].

□

An equivalent formulation of the inequality (5.12) is obtained in the following theorem.

Theorem 5.3.4. et $A, B \in M_n(\mathbb{C})$ such that A and B are positive. Then for $t \geq 1$, $r > 0$, and for the operator norm,

$$\left\|A^{t/2}B^{t/2}\right\|^2 \leq \| \left|AB\right|^r \| . \left\| \left|A^{t-1}B^{t-1}\right|^r \right\| \leq \left\| \left|A^t B^t\right|^r \right\| .$$

Proof. From Theorem 5.3.3, we have

$$\left\| \left|A^{t/2}B^{t/2}\right|^r \right\|^2 \leq \left\| \left|A^{1/t}B^{1/t}\right|^r \right\| . \left\| \left|A^{1-\frac{1}{t}}B^{1-\frac{1}{t}}\right|^r \right\| \leq \| \left|AB\right|^r \| \qquad \text{for } t \geq 1.$$

Replacing A, B by A^t, B^t respectively, we achieve the proof. □

5.4 Matrix trace inequality

Ando and Hiai have proved in [2] and [3] respectively, that inequality (1.15):

$$Re\ tr\ A^{p_1}B^{q_1} \dots A^{p_k}B^{q_k} \leq tr\ A^{p_1+\dots+p_k}B^{q_1+\dots+q_k}$$

and inequality (1.16):

$$\|A^{p_1}B^{q_1} \dots A^{p_k}B^{q_k}\| \leq \left\|A^{p_1+\dots+p_k}B^{q_1+\dots+q_k}\right\|$$

are valid for all positive semidefinite matrices A, B and those nonnegative real numbers $p_1, q_1, \dots, p_k, q_k$ which satisfy certain additional conditions, and where $k \geq 2$ is a natural number.
Lucijan Plevnik in [25] gave two examples to show that the inequalities proven by Ando and Hiai are not valid for all collections of $p_1, q_1, \dots, p_k, q_k \geq 0$. Moreover, he gave a generalization of a trace inequality of type (1.15).

We start this section with the example which shows that (1.15) is not valid for all collections $p_1, q_1, \dots, p_k, q_k \geq 0$.

Example 5.4.1. Let $A = \begin{bmatrix} 76 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 20 & -14 & 13 \\ -14 & 2880 & 3100 \\ 13 & 3100 & 3380 \end{bmatrix}$.

Then we have $A, B \geq 0$ and

$$tr\ A^4BAB^4 = 7608677695167720100 > 7566365725138281700 = tr\ A^5B^5. \tag{5.13}$$

Remark 5.4.1. Example 5.4.1 shows that (1.7) does not hold for matrices $A^5, B^5 \geq 0$ and $t = \frac{4}{5}$. However, we haven't been able to answer the following natural question.

Question 5.4.1. Is the interval $[\frac{1}{4}, \frac{3}{4}]$ the maximal one on which the inequality (1.7) holds?

Corollary 5.4.0.1. If A and B are as in Example 5.4.1, then

$$\left\| A^{\frac{3}{2}} BAB^{\frac{3}{2}} \right\|_2 > \left\| A^{\frac{5}{2}} B^{\frac{5}{2}} \right\|_2 .$$

In particular, (1.16) does not hold for all collections $p_1, q_1, \ldots, p_k, q_k \geq 0$.

Proof. It is well-known that for any two matrices X and Y of appropriate sizes we have

$$|tr\ X^\star Y| \leq \frac{tr\ X^\star X + tr\ Y^\star Y}{2}$$

I found that this is true because of the following:
The map $\langle X, Y \rangle \longrightarrow tr\ (X^\star Y)$ is an inner product because it satisfies the properties of an inner product. (Property 2.1.1).

- $\langle X, Y \rangle = tr\ (X^\star Y) = tr((Y^\star X)^\star) = \overline{tr(Y^\star X)} = \overline{\langle X, Y \rangle}$
- $\langle X + Y, Z \rangle = tr\ ((X^\star + Y^\star)Z) = tr\ (X^\star Z + Y^\star Z) = tr\ (X^\star Z) + tr\ (Y^{|s} Z) = \langle X, Z \rangle + \langle Y, Z \rangle$
- $\langle aX, Y \rangle = tr\ (X^\star \overline{a} Y) = \langle X, \overline{a} Y \rangle = a \langle X, Y \rangle$
- $\langle X, X \rangle = tr\ (X^\star X) \geq 0$ since $X^\star X$ is p.s.d and

$$\begin{aligned}\langle X, X \rangle = 0 \Rightarrow tr\ (X^\star X) = 0 \Rightarrow & \sum_{i=1}^{n} \lambda_i (X^\star X) = 0 \\ & \Rightarrow \lambda_i (X^\star X) = 0\ \ \forall i \quad since\ Sp(X^\star X) \subseteq \mathbb{R}^+ \\ & \Rightarrow X^\star X = 0 \Rightarrow X = 0.\end{aligned}$$

By Cauchy-Schwarz inequality (Theorem 2.1.1) we have:

$$\begin{aligned}|tr\ (X^\star Y)|^2 &\leq tr\ (X^\star X)\ tr\ (Y^\star Y) \\ |tr\ (X^\star Y)| &\leq \sqrt{tr\ (X^\star X)\ tr\ (Y^\star Y)} \\ 2|tr\ (X^\star Y)| &\leq 2\sqrt{tr\ (X^\star X)\ tr\ (Y^\star Y)}.\end{aligned}$$

But since $2\sqrt{a}\sqrt{b} \leq a + b$ for $a, b \geq 0$, then

$$|tr\ X^\star Y| \leq \frac{tr\ X^\star X + tr\ Y^\star Y}{2}.$$

Back to the proof of the corollary, by applying this inequality for $X = A^{\frac{5}{2}} B^{\frac{5}{2}}$ and $Y = A^{\frac{3}{2}} BAB^{\frac{3}{2}}$ we get

$$|tr\ X^\star Y| = |tr\ A^4 BAB^4| \leq \frac{\left\| A^{\frac{5}{2}} B^{\frac{5}{2}} \right\|_2^2 + \left\| A^{\frac{3}{2}} BAB^{\frac{3}{2}} \right\|_2^2}{2}.$$

By Example 5.4.1 we have $tr\ A^5 B^5 < tr\ A^4 BAB^4$. But since $tr\ A^5 B^5 = \left\| A^{\frac{5}{2}} B^{\frac{5}{2}} \right\|_2^2$, we get

$$\left\| A^{\frac{5}{2}} B^{\frac{5}{2}} \right\|_2^2 < \frac{\left\| A^{\frac{5}{2}} B^{\frac{5}{2}} \right\|_2^2 + \left\| A^{\frac{3}{2}} BAB^{\frac{3}{2}} \right\|_2^2}{2},$$

which implies the required results.

Now we give an example that disproves this an estimate in the other direction as well,

$$tr \left(A^{\frac{p_1 + \cdots + p_k}{k}} B^{\frac{q_1 + \cdots + q_k}{k}} \right)^k \leq |tr A^{p_1} B^{q_1} \ldots A^{p_k} B^{q_k}|.$$

∎

Example 5.4.2. Let $C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 6 & 6 \\ -2 & 6 & 15 \end{bmatrix}$.
Then we hace $C, D \geq 0$ and

$$tr\ (C^3D^3)^2 = 52607744 > 25527680 = tr\ C^5DCD^5.$$

We finish this section with a propositions that generalize (1.17).
We recall that inequality (1.17) is

$$tr\left(AB^{\frac{p+q}{2}}\right)^2 \leq tr\ AB^pAB^q \leq tr\ A^2B^{p+q}.$$

for all $A, B \geq 0$ and $p, q \geq 0$.
For this generalization we are going to need the definition and properties of the antisymmetric tensor product.

Definition 5.4.1. The k-th antisymmetric tensor power of A denoted by $\wedge^k A$ is the restriction of the k-th tensor power of A denoted by $\otimes^k A$, to the totally antisymmetric subspace of $(\mathbb{C}^n)^{\otimes k}$.

An important information is that for $A \geq 0$ the largest eigenvalue of $\wedge^k A$ is given by

$$\lambda_1(\wedge^k A) = \lambda_1(A)\lambda_2(A)...\lambda_k(A).$$

Property 5.4.1. For any A and B two matrices in $M_n(\mathbb{C})$ we have:

1. $\wedge^k(AB) = \wedge^k A \wedge^k B$.
2. $(\wedge^k A)^{-1} = \wedge^k A^{-1}$.
3. $(\wedge^k A)^{\frac{1}{2}} = \wedge^k A^{\frac{1}{2}}$.
4. If A is positive semidefinite matrix then so is $\wedge^k A$.

Proposition 5.4.1. Let A and B be two positive semidefinite matrices in $M_n(\mathbb{C})$.

$$\lambda(A) \prec_{log} \lambda(B) \text{ if and only if } \begin{cases} det(A) = det(B) \\ \lambda_1(\wedge^k A) \leq \lambda_1(\wedge^k B) \qquad 1 \leq k \leq n \end{cases}$$

The generalization of (1.17) is giving by the following proposition.

Proposition 5.4.2. Let $A, B \geq 0$ and $p, q \geq 0$. Then we have

$$\lambda\left(AB^{\frac{p+q}{2}}\right)^2 \prec_{w\,log} \lambda(AB^pAB^q) \prec_{w\,log} \lambda(A^2B^{p+q}).$$

in particular,

$$tr\left(AB^{\frac{p+q}{2}}\right)^{2r} \leq tr\ (AB^pAB^q)^r \leq tr\ (A^2B^{p+q})^r$$

for all $r \geq 0$.

Proof. It is enough to show that

$$\lambda_1\left(AB^{\frac{p+q}{2}}\right)^2 \leq \lambda_1(AB^pAB^q) \leq \lambda_1(A^2B^{p+q}). \tag{5.14}$$

This is true because,

$$\wedge^k\left(AB^{\frac{p+q}{2}}\right)^2 = (\wedge^k A)^2(\wedge^k B)^{p+q} \text{ and } \wedge^k\ (AB^pAB^q) = (\wedge^k A)(\wedge^k B)^p(\wedge^k A)(\wedge^k B)^q$$

and since $\wedge^k A$ is also p.s.d when A is p.s.d, thus we can replace A and B by $\wedge^k A$ and $\wedge^k B$ respectively. So we get $\lambda_1\left(\wedge^k(AB^{\frac{p+q}{2}})^2\right) \leq \lambda_1(\wedge^k(AB^pAB^q))$ which by definition yields to the desired result. The same holds true for the second inequality.

There is nothing to prove if $p = q = 0$, so assume that $p + q > 0$.
Let us first prove the second inequality in (5.14). After multiplying A and B with the same positive scalar, if necessary, we may assume that $\lambda_1(A^2B^{p+q}) = 1$ so $\lambda_i(A^2B^{p+q}) \leq 1 \; \forall i$.
Therefore for any eigenvector x we have

$$\begin{aligned}(A^2B^{p+q})x \leq x &\Rightarrow (A^2B^{p+q}) \leq I \\ &\Rightarrow B^{p+q} \leq A^{-2}\end{aligned} \tag{5.15}$$

By taking (5.15) to the power $\frac{p}{p+q}$ we get

$$B^p \leq A^{-\frac{2p}{p+q}} \tag{5.16}$$

And by taking (5.16) to the power $\frac{q}{p}$ we get

$$A^{\frac{2q}{p+q}} \leq B^{-q} \tag{5.17}$$

So by using inequalities (5.15) and (5.16) we get

$$\lambda_1(AB^pAB^q) = \lambda_1\left(B^{\frac{q}{2}}AB^pAB^{\frac{q}{2}}\right) \leq \lambda_1\left(B^{\frac{q}{2}}A^{\frac{2q}{p+q}}B^{\frac{q}{2}}\right) \leq \lambda_1(I) = 1$$

as desired. We proceed with the proof of the first inequality in (5.14).
If $\|.\|$ is the operator norm, then we have

$$\lambda_1\left(AB^{\frac{p+q}{2}}\right)^2 = \lambda_1\left(B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}\right)^2 = \left\|B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}\right\|^2.$$

Because $B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}$ is Hermitian, the latter is not larger than

$$\left\|B^{\frac{q}{2}}AB^{\frac{p}{2}}\right\|^2 = \left\|\left(B^{\frac{q}{2}}AB^{\frac{p}{2}}\right)\left(B^{\frac{p}{2}}AB^{\frac{q}{2}}\right)\right\| = \lambda_1\left(B^{\frac{q}{2}}AB^pAB^{\frac{q}{2}}\right) = \lambda_1\left(AB^pAB^q\right).$$

Therefore $\lambda_1\left(AB^{\frac{p+q}{2}}\right)^2 \leq \lambda_1\left(AB^pAB^q\right).$

$\square$

Remark 5.4.2. After going through the preceding proposition in [25], I spotted an improvement by proving that we actually have a log-majorization and not just a weak log-majorization.
We already proved

$$\lambda_1\left(AB^{\frac{p+q}{2}}\right)^2 = \lambda_1\left(B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}\right)^2 \leq \lambda_1\left(B^{\frac{q}{2}}AB^pAB^{\frac{q}{2}}\right) = \lambda_1\left(AB^pAB^q\right).$$

Since $\left(B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}\right)^2$ and $B^{\frac{q}{2}}AB^pAB^{\frac{q}{2}}$ are Hermitian matrices and since $det\left(B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}\right)^2 = det(B^{\frac{q}{2}}AB^pAB^{\frac{q}{2}})$, then by Proposition 5.4.1 we have

$$\lambda\left(AB^{\frac{p+q}{2}}\right)^2 \prec_{log} \lambda(AB^pAB^q).$$

We proceed similarly for the second inequality of (5.14).

By the end of this chapter and throughout this book, we were able to derive a refinement of Cauchy-Schwarz inequality just like we got some related inequalities along the way. Moreover, two counterexamples of the inequalities of Ando and Hiai were given along with a generalization of a trace inequality. Inequalities related to Heinz mean were also presented in previous chapters and Bourin's question was partially answered.

Despite all the difficulties that researchers are facing in order to give a definite answer to Bourin's question regarding subadditivity inequalities, and despite the many failures in this quest, researchers have not given up hope yet, that one day, their work, no matter how incomplete it might have been, will lead to an answer that satisfies them and Bourin specifically.

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