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Note on the game chromatic index of wheels

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Abstract

We prove that the game chromatic index of n-wheels is n for $n \geq 6$.

1 Introduction

We consider the following games, played on an — initially uncoloured — graph G with a colour set C. Two players, Alice and Bob, alternately colour an uncoloured edge of G with a colour from C, so that adjacent edges receive distinct colours. In the first game we consider, Alice has the first move, in the second game, Bob begins. The respective game ends when no move is possible any more. If at the end every edge is coloured, Alice wins, otherwise Bob wins. The smallest size of a colour set C with which Alice has a winning strategy in the game played on G is called *game chromatic index* of G and denoted by $\chi'_{g_A}(G)$ for the first game and $\chi'_{g_B}(G)$ for the second game.

The game chromatic index $\chi'_{g_A}(G)$ was introduced by Cai and Zhu [8] resp. Lam et al. [12] and is denoted usually as $\chi'_g(G)$. It is the edge colouring variant of the more general game chromatic number introduced by Bodlaender [7]. The game chromatic number is based on a vertex colouring instead of edge colouring game. Variants of the game chromatic number — besides the game chromatic index — are, e.g., the game colouring number [13], and the incidence game chromatic number [2].

Initiated by the paper of Faigle et al. [10] there have been a lot of attempts to bound or determine the game chromatic number of several classes of graphs. For a recent survey on this topic see [5]. A regularly updated list of references can be found at [3]. We will focus here on the results concerning the game chromatic number of line graphs, i.e. the results on the game chromatic index of graphs. Cai and Zhu [8] proved that the game chromatic index of k-degenerate graphs with maximum degree Δ is at most $\Delta+3k-1$, which implies the bound $\Delta+2$ for forests. In the case of forests with maximum degree $\Delta \geq 5$ this bound was tightened to the value $\Delta+1$ by work of Erdös et al. [9] and Andres [1]. For these results it does not matter whether we consider the first or the second game. Bartnicki and Grytczuk [4] improved the result of Cai and Zhu and showed that the game chromatic index even of graphs of arboricity k with maximum degree Δ is at most $\Delta + 3k - 1$. An interesting question was for a long time whether there is a constant c, so that, for every graph G with maximum degree Δ, the game chromatic index of G is at most $\Delta + c$. Beveridge et al. [6] answered this question to the negative.

An *n-wheel*, $n \geq 3$, is a graph with $n+1$ vertices, one of which, say v_0, is adjacent to every other vertex, and if v_0 and its edges are deleted, the remaining graph is an n-cycle. The edges adjacent to v_0 are called *spokes* and the edges of the n-cycle are called *rim edges*. Obviously, the game chromatic index of an n-wheel is at least n. Lam et al. [12] proved that, if Alice begins, the game chromatic index of an n-wheel, $n \geq 4$, is at most $n+1$. In this paper we tighten this upper bound, moreover we prove

Theorem 1. *Let W_n be the n-wheel. Then*

(a) $\chi'_{g_A}(W_n) = n$ *if* $n \geq 6$,

(b) $\chi'_{g_B}(W_n) = n$ *if* $n \geq 3$.

By easy calculations, one observes $\chi'_{g_A}(W_3) = 5$, $\chi'_{g_A}(W_4) = 5$, and $\chi'_{g_A}(W_5) = 6$. Therefore by Theorem 1 the problem of determining the game chromatic index of wheels is completely solved. In particular, for large wheels, the game chromatic index equals to the trivial lower bound n for the game chromatic index. Note that there is a similar result for the incidence game chromatic number of wheels by Kim [11]: the incidence game chromatic number of large wheels equals to the trivial lower bound $\lceil \frac{3n}{2} \rceil$ for the incidence game chromatic number of graphs with maximum degree n.

2 Proof of Theorem 1 (b)

We describe a winning strategy for Alice for the second game played on W_n, $n \geq 3$, with n colours. We number the spokes s_i and the rim edges r_i cyclically in such a way that s_i is adjacent to r_{i+1} and r_{i+2} where we take the indices modulo n. Therefore the spoke s_i and the rim edge r_i are independent for any $i = 0, \ldots, n-1$, since $n \geq 3$.

Alice's strategy is that after each of her moves, for any i, either s_i and r_i are coloured both, or none of them is coloured. She achieves this goal by matching moves. In a *matching move*, if Bob colours r_i (resp. s_i) with a new colour, then Alice colours its partner s_i (resp. r_i) with the same colour, and if Bob colours r_i with an old colour, then Alice colours s_i with a new colour. Note that Bob cannot colour a spoke with an old colour, since by this strategy the set of colours of the rim edges is a subset of the set of colours of the spokes. After Alice's k-th move, exactly k colours are used for spokes. Thus Alice wins.

3 Proof of Theorem 1 (a)

We describe a winning strategy for Alice for the first game played on W_n, $n \geq 6$, with n colours. Here the situation is more complex since Alice has the disadvantage of the first move. However, Alice tries to act much in the way as in the strategy of the previous section.

Again, we number the spokes s_i and the rim edges r_i cyclically in such a way that s_i is adjacent to r_{i+1} and r_{i+2} where we take the indices modulo n, so that the spoke s_i and the rim edge r_i are independent for any $i = 0, \ldots, n-1$, since $n \geq 3$. During the game, Alice will keep in mind one special index i_0 and possibly change the special index several times. We denote s_{i_0} by s and r_{i_0} by r.

Alice's strategy is two-fold. The first part of Alice's strategy will consist of the first $n-3$ moves of Alice and the first $n-4$ moves of Bob. The second part concerns the end-game of colouring the last seven edges.

In her first move, Alice chooses an index as special index and colours the spoke s. In the next $n-4$ moves, she reacts on Bob's play in the following way: If Bob colours a spoke $s_i \neq s$ or a rim edge $r_i \neq r$, Alice answers by a matching move. If Bob colours r with a colour c, Alice chooses a new special index i_0, so that s_{i_0} is uncoloured and not adjacent to the old r, and colours s_{i_0} with c if c was a new colour before Bob's move, otherwise with a new colour. Note that there is such an index i_0, since the colour c at rim r can block at most two spokes, but before Alice plays her move there are still at least four uncoloured spokes. By playing in this way, after Alice's k-th move, exactly k colours are used for spokes and at most $k-1$ colours are used for rim edges, and the set of colours of the rim edges is a subset of the colours of the spokes. At this time, there are three uncoloured spokes and four uncoloured rim edges.

In the end-game, Alice has to avoid the situation that the last two uncoloured spokes are blocked by a new colour on the rim edge adjacent to both spokes or that the last uncoloured spoke is blocked by a new colour. The next lemma shows that in certain situations when there are only five uncoloured edges left, Alice has a winning strategy. After the proof of the lemma we will describe how Alice can mostly achieve one of these situations in her $n-2$-nd move and how she reacts otherwise.

Lemma 2. *If there are only two uncoloured spokes s_{i_1} and s_{i_2} and three uncoloured rim edges e_1, e_2, e_3 left, and e_1 is not adjacent to s_{i_2} (but may be adjacent to s_{i_1}), e_2 is not adjacent to s_{i_1} (but may be adjacent to s_{i_2}), and e_3 is neither adjacent to s_{i_1} nor to s_{i_2}, and there are two unused colours, then Alice has a winning strategy.*

Proof. We may assume that e_1 is adjacent to s_{i_1}.

It is Bob's turn at the beginning. An *old colour* is a colour used before Bob's turn. We distinguish several cases.

If Bob colours a spoke, Alice colours the other spoke, and the last three rim edges can be coloured since $n \geq 5$.

If Bob colours e_1 (resp. e_2), then Alice colours s_{i_2} (resp. s_{i_1}), preferably with the same colour, otherwise with a new colour. Here, the remaining uncoloured spoke can be coloured with the last colour in any case.

The last case is that Bob colours e_3. In this case, Alice colours the rim edge e_1 with an old colour. This is possible, since e_1 has only three coloured adjacent edges, there are two new colours, and $n \geq 6$, so there is at least a fourth old colour. The three remaining uncoloured edges form a path of three edges, or a path of two edges and a single rim edge. It is easy to see that Alice has a winning strategy on the remaining uncoloured path consisting of two or three edges since there are still two colours unused for the adjacent edges of this path. (One colour might have been used for e_3.)

Thus, in any case, Alice wins. □

Now we consider the situation described above: there are three spokes x, y, z and four rim edges a, b, c, d left and it is Bob's turn. We distinguish three cases:

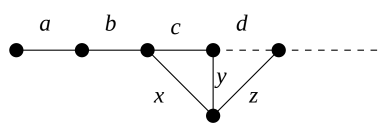

Figure 1: Case 1

Case 1: x, y, z are subsequent spokes, i.e. there is an index i, so that $x = s_i$, $y = s_{i+1}$, and $z = s_{i+2}$ (indices modulo n).

By Alice's moves played so far we may assume that $a = r_i$, $b = r_{i+1}$, $c = r_{i+2}$, and d is an arbitrary other rim edge, see Fig. 1. Note that a is not adjacent to z.

If Bob colours b, c, y, or z, Alice answers by a matching move. If Bob colours d, then Alice colours x. And vice versa, if Bob colours x, then Alice colours d. In all these cases, if Bob uses a new colour, Alice uses the same colour, if he uses an old colour, she uses a new colour. Playing this way, there are still two unused colours after Alice's move, and the situation is exactly as in the preconditions of Lemma 2. By Lemma 2, Alice wins.

We are left with the case that Bob colours a. Then Alice colours c with an old colour. After that the three uncoloured spokes x, y, z can still be coloured with three colours unused so far, except possibly for a. It is easy to see that Alice has a winning strategy: If Bob colours b, then Alice colours y by a matching move. If Bob colours d, then Alice colours x, preferably with the same colour. If Bob colours x, then Alice colours d, preferably with the same colour. If Bob colours y or z, then Alice colours d. If d is adjacent to y and/or z, then Alice uses a colour already used for a spoke, which is possible since $n \geq 6$ and d has at most three coloured adjacent edges. The remaining path with two or three edges can be coloured with two colours if Alice takes care that the middle edge is coloured after her last move. So, also in this case, Alice wins.

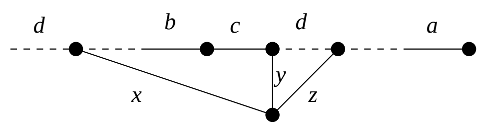

Figure 2: Case 2

Case 2: x is a single spoke, and y, z are subsequent, i.e. there are indices i, j, so that $x = s_i$, $y = s_j$, and $z = s_{j+1}$, and $|j - i| \geq 2$ and $|i - (j+1)| \geq 2$.

Assume that $a = r_i$, $b = r_j$, and $c = r_{j+1}$. Then a may be adjacent to z, but not to x or y, b may be adjacent to x, but not to y or z, and c is adjacent to y, but not to x or z. Note that, since $n \geq 6$, if a is adjacent to z, then b is not adjacent to x. d may be adjacent to either y and z, or z, or x or to none of them. See Fig. 2.

If Bob colours x, then Alice colours d with the same colour. If Bob colours y, then Alice colours a, b or d with the same colour. She chooses the edge to colour in such a way that after her move there is no spoke left with two adjacent uncoloured rim edges. This is possible because of the remarks above. If Bob colours z or c, Alice answers by a matching move. If Bob colours d, Alice colours a spoke not adjacent to d, preferably one of subsequent spokes, preferably with the same colour. Now consider the case that Bob colours a. If d and b are adjacent to x, then Alice colours x, preferably with the same colour. In all other cases (d or b are not adjacent to x), Alice colours y, preferably in the same colour as a. In all cases, after Alice's move we are in a situation as in the precondition of Lemma 2, and by Lemma 2 Alice wins.

We are left with the case that Bob colours b. Then, by a matching move, Alice colours y. Now we are either in the situation of Lemma 2 or z is adjacent to a and d. The latter implies that x is not adjacent to any uncoloured rim edge. No matter what Bob does, Alice can ensure in her next move that z is coloured which will give her a win.

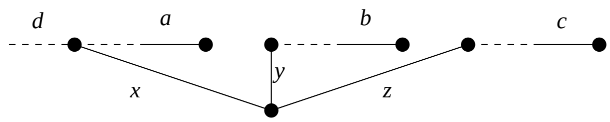

Figure 3: Case 3

Case 3: x, y, z are single spokes.

We may assume that a and d are neither adjacent to y nor to z, b is neither adjacent to x nor to z, c is neither adjacent to x nor to y, see Fig. 3. Hence y resp. z are adjacent to at most one coloured rim edge.

If Bob colours x, then Alice colours b with the same colour. If Bob colours y, then Alice colours d with the same colour. If Bob colours a, then Alice colours y, preferably with the same colour. If Bob colours b, then Alice colours x, preferably with the same colour. By reasons of symmetry we may restrict ourselves to these moves of Bob. After that, Alice wins by Lemma 2.

This proves Theorem 1 (b).

4 Final remark

By the results concerning wheels one might be mislead to conjecture that in the edge colouring game beginning is always a disadvantage for Alice. This is not true. Consider $K_4 - e$, the complete graph on 4 vertices in which one edge is missing. Then $\chi'_{g_A}(K_4 - e) = 3$, but $\chi'_{g_B}(K_4 - e) = 4$. Here, beginning is a real advantage for Alice, she can ensure in her first move, that the edge adjacent to all other edges is coloured.

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