



DISKRETE MATHEMATIK UND OPTI-
MIERUNG

Stephan Dominique Andres:

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Technical Report feu-dmo021.10

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2000 Mathematics Subject Classification: 05C15, 91A43

Keywords: graph coloring game, game chromatic number, game coloring number, perfect graph, B -perfect graph, g -perfect graph

On Characterizing Game-Perfect Graphs by Forbidden Induced Subgraphs

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May 3, 2010

Abstract

A graph G is called g -perfect if, for any induced subgraph H of G , the game chromatic number of H equals the clique number of H . A graph G is called g -col-perfect if, for any induced subgraph H of G , the game coloring number of H equals the clique number of H . In this paper we characterize the classes of g -perfect resp. g -col-perfect graphs by a set of forbidden induced subgraphs and explicitly. Moreover, we study similar notions for variants of the game chromatic number, namely B -perfect and $[A, B]$ -perfect graphs, and for several variants of the game coloring number, and characterize the classes of these graphs.

1 Introduction

A well-known maker-breaker game is one of Bodlaender's graph coloring games [9]. We are given an initially uncolored graph G and a color set C . Two players, Alice and Bob, move alternately with Alice beginning. A move consists in coloring an uncolored vertex with a color from C in such a way that adjacent vertices receive distinct colors. The game ends if no move is possible any more. The maker Alice wins if the vertices of the graph are completely colored, otherwise, i.e. if there is an uncolored vertex surrounded

by colored vertices of each color, the breaker Bob wins. For a graph G , the *game chromatic number* $\chi_g(G)$ of G is the smallest cardinality of a color set C such that Alice has a winning strategy in the game described above.

During the last 17 years, initiated by the paper of Faigle et al. [14], there have been a lot of attempts to obtain good upper bounds for the game chromatic number of the members of interesting classes of graphs, such as trees, forests, and interval graphs [14], outerplanar graphs [16], planar graphs [19, 12, 25, 18, 27], (a, b) -pseudo partial k -trees and graphs embeddable into some surface [26], special graphs embeddable into some surface [17, 22, 4], cactuses [21], Halin graphs [24], line graphs of k -degenerate graphs [10], line graphs of graphs of arboricity k [7], line graphs of forests of maximum degree $\Delta \neq 4$ [10, 13, 1, 2], line graphs of wheels [5], line graphs of certain Cartesian product graphs [6], etc. Obviously, for any graph G , we have

$$\omega(G) \leq \chi(G) \leq \chi_g(G), \quad (1)$$

where $\omega(G)$ denotes the clique number and $\chi(G)$ denotes the chromatic number of G . In most cases the above-mentioned upper bounds for $\chi_g(G)$ are much larger than $\omega(G)$, even larger than $\chi(G)$. For any $n \in \mathbb{N}$, there are bipartite graphs with $\chi_g(G) = n$ (see [18]), so that the gap between $\omega(G)$ (resp. $\chi(G)$) and $\chi_g(G)$ may be arbitrarily large. In this paper we are interested in the other extremal case: those graphs for which $\omega(G) = \chi_g(G)$. It turns out that the class of these graphs does not have a nice structure, so we modify our task as follows. A graph G is called *g -perfect* if $\omega(H) = \chi_g(H)$ holds for any induced subgraph H of G . This notion of game-perfectness was introduced in [3]. It is a game-theoretic analogon of perfectness of graphs, moreover by (1) g -perfect graphs are in particular perfect.

The characterization of perfect graphs by forbidden induced subgraphs was given by the Strong Perfect Graph Theorem of Chudnovsky et al. [11], which proves a conjecture of Berge [8].

Theorem 1 (Chudnovsky, Robertson, Seymour, Thomas (2006)). *A graph is perfect if, and only if, it does neither contain an odd hole nor an odd antihole as induced subgraph.*

We will give a similar characterization of the class of g -perfect graphs, which is, of course, more easy since we will see that the class of g -perfect graphs is a very small subclass of the class of perfect graphs. However, in order to prove the characterization, we need an explicit characterization

which makes use of another notion of game-perfectness, namely of B -perfect graphs. So we will also consider other variants of Bodlaender’s game that will lead to other variants of game-perfectness.

Let “ A ” denote “Alice”, and “ B ” denote “Bob”, and “ $-$ ” denote “none of the players”. Let $X \in \{A, B\}$ and $Y \in \{A, B, -\}$. The game $[X, Y]$ is defined as follows. As in Bodlaender’s game we are given an uncolored graph G and a color set C . The players, A and B , move alternately. Player X has the first move. Y may miss one or several turns, this possibly includes the right to pass in the first move. However the other player(s) (which are not Y) must always move. A move consists in coloring an uncolored vertex with a color from C which is different from the colors of its neighbors, as in Bodlaender’s game, until this is not possible any more. Alice wins if the graph is completely colored in the end, otherwise Bob wins. For a graph G , the $[X, Y]$ -game chromatic number $\chi_{[X, Y]}(G)$ of G is the smallest cardinality of a color set, so that Alice has a winning strategy for this game. A graph G is $[X, Y]$ -perfect if, for any induced subgraph H of G , $\omega(H) = \chi_{[X, Y]}(H)$.

So Bodlaender’s game is the game $[A, -]$ and g -perfectness is the same as $[A, -]$ -perfectness. With a slight abuse of notation we define the following abbreviations

$$A := [A, A], \quad B := [B, B], \quad g := g_A := [A, -], \quad g_B := [B, -].$$

Then our notation matches with the notations of all previous references.

The following observation was shown in [2].

Observation 2. For any graph H ,

$$\omega(H) \leq \chi(H) \leq \chi_A(H) \left\{ \begin{array}{l} \leq \chi_g(H) \leq \chi_{[A, B]}(H) \leq \\ \leq \chi_{[B, A]}(H) \leq \chi_{g_B}(H) \leq \end{array} \right\} \chi_B(H).$$

In particular, B -perfect graphs are $[A, B]$ -perfect, $[A, B]$ -perfect graphs are g -perfect, g -perfect graphs are A -perfect, and A -perfect graphs are perfect.

In this paper we will characterize B -perfect graphs (in Section 2), and $[A, B]$ -perfect and g -perfect graphs (in Section 3) by forbidden induced subgraphs and explicitly. The characterization of A -perfect, g_B -perfect and $[B, A]$ -perfect graphs is still open, some partial results are given in Section 4. In the Section 5 we examine a similar notion to game-perfectness, namely

game-col-perfectness, which is based on game coloring number instead of game chromatic number. For all variants of the underlying marking game we characterize the class of game-col-perfect graphs.

2 Characterizing B -perfect graphs

We start with the smallest class of game-perfect graphs: the class of B -perfect graphs. Its characterization is given in the next theorem.

Theorem 3. *Let G be a (nonempty) graph. Then the following conditions are equivalent:*

- (i) G is B -perfect.
- (ii) G does neither contain a C_4 , nor a P_4 , nor a split 3-star, nor a double fan as induced subgraph (see Fig. 1).
- (iii) For every component H of G , there is $k \geq 0$, so that

$$H = K_1 \vee (H_0 \cup H_1 \cup \dots \cup H_k),$$

where the H_i are complete graphs for $i \geq 1$, and H_0 is either empty or there are $p, q, r \in \mathbb{N}$, so that $H_0 = K_p \vee K_r \vee K_q$ (see Fig. 2).

Proof. (i) \implies (ii): Winning strategies for Bob in game B with 2 colors on C_4 resp. P_4 are obvious. (He has to make use of his right to miss the first turn.) On the split 3-star, Bob wins with 3 colors, since after his second move he can achieve a situation in which two vertices of degree 2 are differently colored. We now prove that Bob wins on the double fan with 3 colors. Note that, in any proper coloring of the double fan, the two upper resp. the two lower vertices of degree 2 must be colored with the same color. Bob has the following winning strategy: He colors an upper vertex of degree 2 with color 1. Then Alice is forced to color the other upper vertex of degree 2

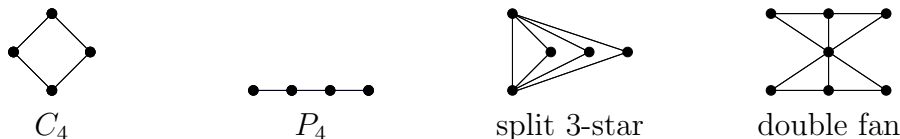


Figure 1: 4 forbidden induced subgraphs for B -perfect graphs

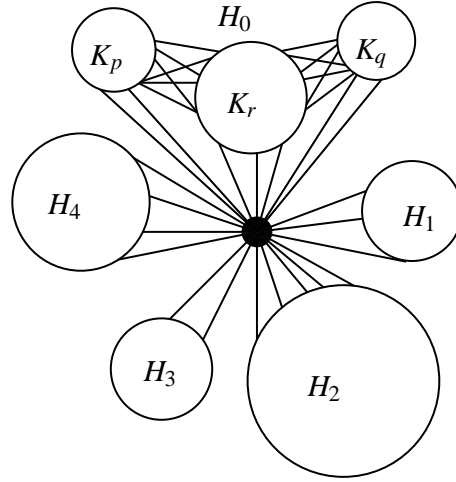


Figure 2: Structure of a component according to (iii)

with the same color. Now Bob colors a lower vertex of degree 2 with color 2. Again, Alice is forced to color the other lower vertex of degree 2 with color 2. Now Bob colors a vertex with degree 3 with color 3, hence he wins.

(iii) \implies (i): We describe a winning strategy for Alice with $\omega(G)$ colors on a graph G as in (iii). This is sufficient since every induced subgraph of G is of the same type as described in (iii). For $H_0 = K_p \vee K_r \vee K_q$ let the K_p and the K_q be the *ears*. Alice always responds to Bob's moves in the same component H (if Bob passes, in an arbitrary component). As long as Bob does not play in an ear, Alice does not play in an ear; she first colors the universal vertex of H . If Bob plays in an ear K_p , Alice colors a vertex in the corresponding ear K_q with the same color (in case there is no uncolored vertex she uses the strategy described before). If Alice is forced to start coloring an ear, then all non-ear-vertices are colored, so a coloring of the ears is possible without creating danger for a non-ear-vertex.

(ii) \implies (iii): We examine the structure of a graph G without induced P_4 , C_4 , split 3-star, double fan. Let H be a component of G . We use the following lemma of Wolk [23].

Lemma 4 (Wolk (1965)). *A connected graph without induced C_4 and P_4 (a so-called trivially perfect graph [15]) has a universal vertex.*

So, H has a universal vertex v . Let H_0, \dots, H_n be the components of $H \setminus v$.

Claim 5. *At most one of the H_i is not complete.*

Proof. Assume H_1, H_2 are not complete. Then both contain a P_3 . So H contains a double fan, which contradicts (ii). \square

Let H_0 be the (only) component of $H \setminus v$ which is not complete. Let K be the largest clique of H_0 . We are done if we show:

Claim 6. *$H_0 \setminus K$ induces a clique.*

Claim 7. *$H_0 \setminus K$ induces a module of H_0 (i.e. if $x \in K$, either x is adjacent to all $y \in H_0 \setminus K$ or to none.)*

Proof of Claim 6. Assume there are non-adjacent vertices $x, y \in H_0 \setminus K$. Since K is a maximal clique, there are $z, z' \in K$ such that neither x, z nor y, z' are adjacent. We note that, again by Lemma 4, H_0 has a universal vertex $w \in K$. If y, z are not adjacent, x, y, z, w, v induce a split 3-star, contradicting (ii). So we may assume that y, z are adjacent and, by symmetry, x, z' are adjacent. This implies that $z \neq z'$ and x, y, z, z' induce a P_4 , contradicting (ii). \square

Proof of Claim 7. Assume that there are $x \in K, s, t \in H_0 \setminus K$, so that s, x are not adjacent, but t, x are adjacent. By Claim 6, s, t are adjacent. Since K is a maximal clique, there is $y \in K$, so that t, y are not adjacent. This implies that s, t, x, y induces either a P_4 or a C_4 , which contradicts (ii). \square

By Claim 6 and Claim 7, $H_0 \setminus K$ corresponds to the K_p , its neighbors correspond to the K_r , and the rest of H_0 corresponds to the K_q . Thus G has the structure as described in (iii). This completes the proof of Theorem 3. \square

3 $[A, B]$ - and g -perfect graphs

Theorem 8. *Let G be a (nonempty) graph. Then the following conditions are equivalent:*

- (i) G is $[A, B]$ -perfect
- (ii) G is g -perfect.

- (iii) G does neither contain a C_4 , nor a P_4 , nor a triangle star, nor a Ξ -graph nor a graph with two components which are a split 3-star or/and a double fan as induced subgraph (see Fig. 3).
- (iv) Let G_1, G_2, \dots, G_m be the components of G . Then G_2, \dots, G_m are B -perfect and G_1 contains a universal vertex v , so that every component of $G_1 \setminus v$ is B -perfect.

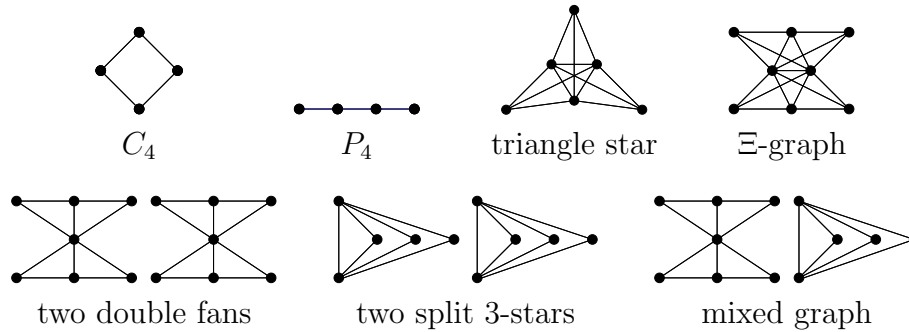


Figure 3: The seven forbidden configurations for g -perfectness

Proof. (i) \implies (ii): This follows directly from Observation 2.

(ii) \implies (iii): Consider the game g , Bodlaender's original game. Obviously, Bob wins on the C_4 and P_4 with 2 colors. Bob wins on two double fans, two split 3-stars resp. the mixed graph with 3 colors since, after Alice has played in a certain component, Bob can follow his strategy as in the proof of Theorem 3 in the other component. Bob also wins on the triangle star with 4 colors: Bob's goal is to have two vertices of degree 3 colored in two different colors and at least one universal vertex uncolored. Obviously, he can achieve this goal within his first two moves.

Now we prove that Bob wins on the Ξ -graph with 4 colors. If Alice colors a vertex of degree 3, Bob colors the uncolored vertex of degree 3 in the same row with a different color and wins.

If Alice, in her first move, colors a vertex of degree 4 in color 1, Bob colors the second vertex of degree 4 in color 2. If Alice now plays a universal vertex (necessarily with color 3), Bob colors a vertex of degree 3 in color 4 and wins. If, on the other hand, Alice colors a vertex of degree 3, Bob colors another

vertex of degree 3 in color 3 and wins, since for the two universal vertices two new colors would be needed.

Consider the last case, that Alice, in her first move, colors a universal vertex. Then the remaining game reduces to a game $[B, -]$ with 3 colors on a double fan, which Bob will win as in the proof of Theorem 3. Therefore Bob always wins on the Ξ -graph.

(iii) \implies (iv): First we prove that every component H of G has a universal vertex v with the property that, if v is deleted, the components of $H \setminus v$ are B -perfect. This is seen as follows: Since G does not contain a P_4 or a C_4 , every component H has a universal vertex v_H . Assume $H \setminus v_H$ is not B -perfect. Then, by Theorem 3, it has either an induced C_4 or P_4 (which is not possible since G does neither contain a C_4 nor a P_4) or an induced double fan or split 3-star. Assume H contains an induced double fan D . Then the graph induced by the vertices of D and v_H is a Ξ -graph in G , a contradiction. Assume H contains an induced split 3-star S . Then the graph induced by the vertices of S and v_H is a triangle star in G , a contradiction.

Now assume two components H_1, H_2 of G are not B -perfect. This means that H_1 and H_2 each contain either a split 3-star or a double fan. But then G contains either two split 3-stars, two double fans or a mixed graph, a contradiction.

(iv) \implies (i): We describe a winning strategy for Alice in the game $[A, B]$ on a graph as defined in (iv). This is sufficient since every induced subgraph of such a graph is of the same type. Alice's strategy is very simple: in her first move she colors the universal vertex v of the special component G_1 . Now the game is reduced to the game $[B, B]$ on $G \setminus v$, which is a B -perfect graph. So Alice has a winning strategy with $\omega(G \setminus v) \in \{\omega(G), \omega(G) - 1\}$ colors. This proves the theorem. \square

Remark. Note that, in spite of the fact that the classes of $[A, B]$ -perfect and g -perfect graphs are the same this does not mean that, for any graph G , $\chi_g(G) = \chi_{[A, B]}(G)$. Consider the graph $G = C_6 \cup K_1$. Here $\chi_g(G) = 2$ (Alice first colors the isolated vertex, then a vertex at distance 3 from the vertex Bob has colored), but $\chi_{[A, B]}(G) = 3$.

4 Other games

There is not much known about A -perfect graphs. The graphs depicted in Fig. 4 are known to be minimal forbidden induced subgraphs, but there are probably many more. In a recent paper [3], the author proved the following two theorems.

Theorem 9. *A triangle-free graph G is A -perfect if, and only if, every component of G is either K_1 or $K_{m,n}$ or $K_{m,n} - e$, where e is an edge.*

Theorem 10. *Complements of bipartite graphs are A -perfect.*

In view of Observation 2 and the preceding results, the following open problem seems to be more difficult than Theorem 3.

Problem 11. *Characterize A -perfect graphs explicitly and/or by forbidden induced subgraphs.*

Even the following is an open question.

Problem 12. *Is the number of minimal forbidden induced subgraphs for A -perfectness that are different from odd antiholes finite?*

For the other variants of the game, the question of Problem 11 might be interesting, too.

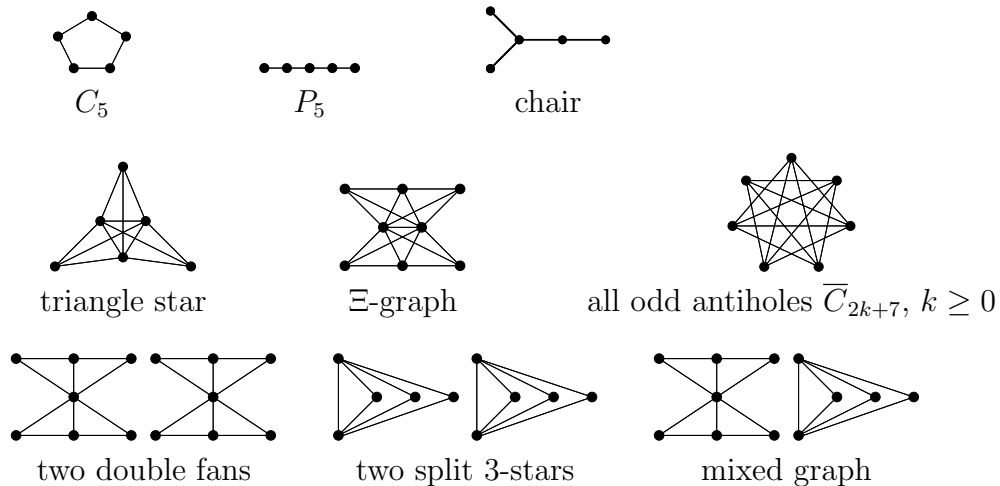


Figure 4: Some forbidden induced subgraphs for A -perfect graphs



Figure 5: Some additional forbidden induced subgraphs for $[B, A]$ -perfect graphs

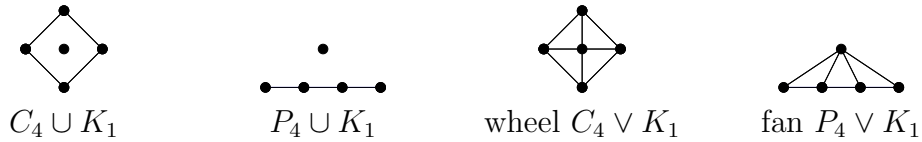


Figure 6: Some additional forbidden induced subgraphs for g_B -perfect graphs

Problem 13. *Characterize g_B -perfect graphs explicitly and/or by forbidden induced subgraphs.*

Problem 14. *Characterize $[B, A]$ -perfect graphs explicitly and/or by forbidden induced subgraphs.*

In Fig. 5 resp. Fig. 6 sets of minimal forbidden induced subgraphs for $[B, A]$ - resp. g_B -perfectness are displayed. It is not known whether these sets (together with the graphs of Fig. 4 resp. the graphs of Fig. 4 and Fig. 5) are complete to characterize game-perfectness for the respective games. Of course the occurrence of the split 3-star and the double fan yields that the triangle star, the Ξ -graph, the two double fans, the two split stars, and the mixed graph are not minimal (for $[B, A]$ - and g_B -perfectness). And the occurrence of the wheel $C_4 \vee K_1$ yields that the odd antiholes \overline{C}_{2k+9} , $k \geq 0$, are not minimal (for g_B -perfectness).

The following theorem is obvious.

Theorem 15. *Either every component of a g_B -perfect graph G is B -perfect or G has only one component.*

Proof. Assume G is g_B -perfect, but not B -perfect. Then a component of G must contain a P_4 or a C_4 . If there is a second component, then G contains a $P_4 \cup K_1$ or a $C_4 \cup K_1$, which are forbidden in g_B -perfect graphs. Thus G is connected. \square

Connected g_B -perfect graphs may have a richer structure which resisted a characterization until now.

We remark that the classes of A -, g -, $[B, A]$ -, g_B - and B -perfect graphs are 5 distinct classes of graphs. This is easily seen as follows. The path P_4 is $[B, A]$ -perfect, g_B -perfect, and A -perfect, but neither g -perfect nor B -perfect. The double fan is A -perfect and g -perfect, but neither $[B, A]$ -perfect nor g_B -perfect. The $P_4 \vee K_1$ is $[B, A]$ -perfect, but not g_B -perfect.

5 Game-col-perfect graphs

In the theory of graph coloring games, a main idea for gaining upper bounds is the so-called *game coloring number* of a graph which was introduced by Zhu [25]. The game underlying this concept is a marking game. In our context we may define some marking games as follows. Let “ A ” denote “Alice”, and “ B ” denote “Bob”, and “ $-$ ” denote “none of the players”. Let $X \in \{A, B\}$ and $Y \in \{A, B, -\}$. The game $[X, Y]$ -col is defined as follows. We are given a graph $G = (V, E)$ whose vertices are unmarked at the beginning. Alice and Bob alternately mark unmarked vertices. Player X begins. Player Y may miss one or several turns (possibly also the first), but the other player(s) always have to move. Whenever a vertex is marked it is assigned a score. The score $s(v)$ of a vertex v is 1 plus the number of neighbors of v which have been marked before v . The score of the game is $S = \max_{v \in V} s(v)$. Alice’s goal is to minimize the score, Bob’s goal to maximize the score. The $[X, Y]$ -*game coloring number* $\text{col}_{[X, Y]}(G)$ is the smallest integer S such that Alice has a strategy to achieve the score S in game $[X, Y]$ -col. Obviously, for a graph G ,

$$\omega(G) \leq \chi(G) \leq \chi_{[X, Y]}(G) \leq \text{col}_{[X, Y]}(G), \quad (2)$$

and we have, in line with Observation 2,

Observation 16. *For any graph H ,*

$$\text{col}_{[A, A]}(H) \left\{ \begin{array}{l} \leq \text{col}_{[A, -]}(H) \leq \text{col}_{[A, B]}(H) \leq \\ \leq \text{col}_{[B, A]}(H) \leq \text{col}_{[B, -]}(H) \leq \end{array} \right\} \text{col}_{[B, B]}(H).$$

The game $[A, -]$ -col is Zhu’s marking game. A graph G is $[X, Y]$ -*col-perfect* if, for any induced subgraph H of G , $\text{col}_{[X, Y]}(G) = \omega(G)$. By (2) a $[X, Y]$ -col-perfect graph is in particular $[X, Y]$ -perfect.

In this section we characterize $[X, Y]$ -col-perfect graphs for any $X \in \{A, B\}$ and $Y \in \{A, B, -\}$. As we shall see, the only significant difference lies in player who is allowed to have the first move.

Theorem 17. *For any graph G ,*

$$(a) \text{ col}_{[B, B]}(G) = \text{col}_{[B, -]}(G) = \text{col}_{[B, A]}(G), \text{ and}$$

$$(b) \text{ col}_{[A, B]}(G) = \text{col}_{[A, -]}(G) = \text{col}_{[A, A]}(G).$$

Proof. We first prove that missing a turn is not an advantage for Bob, then that missing a turn is not an advantage for Alice.

Let $X \in \{A, B\}$. First, assume that Alice has a strategy in game $[X, -]$ -col to achieve a score $S_{[X, -]} \leq \alpha$. We prove that she has a strategy in game $[X, B]$ -col to achieve a score $S_{[X, B]} \leq \alpha$. Alice uses basically her strategy in game $[X, -]$ -col and further sometimes keeps in mind a special unmarked vertex, the *memory vertex*. In case Bob misses his turn and there is a memory vertex, Alice marks the memory vertex, which thereby loses his role as memory vertex. In case Bob misses his turn and there is no memory vertex or if Bob marks the memory vertex, Alice imagines that Bob has marked the unmarked vertex of smallest degree, which now is the new memory vertex, and chooses a vertex for marking by her strategy for game $[X, -]$ -col. If Bob marks any other vertex, Alice replies by using her strategy for the game $[X, -]$ -col. In case the memory vertex is the only unmarked vertex, Alice marks the memory vertex, of course.

Using this strategy the only vertices which might obtain a higher score in game $[X, B]$ -col than in game $[X, -]$ -col are the memory vertices. However, at the time they are chosen, they have smallest degree d among all unmarked vertices, therefore $\alpha \geq d + 1$. Since a memory vertex can obtain at most a score of $d + 1$, the score of the game will be $\leq \alpha$.

Secondly, assume that Bob has a strategy in game $[X, -]$ -col to obtain a score $S_{[X, -]} \geq \alpha$. We prove that he has a strategy in game $[X, A]$ -col to achieve a score $S_{[X, A]} \geq \alpha$. Basically, Bob here uses his strategy in game $[X, -]$ and keeps in mind a memory vertex and proceeds as in Alice's strategy above with the only difference that, if Alice passes and there is no memory vertex or if Alice marks the memory vertex, Bob imagines that Alice has marked the unmarked vertex of largest degree, which is now the new memory vertex.

In this strategy, at any time there has been constructed already a vertex with score α , or there is a memory vertex with degree $d \geq \alpha - 1$, or there is

no memory vertex. If the last unmarked vertex is the memory vertex with degree $d \geq \alpha - 1$, then the score is $\geq d + 1 \geq \alpha$. Otherwise the score of all vertices marked after a memory vertex has been marked (before there is a new memory vertex) is equal to their scores for the game $[X, -]$ -col. Therefore, in any case Bob can achieve a score $\geq \alpha$ in game $[X, A]$ -col. This proves the theorem. \square

The preceding theorem justifies to use the short notations A -col resp. B -col instead of the games $[A, -]$ -col resp. $[B, -]$ -col. Moreover, in view of Observation 16, beginning is always an advantage in the marking game:

Corollary 18. *For any graph G , $\text{col}_A(G) \leq \text{col}_B(G)$.*

Before formulating the theorems on the characterization of game-col-perfect graphs we give a definition. A *diamond* D is the graph K_4 minus one edge. By a $2D$ we mean $D \cup D$, the disjoint union of two diamonds.

Theorem 19. *Let G be a (nonempty) graph. Then the following conditions are equivalent:*

- (i) G is B -col-perfect.
- (ii) G does neither contain a P_4 nor a C_4 nor a D as induced subgraph.
- (iii) Every component of G is of the form

$$K_1 \vee (H_1 \cup H_2 \cup \dots \cup H_m)$$

for some $m \geq 0$, where H_i are complete graphs.

Proof. (i) \implies (ii): Obviously there is a strategy for Bob to obtain a score of 3 resp. 4 in the game B -col played on the P_4 and C_4 resp. the diamond.

(ii) \implies (iii): Let (ii) be true. Since G does neither contain an induced C_4 nor an P_4 , G is trivially perfect, hence by Lemma 4 every component H of G has a universal vertex v_H . Let H be a component of G . Let H' be a component of $H \setminus v_H$. Assume that H' is not complete. Then, since it is connected it contains an induced P_3 . This P_3 together with v is an induced diamond in G , a contradiction to (ii). Thus (iii) holds.

(iii) \implies (i): We show that on G as in (iii) Alice has a strategy to obtain a score of $\omega(G)$. The only dangerous vertices are the universal vertices of

the components of G in case they are the only universal vertex of such a component (otherwise the component is a clique). Alice's strategy is simple. In case Bob marks a vertex in component H , Alice marks a vertex in the same component, preferably the universal vertex of the component. If this is not possible or Bob misses his turn, Alice marks some vertex, preferably a universal vertex of some component. In this way the universal vertices v of the components have score $s(v) \leq 2$, and the score 2 is only achieved if the component is not trivial. Thus the score of the game is $\omega(G)$. \square

Theorem 20. *Let G be a (nonempty) graph. Then the following conditions are equivalent:*

- (i) G is A -col-perfect.
- (ii) G does neither contain a P_4 nor a C_4 nor a $2D$ nor a $K_1 \vee D$ as induced subgraph.
- (iii) Let G_1, G_2, \dots, G_n be the components of G . Then the components G_2, \dots, G_n are B -col-perfect and G_1 contains a universal vertex v , so that every component of $G_1 \setminus v$ is B -col-perfect.

Proof. (i) \implies (ii): Obviously, there is a strategy for Bob to obtain a score of 3 in the game A -col played on the P_4 and C_4 . In the $2D$, Bob can force a score of 4 if in his first move he marks a vertex of degree 2 in a component in which Alice has not marked any vertex before. Now consider the $K_1 \vee D$. This graph has 3 vertices of degree 4 and 2 of degree 3. The score will be 4 if a vertex of degree 4 leaves as the last marked vertex. Bob can achieve this goal if in his first two moves he marks vertices of degree 3 preferably.

(ii) \implies (iii): Let G be as in (ii). Assume there are two components of G that are not B -col-perfect. Since G does neither contain an induced P_4 nor an induced C_4 each of these components contains an induced diamond. So G contains a $2D$, a contradiction.

Now let H be the single component (if any) of G that is not B -col-perfect. Since H is trivially perfect and connected, by Lemma 4 H has a universal vertex v_H . Let H' be a component of $H \setminus v_H$. Assume H' is not B -col-perfect. Then H' contains a diamond D . So, together with v_H , G contains a $K_1 \vee D$, a contradiction. Thus (iii) holds.

(iii) \implies (i): We show that on G as in (iii) in game $[A, B]$ -col Alice has a strategy to obtain a score of $\omega(G)$. In her first move she marks the universal

vertex v of G_1 . Then the game is reduced to a game $[B, B]$ -col on G_2, \dots, G_n and the components of $G_1 \setminus v$, which is a $[B, B]$ -col-perfect graph. We have seen in the previous theorem that Alice now has a winning strategy to obtain score $\omega(G)$. \square

The game coloring number is a game-theoretic analogon of the well-known coloring number $\text{col}(G)$ of a graph G . We might also define col-perfectness and ask: what are the col-perfect graphs? The answer is very easy. Since every cycle of length ≥ 4 is forbidden for col-perfectness, and chordal graphs have a perfect elimination scheme, the class of col-perfect graphs is the class of chordal graphs. This fact was in principle already remarked by Kierstead and Yang [20].

6 Final remarks

Apart from the problems formulated in Section 4 there are some more questions which are related to the classifications of this paper. For example, one might consider the notion of $[X, Y]$ -chi-col-perfectness. A graph G is $[X, Y]$ -chi-col-perfect if, for any induced subgraph H of G , $\chi_{[X, Y]}(H) = \text{col}_{[X, Y]}(H)$. Obviously, the class of $[X, Y]$ -col-perfect graphs is the intersection of the classes of $[X, Y]$ -perfect graphs and $[X, Y]$ -chi-col-perfect graphs.

Problem 21. *For any $X \in \{A, B\}$ and $Y \in \{A, B, -\}$, characterize the class of $[X, Y]$ -chi-col-perfect graphs explicitly and/or by forbidden induced subgraphs.*

The difference between the chromatic number and the game chromatic number is an important parameter in the theory of graph coloring games. So we may define a graph G as $[X, Y]$ -exact if, for any induced subgraph H of G , $\chi(H) = \chi_{[X, Y]}(H)$. Then the class of $[X, Y]$ -perfect graphs is the intersection of the classes of perfect graphs and $[X, Y]$ -exact graphs.

Problem 22. *For any $X \in \{A, B\}$ and $Y \in \{A, B, -\}$, characterize the class of $[X, Y]$ -exact graphs explicitly and/or by forbidden induced subgraphs.*

Acknowledgements

Thanks to Winfried Hochstättler for useful comments, in particular for some hints which helped me to formulate Claim 6 and Claim 7 in a compact way,

and for a hint concerning reference [20]. I also acknowledge a discussion with Stefan Felsner concerning the minimality of the odd antiholes for A -perfectness.

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