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Winfried Hochstättler, Robert Nickel:

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Contact:

winfried.hochstaettler@fernuni-hagen.de

robert.nickel@fernuni-hagen.de

FernUniversität in Hagen
Fakultät für Mathematik und Informatik
Lehrgebiet für Diskrete Mathematik und Optimierung
D – 58084 Hagen

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On the Chromatic Number of an Oriented Matroid

Winfried Hochstättler
FernUniversität in Hagen

Robert Nickel
FernUniversität in Hagen

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Abstract

We prove that the chromatic number of an oriented matroid $\mathcal{O}$ of rank $r \geq 3$ is at most $r+1$ with equality if and only if $\mathcal{O}$ is the oriented matroid of an orientation of K_{r+1}, the complete graph on $r+1$ vertices.

1 Introduction

Recently, two competing definitions for a chromatic number or, dually, a flow number of an oriented matroid have been proposed. The oriented chromatic number χ_o of Goddyn et al. [4, 5] is a generalization of the circular chromatic number and, thus, in general not an integer. Moreover, it is not a matroid invariant (see [4]). The latter is open for the chromatic number χ of Hochstättler and Nešetřil [7] which is always integer. Let $\mathcal{O}$ denote a loopless oriented matroid of rank r. It is immediate from its definition that $\chi_o(\mathcal{O}) \leq r+1$. A simple topological argument, given below, shows that the same holds for $\chi(\mathcal{O})$. Moreover, we will show that, if $r \geq 3$, the complete graph is the only instance where this bound is tight. At least in rank 3 this is not the case for χ_o.

We assume familiarity with oriented matroid theory, the standard reference is Björner et al. [1]. Given a loopless oriented matroid $\mathcal{O}$ of rank r on a finite set E, we denote by $\mathcal{F}_{\mathcal{O}^*}$ the *coflow lattice* of $\mathcal{O}$, i. e. the integer lattice generated by the signed characteristic vectors of signed cocircuits $\mathcal{D}$ of $\mathcal{O}$, where for a signed cocircuit $D = (D^+, D^-) \in \mathcal{D}$ its *signed characteristic vector* $\vec{D} \in \mathbb{Z}^{|E|}$ is defined for any $e \in E$ by

$$\vec{D}(e) := \begin{cases} 1 & \text{if } e \in D^+ \\ -1 & \text{if } e \in D^- \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}_{\mathcal{O}^*} := \operatorname{lat}\{\vec{D} \mid D \in \mathcal{D}\} \subseteq \mathbb{Z}^{|E|}$, where

$$\operatorname{lat}\{v_1, \ldots, v_k\} := \left\{ \sum_{i=1}^{k} \lambda_i v_i \mid \lambda_i \in \mathbb{Z} \right\} \subseteq \mathbb{Z}^n$$

is the integer lattice generated by $\{v_1, \ldots, v_k\} \subset \mathbb{Z}^n$. We call any $x \in \mathcal{F}_{\mathcal{O}^*}$ a *coflow*. Such an x is a *nowhere zero k-coflow* if $0 < |x(e)| < k$ holds for any $e \in E$. The *chromatic number* $\chi(\mathcal{O})$ is the minimal k such that there is a nowhere zero k-coflow. It follows from standard results of the theory of nowhere zero flows (see e.g. [10]) that $\chi(\mathcal{O})$ equals the chromatic number $\chi(G)$ in case $\mathcal{O}$ is the oriented matroid of some orientation of a graph G. Clearly, the chromatic number is reorientation invariant. As a result of [8], $\chi(\mathcal{O})$ is a matroid invariant for uniform or corank 3 oriented matroids. It is unknown whether this is the case in general.

The paper is organized as follows. In the next section we will review the relevant results from Hochstättler and Nešetřil [7] for the rank 2 case. In Section 3 we will found our result in the rank 3 case. Section 4 is devoted to an infinite class of oriented matroids of rank and chromatic number 3. Finally, in Section 5 we will prove the result for arbitrary rank $r \geq 3$.

2 Rank 2

We summarize some results of [7] concerning the lattice $\mathcal{F}_{\mathcal{O}^*}$ of rank 2 oriented matroids.

Proposition 1 ([7])**.** *Let $\mathcal{O}$ be a simple rank 2 oriented matroid on a finite set E and e_i the i^{th} unit vector of $\mathbb{Z}^{|E|}$. Then there is a reorientation ${}_I\mathcal{O}$ of $\mathcal{O}$ such that*

$$\mathcal{F}_{{}_I\mathcal{O}^*} = \begin{cases} \operatorname{lat}\{\mathrm{e}_i \pm \mathrm{e}_j \mid i, j \in E\} & \text{if } |E| \text{ is even} \\ \operatorname{lat}\{\mathrm{e}_i - \mathrm{e}_j \mid i \neq j \in E\} & \text{if } |E| \text{ is odd} \end{cases}$$

The elements of a general loopless rank 2 oriented matroid are partitioned into classes $\overline{e} \subseteq E$, $e \in E$ of elements which are pairwise parallel or antiparallel to e. Let $\chi_{\overline{e}} \in \{0,1\}^{|E|}$ denote the characteristic vector of $\overline{e}$ for some $e \in E$ and let $\overline{E} := \{\overline{e} \mid e \in E\}$ be the set of parallel classes of $\mathcal{O}$.

Corollary 2. *Let $\mathcal{O}$ be a line with parallel classes $\overline{E}$. Then there is a reorientation ${}_I\mathcal{O}$ of $\mathcal{O}$ such that*

$$\mathcal{F}_{{}_I\mathcal{O}^*} = \begin{cases} \operatorname{lat}\{\chi_{\overline{e}} \pm \chi_{\overline{f}} \mid \overline{e}, \overline{f} \in \overline{E}\} & \text{if } |\overline{E}| \text{ is even} \\ \operatorname{lat}\{\chi_{\overline{e}} - \chi_{\overline{f}} \mid \overline{e} \neq \overline{f} \in \overline{E}\} & \text{if } |\overline{E}| \text{ is odd.} \end{cases}$$

As a direct consequence, every loopless rank 2 oriented matroid has a nowhere zero 2-coflow if it consists of an even number of parallel classes and a nowhere zero 3-coflow, otherwise. Figure 1 shows an orientation of (multiple) points on a line. The hyperplanes, i. e. points, are oriented in an alternating manner, which yields an orientation as in Corollary 2.

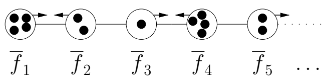

Figure 1: An alternating orientation of a line with parallel classes $\overline{f}_i$.

3 Rank 3

We start this section on the rank 3 case with a result for arbitrary rank. By the Topological Representation Theorem of Folkman and Lawrence [3], each rank r oriented matroid has a topological representation as an arrangement of pseudohyperspheres of S^{r-1}. The maximal cells $\mathcal{T}$ of such an arrangement are called *topes*, they correspond to signed covectors of full support, and the minimal cells, the *vertices*, correspond to the cocircuits of the oriented matroid. The signed cocircuits to vertices of a fixed tope T are *conformal*, i. e. for any two of them, say D_1, D_2, the *separation set* $S(D_1, D_2) := (D_1^+ \cap D_2^-) \cup (D_1^- \cap D_2^+)$ is empty. We say that a set of cocircuits $\{D_1, \ldots, D_k\}$ *spans* a tope T, if $T = D^1 \circ \ldots \circ D^k$, where $X \circ Y$ denotes the *composition* of two signed subsets X, Y (see [1]). By induction it is immediate that a tope $T \in \mathcal{T}$ of an oriented matroid of rank r is spanned by at most r cocircuits. For a signed subset X let $\operatorname{supp}(X) := \{e \in E \mid e \in X^+ \cup X^-\}$ denote its support.

Proposition 3. *Let $\mathcal{O}$ denote a loopless oriented matroid.*

a) If $D_1, \ldots, D_k$ are cocircuits spanning a tope T, then $\chi(\mathcal{O}) \leq k + 1$.

b) If $\mathcal{O}$ is of rank r then $\chi(\mathcal{O}) \leq r + 1$.

Moreover, there is an acyclic reorientation of $\mathcal{O}$ with a positive nowhere zero $(k+1)$-coflow in a) resp. a positive nowhere zero $(r+1)$-coflow in b).

Proof. a) Reorient $\mathcal{O}$ such that T is all positive. Then by conformity, $D_1, \ldots, D_k$ are non-negative and, since $\mathcal{O}$ is loopless,

$$0 < \sum_{i=1}^{k} \vec{D}_i < k + 1.$$

b) Choose a tope T and select $D_1, \ldots, D_r$ that span T. Then the result follows by a).

□

Clearly, this bound is tight for $\mathcal{O}(K_4)$. In the following we will show that in the rank 3 case $\mathcal{O}(K_4)$ is the only worst case example. This will be done by case checking. We start with two easy cases. An oriented matroid of rank 3 is *triangular* if every 2-dimensional face is a triangle. Note that, choosing a horizon, an oriented matroid of rank 3 is representable by a pseudoline arrangement.

Proposition 4. *Let $\mathcal{O}$ be a loopless oriented matroid of rank 3.*

a) If $\mathcal{O}$ is non-triangular, then $\chi(\mathcal{O}) \leq 3$.

b) If $\mathcal{O}$ has a pseudoline $e \in E$ incident to only two vertices, then $\chi(\mathcal{O}) \leq 3$.

Moreover, in both cases there is an acyclic reorientation with a positive nowhere zero 3-coflow.

Proof. a) Choose a tope that is not a triangle and two cocircuits D_1, D_2 corresponding to non-adjacent vertices of that tope. Then D_1 and D_2 span T and the result follows from Proposition 3.

b) Let T be a tope that is incident to a segment of e. By a) we may assume that T is a triangle. Let D_1, D_2, D_3 denote the cocircuits corresponding to the vertices of T such that $\vec{D}_1(e) = \vec{D}_2(e) = 0 \neq \vec{D}_3(e)$. Since e has only two vertices, $\mathrm{supp}(D_1) \cup \mathrm{supp}(D_2) = E \setminus \{e\}$ and $\mathrm{supp}(D_1) \cap \mathrm{supp}(D_2) = \emptyset$. Reorient $\mathcal{O}$ such that T is all positive, then

$$0 < \vec{D}_1 + \vec{D}_2 + \vec{D}_3 < 3$$

implying the assertion.

□

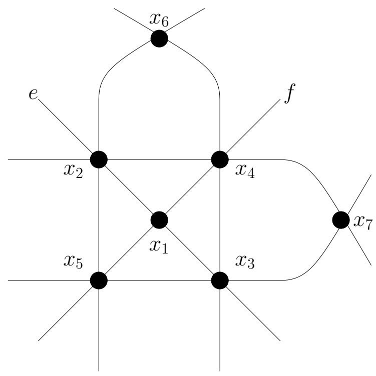

Figure 2: The pseudoline arrangement of $\mathcal{O}(K_4)$.

Thus, in the following we may restrict our analysis to triangular pseudoline arrangements where each pseudoline is incident to at least three vertices. By Sylvester's property (see e. g. [6]), each pseudoline arrangement has a *simple vertex*, i. e. a vertex x_1 which is incident to exactly two pseudolines e and f. We may choose the horizon of our oriented matroid such that none of the four edges starting at x_1 is unbounded. Let x_2 and x_3 denote the neighbors of x_1 on e and x_4, x_5 those on f. Since $\mathcal{O}$ is triangular, each of the pairs (x_2, x_4), (x_2, x_5), (x_3, x_4), and (x_3, x_5) must lie on a common pseudoline, yielding a configuration of 6 pseudolines isomorphic to the arrangement of K_4 (see Figure 2).

If $\mathcal{O}$ is not isomorphic to $\mathcal{O}(K_4)$, there has to be an additional element. Before we can finish our analysis we have to consider one more special case.

Proposition 5. *If $\mathcal{O}$ is the oriented matroid of the non-Fano configuration F_7^- (see Figure 3) then $\chi(\mathcal{O}) = 2$ and moreover, there is an acyclic reorientation ${}_I\mathcal{O}$ such that $\mathbb{1} \in \mathcal{F}_{{}_I\mathcal{O}^*}$.*

Proof. Reorient $\mathcal{O}$ as in Figure 3 i. e. such that the triangular tope formed by x_1, x_2, x_4 is all positive. Let D_{x_i} denote cocircuits corresponding to the vertices x_i. Then we have

$$\begin{aligned} D_{x_1} &= (0, 0, 1, \quad 1, 1, \quad 1, \quad 1)^\top \\ D_{x_6} &= (1, 1, 0, -1, 1, \quad 0, \quad 0)^\top \\ D_{x_8} &= (1, 0, 0, -1, 1, -1, \quad 1)^\top \\ D_{x_9} &= (0, 1, 0, -1, 1, \quad 1, -1)^\top \end{aligned}$$

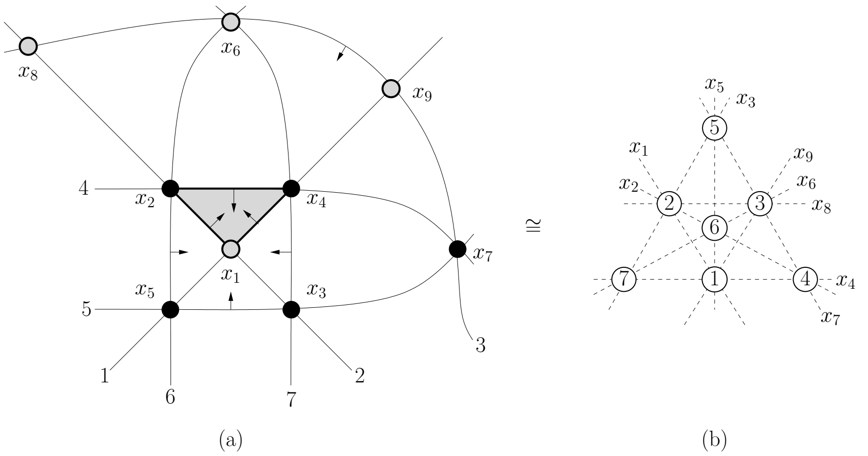

Figure 3: The pseudoline arrangement of the non-Fano configuration. Note that the point configuration in (b) represents a different member of the reorientation class.

and therefore,

$$D_{x_1} + 2D_{x_6} - D_{x_8} - D_{x_9} = (1,1,1,1,1,1,1)^\top$$

is a positive nowhere zero 2-coflow. $\square$

In the remaining cases the following lemma guarantees the existence of a simple vertex that forms a triangle with an edge that is on a pseudoline with at least four vertices.

Lemma 6. *Let $\mathcal{O}$ be a triangular oriented matroid, with at least three vertices on each pseudoline such that its simplification is neither $\mathcal{O}(K_4)$ nor $\mathcal{O}(F_7^-)$ and x_1 a simple vertex. Then $\mathcal{O}$ contains a triangle u, v, x_1 such that the pseudoline through u, v contains at least four vertices.*

Proof. By our analysis preceding Proposition 5, $\mathcal{O}$ contains a K_4 deletion minor with a cocircuit D_{x_1} corresponding to x_1. Let wlog. the lines and vertices of this minor be labeled as in Figure 2. Since the simplification of $\mathcal{O}$ is neither $\mathcal{O}(K_4)$ nor $\mathcal{O}(F_7^-)$, $\mathcal{O}$ must contain an additional pseudoline that does not contain both x_6 and x_7. This line generates an additional vertex either on the pseudolines 4 and 5 or on 6 and 7. $\square$

Theorem 7. *Every loopless oriented matroid of rank 3 not isomorphic to $\mathcal{O}(K_4)$ satisfies $\chi(\mathcal{O}) \leq 3$.*

Proof. By Proposition 3 we may assume that $\mathcal{O}$ is triangular and (wlog.) simple and that each pseudoline is incident to at least three vertices. The regular case is trivial, since $\mathcal{O}(K_4)$ is maximal regular and any minor is 3-colorable. We now assume that $\mathcal{O}$ is non-regular. The case that $\mathcal{O} \cong \mathcal{O}(F_7^-)$ is covered by Proposition 5. Otherwise, by Lemma 6, there is a simple vertex u incident to a triangular tope T spanned by cocircuits D_u, D_f, D_g such that $e \in D_u$, with $e \notin D_f \cup D_g$, is a pseudoline which is incident to at least 4 vertices.

Let $f \in D_g, g \in D_f$ be the two pseudolines incident to u and let wlog. $\mathcal{O}$ be reoriented and embedded such that T is a bounded positive tope (see Figure 4). Since u is simple, we have $\operatorname{supp}(D_u) = E \setminus \{f, g\}$.

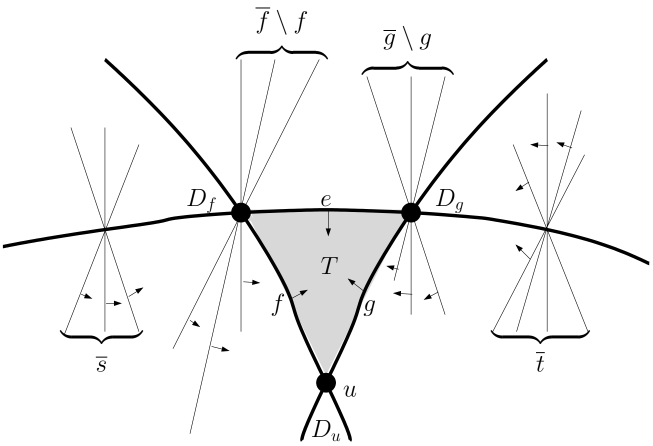

Figure 4: The triangle in the proof of Theorem 7.

Let $\overline{E}$ be the set of parallel classes of $\mathcal{O}/e$. By the choice of e and since vertices of e correspond to parallel classes of $\mathcal{O}/e$, there are classes $\overline{s} \neq \overline{t} \in \overline{E} \setminus \{\overline{f}, \overline{g}\}$ and an embedding such that $\overline{s}$ is the other parallel class next to $\overline{f}$ and $\overline{t}$ the other class next to $\overline{g}$ on the line $\mathcal{O}/e$. Taking the above reorientation into account, $\mathcal{O}/e$ can be represented as in Figure 5.

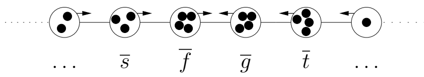

Figure 5: The oriented line $\mathcal{O}/e$.

$\vec{D}_u$ is zero on f, g and $+1$, otherwise. Let $y_1' := \chi_{\overline{s}} + \chi_{\overline{f}}$ and $y_2' := \chi_{\overline{t}} + \chi_{\overline{g}}$ and y_1, y_2 be the extensions of y_1', y_2' to vectors in $\mathbb{Z}^{|E|}$ via $y_i(e) := 0$. By Corollary 2 and the chosen embedding and orientation, $y_1', y_2' \in \mathcal{F}_{(\mathcal{O}/e)^*}$. Since cocircuits of $\mathcal{O}/e$ are the cocircuits of $\mathcal{O}$ not containing e, it follows that $y_1, y_2 \in \mathcal{F}_{\mathcal{O}^*}$ and therefore, $x := \vec{D}_u + y_1 + y_2 \in \mathcal{F}_{\mathcal{O}^*}$. Then x satisfies

$$x(h) = \begin{cases} 2 & \text{if } h \in \overline{s} \cup \overline{t} \cup (\overline{f} \setminus \{f\}) \cup (\overline{g} \setminus \{g\}) \\ 1 & \text{otherwise.} \end{cases}$$

and hence, is a positive nowhere zero 3-coflow. □

4 Examples with $\chi(\mathcal{O}) = 3$

In this section we present a sufficient condition for a pseudoline arrangement not to admit a nowhere zero 2-coflow and an infinite family of line arrangements for which this property holds. A pseudoline arrangement is said to have an *m-star* if it has a vertex which is incident to exactly m pseudolines.

Lemma 8. *Let $\mathcal{O}$ be a rank 3 oriented matroid such that its pseudoline arrangement has a $(2k+1)$-star $(k \geq 3)$ that is incident to all other vertices. Then $\chi(\mathcal{O}) \geq 3$.*

Proof. Since for $F \subseteq E$ the cocircuits of the restriction minor $\mathcal{O}(F)$ are restrictions of cocircuits of $\mathcal{O}$, the coflows of $\mathcal{O}(F)$ are restrictions of coflows of $\mathcal{O}$. The claim follows from Proposition 1 since the restriction to the $(2k+1)$-star is the $(2k+1)$-line. $\square$

See Figure 6 for a triangular and a nontriangular example satisfying the conditions of Lemma 8.

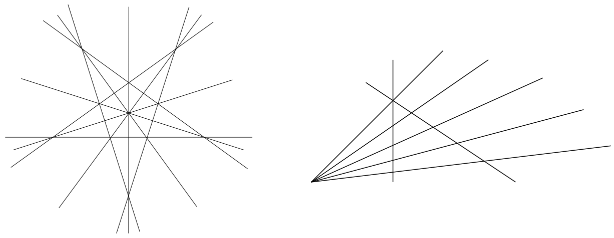

Figure 6: A triangular and a nontriangular example of a non-regular rank 3 oriented matroid with $\chi(\mathcal{O}) = 3$. The first example belongs to an infinite family of triangular line arrangements consisting of the sides of a regular odd n-gon and all lines of mirror symmetry called $\mathcal{R}(2n)$ (see [6]).

5 Higher Rank

We will need the following special case of the Reconstruction Conjecture (see Kelly [9]) which should be known. Since we did not find a reference, we give a proof.

Theorem 9. *Let $\mathcal{M}$ be a simple matroid of rank $r \geq 3$ on a finite set E such that the simplification of any of its contractions $\mathcal{M}/e$ is isomorphic to K_r. Then $\mathcal{M}$ is isomorphic to K_{r+1}.*

Proof. If $\mathcal{M}$ is non-regular then, as $r \geq 3$, it contains an element e such that $\mathcal{M}/e$ still contains a four point line as a minor contradicting our assumptions. If $\mathcal{M}$ is graphic, then the theorem obviously holds. If $\mathcal{M}$ is cographic but not graphic, then $\mathcal{M}/e$ must be planar for all $e \in E$. Thus $\mathcal{M}$ must be the dual of either $K_{3,3}$ or K_5, neither of which satisfies the assumptions. If $\mathcal{M}$ is the matroid R_{10} then the simplification of any contraction has rank 4 and at most 9 elements and, thus, cannot be K_5.

Hence, we may assume that $\mathcal{M}$ is neither graphic, cographic, nor isomorphic to R_{10}. Thus, by Seymour's famous Decomposition Theorem ([11, 13.2.4]), it is the 2- or 3-sum of two matroids one of which is not graphic. The latter remains invariant if we contract an element in the other component. The claim follows. $\square$

Theorem 10. *Let $\mathcal{O}$ be a loopless oriented matroid of rank $r \geq 3$. Then $\chi(\mathcal{O}) \leq r+1$. Moreover, $\chi(\mathcal{O}) = r+1 \iff \mathcal{O} \cong \mathcal{O}(K_{r+1})$.*

Proof. We wlog. assume that $\mathcal{O}$ is simple. By Proposition 3 the inequality holds since every rank r tope is spanned by at most r cocircuits. The reverse direction of the equivalence is obvious. For the forward implication, we prove a slightly stronger result:

> If $\mathcal{O} \ncong \mathcal{O}(K_{r+1})$, then $\mathcal{O}$ has an acyclic reorientation that admits a positive nowhere zero r-coflow.

We prove this by induction on r founded by Theorem 7.

Now let $r > 3$. By Theorem 9, there is an $e \in E$ such that $\mathcal{O}/e \ncong \mathcal{O}(K_r)$. By inductive assumption, there is a positive tope $T' \in \mathcal{T}(\mathcal{O}/e)$ and a positive nowhere zero $(r-1)$-coflow $x' \in \mathcal{F}_{(\mathcal{O}/e)^*} = \mathcal{F}_{\mathcal{O}^* \setminus e}$. Let $x \in \mathbb{Z}^{|E|}$ be defined for all $f \in E$ by

$$x(f) := \begin{cases} x'(f) & \text{for } f \neq e \\ 0 & \text{otherwise.} \end{cases}$$

Since the cocircuits of $\mathcal{O}/e$ are the cocircuits of $\mathcal{O}$ that do not contain e, we have $x \in \mathcal{F}_{\mathcal{O}^*}$. Let T be one of the two topes of $\mathcal{O}$ with facet T'. Reorient $\mathcal{O}$ such that T is positive and let D_e be a cocircuit with $e \in D_e$ that corresponds to a vertex of T. Then $x + \vec{D}_e \in \mathcal{F}_{\mathcal{O}^*}$ is a positive nowhere zero r-coflow of $\mathcal{O}$. □

6 Open Questions

One strategy to prove that $\chi(\mathcal{O})$ is a matroid invariant in the rank 3 case would be to classify the classes with $\chi(\mathcal{O}) = 2$ resp. $\chi(\mathcal{O}) = 3$. Lemma 8 gives an example of a sufficient condition for $\chi(\mathcal{O}) = 3$ that does not depend on the orientation of a matroid. We can show that $\chi(\mathcal{O}) = 2$ holds when the arrangement has a so-called *Gallai triangle* (see [2, Section 3.2]), i. e. if there are three simple vertices that are the intersection of three pseudolines. A classification that uses the structure of simple vertices might be helpful. In this context the following question arose that might be of independent interest:

> Does there exist a pseudoline arrangement with no two simple vertices on the same pseudoline different from $\mathcal{R}(2(2k+1))$?

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented matroids*. Cambridge University Press, Cambridge, 2nd edition, 1999.

[2] P. Erdös and G. Purdy. Extremal problems in combinatorial geometry. In *Handbook of combinatorics (vol. 1)*, pages 809–874. MIT Press, 1995.

[3] J. Folkman and J. Lawrence. Oriented matroids. *Journal of Combinatorial Theory. Series B*, 25(2):199–236, 1978.

[4] L. Goddyn, P. Hliněný, and W. Hochstättler. Balanced signings of oriented matroids and chromatic number. *Combinatorics, Probability and Computing*, 15(4):523–539, 2006.

[5] L. A. Goddyn, M. Tarsi, and C.-Q. Zhang. On (k,d)-colorings and fractional nowhere zero flows. *Journal of Graph Theory*, 28:155–161, 1998.

[6] B. Grünbaum. *Arrangements and spreads.* American Mathematical Society Providence, R.I., 1972.

[7] W. Hochstättler and J. Nešetřil. Antisymmetric flows in matroids. *European Journal of Combinatorics*, 27(7):1129–1134, 2006.

[8] W. Hochstättler and R. Nickel. The flow lattice of oriented matroids. *Contributions to Discrete Mathematics*, 2(1):68–86, 2007.

[9] P. J. Kelly. *On isometric transformations.* PhD thesis, University of Wisconsin, 1942.

[10] J. Nešetřil and A. Raspaud. Duality, nowhere-zero flows, colorings and cycle covers. KAM-DIMATIA Series (99-422), 1999.

[11] J. G. Oxley. *Matroid theory.* The Clarendon Press Oxford University Press, New York, 1992.