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SIMPLE GAMMOIDS OF RANK 3 HAVE POSITIVE COLINES

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ABSTRACT. A positive coline in a matroid is a flat of codimension 2, such that it is a subset of more flats of codimension 1, that consist of only one extra element, than flats of codimension 1, that have more than one extra element. We show that every simple gammoid of rank 3 has at least one positive coline.

1. PRELIMINARIES

First we collect some handy tools from the literature.

Theorem 1. [1, 2] *A matroid $M = (E, \mathrm{rk})$ with $\mathrm{rk}(E) \leq 3$ is a gammoid, if and only if M^* is a transversal matroid.*

Lemma 2. [1] *$M = (E, \mathrm{rk})$, $X \subseteq E$. X is a cyclic flat of M, if and only if $E \backslash X$ is a cyclic flat of M^*.*

Theorem 3. [1, 3] *Let $M = (E, \mathrm{rk})$ be a matroid and $\mathscr{Z} \subseteq 2^E$ be the family of cyclic flats of M. Then M is a transversal matroid, if and only if*

$$\forall \emptyset \neq \mathscr{F} \subseteq \mathscr{Z}: \ \mathrm{rk}\left(\bigcap_{Z \in \mathscr{F}} Z\right) \leq \sum_{\mathscr{F}' \subseteq \mathscr{F}} (-1)^{|\mathscr{F}'|+1} \mathrm{rk}\left(\bigcup_{Z' \in \mathscr{F}'} Z'\right)$$

holds.

It is well known that transversal matroids are representable over the reals. In some sense Theorem 3 asserts that the points in a realization of a transversal matroid are still in "quite" general position or in other words, transversal matroids share some properties with uniform matroids. To make this more precise in the rank 3 case we define

Definition 4. Let $M = (E, \mathrm{rk}, \mathrm{cl})$ be a simple matroid with $\mathrm{rk}(E) = 3$. The family of copoints of M is defined to be

$$\mathscr{K} = \left\{ \mathrm{cl}(X) \mid X \in \binom{E}{2} \right\},$$

and the sub-families of copoints of M with some given nullity $i \in \mathbb{N}$ are defined to be

$$\mathscr{K}_i = \{ K \in \mathscr{K} \mid |K| = i + 2 \}.$$

A copoint $X \in \mathscr{K}$ is called *simple*, if $X \in \mathscr{K}_0$. Otherwise, $X \in \mathscr{K}_{\geq 1} := \bigcup_{i \geq 1} \mathscr{K}_i$ and we say that X is *fat*.

Remark. In the above definition, $\mathrm{rk}(X) = 2$, thus $|\mathrm{cl}(X)| < |E|$ because $\mathrm{rk}(E) = 3$. Thus for all $i \geq |E| - 2$, $\mathscr{K}_i = \emptyset$.

Definition 5. Let $M = (E, \mathrm{rk}, \mathrm{cl})$ be a simple matroid with $\mathrm{rk}(E) = 3$. The family of colines of M is defined to be

$$\mathscr{L} = \left\{ \mathrm{cl}(X) \mid X \in \binom{E}{1} \right\} = \{\{e\} \mid e \in E\}.$$

The set of simple copoints on $G \in \mathscr{L}$ is defined to be $\mathscr{K}_0^G = \{K \in \mathscr{K}_0 \mid G \subseteq K\}$ and the set of fat copoints on G is defined to be $\mathscr{K}_{\geq 1}^G = \{K \in \mathscr{K}_{\geq 1} \mid G \subseteq K\}$. The set of all copoints on G is defined to be $\mathscr{K}^G = \mathscr{K}_0^G \cup \mathscr{K}_{\geq 1}^G$. A coline $G \in \mathscr{L}$ is called *positive*, if

$$\left|\mathscr{K}_0^G\right| > \left|\mathscr{K}_{\geq 1}^G\right|.$$

Clearly, in a uniform matroid all copoints are simple on all colines and hence all colines are positive. What we can save for transversal matroids of rank 3 is that they still have at least one positive coline. In order to apply Theorem 3 we observe, that in this case the cyclic flats coincide with the fat copoints.

Proposition 6. *Let $M = (E, \mathrm{rk}, \mathrm{cl})$ be a simple matroid with $\mathrm{rk}(E) = 3$ and $Z \subsetneq E$. Then Z is a cyclic flat if and only if $Z \in \mathscr{K}_{\geq 1}$.*

Proof. This is immediate from the fact that $Z \subsetneq E$ and M is simple. □

In order to wrap this all up nicely for some matroid $M = (E, \mathrm{rk})$ and its dual matroid $M^* = (E, \mathrm{rk}^*)$, we use the well-known formula for $X \subseteq E$

$$\mathrm{rk}^*(X) = |X| - \mathrm{rk}(E) + \mathrm{rk}(E \backslash X).$$

We set $X := E \backslash K$ for some $K \subseteq E$ and obtain

$$\begin{aligned} \mathrm{rk}^*(E \backslash K) &= |E \backslash K| - \mathrm{rk}(E) + \mathrm{rk}(K) \\ &= (|E| - \mathrm{rk}(E)) - (|K| - \mathrm{rk}(K)). \end{aligned} \tag{1.1}$$

This means that the dual-rank of the complement of a flat equals to the rank of the dual matroid minus the nullity of the flat. Together with Theorems 1, 3, and Lemma 2, this yields:

Corollary 7. *Let $M = (E, \mathrm{rk}, \mathrm{cl})$ be a simple matroid with $\mathrm{rk}(E) = 3$. If M is a gammoid, then for all non-empty families $\emptyset \neq \mathscr{F} \subseteq \mathscr{K}_{\geq 1}$,*

$$\mathrm{rk}^*\left(\bigcap_{K \in \mathscr{F}} E \backslash K \right) \leq \sum_{\mathscr{F}' \subseteq \mathscr{F}} (-1)^{|\mathscr{F}'|+1} \mathrm{rk}^*\left(\bigcup_{K' \in \mathscr{F}'} E \backslash K' \right).$$

2. Simple Gammoids of Rank 3

Theorem 8. *Let $M = (E, \mathrm{rk}, \mathrm{cl})$ be a simple gammoid with $\mathrm{rk}(E) = 3$. Then M has a positive coline.*

Proof. We are going to use the fact that the Corollary 7 imposes an upper bound on the total number of fat copoints together with the assumption, that M has no positive coline, which imposes some sort of a lower bound on the total number of fat copoints. Now, let's assume that M is a simple gammoid of rank 3 with no positive coline.

We are going to dismiss some marginal cases first.

Assume that $\bigcap_{K \in \mathscr{K}_{\geq 1}} \neq \emptyset$. Then all fat copoints lie on the same coline $e \in E$. Thus, every other coline $f \in E \backslash \{e\}$ has at most one fat copoint. f cannot have only one copoint, since that would imply rank 2 for M. If $f \in E\backslash\{e\}$ has more than two copoints, f would be a positive coline. Now take any $f \in E\backslash\{e\}$, then $g \in E\backslash\{e,f\}$ is the only coline which intersects f in a simple copoint. Then either g intersects e in a simple copoint and then g has at least two simple and no fat copoints and, therefore, is a positive coline; or there is another coline $h \in E\backslash\{e,f,g\}$ that intersects e in the same fat copoint as g, and this copoint is different from the copoint in which f and e intersect. But then h must intersect f in a simple copoint on f – a contradiction to f being non-positive.

So we may assume that $\bigcap_{K\in\mathscr{K}_{\geq 1}} = \emptyset$. Now assume that $\bigcup_{K\in\mathscr{K}_{\geq 1}} \neq E$, then there is some coline that has no fat copoint; hence we may assume that $\bigcup_{K\in\mathscr{K}_{\geq 1}} = E$. Clearly, since M is simple, for all $K_1,K_2 \in \mathscr{K}$ we must have that $|K_1 \cap K_2| \leq 1$. Thus, for any $\mathscr{F} \subseteq \mathscr{K}$ with $|\mathscr{F}| > 1$, $\mathrm{rk}\left(\bigcap_{K\in\mathscr{F}} K\right) = \left|\bigcap_{K\in\mathscr{F}} K\right|$, i.e. the nullity of $\bigcap_{K\in\mathscr{F}} K$ is zero.

In particular, we set $\mathscr{F} := \mathscr{K}_{\geq 1}$ in Corollary 7. Clearly,

$$\mathrm{rk}^*\left(\bigcap_{K\in\mathscr{K}_{\geq 1}} E\backslash K\right) = \mathrm{rk}^*\left(E\backslash \bigcup_{K\in\mathscr{K}_{\geq 1}} K\right) = \mathrm{rk}^*\left(E\backslash E\right) = 0 \qquad \text{and}$$

$$\mathrm{rk}^*\left(\bigcup_{K'\in\mathscr{F}'} E\backslash K'\right) = \mathrm{rk}^*\left(E\backslash \bigcap_{K'\in\mathscr{F}'} K'\right) \overset{1.1}{=} (|E|-3) - \left(\left|\bigcap_{K'\in\mathscr{F}'} K'\right| - \mathrm{rk}\left(\bigcap_{K'\in\mathscr{F}'} K'\right)\right).$$

For $\mathscr{F}' \subseteq \mathscr{K}_{\geq 1}$ with $|\mathscr{F}'| > 1$ we get $\mathrm{rk}^*\left(\bigcup_{K'\in\mathscr{F}'} E\backslash K'\right) = |E|-3$, for $i \geq 1$ and $\mathscr{F}' \in \binom{\mathscr{K}_i}{1}$ we get

$$\mathrm{rk}^*\left(\bigcup_{K'\in\mathscr{F}'} E\backslash K'\right) = |E|-3-i.$$

So we may transform the formula from Corollary 7 further:

$$\begin{aligned}
0 &\leq \sum_{\mathscr{F}'\subseteq\mathscr{K}_{\geq 1}} (-1)^{|\mathscr{F}'|+1}\,\mathrm{rk}^*\left(\bigcup_{K'\in\mathscr{F}'} E\backslash K'\right) \\
&= \underbrace{0}_{\mathscr{F}'=\emptyset} + \underbrace{\sum_{i\geq 1}\left(\sum_{K'\in\mathscr{K}_i}(|E|-3-i)\right)}_{|\mathscr{F}'|=1} + \sum_{\substack{\mathscr{F}'\subseteq\mathscr{K}_{\geq 1},\\ |\mathscr{F}'|\geq 2}} \left((-1)^{|\mathscr{F}'|+1}(|E|-3)\right) \\
&= (|E|-3) + \sum_{\mathscr{F}'\subseteq\mathscr{K}_{\geq 1}} \left((-1)^{|\mathscr{F}'|+1}(|E|-3)\right) - \sum_{i\geq 1} i\cdot|\mathscr{K}_i| \\
&= (|E|-3)\left(1 + \sum_{\mathscr{F}'\subseteq\mathscr{K}_{\geq 1}} (-1)^{|\mathscr{F}'|+1}\right) - \sum_{i\geq 1} i\cdot|\mathscr{K}_i| \\
&= (|E|-3)\left(1 - \sum_{k=0}^{|\mathscr{K}_{\geq 1}|} \binom{|\mathscr{K}_{\geq 1}|}{k} 1^{|\mathscr{K}_{\geq 1}|-k}(-1)^k\right) - \sum_{i\geq 1} i\cdot|\mathscr{K}_i| \\
&= (|E|-3)\left(1-(1-1)^{|\mathscr{K}_{\geq 1}|}\right) - \sum_{i\geq 1} i\cdot|\mathscr{K}_i| = |E|-3-\sum_{i\geq 1} i\cdot|\mathscr{K}_i|
\end{aligned}$$

Thus we get some kind of weighted upper bound:

$$\sum_{i\geq 1} i\cdot|\mathscr{K}_i| \quad \leq \quad |E|-3 \tag{2.1}$$

Since $i\cdot|\mathscr{K}_i|\geq 0$ for all $i\geq 0$, and, by assumption, $\sum_{i\geq 1} i\cdot|\mathscr{K}_i|\neq 0$, we may define the vector $\alpha\colon \{1,\ldots,|E|-3\}\to[0,1]$ such that for all $1\leq j\leq |E|-3$,

$$\alpha_j = \frac{j\cdot|\mathscr{K}_j|}{\sum_{i\geq 1} i\cdot|\mathscr{K}_i|}.$$

Then, clearly,

$$\sum_{i=1}^{|E|-3}\alpha_i = 1$$

and inequality 2.1 implies

$$\forall i\in\{1,\ldots,|E|-3\}:\quad i\cdot|\mathscr{K}_i|\leq \alpha_i\cdot(|E|-3)\,. \tag{2.2}$$

Now remember that M may be represented as a pseudo line arrangement. If we fix $e\in E$, the coline $\{e\}$ has to intersect all $E\backslash\{e\}$ other colines in exactly one copoint, i.e. $\sum_{K\in\mathscr{K}^{\{e\}}}|K\backslash\{e\}| = |E|-1$. Counting each intersection between two lines twice, we get

$$\sum_{e\in E}\left(\sum_{K\in\mathscr{K}^{\{e\}}}|K\backslash\{e\}|\right) = \sum_{e\in E}(|E|-1) = |E|\cdot(|E|-1)$$

as the number total ordered intersections in the pseudo line arrangement. On the other hand, we may count this number per copoint, noting that a copoint with k elements gives rise to $k\cdot(k-1)$ ordered intersections between two lines, i.e.

$$|E|\cdot(|E|-1) \quad = \quad \sum_{K\in\mathscr{K}}(|K|\cdot(|K|-1))\,.$$

Furthermore, if we have no positive coline, we have that for every coline $\{e\}$,

$$|\mathscr{K}_0^{\{e\}}|\leq|\mathscr{K}_{\geq 1}^{\{e\}}|.$$

Thus, there is an injective map

$$\iota\colon \mathscr{K}_0\to\{(K,e)\in\mathscr{K}_{\geq 1}\times E\mid e\in K\}\,,$$

where e may be chosen to be one of the intersecting lines. Therefore, we may estimate

$$|\mathscr{K}_0|\leq\sum_{K\in\mathscr{K}_{\geq 1}}|K| = \sum_{i\geq 1}(i+2)\cdot|\mathscr{K}_i|.$$

We obtain

$$\begin{aligned} |E|\cdot(|E|-1) &= \sum_{K\in\mathscr{K}}(|K|\cdot(|K|-1)) \\ &= |\mathscr{K}_0|+\sum_{i\geq 1}(i+2)(i+1)|\mathscr{K}_i| \\ &\leq \sum_{i\geq 1}(i+2)^2\cdot|\mathscr{K}_i|. \end{aligned} \tag{2.3}$$

Substituting $|\mathscr{K}_i|$ in 2.3 using 2.2 and the fact that $|E|\geq 7$ implies

$$\forall i\in\{1,\ldots,|E|-3\}:\quad \frac{(i+2)^2}{i}\leq\frac{(|E|-1)^2}{|E|-3}$$

we obtain

$$\begin{aligned}
|E|\cdot(|E|-1) &\leq (|E|-3)\cdot\sum_{i=1}^{|E|-3}\alpha_i\cdot\frac{(i+2)^2}{i}\\
&\overset{|E|\geq 7}{\leq} (|E|-3)\cdot\frac{(|E|-1)^2}{|E|-3}\\
&= (|E|-1)^2
\end{aligned}$$

Obviously, the deduced inequality $|E| \leq |E|-1$ cannot be satisfied. Thus, there cannot be a simple gammoid with rank 3 and more than 6 elements which has no positive coline. Checking the remaining cases for $4 \leq |E| \leq 6$: We use the following fact

$$\forall i \in \{1,\ldots,|E|-3\}: \quad \frac{(i+2)^2}{i} \leq 9$$

and obtain

$$\begin{aligned}
|E|\cdot(|E|-1) &\leq (|E|-3)\cdot\sum_{i=1}^{|E|-3}\alpha_i\cdot\frac{(i+2)^2}{i}\\
&\leq (|E|-3)\cdot 9
\end{aligned}$$

and the contradiction $0 \leq -|E|^2 + 10\cdot|E| - 27 \in \{-3,-2\}$ establishes the proposition. □

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