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Stephan Dominique Andres:

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Contact: dominique.andres@fernuni-hagen.de

FernUniversität in Hagen
Fakultät für Mathematik und Informatik
Lehrgebiet für Diskrete Mathematik und Optimierung
D – 58084 Hagen

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On kernels in game-perfect digraphs and a characterization of weakly game-perfect digraphs

Stephan Dominique Andres
FernUniversität in Hagen
Fakultät für Mathematik und Informatik
IZ, Universitätsstr. 1, D-58084 Hagen, Germany
dominique.andres@fernuni-hagen.de

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Abstract

We prove that game-perfect digraphs with respect to our digraph version [1] of Bodlaender's original version of a maker-breaker graph colouring game [6] always have a kernel. Furthermore, we introduce weakly game-perfect digraphs related to Yang and Zhu's digraph version [12] of the maker-breaker graph colouring game and characterize the classes of weakly game-perfect digraphs by means of the classes of undirected game-perfect graphs.

1 Introduction

We consider the following game, played by two players, Alice (A) and Bob (B), with a finite digraph D and colour set C. At the beginning, every vertex is uncoloured. The players alternately choose an uncoloured vertex of D and colour it with a colour from C that is different from the colours of every in-neighbour of the vertex actually to be coloured. The game ends when no further move is possible. Alice wins if every vertex is coloured at the end, otherwise Bob wins. Thus Bob wins if there exists a vertex v, surrounded by vertices of all colours with arcs pointing towards v.

We consider six variants of this game, denoted by the symbol $g := [X, Y]$, with $X \in \{A, B\}$ and $Y \in \{A, B, -\}$. X denotes the player who has the first move. Y denotes a player that has the right to miss one or several turns at any time of the game, including the first move; $Y = -$ means that none of the players has this right. We also use the abbreviations $g_A := [A, -]$ and $g_B := [B, -]$. The game g_A is the digraph version of a maker-breaker graph colouring game introduced by Bodlaender [6].

For any variant g of the game we define the *game chromatic number* $\chi_g(D)$ of the digraph D as the smallest size of a colour set C such that Alice has a

winning strategy for the game played on D with $|C|$ colours. This parameter is well-defined, since Alice always has winning strategy with $1 + \Delta^-(D)$ colours.

A digraph D is *(strongly) game-perfect* with respect to the game variant g, or, *g-perfect* for short, if, for every induced subdigraph H of D, we have

$$\chi_g(H) = \omega(H),$$

where $\omega(H)$ denotes the *clique number* of H, i.e. the size of the largest symmetric clique in H.

All digraphs we consider are assumed to have no loops. For standard notions concerning digraphs we refer to the book of Bang-Jensen and Gutin [5]. By D^C we denote the complement of a digraph D. Undirected graphs are considered as *symmetric digraphs*, i.e. those digraphs where for any arc (v, w) there also exists the arc (w, v). A pair $\{(v, w), (w, v)\}$ with $v \neq w$ is called an *edge*. An arc (v, w) such that (w, v) is not an arc is called *single arc*. For a digraph $D = (V, A)$, the symmetric digraph $S(D) := (V, A')$, where A' is the union of all edges of D, is called *symmetric part* of D.

In [2] resp. [4, 9] the classes of undirected $[B, B]$-, $[A, B]$- and g_A- resp. g_B-perfect graphs are characterized by sets of forbidden induced subdigraphs and by explicit structural descriptions. For the game $[B, A]$ a (possibly not complete) set of 16 forbidden structures is known [9], for the game $[A, A]$ (a possibly not complete) set of finitely many forbidden structures plus an infinite series of forbidden structures (the odd antiholes) is known [2, 9].

Game-perfect digraphs are a game-theoretic analog of perfect digraphs introduced in [3]. This concept is based on the digraph colouring parameter *dichromatic number* $\chi(D)$ which was first introduced by Neumann-Lara [11]. We have the following obvious relations of colouring parameters.

Observation 1. $\omega(D) \leq \chi(D) \leq \chi_{[X,Y]}(D)$ *for any game* $[X, Y]$.

Observation 2. $\chi_{g_A}(D) \geq \chi_{[A,A]}(D)$.

The proof of the following theorem uses the Strong Perfect Graph Theorem [8]. A *filled odd hole* resp. *filled odd antihole* is a digraph the symmetric part of which is an odd hole resp. an odd antihole.

Theorem 3 (Andres and Hochstättler (2015) [3])**.** *A digraph D is perfect if and only if D contains no induced filled odd hole, no induced filled odd antihole and no induced directed cycle of length ≥ 3.*

The proof of the following theorem uses Lovász' (Weak) Perfect Graph Theorem [10] and a major result of Boros and Gurvich [7].

Theorem 4 (Andres and Hochstättler (2015) [3])**.** *For any perfect digraph D, the complement D^C has a kernel.*

In [1] some simple subclasses of game-perfect digraphs are characterized. Among these results we cite the following. We start with a definition. A *superorientation* of an undirected graph G is the digraph created by replacing every edge of the graph by either an edge or a single arc. The results mainly characterize game-perfect superorientations of complete graphs K_n.

Theorem 5 (Andres (2012) [1]). *(a) For $n \geq 5$ there is no superorientation D of K_n with $\omega(D) \leq 2$ such that D is $[A, A]$-perfect.*

(b) For $n = 4$, there is a single superorientation of K_4 that is $[A, A]$-perfect and has clique number at most 2, namely the complement $\vec{C}_4^C$ of a directed 4-cycle.

Theorem 6 (Andres (2012) [1]). *For $n \geq 4$ there is no superorientation D of K_n with $\omega(D) \leq 2$ such that D is g_A-perfect (resp. $[A, B]$-perfect).*

Our first main result (Corollary 9) will conclude from the facts mentioned in the introduction that game-perfect digraphs (with respect to the most common game g_A) always have a kernel.

2 Existence of kernels in game-perfect digraphs

Theorem 7. *Let D be an $[A, A]$-perfect digraph that does not contain an induced $\vec{C}_4^C$. Then D is the complement of a perfect digraph.*

Proof. D is a perfect digraph, since every game-perfect digraph is perfect by Observation 1. By Theorem 3, D has neither an induced filled odd hole nor an induced filled odd antihole. By Theorem 3, D has no induced complement of a directed 3-cycle (since a directed 3-cycle is self-complementary). By precondition, D has no induced complement of a directed 4-cycle. By Theorem 5, D has no complement of a directed n-cycle, $n \geq 5$. (Indeed, any four consecutive vertices in a directed cycle $\vec{C}_n$ of length $n \geq 5$ would induce the complement of a directed path $\vec{P}_4$ in the complement $\vec{C}_n^C$ of $\vec{C}_n$, which is a superorientation of K_4 with clique number 2 that, by Theorem 5 (b), is not $[A, A]$-perfect.) Therefore D^C is a digraph without induced filled odd and without induced filled odd antihole and without induced directed cycles of length ≥ 3. From Theorem 3 follows that D^C is perfect. □

Corollary 8. *Let D be an $[A, A]$-perfect digraph that does not contain an induced $\vec{C}_4^C$. Then D has a kernel.*

Proof. Let D be a game-perfect digraph (with respect to game $[A, A]$) that does not contain an $\vec{C}_4^C$. By Theorem 7, D^C is perfect. Therefore, by Theorem 4, D has a kernel. □

Corollary 9. *Let D be an g_A-perfect digraph. Then D has a kernel.*

Proof. By Observation 2, D is $[A, A]$-perfect. By Theorem 6, D contains no induced $\vec{C}_4^C$. Therefore, by Corollary 8, D has a kernel. □

3 Weakly game-perfect digraphs

In this section we consider a notion of game-perfectness related to the weak game chromatic number $\chi_{wg}(D)$ of a digraph D. The weak game chromatic number and its underlying game were introduced by Yang and Zhu [12].

The rules of this so-called *weak colouring game* wg are the following. Two players, Alice and Bob, alternately colour vertices of a given digraph D with a colour from a given colour set C, so that no monochromatic cycles (of length ≥ 2) are created. The game ends when such move is not possible any more. Alice wins if every vertex is coloured at the end of the game. Otherwise Bob wins.

As for the game g, we may specify the player $X \in \{A, B\}$ who starts and the player $Y \in \{A, B, -\}$ who has the right to miss one or several turns (including the first one), where "A" resp. "B" denote Alice resp. Bob and "$-$" denotes "none of the players". This defines six different games of the form $wg = w[X, Y]$ for digraphs with respective *associated game* $g = [X, Y]$ for undirected graphs.

The *weak game chromatic* $\chi_{wg}(D)$ number w.r.t. the game wg of a loopless digraph is the smallest size $|C|$ of a colour set C such that Alice has a winning strategy for the game wg played on D with colour set C. The digraph D is *weakly game-perfect* w.r.t. the game wg (or, *wg-perfect* for short) if for any induced subdigraph H of D, $\chi_{wg}(H) = \omega(H)$.

Note that in the case of symmetric digraphs (which we can consider as undirected graphs) the games wg and g coincide. The same holds for the weak game chromatic number and the game chromatic number resp. for the notion of weakly game-perfect and (strongly) game-perfect.

As we shall see in Theorem 11, the characterization of weakly game perfect digraphs w.r.t. a game wg can be easily reduced to the characterization of game-perfect (undirected) graphs w.r.t. the associated game g.

The proof of the Theorem 11 uses basically the same idea as the main theorem for characterizing perfect digraphs by means of perfect graphs in [3]. We start with an obvious lemma.

Lemma 10. *In a digraph D that does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as an induced subdigraph, every directed cycle has a (symmetric) edge as a chord.*

Theorem 11. *For a weak game wg and the associated undirected game g, a digraph D is wg-perfect if and only if its symmetric part $S(D)$ is a g-perfect graph and D does not contain any directed cycle $\vec{C}_n$ with $n \geq 3$ as an induced subdigraph.*

Proof. First, let $S(D)$ be a non-g-perfect graph. This means that there exists a subdigraph H of D such that Bob has a winning strategy for the game g on $S(H)$. He uses the same strategy for the game wg on the digraph H. The single arcs do not restrict him in doing so, since whenever he would close a monochromatic directed cycle by his strategy, by Lemma 10 this cycle would either be a monochromatic edge or have a monochromatic edge as a chord, which both is not possible in his strategy for the game g on $S(H)$. Therefore

every move of Bob is feasible in the game wg until he creates a win for him or forces Alice to do so, which will happen, since his strategy is a winning strategy on $S(H)$. Therefore D is not wg-perfect.

Let D contain any directed cycle of length at least 3. Then D is not perfect, thus it is not wg-perfect.

Second, let $S(D)$ be a g-perfect graph and D not contain any induced directed cycle of length ≥ 3. Let H be an induced subdigraph of D. Then $S(H)$ is an induced subdigraph of $S(D)$. Thus Alice has a winning strategy for the game g on $S(H)$. She uses the same strategy for the game wg on H. Whenever she would close a monochromatic directed cycle by her strategy, by Lemma 10 this cycle would either be a monochromatic edge or have a monochromatic edge as a chord, which both is not possible in her strategy for the game g on $S(H)$. Therefore every move of Alice is feasible in the game wg until she creates a win for her, i.e. every vertex of H is coloured, which will happen, since her strategy is a winning strategy on $S(H)$. Since H was arbitrarily chosen, D is wg-perfect. □

By the characterizations mentioned in the introduction [2, 4, 9] and Theorem 11 we obtain explicit characterizations for the four classes of weakly $[B, B]$-, $[A, B]$-, g_A- and g_B-perfect digraphs.

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