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# Joins of Oriented Matroids

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**Abstract**

We define series/parallel/2-sum connection of two oriented matroids in terms of various axiom systems and an oriented modular join and sum operation by means of signed cocircuits and covectors.

## 1 Introduction

Parallel, series, 2-sum, and generalized parallel connection of two (non-oriented) matroids are well known operations in matroid theory (see Brylawski [3, Chapter 7]). Although a generalization to oriented matroids is natural and meaningful, it appeared in the literature only partially and very recently in independent papers [5] and [8].

Dong [5] defined a parallel connection $\mathcal{O}_1 \oplus_{\mathrm{P}} \mathcal{O}_2$ in terms of covectors and Hochstättler and Nickel [8] defined a 2-sum $\mathcal{O}_1 \oplus_2 \mathcal{O}_2$ via the sets of circuits. The purpose of this paper is to prove that these definitions are compatible, i. e.

$$\begin{aligned} \mathcal{O}_1 \oplus_2 \mathcal{O}_2 &= (\mathcal{O}_1 \oplus_{\mathrm{P}} \mathcal{O}_2) \setminus g \text{ and} \\ \mathcal{O}_1 \oplus_2 \mathcal{O}_2 &= (\mathcal{O}_1 \oplus_{\mathrm{S}} \mathcal{O}_2) \,/\, g. \end{aligned}$$

and to work out how 2-sum, series and parallel connection (see Figure 1) act on the different cryptomorphic axiom systems of oriented matroids. To take a leaf out of Brylawski's book [3], we will formulate these operations in terms of circuits, vectors, cocircuits, covectors, chirotopes, and the convex closure operator. Finally, we define a modular join and a modular sum of two oriented matroids in terms of cocircuits and covectors enabling us to glue together oriented matroids at suitable flats of arbitrary dimension.

## 2 Definitions and Notation

We assume familiarity with oriented matroid theory and freely use the notation defined in [1]. Let $\mathcal{O}_1, \mathcal{O}_2$ be two oriented matroids of rank $r_1$ resp. $r_2$ and element sets $E_1, E_2$. Let furthermore $\chi_i$, $\mathcal{C}_i$, $\mathcal{V}_i$ be the chirotope, set of signed circuits resp. set of signed vectors of $\mathcal{O}_i$ and $\mathcal{D}_i$ and $\mathcal{L}_i$ the sets of signed cocircuits resp. covectors for $i = 1, 2$. We denote by $\mathcal{M}_i := \underline{\mathcal{O}_i}$ the underlying

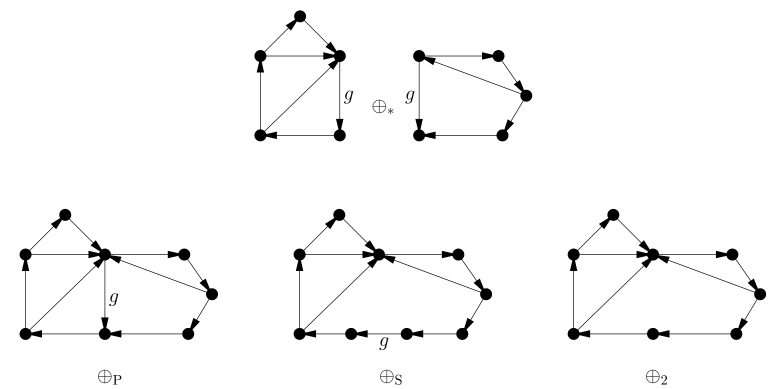


Figure 1: parallel, series, and 2-sum connection of two digraphs.

matroid of $\mathcal{O}_i$ and by $\mathfrak{B}_i$, $\mathfrak{F}_i$, $\mathfrak{D}_i$, $\mathfrak{r}_i$, resp. $\mathfrak{cl}_i$ its set of bases, flats, cocircuits, its rank function resp. matroid closure operator.

In the next section we will introduce the parallel, series, and 2-sum connection of $\mathcal{O}_1$ and $\mathcal{O}_2$ denoted by $\mathcal{O}_1 \oplus_\mathrm{P} \mathcal{O}_2$ (resp. $\oplus_\mathrm{S}, \oplus_2$). Occasionally, we will use $\oplus_*$ as a placeholder for $\oplus_\mathrm{P}, \oplus_\mathrm{S}, \oplus_2$. If we e. g. write $\mathcal{C}_1 \oplus_* \mathcal{C}_2$ or $\mathcal{C}_{\oplus_*}$ we actually mean $\mathcal{C}(\mathcal{O}_1 \oplus_* \mathcal{O}_2)$ (resp. $\mathfrak{B}, \mathcal{V}, \mathcal{L}, \mathcal{D}$).

For an arbitrary signed subset $F$ let $\underline{F} = \{e : F(e) \neq 0\}$ be the support of $F$, $z(F) := E \setminus \underline{F}$ its zero set, and ${}_{-}^{g}F := (F^+ \Delta(\{g\} \cap \underline{F}), F^- \Delta(\{g\} \cap \underline{F}))$ the *reorientation* on $g$, where $\Delta$ denotes the symmetric difference. For a family of signed subsets $\mathcal{F}$ let $\underline{\mathcal{F}} = \{\underline{F} \mid F \in \mathcal{F}\}$ be the family of supports and ${}_{-}^{g}\mathcal{F} := \{{}_{-}^{g}F : F \in \mathcal{F}\}$ the reorientation on $g$. If $T \subseteq E$ we write $F \cap T := (F^+ \cap T, F^- \cap T)$. For two signed subsets $F_1, F_2$ of $E_1, E_2$ let $F_1 \circ F_2$ denote the *composition* defined by

$$(F_1 \circ F_2)_e := \begin{cases} F_1(e) & \text{if } F_1(e) \neq 0 \\ F_2(e) & \text{otherwise.} \end{cases}$$

Let $T \subseteq E$ and $F_1 \cap T = F_2 \cap T$. Then we furthermore define the nonstandard operation *composition with $T$-deletion* by

$$F_1 \oslash_\mathrm{T} F_2 := (F_1 \circ F_2) \setminus T.$$

For two families $\mathcal{F}_i \subseteq 2^{E_i}$, $i = 1, 2$ let

$$\begin{aligned} \mathcal{F}_1 \circ \mathcal{F}_2 &:= \{F_1 \circ F_2 \mid F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2\} \text{ and} \\ \mathcal{F}_1 \circ_\mathrm{T} \mathcal{F}_2 &:= \{F_1 \circ F_2 \mid F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2,\ F_1 \cap T = F_2 \cap T \neq \emptyset\} \\ \mathcal{F}_1 \oslash_\mathrm{T} \mathcal{F}_2 &:= \{F_1 \oslash_\mathrm{T} F_2 \mid F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2,\ F_1 \cap T = F_2 \cap T \neq \emptyset\}. \end{aligned}$$

If $T = \{g\}$ for some $g \in E$ we write $X_1 \oslash_\mathrm{g} X_2$, $\mathcal{F}_1 \circ_\mathrm{g} \mathcal{F}_2$, resp. $\mathcal{F}_1 \oslash_\mathrm{g} \mathcal{F}_2$ instead. If

$\mathcal{F}$ is a family of signed subsets of $E$ and $T \subseteq E$, then we define

$$\begin{aligned} \mathcal{F}^{\setminus T} &:= \{F \in \mathcal{F} \mid F \cap T = \emptyset\} \\ \mathcal{F}^{T} &:= \{F \in \mathcal{F} \mid F \cap T \neq \emptyset\} \end{aligned}$$

and, again, write $\mathcal{F}^{\setminus g}$ resp. $\mathcal{F}^{g}$ instead of $\mathcal{F}^{\setminus \{g\}}$ resp. $\mathcal{F}^{\{g\}}$.

We need similar definitions for families $\mathfrak{X}$ of unsigned subsets of a finite set $E$ as well:

$$\begin{aligned} \mathfrak{X}^{\setminus T} &:= \{X \in \mathfrak{X} : X \cap T = \emptyset\} \\ \mathfrak{X}^{T} &:= \mathfrak{X} \setminus \mathfrak{X}^{\setminus T} \\ \mathfrak{X}_1 \circ \mathfrak{X}_1 &:= \{X \cup Y : X \in \mathfrak{X}_1, Y \in \mathfrak{X}_2\} \\ \mathfrak{X}_1 \circ_{\mathrm{T}} \mathfrak{X}_1 &:= \{X \cup Y : X \in \mathfrak{X}_1, Y \in \mathfrak{X}_2,\ X \cap T = Y \cap T \neq \emptyset\} \\ \mathfrak{X}_1 \oslash_{\mathrm{T}} \mathfrak{X}_1 &:= \{(X \cup Y) \setminus T : X \in \mathfrak{X}_1, Y \in \mathfrak{X}_2,\ X \cap T = Y \cap T \neq \emptyset\}. \end{aligned}$$

To connect two sets of bases we furthermore define

$$\mathfrak{B}_1 \vee_{\mathrm{g}} \mathfrak{B}_2 := \{(B_1 \cup B_2) \setminus g \mid B_1 \in \mathfrak{B}_1, B_2 \in \mathfrak{B}_2, g \in B_1 \Delta B_2\}.$$

### The Convex Closure Operator

The convex closure operator was defined by Folkman and Lawrence [6] and uses a slightly different notion of an oriented matroid. There, the oriented matroid acts on a set $\mathbb{E}$ with an involution $* : \mathbb{E} \to \mathbb{E}$. Actually, any element $e \in E$ is contained in $\mathbb{E}$ together with its "copy" $e^*$. With this notation, analog circuit axioms (see [4, Theorem 3]) characterize entire reorientation classes of oriented matroids but not single orientations. We use a modified version of this notation that appears in [7] and makes the convex closure more compatible with standard notation.

Let $\pm E := \{+e, -e \mid e \in E\}$ and for some $A \subseteq \pm E$ let $\sigma A^{\tau} := \{\sigma e \mid \tau e \in A\}$ for $\sigma, \tau \in \{+, -\}$ (partitioning $\mathbb{E} = +E \dot\cup -E$ this way chooses a particular orientation). We will always refer to an element of $\pm E$ together with its sign. For some signed subset $F = (F^+, F^-)$ of $E$ and a set $A \subseteq \pm E$ we write $F \subseteq A$ if $+F^+ \cup -F^- \subseteq A$ and we abbreviate $A \setminus e := A \setminus \{+e, -e\}$.

Büchi and Fenton [4] explicitly state the definition of an oriented matroid in terms of a convex closure operator (see also [1, Exercise 3.11]):

**Definition 1.** *A function* $\operatorname{conv} : 2^{\pm E} \to 2^{\pm E}$ *is called the* convex closure operator of an oriented matroid *if it satisfies*
*(CV1)* $\operatorname{conv}(\emptyset) = \emptyset$.
*(CV2)* $A \subseteq \operatorname{conv}(A) = \operatorname{conv}(\operatorname{conv}(A))$.
*(CV3)* $A \subseteq B \Rightarrow \operatorname{conv}(A) \subseteq \operatorname{conv}(B)$.
*(CV4)* $\operatorname{conv}(-A) = -\operatorname{conv}(A)$.
*(CV5)* $\sigma e \in \operatorname{conv}(A \cup -\sigma e) \Rightarrow \sigma e \in \operatorname{conv}(A)$.
*(CV6)* $\sigma e \in \operatorname{conv}(A \cup -\tau f)$ *and* $\sigma e \notin \operatorname{conv}(A) \Rightarrow \tau f \in \operatorname{conv}(A \setminus \tau f \cup -\sigma e)$.

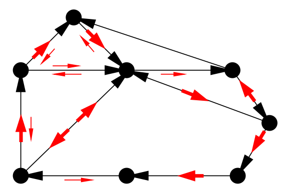

Figure 2: In case of a digraph, an arc $f$ is in $\mathrm{conv}(A)$ with the respective sign iff there is a dipath in $A$ connecting its endpoints. Bold arrows indicate that an arc is contained in $A$ with the respective sign and thin arrows beside an arc indicate that the arc is in $\mathrm{conv}(A) \setminus A$.

It is shown in [4, 7] that the convex closure operator yields a cryptomorphic characterization of oriented matroids. In particular

$$\begin{aligned}\mathcal{C}_{\mathrm{conv}} := \{&(A^+, A^-) \mid \\ &A = +A^+ \dot\cup -A^- \text{ is a minimal nonempty set with } -A \subseteq \mathrm{conv}(A)\}\end{aligned}$$

satisfies the circuit axioms of an oriented matroid $\mathcal{O}_{\mathrm{conv}}$ and

$$\mathrm{conv}(A) = A \cup \{\tau f \in \pm E \mid \exists C \in \mathcal{C}_{\mathrm{conv}} : -\tau f \in C \subseteq A \cup -\tau f\}.$$

# 3 Series, Parallel and 2-Sum Connection

In this section let $E_1 \cap E_2 = \{g\}$. The definition of the connections of $\mathcal{O}_1$ and $\mathcal{O}_2$ along $g$ will be given in terms of circuits, vectors, cocircuits, covectors, signed bases (chirotopes), and the convex closure operator.

## 3.1 Circuits and Vectors

Hochstättler and Nickel [8] introduced a 2-sum $\mathcal{O}_1 \oplus_2 \mathcal{O}_2$ via the sets of signed circuits:

$$\mathcal{C}_1 \oplus_2 \mathcal{C}_2 := \mathcal{C}_1^{\setminus g} \cup \mathcal{C}_2^{\setminus g} \cup (\mathcal{C}_1 \oslash_{\mathrm{g}} {\overset{g}{-}}\mathcal{C}_2).$$

At first we derive compatible series and parallel connection.

**Proposition 2.**

$$\begin{aligned}\mathcal{C}_1 \oplus_{\mathrm{P}} \mathcal{C}_2 &:= \mathcal{C}_1 \cup \mathcal{C}_2 \cup (\mathcal{C}_1 \oslash_{\mathrm{g}} {\overset{g}{-}}\mathcal{C}_2) \\ \mathcal{C}_1 \oplus_{\mathrm{S}} \mathcal{C}_2 &:= \mathcal{C}_1^{\setminus g} \cup \mathcal{C}_2^{\setminus g} \cup (\mathcal{C}_1 \circ_{\mathrm{g}} {\overset{g}{-}}\mathcal{C}_2)\end{aligned}$$

*Proof.* It is straightforward to verify that $\mathcal{C}_1 \oplus_* \mathcal{C}_2$ satisfy the circuit axioms of oriented matroid theory. We give the details only for $\mathcal{C}_1 \oplus_{\mathrm{S}} \mathcal{C}_2$ (see also [8, Proposition 6.1]). Obviously, $\mathcal{C}_{\oplus_{\mathrm{S}}}$ is antisymmetric and its support forms a clutter. It suffices to verify oriented circuit elimination for $C_1, C_2 \in \mathcal{C}_{\oplus_{\mathrm{S}}}$ with $e \in C_1^+ \cap C_2^-$. This is straightforward if at least one of $C_i$ is in $\mathcal{C}_i^{\backslash g}$. Thus, let $C_1 = C_{11} \circ C_{12}$ and $C_2 = C_{21} \circ C_{22}$ for circuits $C_{ij} \in \mathcal{C}_j^g$ $(i = 1, 2)$ and wlog. $e \in C_{11}^+ \cap C_{21}^- \subseteq E_1$. If $C_{11}(g) = -C_{21}(g)$ then also $C_{12}(g) = -C_{22}(g)$ and eliminating $g$ between $C_{12}$ and $C_{22}$ we find $C_3 \in \mathcal{C}_2^{\backslash g} \subseteq \mathcal{C}_{\oplus_{\mathrm{S}}}$ as required. Otherwise, $C_{11}(g) = C_{21}(g)$ and hence, $e \neq g$. Using circuit elimination in $\mathcal{O}_1$ by fixing $g$ we find a circuit $C_{31}$ with $C_{31} \subseteq (C_{11}^+ \cup C_{21}^+) \setminus e$ and $C_{31}^- \subseteq (C_{11}^- \cup C_{21}^-) \setminus e$. Now, $C_3 := C_{31} \circ C_{12}$ is a circuit as required. □

**Corollary 3.**

$$\mathcal{C}_1 \oplus_2 \mathcal{C}_2 = (\mathcal{C}_1 \oplus_{\mathrm{P}} \mathcal{C}_2) \setminus g = (\mathcal{C}_1 \oplus_{\mathrm{S}} \mathcal{C}_2) \,/\, g.$$

We will now determine $\mathcal{V}_{\oplus_*}$ from $\mathcal{C}_{\oplus_*}$.

**Corollary 4.**

$$\begin{aligned}
\mathcal{V}_1 \oplus_{\mathrm{P}} \mathcal{V}_2 &= \mathcal{V}_1 \circ \mathcal{V}_2 \cup \mathcal{V}_2 \circ \mathcal{V}_1 \cup (\mathcal{V}_1 \not\circ_{\mathrm{g}} \overset{g}{-} \mathcal{V}_2) \\
\mathcal{V}_1 \oplus_{\mathrm{S}} \mathcal{V}_2 &= \mathcal{V}_1^{\backslash g} \circ \mathcal{V}_2^{\backslash g} \cup (\mathcal{V}_1 \circ_{\mathrm{g}} \overset{g}{-} \mathcal{V}_2) \\
\mathcal{V}_1 \oplus_2 \mathcal{V}_2 &= \mathcal{V}_1^{\backslash g} \circ \mathcal{V}_2^{\backslash g} \cup (\mathcal{V}_1 \not\circ_{\mathrm{g}} \overset{g}{-} \mathcal{V}_2)
\end{aligned}$$

*Proof.* We work out details of the parallel connection only, the other cases being similar. "$\supseteq$" is trivial. For "$\subseteq$" let $V \in \mathcal{V}_{\oplus_{\mathrm{P}}}$. We know from Proposition 3.1 that $\mathcal{V}_{\oplus_{\mathrm{P}}}$ is the set of vectors of an oriented matroid. Therefore (see [1, Corollary 3.7.6]), $V$ is a *conformal* composition of circuits, i. e. $V = C_1 \circ \ldots \circ C_k$ with $C_i(e)C_j(e) \in \{0, +\}$ for all $e \in E$. If $g \notin V$ then $C_i \in \mathcal{C}_1^{\backslash g} \cup \mathcal{C}_2^{\backslash g} \cup (\mathcal{C}_1 \not\circ_{\mathrm{g}} \overset{g}{-} \mathcal{C}_2)$ and (since $\mathcal{V}_1^{\backslash g} \circ \mathcal{V}_2^{\backslash g} \subseteq \mathcal{V}_1 \circ \mathcal{V}_2$) $V \in \mathcal{V}_1 \circ \mathcal{V}_2 \cup (\mathcal{V}_1 \not\circ_{\mathrm{g}} \overset{g}{-} \mathcal{V}_2)$. Otherwise, assume wlog. $V(g) = +$. Replacing each circuit $C_j \in (\mathcal{C}_1 \not\circ_{\mathrm{g}} \overset{g}{-} \mathcal{C}_2)$ by $C_j^{i_1} \circ C_j^{i_2}$ with $i_1, i_2 \in \{1, 2\}$, $i_1 \neq i_2$ such that $(C_j^{i_1} \circ C_j^{i_2})(g) = +$, we may assume that $V = D_1 \circ \ldots \circ D_l$ with $D_i \in \mathcal{C}_1 \cup \mathcal{C}_2$. Note that this is conformal up to $g$. Let $i$ be the first index with $D_i(g) \neq 0$ (hence, $D_i(g) = +$) and wlog. $D_i \in \mathcal{C}_1^g$. Then we can rearrange the circuits $D_j$ so that

$$V = (D_{i_1} \circ \ldots \circ D_{i_m}) \circ (D_{j_1} \circ \ldots \circ D_{j_{m'}})$$

with $D_{i_\ell} \in \mathcal{C}_1$ $(\ell \in \{1, \ldots, m\})$ and $D_{j_\ell} \in \mathcal{C}_2$ $(\ell \in \{1, \ldots, m'\})$. Hence, $V = V_1 \circ V_2 \in \mathcal{V}_1 \circ \mathcal{V}_2$. □

## 3.2 Covectors and Cocircuits

The parallel connection in terms of covectors as proposed by Dong [5]

$$\mathcal{L}_1 \oplus_{\mathrm{P}} \mathcal{L}_2 := \mathcal{L}_1^{\backslash g} \circ \mathcal{L}_2^{\backslash g} \cup (\mathcal{L}_1 \circ_{\mathrm{g}} \mathcal{L}_2)$$

is compatible with the 2-sum of Hochstättler and Nickel [8]:

**Proposition 5.** *$\mathcal{L}_1 \oplus_{\mathrm{P}} \mathcal{L}_2$ is the set of covectors of $\mathcal{C}_1 \oplus_{\mathrm{P}} \mathcal{C}_2$.*

*Proof.* Dong [5, Proposition 4.2] proved for affine oriented matroids that $\underline{\mathcal{O}_1 \oplus_{\mathrm{P}} \mathcal{O}_2}$ is the parallel connection of $\underline{\mathcal{O}}_1$ and $\underline{\mathcal{O}}_2$, but the proof does not use affinity. To show that the signs are correct, it remains to verify that each covector is orthogonal to each circuit. Obviously, $\mathcal{L}_1^{\setminus g} \circ \mathcal{L}_2^{\setminus g}$ is orthogonal to $\mathcal{C}_{\oplus_{\mathrm{P}}}$ and $\mathcal{C}_i$ is orthogonal to $\mathcal{L}_{\oplus_{\mathrm{P}}}$ for $i = 1, 2$.

Now let $C = C_1 \oslash_{\mathrm{g}} C_2 \in \mathcal{C}_1 \oslash_{\mathrm{g}} {}^{\underline{g}}\mathcal{C}_2$ and $L = L_1 \circ L_2 \in \mathcal{L}_1 \circ_{\mathrm{g}} \mathcal{L}_2$, wlog. $C(g) = L(g) = +$, and assume for a contradiction that $L \not\perp C$. It follows that

$$
\begin{aligned}
&C_1(e)L_1(e) \in \{\sigma, 0\} \ \forall e \in E_1 \setminus g \text{ and} \\
&C_2(e)L_2(e) \in \{\sigma, 0\} \ \forall e \in E_2 \setminus g.
\end{aligned}
$$

for some $\sigma \in \{+, -\}$. Because of $L_i \perp C_i$ for $i = 1, 2$ we must have

$$
+ = C_1(g)L_1(g) = -\sigma = C_2(g)L_2(g) = -.
$$

□

As a direct consequence we can determine the set of signed covectors of the 2-sum by deletion of $g$ (resp. contraction of $g$ in the dual).

**Corollary 6.** *$\mathcal{L}_1 \oplus_2 \mathcal{L}_2 := \mathcal{L}_1^{\setminus g} \circ \mathcal{L}_2^{\setminus g} \cup (\mathcal{L}_1 \oslash_{\mathrm{g}} \mathcal{L}_2)$ is the set of signed covectors of $\mathcal{O}_1 \oplus_2 \mathcal{O}_2$.*

Corollary 4 together with Proposition 5 yield a nice analog to the duality of series and parallel connection (i. e. $(\underline{\mathcal{O}}_1 \oplus_{\mathrm{P}} \underline{\mathcal{O}}_2)^* = \underline{\mathcal{O}}_1^* \oplus_{\mathrm{S}} \underline{\mathcal{O}}_2^*$). In case of oriented matroids the sign of $g$ switches under this duality. For that purpose let ${}^{\underline{g}}\mathcal{O}_i$ be the reorientation of $\mathcal{O}_i$ with respect to $g$. Then

**Corollary 7.**

$$
\begin{aligned}
(\mathcal{O}_1 \oplus_{\mathrm{P}} \mathcal{O}_2)^* &= \mathcal{O}_1^* \oplus_{\mathrm{S}} {}^{\underline{g}}\mathcal{O}_2^* \quad \textit{and} \\
(\mathcal{O}_1 \oplus_{\mathrm{S}} \mathcal{O}_2)^* &= \mathcal{O}_1^* \oplus_{\mathrm{P}} {}^{\underline{g}}\mathcal{O}_2^*
\end{aligned}
$$

and, as a direct consequence, we get for the covectors of the series connection

**Corollary 8.** *The set of covectors of $\mathcal{C}_1 \oplus_{\mathrm{S}} \mathcal{C}_2$ is given by*

$$
\mathcal{L}_1 \oplus_{\mathrm{S}} \mathcal{L}_2 \;=\; \mathcal{L}_1 \circ {}^{\underline{g}}\mathcal{L}_2 \,\cup\, {}^{\underline{g}}\mathcal{L}_2 \circ \mathcal{L}_1 \,\cup\, (\mathcal{L}_1 \oslash_{\mathrm{g}} \mathcal{L}_2).
$$

Considering that cocircuits are covectors of minimal support we get

**Corollary 9.**

$$
\begin{aligned}
\mathcal{D}_1 \oplus_{\mathrm{P}} \mathcal{D}_2 &= \mathcal{D}_1^{\setminus g} \cup \mathcal{D}_2^{\setminus g} \cup (\mathcal{D}_1 \circ_{\mathrm{g}} \mathcal{D}_2) \\
\mathcal{D}_1 \oplus_{\mathrm{S}} \mathcal{D}_2 &= \mathcal{D}_1 \;\cup\; {}^{\underline{g}}\mathcal{D}_2 \;\cup (\mathcal{D}_1 \oslash_{\mathrm{g}} \mathcal{D}_2) \\
\mathcal{D}_1 \oplus_2 \mathcal{D}_2 &= \mathcal{D}_1^{\setminus g} \cup \mathcal{D}_2^{\setminus g} \cup (\mathcal{D}_1 \oslash_{\mathrm{g}} \mathcal{D}_2)
\end{aligned}
$$

### 3.3 Chirotopes

The set of bases of a matroid which is the parallel connection of matroids with bases $\mathfrak{B}_1$ and $\mathfrak{B}_2$ (see Brylawski [3]) is given by

$$\begin{aligned}\mathfrak{B}_1 \oplus_{\mathrm{P}} \mathfrak{B}_2 &= \mathfrak{B}_1^g \circ \mathfrak{B}_2^g \;\cup\; \mathfrak{B}_1^{\backslash g} \vee_{\mathrm{g}} \mathfrak{B}_2^g \;\cup\; \mathfrak{B}_1^g \vee_{\mathrm{g}} \mathfrak{B}_2^{\backslash g} \\ &= \mathfrak{B}_1^g \circ \mathfrak{B}_2^g \;\cup\; \mathfrak{B}_1 \vee_{\mathrm{g}} \mathfrak{B}_2.\end{aligned}$$

The signed analog of bases are called chirotopes. Recall that $\chi_i : \mathfrak{B}_i \to \{+,-,0\}$ are the chirotopes of $\mathcal{O}_i$. From now on assume that bases are always given with an ordering. If $B = B_1 \cup B_2$, $B_i \in \mathfrak{B}_i$ then let $B_i$ have the ordering induced by $B$.

**Lemma 10.** *The function $\chi_{\oplus_{\mathrm{P}}} : \mathfrak{B}_{\oplus_{\mathrm{P}}} \to \{+,-,0\}$ defined by*

$$\chi_{\oplus_{\mathrm{P}}}(B) := \chi_1(B_1)\chi_2(B_2),$$

*where $B \in \mathfrak{B}_{\oplus_{\mathrm{P}}}$ and therefore, $B = B_1 \cup B_2$ with $g \in B_1 \cap B_2$ resp. $B = (B_1 \cup B_2) \setminus g$ with $g \in B_i$ for exactly one $i \in \{1,2\}$ for some $B_1 \in \mathfrak{B}_1, B_2 \in \mathfrak{B}_2$, is the chirotope of $\mathcal{O}_1 \oplus_{\mathrm{P}} \mathcal{O}_2$*

*Proof.* We prove that $\chi_{\oplus_{\mathrm{P}}}$ defined as above is a proper basis orientation of $\mathfrak{B}_{\oplus_{\mathrm{P}}}$ (see e. g. [1, Definition 3.5.1]). Let $B, B' \in \mathfrak{B}_{\oplus_{\mathrm{P}}}$ with $|B \cap B'| = r_1 + r_2 - 2$ and

$$\begin{aligned} e_1 &:= B \setminus B' \\ e_2 &:= B' \setminus B \\ X_1 &:= (B \setminus e_1) \cap E_1 = (B' \setminus e_2) \cap E_1 \\ X_2 &:= (B \setminus e_1) \cap E_2 = (B' \setminus e_2) \cap E_2 \quad \text{and thus} \\ B &= e_1 \cup X_1 \cup X_2 \quad \text{and} \\ B' &= e_2 \cup X_1 \cup X_2 \end{aligned}$$

and let $C$ be the (up to sign reversal unique) circuit in $\{e_1, e_2\} \cup X_1 \cup X_2$. We have to show that

$$\chi_{\oplus_{\mathrm{P}}}(B') = -C(e_1)C(e_2)\chi_{\oplus_{\mathrm{P}}}(B).$$

We consider the following cases:

$X_1 \cap X_2 = \{g\}$ : Hence, $B, B' \in \mathfrak{B}_1^g \circ \mathfrak{B}_2^g$. Since $|X_1 \cap X_2| = 1$ and $|X_1 \cup X_2| = r_1 + r_2 - 2$ we may assume wlog. that $|X_2| = r_2$. Thus, $X_2 \in \mathfrak{B}_2^g$ and since $X_2 \cup e_i$ is independent for $i = 1, 2$, we must have $e_1, e_2 \in E_1 \setminus g$. The claim follows since $\chi_1$ is a proper basis orientation of $\mathfrak{B}_1$.

$X_1 \cap X_2 = \emptyset$ : If $e_1, e_2 \in E_i$ for some $i$, again the claim follows since $\chi_i$ is a proper basis orientation of $\mathfrak{B}_i$. Otherwise, let wlog. $e_i \in E_i$ and $B \in \mathfrak{B}_1^{\backslash g} \vee_{\mathrm{g}} \mathfrak{B}_2^g$ implying that $X_1 \cup e_1$ and $X_2 \cup g$ are bases. Then $B' \in \mathfrak{B}_{\oplus_{\mathrm{P}}}$ implies that $B'$ must be in $\mathfrak{B}_1^g \vee_{\mathrm{g}} \mathfrak{B}_2^{\backslash g}$ and therefore, $X_1 \cup g$ and $X_2 \cup e_2$ must be bases as well. Since $C$ must contain $e_1$ and $e_2$, we have that

$C = C_1 \circ_{\mathrm{g}} C_2$ for circuits $C_1 \in \mathcal{C}_1^g$ and $C_2 \in \mathcal{C}_2^g$ with $C_1(g) = -C_2(g)$. Hence, we can exchange $e_1$ with $g$ and on the other side $g$ with $e_2$ and obtain

$$\begin{aligned}
\chi_{\oplus_{\mathrm{P}}}(B') &= \chi_1(X_1 \cup g)\chi_2(X_2 \cup e_2) \\
&= (-C_1(g)C_1(e_1)\chi_1(X_1 \cup e_1)) \cdot \\
&\quad \cdot (-C_2(e_2)C_2(g)\chi_2(X_2 \cup g)) \\
&\overset{C_1(g)C_2(g)=-}{=} -C_1(e_1)C_2(e_2)\chi_1(X_1 \cup e_1)\chi_2(X_2 \cup g) \\
&= -C(e_1)C(e_2)\chi_{\oplus_{\mathrm{P}}}(B).
\end{aligned}$$

□

Deleting $g$ from $\mathcal{O}_1 \oplus_{\mathrm{P}} \mathcal{O}_2$ we conclude that (c. f. [3])

$$\mathfrak{B}_1 \oplus_2 \mathfrak{B}_2 = \mathfrak{B}_1^{\setminus g} \vee_{\mathrm{g}} \mathfrak{B}_2^g \, \cup \, \mathfrak{B}_1^g \vee_{\mathrm{g}} \mathfrak{B}_2^{\setminus g}$$

and

**Corollary 11.** *The chirotope $\chi_{\oplus_2}$ of $\mathcal{O}_1 \oplus_2 \mathcal{O}_2$ is given by*

$$\chi_{\oplus_2}(B) = \chi_1(B_1)\chi_2(B_2),$$

*where $B \in \mathfrak{B}_{\oplus_2}$ with $B = (B_1 \cup B_2) \setminus g$ for $B_i \in \mathfrak{B}_i,\ i = 1, 2$ and $g \in B_i$ for exactly one $i$.*

The basis orientation of the series connection is governed by the set of bases of the underlying matroid (see [3]) as well:

$$\mathfrak{B}_1 \oplus_{\mathrm{S}} \mathfrak{B}_2 = \mathfrak{B}_1^{\setminus g} \circ \mathfrak{B}_2^{\setminus g} \, \cup \, \mathfrak{B}_1^{\setminus g} \circ \mathfrak{B}_2^g \ \ \cup \, \mathfrak{B}_1^g \circ \mathfrak{B}_2^{\setminus g}.$$

**Proposition 12.** *The chirotope $\chi_{\oplus_{\mathrm{S}}}$ of $\mathcal{O}_1 \oplus_{\mathrm{S}} \mathcal{O}_2$ is given by*

$$\chi_{\oplus_{\mathrm{S}}}(B) = \chi_1(B_1)\chi_2(B_2),$$

*where $B \in \mathfrak{B}_{\oplus_{\mathrm{S}}}$ and therefore, $B = B_1 \dot{\cup} B_2$ for some $B_1 \in \mathfrak{B}_1, B_2 \in \mathfrak{B}_2$.*

*Proof.* If $e_1, e_2 \in E_i$ for some $i$, the claim follows since $\chi_i$ is a proper basis orientation of $\mathcal{O}_i$.

Let $B, B' \in \mathfrak{B}_{\oplus_{\mathrm{S}}}$ with $|B \cap B'| = r_1 + r_2 - 2$ and $e_1, e_2, X_1, X_2, C$ be defined as in Lemma 10. If $g \notin X_1 \cup X_2$, then we conclude $e_1, e_2 \in E_i$ for $i \in \{1, 2\}$ as follows. If $g \notin \{e_1, e_2\}$, this is immediate. Otherwise, if $e_1 = g$, we may wlog. assume that $X_1 \cup g$ and $X_2$ are bases and hence, $e_1, e_2 \in E_1$.

The only remaining case is $g \in X_1 \cup X_2$ and wlog. $e_i \in E_i$ for $i = 1, 2$. By the definition of the $X_i$ we have $g \in X_1 \cap X_2$. Because $B \in \mathfrak{B}_1^{\setminus g} \circ \mathfrak{B}_2^{\setminus g}$ implies $e_1, e_2 \in E_i$ for some $i$, we may, by symmetry, assume $B \in \mathfrak{B}_1^{\setminus g} \circ \mathfrak{B}_2^g$ and

$B' \in \mathfrak{B}_1^g \circ \mathfrak{B}_2^{\setminus g}$. Thus, $X_1, X_1 \setminus g \cup e_1, X_2$, and $X_2 \setminus g \cup e_2$ are bases of $\underline{\mathcal{O}}_1$ resp. $\underline{\mathcal{O}}_2$. Hence,

$$\begin{aligned}
\chi_{\oplus_S}(B') &= \chi_1(X_1)\chi_2(X_2 \setminus g \cup e_2) \\
&= (-C_1(g)C_1(e_1)\chi(X_1 \setminus g \cup e_1)) \cdot \\
&\quad \cdot (-C_2(e_2)C_2(g)\chi_2(X_2)) \\
&\overset{C_1(g)C_2(g)=-}{=} C_1(e_1)C_2(e_2)\chi_1(X_1 \setminus g \cup e_1)\chi_2(X_2) \\
&= -C(e_1)C(e_2)\chi_{\oplus_S}(B).
\end{aligned}$$

□

**Remark 13.** *Björner et al. [1, Section 7.6] considered two special cases of oriented matroid union, i. e. disjoint ground sets (direct sum) and equal ground sets. As (unoriented) series connection is a special case of matroid union (see e. g. [9, Proposition 12.3.6]), Proposition 12 is another special case of oriented matroid union.*

## 3.4 Convex Closure

We are now going to determine the convex hull operators $\mathrm{conv}_{\oplus_*}$ of $\mathcal{O}_{\oplus_*}$ from the sets of circuits. For $\mathcal{O}_i$ let $\mathrm{conv}_i := \mathrm{conv}_{\mathcal{C}_i}$ $(i = 1, 2)$ and $\mathrm{conv}_{\oplus_*} := \mathrm{conv}_{\mathcal{C}_1 \oplus_* \mathcal{C}_2}$. We identify $\mathrm{conv}_i(A)$ with $\mathrm{conv}_i(A \cap E_i)$ for arbitrary sets $A$ $(i = 1, 2)$ simplifying notation.

First we consider the parallel connection:

**Theorem 14.** *Let*

$$\begin{aligned}
G_1 &:= \{+g, -g\} \cap \mathrm{conv}_2(A) \quad \textit{and} \\
G_2 &:= \{+g, -g\} \cap \mathrm{conv}_1(-A).
\end{aligned}$$

*Then* $\mathrm{conv}_{\oplus_P}(A) = \mathrm{conv}_1(A \cup G_1) \cup \mathrm{conv}_2(A \cup G_2)$.

*Proof.* "$\subseteq$" Let $\tau f \in \mathrm{conv}_{\oplus_P}(A)$ for some $\tau \in \{+, -\}$ and wlog. $\tau = +$. The case $+f \in A$ is trivial and otherwise, there is a circuit $C \in \mathcal{C}_{\oplus_P}$ such that $-f \in C \subseteq A \cup -f$. If $C \in \mathcal{C}_i$ for $i = 1$ or 2, then $+f \in \mathrm{conv}_i(A) \subseteq \mathrm{conv}_i(A \cup G_i)$. Otherwise, there is a circuit $C = C_1 \circ_g C_2 \in \mathcal{C}_1 \circ_g \overset{g}{-} \mathcal{C}_2$. If $-f \in C_1$ and $C_1(g) = \sigma$, we have the implication

$$\begin{aligned}
-f \in C_1 &\Rightarrow C_2 \setminus g \subseteq A \\
&\Rightarrow \sigma g \in \mathrm{conv}_2(A) \\
&\Rightarrow \sigma g \in G_1 \\
&\Rightarrow -f \in C_1 \subseteq A \cup G_1 \cup -f \\
&\Rightarrow +f \in \mathrm{conv}_1(A \cup G_1).
\end{aligned}$$

An analogous argument verifies $-f \in C_2 \Rightarrow +f \in \mathrm{conv}_2(A \cup G_2)$.

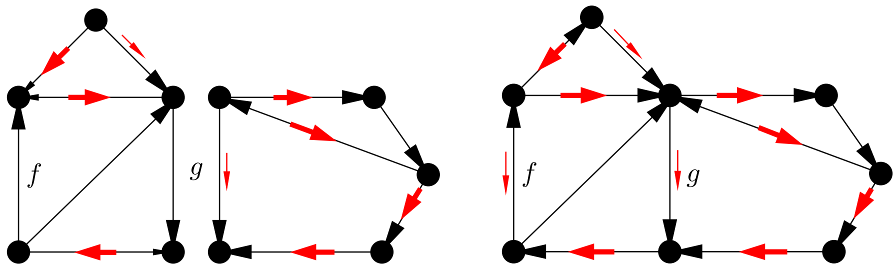


Figure 3: $-f$ is contained in $\mathrm{conv}_{\oplus_\mathrm{P}}(A)$ but not in $\mathrm{conv}_1(A)$

"$\supseteq$" Let $\tau f \in \mathrm{conv}_1(A \cup G_1)$. Again we may assume $\tau = +$. If $+f \in \mathrm{conv}_i(A)$ or $G_i = \emptyset$ then $+f \in \mathrm{conv}_{\oplus_\mathrm{P}}(A)$ follows immediately. If $+f \in \mathrm{conv}_1(A \cup G_1) \setminus \mathrm{conv}_1(A)$ and $\sigma g \in G_1$ then there is a circuit $C_1 \in \mathcal{C}_1^g$, $C_1(g) = \sigma$ so that $-f \in C_1 \subseteq A \cup G_1 \cup -f$ and a circuit $C_2 \in \mathcal{C}_2^g$, $C_2(g) = -\sigma$, satisfying $C_2 \subseteq A \cup -\sigma g$. It follows that $-f \in C_1 \not\circ_\mathrm{g} C_2 \subseteq A \cup -f$ and therefore, $+f \in \mathrm{conv}_{\oplus_\mathrm{P}}(A)$.

The case $\tau f \in \mathrm{conv}_2(A \cup G_2)$ is analogous.

$\square$

In Figure 3 you see an example for the convex closure operator of the parallel connection where an edge $f$ is contained in the convex closure of the parallel connection but not in $\mathrm{conv}_i(A)$. Red arrows indicate that an arc is in the convex closure of $A$ with the respective sign while bold red arcs are the elements of $A$.

We will derive the convex closure operator of the 2-sum $\mathcal{O}_1 \oplus_2 \mathcal{O}_2$ from the operator of the series connection.

**Theorem 15.** *Let*

$$\begin{aligned} G_1 &:= \{+g, -g\} \cap A \cap \mathrm{conv}_2(A \setminus g) \\ G_2 &:= \{+g, -g\} \cap A \cap \mathrm{conv}_1(-A \setminus g). \end{aligned}$$

*Then* $\mathrm{conv}_{\oplus_\mathrm{S}}(A) = A \cup \mathrm{conv}_1(A \setminus g \cup G_1) \cup \mathrm{conv}_2(A \setminus g \cup G_2)$.

*Proof.* "$\subseteq$" Let $\tau f \in \mathrm{conv}_{\oplus_\mathrm{S}}(A)$. If $\tau f \in A$ or $\tau f \in \mathrm{conv}_i(A \setminus g)$ for $i = 1$ or 2 the claim is true. It remains to consider

$$\tau f \in \mathrm{conv}_{\oplus_\mathrm{S}}(A) \setminus (A \cup \mathrm{conv}_1(A \setminus g) \cup \mathrm{conv}_2(A \setminus g))$$

and wlog. $\tau = +$. Then there is a circuit $C = C_1 \circ C_2 \in \mathcal{C}_1 \circ_\mathrm{g} \overset{g}{-} \mathcal{C}_2$ (wlog. $C(g) = +$) satisfying $-f \in C \subseteq A \cup -f$. It follows that $+g \in A$ and hence, $f \in \mathrm{conv}_1(A \setminus g \cup G_1)$. If $f \in C_1$, then $-g \in C_2 \subseteq A \cup -g$ and hence, $+g \in \mathrm{conv}_2(A \setminus g)$.

Note that

$$\begin{aligned} -G_1 &= \{+g, -g\} \cap A \cap \mathrm{conv}_2(-A \setminus g) \quad \text{and} \\ -G_2 &= \{+g, -g\} \cap A \cap \mathrm{conv}_1(A \setminus g). \end{aligned}$$

Hence, the case $f \in C_2$ follows by symmetry, reorienting $g$.

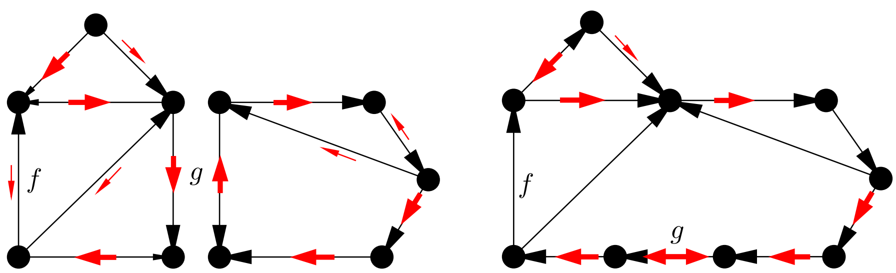


Figure 4: $-f$ is contained in $\mathrm{conv}_1(A)$ but not in $\mathrm{conv}_{\oplus_S}(A)$

"⊇" Let $\tau f \in \mathrm{conv}_1(A \setminus g \cup G_1)$. The cases $\tau f \in A \cup \mathrm{conv}_1(A \setminus g)$ and $G_1 = \emptyset$ are trivial. Thus,

$$\tau f \in \mathrm{conv}_1(A \setminus g \cup G_1) \setminus (A \cup \mathrm{conv}_1(A \setminus g)),$$

wlog. $\tau = +$, and $\sigma g \in G_1$. We consider the case $f \notin \{\pm g\}$ first. Hence, there is a circuit $C_1 \in \mathcal{C}_1^g$ with $C_1(g) = \sigma$ and $-f \in C_1 \subseteq A \setminus g \cup G_1 \cup -f$. Since $\sigma g \in G_1 \subseteq \mathrm{conv}_2(A \setminus g)$, we also have a circuit $C_2 \in \mathcal{C}_2^g$ satisfying $C_2(g) = -\sigma$ and $C_2 \subseteq A \cup -\sigma g$. It follows that $-f \in C_1 \circ C_2 \subseteq A \cup -f$ and $+f \in \mathrm{conv}_{\oplus_S}(A)$.

Now let $f = g$ and hence, $+g \in \mathrm{conv}_1(A \setminus g \cup G_1) \setminus \mathrm{conv}_1(A \setminus g)$. Then $+g \in G_1 \subseteq A$ and hence, $+g \in \mathrm{conv}_{\oplus_S}(A)$ completing the proof.

□

Figure 4 shows how arcs can be contained in the convex closure of one of the graphs but not in the convex closure of their series connection if $g \notin \mathrm{conv}_i(A \setminus g)$. By contraction we obtain the result for the 2-sum.

**Corollary 16.** *Let*

$$\begin{aligned} G_1 &:= \{+g, -g\} \cap \mathrm{conv}_2(A \setminus g) \\ G_2 &:= \{+g, -g\} \cap \mathrm{conv}_1(-A \setminus g). \end{aligned}$$

*Then* $\mathrm{conv}_{\oplus_2}(A) = \mathrm{conv}_{\oplus_S}(A) \setminus g = (\mathrm{conv}_1(A \setminus g \cup G_1) \cup \mathrm{conv}_2(A \setminus g \cup G_2)) \setminus g$.

# 4 Generalized Parallel Connection, Modular Join, and Modular Sum

During this section $\mathcal{O}_1, \mathcal{O}_2$ are oriented matroids on the ground sets $E_1$ resp. $E_2$ such that $E_1 \cap E_2 = T$ and $\mathcal{O}_1[T] = \mathcal{O}_2[T]$. The underlying matroids are $\mathcal{M}_i := \underline{\mathcal{O}}_i$ and have the set of flats $\mathfrak{F}_i$, rank function $\mathfrak{r}_i$ and matroid closure operator $\mathfrak{cl}_i$ for $i = 1, 2$.

**Definition 17.** *Let $\mathfrak{F}$ denote the family of flats of a matroid $\mathcal{M}$ with rank function $\mathfrak{r}$. We call two flats $X, Y \in \mathfrak{F}$ a* modular pair *if*

$$\mathfrak{r}(X) + \mathfrak{r}(Y) = \mathfrak{r}(X \cup Y) + \mathfrak{r}(X \cap Y).$$

*A flat $T$ is* modular *if for all $X \in \mathfrak{F}$ $X, T$ is a modular pair of flats.*

We will introduce the modular join of $\mathcal{O}_1$ and $\mathcal{O}_2$ as an oriented version of a special case of the generalized parallel connection from matroid theory (see e. g. [3]). First we review the basics of the generalized parallel connection from matroid theory including some seemingly new observations.

**Proposition 18** ([3])**.** *If $T \in \mathfrak{F}_1$ is a modular flat of $\mathcal{M}_1$ and $T \in \mathfrak{F}_2$ then the set*

$$\mathfrak{F}_{\oplus_T} := \{F : F \cap E_i \in \mathfrak{F}_i \text{ for } i = 1, 2\}$$

*is the set of flats of a matroid, called the* generalized parallel connection *of $M_1$ and $M_2$ denoted by $\mathcal{M}_{\oplus_T}$.*

**Remark 19.** *If $T$ is not a flat in $\mathcal{M}_2$ then one can extend $\mathcal{M}_1$ by the elements $\mathfrak{cl}_2(T) \backslash T$ via the modular cut $\{T\}$ yielding a matroid $\hat{\mathcal{M}}_1$ in which $\hat{T} := \mathfrak{cl}_2(T)$ is a modular flat. The generalized parallel connection of $\mathcal{M}_1$ and $\mathcal{M}_2$ is defined to be the generalized parallel connection of $\hat{\mathcal{M}}_1$ and $\mathcal{M}_2$ with respect to the common flat $\hat{T}$. For details on modular cuts and single element extensions we refer the reader to [9].*

The rank function of the generalized parallel connection $\mathcal{M}_{\oplus_T}$ is given by the following proposition.

**Proposition 20** ([2, Proposition 5.5])**.** *If $\mathfrak{r}_{\oplus_T}, \mathfrak{r}_1, \mathfrak{r}_2$ are the rank functions of $\mathcal{M}_{\oplus_T}, \mathcal{M}_1, \mathcal{M}_2$ respectively, then for any $F \in \mathfrak{F}_{\oplus_T}$ we have*

$$\mathfrak{r}_{\oplus_T}(F) = \mathfrak{r}_1(F \cap E_1) + \mathfrak{r}_2(F \cap E_2) - \mathfrak{r}_1(F \cap T).$$

As a direct consequence, the rank of the generalized parallel connection is $\operatorname{rank}(\mathcal{O}_1) + \operatorname{rank}(\mathcal{O}_2) - \mathfrak{r}_{\oplus_T}(T)$.

**Proposition 21** ([2, Proposition 5.10])**.**

$$E_1 \in \mathfrak{F}_{\oplus_T} \iff T \in \mathfrak{F}_{\oplus_T} \iff T \in \mathfrak{F}_2.$$

Hence under the above assumption, $E_1$, $E_2$, and $T$ are flats of $\mathcal{M}_{\oplus_T}$. From now on we, additionally, assume that $T$ is a common modular flat of $\mathcal{M}_1$ and $\mathcal{M}_2$. As a preparatory step to defining the modular join for oriented matroids, first we derive the modular join of two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ in terms of its cocircuits.

**Proposition 22.** *The set of cocircuits of the modular join $\mathcal{M}_{\oplus_T} = \mathcal{M}_1 \oplus_T \mathcal{M}_2$ is*

$$\mathfrak{D}_{\oplus_T} = \mathfrak{D}_1^{\backslash T} \cup \mathfrak{D}_2^{\backslash T} \cup (\mathfrak{D}_1^T \circ_{\mathrm{T}} \mathfrak{D}_2^T).$$

*Proof.* By Proposition 20 and since $T$ is a modular flat in $\mathcal{M}_1$ and $\mathcal{M}_2$, we have for any flat $H \in \mathfrak{F}_{\oplus_T}$ and $H_i := H \cap E_i$, $i = 1, 2$

$$\begin{aligned}\mathfrak{r}_{\oplus_T}(H) &= \mathfrak{r}_{\oplus_T}(H_1) + \mathfrak{r}_{\oplus_T}(H_2) - \mathfrak{r}_{\oplus_T}(H_1 \cap H_2)\\ &= \mathfrak{r}_{\oplus_T}(H_1) - \mathfrak{r}_{\oplus_T}(H_1 \cap T) + \mathfrak{r}_{\oplus_T}(H_2)\\ &= \mathfrak{r}_{\oplus_T}(H_1 \cup T) - \mathfrak{r}_{\oplus_T}(T) + \mathfrak{r}_{\oplus_T}(H_2),\end{aligned}$$

and by symmetry

$$\mathfrak{r}_{\oplus_T}(H) = \mathfrak{r}_{\oplus_T}(H_1) + \mathfrak{r}_{\oplus_T}(H_2 \cup T) - \mathfrak{r}_{\oplus_T}(T).$$

Again by Proposition 20, $\mathfrak{r}_{\oplus_T}$ equals $\mathfrak{r}_i$ when restricted to $E_i$. Hence, a closed set $H \in \mathfrak{F}_{\oplus_T}$ is a hyperplane (i. e. has rank $r_1 + r_2 - \mathfrak{r}_{\oplus_T}(T) - 1$) if and only if

$$\begin{aligned}r_1 + r_2 - 1 &= \mathfrak{r}_1(H_1 \cup T) + \mathfrak{r}_2(H_2)\\ &= \mathfrak{r}_1(H_1) + \mathfrak{r}_2(H_2 \cup T),\end{aligned}$$

meaning that exactly one of the following cases applies:

(1) $H_1 = E_1$ and $H_2$ is a hyperplane of $\mathcal{M}_2$ completely containing $T$,

(2) $H_2 = E_2$ and $H_1$ is a hyperplane of $\mathcal{M}_1$ completely containing $T$,

(3) $H_i$ are hyperplanes in $\mathcal{M}_i$ for $i = 1, 2$ which do not contain $T$ completely.

In case (1) resp. (2) $H$ is a hyperplane whose complement is a cocircuit in $\mathcal{M}_2$ resp. $\mathcal{M}_1$ and in case (3) $H_1$ and $H_2$ are complements of cocircuits $D_1, D_2$ with $D_1 \cap T = D_2 \cap T \neq \emptyset$. □

We are now aiming to define an oriented modular join with respect to a common modular flat $T$ as an oriented analogue of Proposition 22. We will prove that this is well defined in Theorem 25 and start with some observations.

**Proposition 23.** *Let $T$ be a modular flat of a matroid $\mathcal{M}$ and $C$ a cocircuit such that $C \cap T \notin \{\emptyset, T\}$. Then $C \cap T$ is a cocircuit of $\mathcal{M}[T]$.*

*Proof.* Let $\mathfrak{r}$ be the rank function of $\mathcal{M}$. By modularity,

$$\begin{aligned}\mathfrak{r}(z(C) \cap T) &= \mathfrak{r}(z(C)) + \mathfrak{r}(T) - \mathfrak{r}(z(C) \cup T)\\ &= \operatorname{rank}(\mathcal{M}) - 1 + \mathfrak{r}(T) - \operatorname{rank}(\mathcal{M})\\ &= \mathfrak{r}(T) - 1.\end{aligned}$$

□

The following observation will be crucial for an inductive proof of the correctness of our join operation.

**Lemma 24.** *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be simple oriented matroids on the ground sets $E_1 \cap E_2 = T$ such that $\mathcal{O}_1[T] = \mathcal{O}_2[T]$ and $T$ is a common modular flat of rank 2. Then*

$$\mathcal{D}_{\oplus_T} := \mathcal{D}_1^{\setminus T} \cup \mathcal{D}_2^{\setminus T} \cup (\mathcal{D}_1^T \circ_{\mathrm{T}} \mathcal{D}_2^T)$$

*is the family of cocircuits of an oriented matroid.*

*Proof.* Wlog. $\mathcal{O}$ is reoriented such that every $X \in \mathcal{D}$ restricted to the elements of $T$ has one of the sign patterns $0 \ldots 0$, $- \ldots - 0 + \ldots +$, or $+ \ldots + 0 - \ldots -$ wrt. a fixed linear ordering of the elements of $T$. Let $C \neq -D$ be elements of $\mathcal{D}_{\oplus_T}$ such that $e \in C^+ \cap D^-$. We proceed by case study:

(1) $e \in T$: Then $C = C_1 \circ C_2, D = D_1 \circ D_2 \in \mathcal{D}_1^T \circ_{\mathrm{T}} \mathcal{D}_2^T$. If $C \cap T = -D \cap T$ elimination between $C_1$ and $D_1$ yields a cocircuit $F_1 \in \mathcal{D}_1^{\setminus T}$ such that $F_1^\sigma \subseteq (C_1^\sigma \cup D_1^\sigma) \setminus e$ for $\sigma = +, -$. Otherwise, $\underline{C \cap T} \neq \underline{D \cap T}$ and there is some $f \in T \cap ((C_1^+ \setminus D_1^-) \cup (C_1^- \setminus D_1^+))$ and we can perform strong cocircuit elimination between $C_i, D_i$ for $i = 1, 2$ with respect to $e$ by fixing $f$ which yields cocircuits $F_i \in \mathcal{D}_i^T$ satisfying $F_1 \cap T = F_2 \cap T$ since $T$ is a modular line. Hence, $F_1 \circ F_2 \in \mathcal{D}_1^T \circ_{\mathrm{T}} \mathcal{D}_2^T$ is a cocircuit as required.

(2) Wlog. $e \in E_1 \setminus T$: Let $F_1 \in \mathcal{D}_1$ be a cocircuit satisfying $F_1^\sigma \subseteq (C_1^\sigma \cup D_1^\sigma) \setminus e$. We are done if $F_1 \in \mathcal{D}_1^{\setminus T}$. Otherwise, let $f_C, f_D, f_F$ be the unique elements in $z(C_1) \cap T, z(D_1) \cap T$ resp. $z(F_1) \cap T$.

   (i) $f_F = f_C = f_D$: Then $F_1 \circ C_2$ or $F_1 \circ D_2$ is a cocircuit as required.

   (ii) $f_F = f_C \neq f_D$: Assume $F_1 \cap T = -C_1 \cap T$. Then $F_1 \cap T \subseteq D_1 \cap T$, a contradiction, as $F_1(f_D) \neq 0$. Hence, $F_1 \cap T = C_1 \cap T$ and $F_1 \circ C_2$ is a cocircuit as required.

   (iii) $|\{f_C, f_D, f_F\}| = 3$: By cocircuit elimination we necessarily must have $C(f_F) = -D(f_F) \neq 0$. We eliminate $f_F$ between $C_2$ and $D_2$ in $\mathcal{O}_2$ and get a cocircuit that either is in $\mathcal{D}_2^{\setminus T}$ as required or satisfies $F_2(f_F) = 0$. Since $F_1(f_C) = F_2(f_C) = D(f_C)$, we must have $F_1 \cap T = F_2 \cap T$ and $F_1 \circ F_2$ is a cocircuit as required.

□

**Theorem 25.** *Let $\mathcal{O}_1, \mathcal{O}_2$ be oriented matroids with a common modular flat $T = E_1 \cap E_2$. Then*

$$\mathcal{D}_{\oplus_T} := \mathcal{D}_1^{\setminus T} \cup \mathcal{D}_2^{\setminus T} \cup (\mathcal{D}_1^T \circ_{\mathrm{T}} \mathcal{D}_2^T)$$

*is the family of signed cocircuits of an oriented matroid, called the* modular join *of $\mathcal{O}_1$ and $\mathcal{O}_2$, denoted by $\mathcal{O}_{\oplus_T}$.*

*Proof.* We may wlog. assume that $\mathcal{O}_1$ and $\mathcal{O}_2$ are simple. We prove the theorem by induction on $|T|$. For $|T| \in \{0, 1\}$ the statement corresponds to the signed cocircuits of direct sum resp. parallel connection (empty set and single edges are always modular flats). Now let $|T| \geq 2$ and $C, D \in \mathcal{D}_{\oplus_T}$ such that $C \neq -D$ and $e \in C^+ \cap D^-$. If there exists some $f \in z(C) \cap z(D) \cap T$ then $C, D \in \mathcal{D}_{\oplus_T} / f$

and by inductive assumption, there is some $F \in \mathcal{D}_{\oplus_T} / f \subset \mathcal{D}_{\oplus_T}$ satisfying $F^+ \subseteq (C^+ \cup D^+) \setminus e$ and $F^- \subseteq (C^- \cup D^-) \setminus e$. We now may assume that $z(C) \cap z(D) \cap T = \emptyset$.

Since $z(C_1) \cap T$ is a modular flat in $\mathcal{O}_1$, we have

$$\begin{aligned} 0 = \mathfrak{r}(z(C_1) \cap z(D_1) \cap T) &= \mathfrak{r}((z(C_1) \cap T) \cap (z(D_1) \cap T)) \\ &= \mathfrak{r}(z(C_1) \cap T) + \mathfrak{r}(z(D_1) \cap T) \\ &\quad - \mathfrak{r}((z(C_1) \cap T) \cup (z(D_1) \cap T) \\ &= \mathfrak{r}(T) - 2. \end{aligned}$$

It thus suffices to consider the case that $\mathfrak{r}(T) = 2$ which was done in Lemma 24. $\square$

**Corollary 26.**

$$\mathcal{L}_1 \oplus_T \mathcal{L}_2 = \mathcal{L}_1^{\setminus T} \circ \mathcal{L}_2^{\setminus T} \cup (\mathcal{L}_1 \circ_{\mathrm{T}} \mathcal{L}_2).$$

Please note the analogy to the parallel connection. Furthermore, it is now immediate to define the modular sum of two oriented matroids as a generalization of 2-sum.

**Definition 27.** *Let $\mathcal{O}_1, \mathcal{O}_2$ be oriented matroids on the ground sets $E_1 \cap E_2 = T$ so that $T$ is a common modular flat. The* modular sum $\mathcal{O}_1 \oplus_{\setminus T} \mathcal{O}_2$ *is defined via its set of cocircuits $\mathcal{D}_{\oplus_{\setminus T}}$:*

$$\mathcal{D}_{\oplus_{\setminus T}} := \mathcal{D}_1^{\setminus T} \cup \mathcal{D}_2^{\setminus T} \cup (\mathcal{D}_1 \not\circ_{\mathrm{T}} \mathcal{D}_2).$$

# 5 Concluding Remarks

While parallel, series, and 2-sum connection have been studied involving the most important axiom systems, this is left open for the operations of modular join and modular sum as e. g. the set of circuits of the generalized parallel connection is not an immediate analogue to the parallel connection. Furthermore, if $T$ contains more than one element, generalized parallel connection lacks of a meaningful dual operation which in the case of $T = 1$ is the series connection and corresponds to matroid union if the ground sets intersect in $T$. This does not hold for larger $T$ as well.

The generalized parallel connection of a matroid is well defined as soon as $T$ is a modular flat of $\mathcal{O}_1$. We leave it as an open question whether the equation in Corollary 26 yields an oriented matroid if $T$ is not a modular flat of $\mathcal{O}_2$. Note that the unoriented analogue holds.

# References


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