Lightness of Digraphs in Surfaces and Directed Game Chromatic Number

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Abstract

The *lightness* of a digraph is the minimum arc value, where the value of an arc is the maximum of the in-degrees of its terminal vertices. We determine upper bounds for the lightness of simple digraphs with minimum in-degree at least 1 (resp., graphs with minimum degree at least 2) and a given girth k, and without 4-cycles, which can be embedded in a surface S. (Graphs are considered as digraphs each arc having a parallel arc of opposite direction.) In case $k \geq 5$, these bounds are tight for surfaces of nonnegative Euler characteristics. This generalizes results of He et al. [11] concerning the lightness of planar graphs. From these bounds we obtain directly new bounds for the game coloring number, and thus for the game chromatic number of (di)graphs with girth k and without 4-cycles embeddable in S. The game chromatic resp. game coloring number were introduced by Bodlaender [3] resp. Zhu [22] for graphs. We generalize these notions to arbitrary digraphs. We prove that the game coloring number of a directed simple forest is at most 3.

Key words: lightness, girth, game coloring number, forest, planar digraph, torus, projective plane, Klein bottle, game chromatic number
MSC: 05C20, 05C10, 91A43, 05C15

1 Introduction

Several graph parameters which result from the vertex-edge incidence structure have been widely discussed in the literature, such as the maximum vertex degree, the minimum vertex degree, the Szekeres-Wilf number or the maximum edge degree. Recently, in some publications in the field of graph coloring

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games [22,11], a new concept has been proven of value, the concept of light edges. A *light edge* is an edge which has very few neighbors of each of its two terminal vertices. He, Hou, Lih, Shao, Wang and Zhu [11] formalized this notion in the following way. For a graph G, they defined a parameter $M^*(G)$ which is the minimum of $M(e)$ over all edges e of G, where $M(e)$ is the maximum of the degrees of the terminal vertices of e.

In the spirit of their ideas we define a more general parameter for directed graphs. For a digraph $D = (V, E)$ and an arc $e = (v, w) \in E$, let $L^+(e) = \max\{d^+(v), d^+(w)\}$, where $d^+(u)$ denotes the in-degree of vertex u. We call

$$L^+(D) = \min_{e \in E} L^+(e)$$

positive lightness or simply *lightness* of D. The *negative lightness* $L^-(D)$ of D is defined in the same way by considering the out-degrees instead of the in-degrees. It will be denoted by $L(D)$ whenever $L^+(D) = L^-(D)$.

For the following, all digraphs are assumed to have neither multiple arcs nor loops. However, pairs of antiparallel arcs are allowed. We mainly consider two classes of digraphs, i.e., simple digraphs (without antiparallel arcs), and graphs (where for each arc there is an antiparallel arc). In this way, for a graph G, $L(G)$ is the same as the parameter $M^*(G)$ defined by He et al. [11].

The lightness of a digraph D seems to be closely related to another graph parameter, the *weight* $w(D)$. It is defined as the minimum arc weight, where the *weight* of an arc (v, w) is the sum $d^+(v) + d^+(w)$. Obviously,

$$\frac{1}{2}w(D) \leq L^+(D) \leq w(D) - \delta^+(D), \tag{1}$$

where $\delta^+(D)$ denotes the minimum in-degree of D. However, both estimations may be proper, see Section 7.

Determining the weight of certain kinds of planar graphs has been considered since some time. Let G_3 be a 3-connected planar graph, and G_2 be a planar graph with minimum degree $\delta \geq 2$. By the righthand side of (1), a result of Kotzig [14] concerning the weight of G_3 implies $L(G_3) \leq 10$. Similarly, if G_3 has no 4-cycles, then $L(G_3) \leq 7$, and if G_3 has girth 5, then $L(G_3) \leq 5$, both by a result of Borodin [5]. Planar graphs G_2 with minimum degree 2 and without a certain kind of "alternating" even cycles have $L(G_2) \leq 13$ by another result of Borodin [4] together with (1).

He et al. [11] consider the case of planar graphs with minimum degree $\delta \geq 2$ and without 4-cycles. They determined upper bounds for the lightness of these graphs which depend on the (undirected) girth k, and which are best-possible if $k \geq 5$. Our main aim is to generalize these results to (planar) simple digraphs, and to graphs resp. simple digraphs which are embeddable in other surfaces, with the same restrictions on minimum degree, cycles, and girth in the case of graphs, resp., in the case of simple digraphs restricted to those with minimum in-degree $\delta^+ \geq 1$, without 4-cycles, and prescribed girth k. In Section 2 we determine upper bounds for the lightness of such simple digraphs embeddable in a surface S, whereas Section 3 is devoted to the case of graphs in S. S may be either one of the orientable surfaces S_γ, $0 \leq \gamma \leq 6$, or one of the non-orientable surfaces $N_{\overline{\gamma}}$, $1 \leq \overline{\gamma} \leq 9$, possibly even some other surface.

Whenever $k \geq 5$, the bounds are tight for the surfaces of nonnegative Euler characteristics, i.e., for the sphere, the torus, the projective plane and the Klein bottle, as shown in Section 5. In the case of other surfaces, the bounds depend on a topological parameter which is not exactly known for any of these surfaces (exept the double torus). It is the minimum number of edges a graph can have which is embeddable in that surface but not embeddable in a surface of lower genus resp. lower crosscapnumber. For the torus and the projective plane this parameter is 9 by the Kuratowski theorem. It is 15 for the Klein bottle by [8,2] and 18 for the double torus by [17]. In general, the better the topological parameter can be estimated, the better will be the bounds for lightness, and the more surfaces our results will apply to.

If $k = 3$, our bound in the planar simple digraph case is 4. We present an example of a planar simple digraph with minimum in-degree 1 and without 4-cycles which has lightness 3, thus nearly reaching our bound.

1.1 Digraph coloring games

Work on lightness was motivated by its applications concerning graph coloring games. The first version of these games was introduced by Bodlaender [3]. As we are not only interested in graphs, we will generalize his game to arbitrary digraphs in the following way.

Two players, *Alice* and *Bob*, are given an initially uncolored digraph D and a number k of colors. During the game, they alternately color an uncolored vertex with a color not used before for any of its in-neighbors. w is an in-neighbor of v if there is an arc (w, v). The game ends when no further move is possible. Alice wins if the graph is completely colored at the end, otherwise Bob wins. One may assume that Alice has the first move, and passing is not permitted, as in Bodlaender's game. We denote this game by $[A, -]$. Another version allows Bob to play first and to miss one or several turns, which generalizes a game proposed in [1]. The game defined hereby is called $[B, B]$.

For such a version g of the game, the smallest number k of colors for which Alice has a winning strategy for g is called *(directed) game chromatic number* $\chi_g(D)$ of D for g. It is easy to see that $\chi_{[A,-]}(D) \leq \chi_{[B,B]}(D)$, as for undirected graph coloring games [1]. Note that the color classes created by the directed coloring game will be acyclic, i.e., they do not induce directed cycles. Indeed, the concept of directed game chromatic number is a combination of Bodlaender's game chromatic number for undirected graphs [3] and the dichromatic number of a digraph which was introduced in 1982 by Neumann-Lara [19]. Since the directed game chromatic number of a graph is its game chromatic number, in order to simplify notation we will omit "directed" even when talking about digraphs.

The directed game chromatic number is not related to the oriented game chromatic number introduced by Nešetřil and Sopena [18]. This number is based on the same type of two-player game, however, the coloring created by the players must be an oriented coloring. The color classes in an oriented coloring of a digraph have to be independent and must have the property, whenever there are two equally-colored vertices v, w and arcs (v,x), (w,y), then x and y have to be colored distinctly. On the other hand, in a directed coloring defined by our game, the color classes are not necessarily independent.

Upper bounds for the game chromatic number of several classes of graphs have been achieved, e.g., by Faigle et al. [7] who determined the bound 4, which is best possible, for forests, or, in a series of papers, by Kierstead and Trotter [13], by Dinski and Zhu [6], by Zhu [22], by Kierstead [12], and by Zhu [24] who reduced the upper bound for planar graphs to the value 17. Improving a result of Zhu [23], Kierstead [12] has shown that $\chi_g(G) \le \lfloor \frac{1}{4}(3\sqrt{73+96\gamma}+41) \rfloor$ holds for graphs G embeddable in the orientable surface S_γ.

Often, upper bounds for the game chromatic number of graphs are not obtained directly, but by estimating the so-called *coloring number* introduced by Zhu [22]. In the general case of digraphs we may define a similar parameter by the following *directed marking game*. It is played by Alice and Bob, given a digraph D and a number k. Alternately, the players choose a vertex which has $k-1$ chosen in-neighbors at most. When no further move is possible, Alice wins if every vertex has been chosen, otherwise Bob wins. Different versions of the game are denoted as before. Note that, if Alice has a winning strategy for the directed marking game, she wins the corresponding version of the coloring game with k colors, as well, by choosing the vertices to be colored according to her winning strategy for the directed marking game. For such a version g of the directed marking game, the smallest number k for which Alice has a winning strategy is called *(directed) game coloring number* $\mathrm{col}_g(D)$ of D. By the same reasons as before we may omit "directed". We remark the fundamental estimation

$$\chi_g(D) \le \mathrm{col}_g(D). \tag{2}$$

For undirected graphs, it was used by some authors [7,10] even before the work of Zhu [22].

By results of Zhu [22] and He et al. [11] upper bounds for the lightness imply upper bounds for the game coloring number, and thus, for the game chromatic number. The upper bounds that we will obtain in Section 6 for the special classes of graphs embeddable in a surface S with large girth are considerably better than the previously known upper bounds for the game chromatic number of graphs embeddable in S without restriction to the girth.

In Section 6 we will see that our results concerning the lightness of classes of graphs lead to bounds for the game coloring number of certain simple digraphs as well. We indicate and conjecture that these bounds may be even tightened by using the idea of Section 7 and our main results concerning the lightness of classes of simple digraphs.

2 The structure of digraphs in surfaces

Every finite connected graph can be cellularly embedded in one of the nonorientable surfaces $N_{\overline{\gamma}}$ where $\overline{\gamma} \geq 1$ is the *crosscapnumber* and in one of the orientable surfaces S_γ where $\gamma \geq 0$ is the genus of the surface, cf. [9]. Let $N_0 = S_0$. The genus (resp., crosscapnumber) of a graph G is the smallest $\gamma \geq 0$ for which G embeds in S_γ (resp., N_γ). The *Euler characteristic* $\chi(S)$ of a surface S is defined by the invariant $\#V - \#E + \#F$, where F denotes the set of faces in a 2-cell embedding of a finite graph $G = (V, E)$ in S. It is well-known that $\chi(S_\gamma) = 2 - 2\gamma$, and $\chi(N_{\overline{\gamma}}) = 2 - \overline{\gamma}$. The surfaces of nonnegative Euler characteristic are the *sphere* S_0, the *torus* S_1, the *projective plane* N_1, and the *Klein bottle* N_2.

In Section 3 resp. this section we will generalize Theorems 2.1. and 2.2. in He et al. [11] which examine the lightness of planar graphs to other surfaces resp. to digraphs embeddable in surfaces.

$\overline{\gamma}$	1	2	3	4	5	6	7	8	9	10	11	12	...
$M(N_{\overline{\gamma}}) \geq$	9	15	15	15	15	15	15	15	15	15	15	15	
$M(N_{\overline{\gamma}}) \leq$	9	15	19	23	25	29	33	35	39	41	45	47	

Table 1
Bounds for M (nonorientable case)

For surfaces of negative Euler characteristic, a crucial parameter in our considerations is the minimal edge number of a graph with genus γ resp. crosscapnumber $\overline{\gamma}$ which we denote by $M(S)$ where $S = S_\gamma$ resp. $S = N_{\overline{\gamma}}$. To our knowledge, $M(S)$ has been determined for the surfaces S of nonnegative Euler characteristics and the double torus only: $M(S_1) = M(N_1) = 9$ by Kuratowski's Theorem, $M(N_2) = 15$ by results of Glover, Huneke and Wang [8] and Archdeacon [2], and $M(S_2) = 18$ by work of Myrvold [17]. Note that it is not even known whether $M(S_{\gamma+1}) = M(N_{2\gamma+1})$. The upper bounds for $M(S)$ in Tables 1 resp. 2 are given by examples of $K_{m,n} - M_k$ the crosscapnumber resp. the genus of which are well-known [16,15]. (M_k denotes a set of k independent edges.)

γ	1	2	3	4	5	6	7	8	9	10	11	12	...
$M(S_\gamma) \geq$	9	18	19	20	21	22	23	24	25	26	27	28	
$M(S_\gamma) \leq$	9	18	25	33	39	45	49	55	61	67	71	77	

Table 2
Bounds for M (orientable case)

Let us fix some terms. All digraphs we consider will neither have multiple arcs, nor loops. *Graphs* are digraphs where for each arc there is an opposite arc. A pair $\{(v,w),(w,v)\}$ of opposite arcs will be called *edge* vw. *Simple digraphs* do not have any pair of opposite arcs. For a digraph $D = (V,E)$, let $d(v) = d^+(v)$ resp. $d^-(v)$ be the number of in-arcs resp. out-arcs of vertex v, and

$$\delta(D) = \delta^+(D) = \min_{v \in V} d^+(v)$$

the minimum (in-)degree, and $\Delta^+(D) = \max_{v \in V} d^+(v)$, and

$$L^+(e) = \max\{d^+(v), d^+(w)\}$$

if $e = (v,w)$ is an arc, and

$$L(D) = L^+(D) = \min_{e \in E} L^+(e)$$

the *(positive) lightness* of D. Let

$$\delta^\pm(D) = \min_{v \in D} d^+(v) + d^-(v).$$

The *girth* $g(D)$ of D is the length of its shortest undirected cycle (or infinity if there is no cycle), where we allow a cycle to pass only one arc of each pair of opposite arcs.

We further define for a nonnegative integer k and a surface S

$$F_S(k) = \frac{M(S)k + M(S)}{(2\chi(S) + M(S))\,k + 2\chi(S) - 3M(S)}, \quad H_S = \frac{5M(S)}{10\chi(S) + M(S)}.$$

These parameters $F_S(k)$ and H_S (whenever well-defined and positive) will be main part of the upper bounds discussed in Theorems 1, 2, 3, and 4. In order to simplify the notation we will write M resp. χ instead of $M(S)$ resp. $\chi(S)$ when there is only one surface S. Clearly, $F_S(k)$ is non-increasing when $k \longrightarrow \infty$ for $2\chi + M > 0$ and $k \geq k_0 > \frac{3M-2\chi}{2\chi+M}$.

Theorem 1 *Let S be a surface of Euler characteristic $\chi(S)$ and $D = (V, E)$ be a simple digraph embeddable in S with $\delta^+(D) \geq 1$ and $g(D) \geq k$ for odd $k \geq 5$.*

(a) If $\chi(S) > 0$, then

$$L^+(D) \leq \left\lceil \frac{4}{k-3} \right\rceil .$$

(b) If $\chi(S) \leq 0$, and $M(S) + 2\chi(S) > 0$, and $k > \frac{3M(S)-2\chi(S)}{2\chi(S)+M(S)}$, then

$$L^+(D) \leq \lfloor F_S(k) \rfloor .$$

Proof. Assume $D = (V, E)$ is a counterexample. W.l.o.g. D is connected. So $g(D) \geq k$ for odd $k \geq 5$ and $L^+(D) \geq c + 1$ where

$$c = \left\lceil \frac{4}{k-3} \right\rceil \qquad \text{in case (a), resp.,} \tag{3}$$

$$c = \lfloor F_S(k) \rfloor \qquad \text{in case (b),} \tag{4}$$

and there is a 2-cell embedding which embeds D in S. By deleting all vertices v with $d^+(v) + d^-(v) = 1$ succesively and subdividing each arc (v, w) with $d^+(v) \geq c + 1$ and $d^+(w) \geq c + 1$ once (and maintaining the orientation) we obtain an auxiliary digraph $\overline{D} = (\overline{V}, \overline{E})$. Let

$$V_i := \left\{ v \in \overline{V} \,\middle|\, d^+(v) = i \right\}, \qquad i = 1, \ldots, c$$

$$V_{c+1} := \left\{ v \in \overline{V} \,\middle|\, d^+(v) \geq c + 1 \right\}.$$

Further $n_i := \#V_i$, $m_i := \#\{(v, w) \in \overline{E} \,|\, w \in V_i\}$, $n := \#\overline{V}$, $m := \#\overline{E}$. $\overline{D}$ is bipartite with V_{c+1} forming one of the partite sets. Thus $g(\overline{D}) \geq k + 1$ since k is odd. By the construction, $L^+(\overline{D}) \geq c + 1$, $\delta^+(\overline{D}) \geq 1$, and

$$\delta^{\pm}(\overline{D}) \geq 2. \tag{5}$$

Like D, $\overline{D}$ embeds in S, and for a fixed 2-cell embedding we have

$$f \leq \frac{2}{k+1} m$$

where f denotes the number of faces of $\overline{D}$.

Obviously,

$$\sum_{i=1}^{c+1} m_i = m. \tag{6}$$

In view of (5), $m_1 \leq m_{c+1}$, so that

$$-m_{c+1} \leq -m_1, \quad m_1 \leq \frac{m}{2}. \tag{7}$$

Hence

$$\begin{aligned}
n &= \sum_{i=1}^{c+1} n_i \quad \leq \quad m_1 + \sum_{i=2}^{c} \frac{m_i}{i} + \frac{m_{c+1}}{c+1} \\
&\leq \frac{m_1}{2} + \frac{m_1 + m_2 + \ldots + m_{c+1}}{2} + \frac{-c+1}{2c+2} m_{c+1} \\
&\overset{(6),(7)}{\leq} \frac{c+2}{2c+2} m \quad = \quad \frac{1}{2}\left(1 + \frac{1}{c+1}\right) m.
\end{aligned}$$

In case (a), by (3), $1/(c+1) \leq (k-3)/(k+1)$, which implies

$$n - m + f \leq \left(\frac{k-1}{k+1} - 1 + \frac{2}{k+1}\right) = 0,$$

and contradicts $\chi \geq 1$.

In case (b), by the preconditions, $F_S(k) > 0$. Then, by (4), $1/(c+1) < 1/F_S(k)$, hence

$$\begin{aligned}
n - m + f &< \left(\frac{1}{2} + \frac{(2\chi + M)k + 2\chi - 3M}{2Mk + 2M} - 1 + \frac{2}{k+1}\right) m \\
&= \chi \frac{m}{M} \leq \chi.
\end{aligned}$$

The last estimation holds since $\chi \leq 0$, and $0 < M \leq m$ as we may assume w.l.o.g. that D (and so $\overline{D}$) does not embed in a surface of lower genus resp. lower crosscapnumber than S. On the other hand this is a contradiction, because by definition of Euler characteristics $n - m + f \geq \chi$. $\square$

If we drop the prerequisite $\delta^+(D) \geq 1$, it is easy to see that the parameter $L^+(D)$ is not bounded by any constant. Think of a star, for example. The same problem occurs if we allow 4-cycles: for each $n \geq 1$, there are planar bipartite digraphs $\vec{K}_{2,2n}$ with $\delta^+(\vec{K}_{2,2n}) \geq 1$ but $L^+(\vec{K}_{2,2n}) \geq n$. However, we may permit 3-cycles, as stated in the following theorem.

Theorem 2 *Let S be a surface of Euler characteristic $\chi(S)$ and D be a simple digraph embeddable in S with $\delta^+(D) \geq 1$ and $g(D) \geq 3$ which does not contain any 4-cycles.*

(a) If $\chi(S) > 0$, then $L^+(D) \leq 4$.
(b) If $\chi(S) \leq 0$ and $M(S) > -10\chi(S)$, then $L^+(D) \leq \lfloor H_S \rfloor$.

Proof. Assume the theorem is false. W.l.o.g. we may assume that there is a connected counterexample $D = (V, E)$ with $\delta^{\pm}(D) \geq 2$ (cf. the proof of the preceding theorem). Let $c = 4$ in case (a), and $c = \left\lfloor \frac{5M(S)}{10\chi(S)+M(S)} \right\rfloor$ in case (b). We define

$$V_{c+1} := \{v \in V \mid d^+(v) \geq c+1\},$$
$$T := \{(v, w) \in E \mid v \in V_{c+1} \wedge w \in V_{c+1}\},$$

and $V_i := \{v \in V \mid \delta^+(v) = i\}$ for $1 \leq i \leq c$, $E_i := \{(v, w) \in E \mid w \in V_i\}$ for $1 \leq i \leq c+1$. $n_i := \#V_i$, $m_i := \#E_i$, $n := \#V$, $m := \#E$, $t := \#T$. Let f_i be the number of i-faces, i.e., of faces bounded by exactly i arcs, and f the number of faces. Since $f_4 = 0$ we have

$$3f_3 + 5f_5 + 6f_6 + 7f_7 + \ldots = 2m,$$

and further

$$f = \frac{1}{5} \cdot \sum_{i \geq 3} 5f_i \leq \frac{1}{5}\left(2f_3 + \sum_{i \geq 3} if_i\right) = \frac{2}{5}m + \frac{2}{5}f_3. \tag{8}$$

As in the preceding proof we will also consider the digraph $\overline{D} = (\overline{V}, \overline{E})$ obtained from D by subdividing each arc from T once by maintaining the orientation. We define $\overline{n}_i$ resp. $\overline{m}_i$ by replacing V resp. E by $\overline{V}$ resp. $\overline{E}$ in the definitions leading to n_i resp. m_i. Obviously, $\overline{n}_1 = n_1 + t$, $\overline{n}_i = n_i$ for $i \geq 2$, $\overline{m} = m + t$. As above we state that

$$\sum_{i=1}^{c+1} \overline{m}_i = \overline{m}, \tag{9}$$

$$-\overline{m}_{c+1} \leq -\overline{m}_1, \tag{10}$$

from which

$$\overline{m}_1 \leq \overline{m}/2 \tag{11}$$

follows.

Thus we conclude

$$\begin{aligned}
n &= \sum n_i = \sum \overline{n}_i - t \leq \sum \frac{\overline{m}_i}{i} - t \\
&\leq \frac{\overline{m}_1}{2} + \frac{\overline{m}_1 + \overline{m}_2 + \ldots + \overline{m}_{c+1}}{2} + \left(\frac{1}{c+1} - \frac{1}{2}\right)\overline{m}_{c+1} - t \\
&\overset{(9)}{=} \frac{\overline{m}}{2} + \frac{\overline{m}_1}{2} - \frac{c-1}{2(c+1)}\overline{m}_{c+1} - t \\
&\overset{(10),(11)}{\leq} \frac{c+2}{2(c+1)}\overline{m} - t = \frac{1}{2}\left(1 + \frac{1}{c+1}\right)m - \frac{c}{2(c+1)}t.
\end{aligned} \tag{12}$$

Note further that $f_3 \leq t$ since every 3-face of D contains an arc from T and there are no adjacent 3-faces (otherwise there would be a 4-cycle). Combining this with (8) and (12) yields in case (a)

$$n - m + f \leq \frac{2}{5}(f_3 - t) \leq 0$$

since $1/(c+1) = 1/5$. In case (b), by the prerequisite $H_S > 0$, therefore we have $1/(c+1) < 1/H_S$. Furthermore, $c \geq 4$ because $M > 0$, which implies $-c/(2c+2) \leq -2/5$. W.l.o.g. we may assume again that D does not embed in a surface of lower genus resp. lower crosscapnumber, so $M \leq m$. We conclude

$$n - m + f < \left(\frac{1}{2} + \frac{10\chi + M}{10M} - 1 + \frac{2}{5}\right) m + \frac{2}{5}(f_3 - t) \leq \chi \frac{m}{M} \leq \chi.$$

In both cases we obtain a contradiction against the definition of Euler characteristics. □

3 The structure of graphs in surfaces

The following theorem generalizes Theorem 2.1. in He et al. [11], which examines the lightness of planar graphs, to graphs in arbitrary surfaces.

Theorem 3 *Let S be a surface of Euler characteristic $\chi(S)$ and $G = (V, E)$ be a graph embeddable in S with $\delta(G) \geq 2$ and $g(G) \geq k$ for odd $k \geq 5$.*

(a) If $\chi(S) > 0$, then

$$L(G) \leq \left\lceil \frac{k+5}{k-3} \right\rceil.$$

(b) If $\chi(S) \leq 0$ and $M(S) + 2\chi(S) > 0$ and $k > \frac{3M(S) - 2\chi(S)}{2\chi(S) + M(S)}$, then

$$L(G) \leq \lfloor 2F_S(k) \rfloor.$$

Proof. (a) has been proven by He et al. [11] for planar graphs. The same proof holds for graphs embeddable in the projective plane. We are left to consider (b). Again we assume a connected graph $G = (V, E)$ is a counterexample. Hence $g(G) \geq k \geq 5$ for odd k and $L(G) \geq c + 1$ where

$$c = \lfloor 2F_S(k) \rfloor, \tag{13}$$

and there is a 2-cell embedding which embeds G in S. By subdividing each edge vw with $d(v) \geq c + 1$ and $d(w) \geq c + 1$ once we obtain (as in the preceding proof) an auxiliary graph $\overline{G} = (\overline{V}, \overline{E})$ with $g(\overline{G}) \geq k + 1$ (since k is odd), $L(\overline{G}) \geq c + 1$, and $\delta(\overline{G}) \geq 2$. Again this construction produces a bipartite graph with partite sets

$$V_1 := \left\{ v \in \overline{V} \,\middle|\, d(v) \leq c \right\}, \text{ and}$$

$$V_2 := \left\{ v \in \overline{V} \,\middle|\, d(v) \geq c+1 \right\}.$$

Further $n_i := \#V_i$, $n := \#\overline{V}$, $m := \#\overline{E}$. W.l.o.g. $\overline{G}$ does not embed in a surface of lower genus resp. lower crosscapnumber than S. Since $\overline{G}$ is homeomorphic to G, $\overline{G}$ embeds in S, and for a fixed 2-cell embedding we have

$$f \leq \frac{2}{k+1} m$$

where f denotes the number of faces of $\overline{G}$. The number of vertices is bounded by

$$n = n_1 + n_2 \leq \left(\frac{1}{2} + \frac{1}{c+1} \right) m,$$

c.f. He et al. [11]. By the preconditions and by (13), $1/(c+1) < 1/\left(2F_S(k)\right)$. As in the proof of Theorem 1 we obtain the contradiction $n - m + f < \chi$. $\square$

By the same refinement which extends the proof of Theorem 1 to a proof of Theorem 2 the proof of Theorem 3 may be modified to prove the following

Theorem 4 *Let S be a surface of Euler characteristic $\chi(S)$ and G be a graph embeddable in S with $\delta^+(G) \geq 2$ and $g(G) \geq 3$ which does not contain any 4-cycles.*

(a) If $\chi(S) > 0$, then $L(G) \leq 9$.
(b) If $\chi(S) \leq 0$ and $M(S) > -10\chi(S)$, then $L(G) \leq \lfloor 2H_S \rfloor$.

The proof is left to the reader. Note that He et al. [11] achieved the tighter bound 8 for planar graphs using special properties of cycles in planar embeddings.

4 On the parameter M

In a series of papers Glover, Huneke, Wang [8], and Archdeacon [2] classified the irreducible graphs for the projective plane, i.e. those graphs which cannot be embedded in the projective plane but every subgraph can (up to homeomorphisms). From their results follows

Theorem 5 (Glover, Huneke and Wang [8]; Archdeacon [2])
$M(N_2) = 15$.

Corollary 6 *$M(N_{\overline{\gamma}}) \geq 15$ for $\overline{\gamma} \geq 2$.*

Thus, we have $M + 2\chi \geq 15 + 2\chi > 0$ for $\chi \geq -7$, i.e. Theorems 1 (b) and 3 (b) apply at least to the surfaces $N_{\overline{\gamma}}$, $2 \leq \overline{\gamma} \leq 9$. Probably they apply to much more surfaces since $M(N_{\overline{\gamma}})$ increases with $\overline{\gamma}$, cf. Table 1 for the possible ranges. Theorems 2 (b) and 4 (b) apply at least to the surfaces N_2 and N_3, since $M \geq 15 > -10\chi$ then.

Myrvold [17] gives a classification of all irreducible graphs for the torus with at most 11 vertices. These graphs have at least 18 edges.

Theorem 7 (Myrvold [17])
$M(S_2) = 18$.

Lemma 8 *For any orientable surface S_γ with $\gamma \geq 2$, $M(S_\gamma) \geq 16 + \gamma$.*

Proof. This is an obvious induction on γ. For $\gamma = 2$ the statement is true by Myrvold's theorem [17]. Note that the deletion of an edge in a graph reduces the genus by at most one, which implies the rest. □

By Lemma 8 Theorems 1 (b) resp. 3 (b) apply at least to the surfaces S_γ, $1 \leq \gamma \leq 6$, since then $M + 2\chi \geq 20 - 3\gamma > 0$. Theorems 2 (b) and 4 (b) apply at least to the torus. Again, for the orientable surfaces, the parameter M will be probably greater, so that the bounds can be significantly tightened, cf. Table 2.

Without any further knowledge of M we already obtain the upper bounds for the lightness of graphs in surfaces which are given in the Appendix. In order to obtain tighter bounds for the lightness resp. positive lightness of graphs resp. digraphs in surfaces, a main subject of future research has to be finding better lower bounds for $M(S)$.

5 Tightness of the bounds

There is a series of corollaries from Theorems 1 and 3. Note that, if $\chi = 0$, the parameter M cancels out in the expression $F_S(k)$.

Corollary 9 *Let D resp. G be a simple digraph with $\delta^+(D) \geq 1$ resp. a graph with $\delta(G) \geq 2$ embeddable in the sphere or the projective plane. Then*

a) $L(G) \leq 5$ *if* $g(G) \geq 5$,
b) $L(G) \leq 3$ *if* $g(G) \geq 7$,
c) $L(G) \leq 2$ *if* $g(G) \geq 11$,
a') $L^+(D) \leq 2$ *if* $g(D) \geq 5$,
b') $L^+(D) \leq 1$ *if* $g(D) \geq 7$.

Corollary 9 is tight in the projective plane case: For a) consider the complete graph K_6 and subdivide each edge once. By the result of Ringel and Youngs [20] K_6, and thus the resulting graph, can be embedded in the projective plane. It has girth 6, minimum degree 2, and lightness 5. For b) consider the graph depicted in Fig. 1 (a). For a') take the example from a) and orient the edges in such a way that $d^+(v) \geq 1$ for every vertex v and every vertex of degree 5 has in-degree at least 2. The tightness in the undirected sphere case was already proven in [11], in the directed sphere case a') edges are oriented suitably as above.

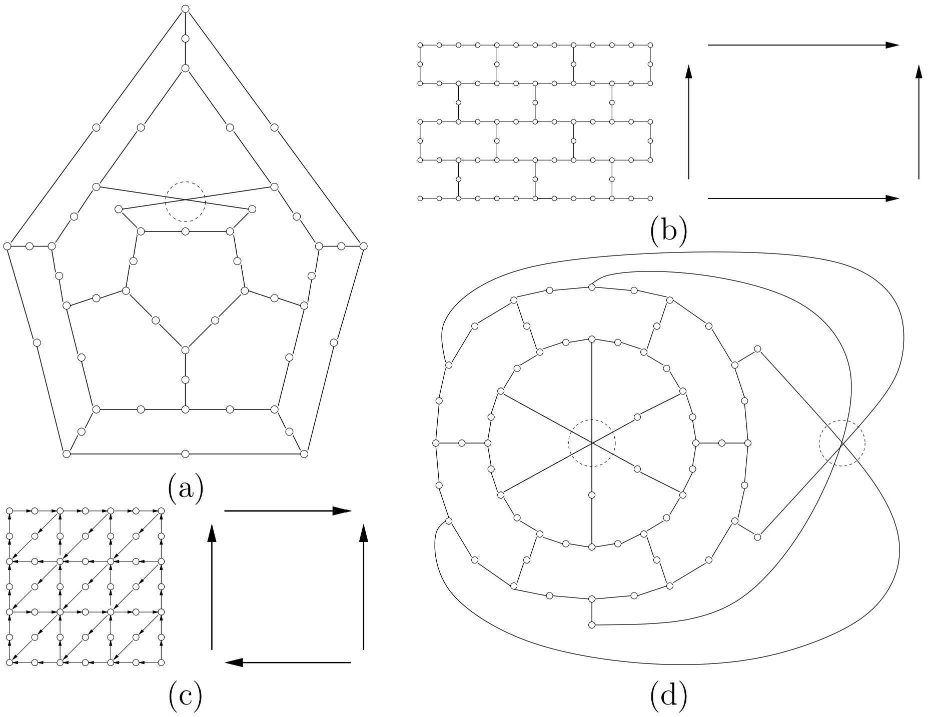

Fig. 1. (a) projective twisted dodecahedron (b) graph with torus-identification (c) digraph with Klein-bottle identification (d) double-twisted doubleclock

Corollary 10 *Let D resp. G be a simple digraph with $\delta^+(D) \geq 1$ resp. a graph with $\delta(G) \geq 2$ embeddable in the torus or the Klein bottle. Then*

a) $L(G) \leq 6$ *if* $g(G) \geq 5$,
b) $L(G) \leq 4$ *if* $g(G) \geq 7$,
c) $L(G) \leq 3$ *if* $g(G) \geq 9$,
d) $L(G) \leq 2$ *if* $g(G) \geq 13$,
a') $L^+(D) \leq 3$ *if* $g(D) \geq 5$,
b') $L^+(D) \leq 2$ *if* $g(D) \geq 7$,
c') $L^+(D) \leq 1$ *if* $g(D) \geq 9$.

Corollary 10 is tight in the torus case: For a) consider the complete graph K_7 and subdivide each edge once. The resulting graph can be embedded in the torus and has girth 6, minimum degree 2, and lightness 6. For b) consider

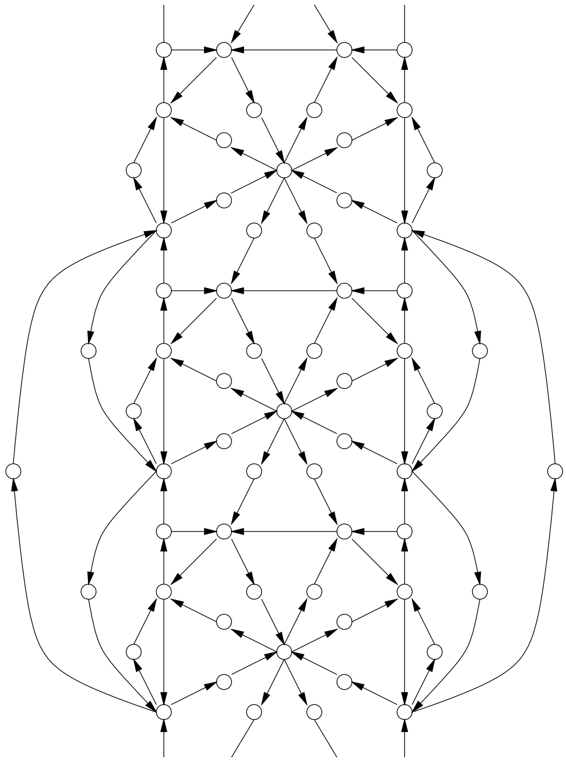

Fig. 2. The upper and the lower border of this illustration have to be glued together, so that a planar graph is obtained. (Imagine rolling the rectangle on a cylinder.)

the graph obtained from subdividing each edge once in either the complete bipartite graph $K_{4,4}$ or the 4-dimensional hypercube. Both examples have genus 1, girth 8, minimum degree 2 and lightness 4. Furthermore, c) is tight since the graph G of Fig. 1 (b) with $g(G) = 12$, $\delta(G) = 2$, and $L(G) = 3$ can be embedded in the torus. For a') resp. b') we may take the same examples as for a) resp. b) and orient the edges in such a way that the minimum in-degree is 1 and the positive lightness 3 resp. 2. Such orientations are easily found.

Corollary 10 is tight in the Klein bottle case: For a) consider the graph G of Fig. 1 (c) (without the orientation) with $g(G) = 6$, $\delta(G) = 2$, and $L(G) = 6$. An example for the tightness of b) is obtained from $K_{4,4}$ by subdividing each edge once. This graph has crosscapnumber 2, girth 8, minimum degree 2, and lightness 4. For c) consider the double-twisted doubleclock which is depicted in Fig. 1 (d). For a') consider the digraph of Fig. 1 (c) again, for b') the subdivision of $K_{4,4}$ with an obvious orientation.

We do not know whether the result of Theorem 2 is tight, not even in the case of planar digraphs, i.e. whether there exists a planar digraph D with $\delta^+(D) \geq 1$ and which does not contain 4-cycles, however having $L^+(D) = 4$. Figure 2 depicts a planar digraph D obeying the preconditions of Theorem 2 with $L^+(D) = 3$.

6 Application to graph coloring and ordering games

The application of our results to game coloring numbers is based on a simple but important observation of Zhu [22] on edge partitions. Let $G = (V, E)$, $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be graphs with the same vertex set. $G_1|G_2$ is an *edge partition* of G if $E = E_1 \dot{\cup} E_2$.

Observation 11 (Zhu [22]; Guan and Zhu [10])
If a graph G has an edge partition $G_1|G_2$, $\mathrm{col}_g(G) \leq \mathrm{col}_g(G_1) + \Delta(G_2)$, for any version g of the marking game.

We may define an *arc partition* $D_1|D_2$ of a digraph $D = (V, E)$ in line, i.e. if $D_1 = (V, E_1)$, $D_2 = (V, E_2)$, and $E = E_1 \dot{\cup} E_2$.

Observation 12 *If a digraph D has an arc partition $D_1|D_2$,*

$$\mathrm{col}_g(D) \leq \mathrm{col}_g(D_1) + \Delta^+(D_2),$$

for any version g of the directed marking game. □

A graph G is called *i-hereditary* if, for every subgraph H of G,

$$\delta(H) \leq 1 \quad \text{or} \quad L(H) \leq i.$$

Let $u(S, k)$ be an upper bound for the lightness of graphs embeddable in a surface S with girth at least k and minimum degree at least 2. Possibly, $u(S, k) = \infty$. Since every subgraph of a graph G embeddable in S with girth at least k embeds in S and has girth at least k, too, G is $u(S, k)$-hereditary.

He et al. proved the following

Lemma 13 (He, Hou, Lih, Shao, Wang and Zhu [11])
If a graph G is i-hereditary, G has an edge partition $G_1|G_2$, so that G_1 is a forest and $\Delta(G_2) \leq i - 1$.

By a result of Faigle et al. [7], the game coloring number of a forest is at most 4, a statement which we can combine with Observation 11 and Lemma 13 to obtain

Corollary 14 *For a graph G embeddable in a surface S with girth at least k,*

$$\mathrm{col}_g(G) \leq u(S, k) + 3,$$

for any version g of the marking game.

In Table 5 resp. Table 6 these bounds which result from Theorem 3 and Corollary 14 are given explicitly for the surfaces S_j, $0 \le j \le 3$, resp., N_j, $1 \le j \le 8$. For reasons of clarity and space, the bounds for N_9 are omitted. By (2), these numbers are bounds for the respective game chromatic numbers, too.

One method, to obtain bounds for the directed game coloring number, i.e. for the simple digraph case, is to use Lemma 13 again in conjunction with Observation 12 and the following

Theorem 15 *For an orientation $\vec{F}$ of a forest F and any version g of the directed marking game,*

$$\mathrm{col}_g(\vec{F}) \le \mathrm{col}_{[B,B]}(\vec{F}) \le 3.$$

Proof. The first estimation is obvious, cf. [1]. We are left to prove that Alice has a winning strategy for the directed marking game on the digraph $\vec{F}$ which guarantees that the players never create an unchosen vertex with more than 2 incoming neighbors. A move of the game can be regarded as splitting a subtree of the forest into several subtrees with the property that, initially, the chosen vertex belongs to each new subtree and is a leaf in those subtrees. If this leaf has in-degree 1, it is erased in the respective subtree, because, for the rest of the game, it is no danger for its neighbor. (Other vertices than the chosen one are not splitted in that move.) With respect to a certain situation of the game, we call the subtrees *independent subtrees.*

Alice's winning strategy consists in playing in such a way that after each of her moves each independent subtree contains at most one chosen vertex. Therefore Bob always has a move as long as there is an unchosen vertex. Consider the case that Bob has chosen a vertex. If, after his move, each independent subtree has at most one chosen vertex, Alice may simply choose the neighbor of a chosen vertex in an independent subtree or any vertex in an independent subtree with no chosen vertex. Otherwise, Bob has created at most one independent subtree with 2 chosen vertices v and w. Then, in order to reinstall her strategy, Alice considers the path P from v to w and chooses a vertex with in-degree 2 (in P), which obviously exists. By induction, at the end of the game each independent subtree will consist of a single vertex. □

Corollary 16 *For a digraph D which is the orientation of a graph embeddable in a surface S with girth at least k,*

$$\mathrm{col}_g(D) \le u(S,k) + 2,$$

for any version g of the ordering game.

Proof. In Observation 12 we can estimate $\Delta^+(D_2) \le \Delta(D_2)$. □

Hence, by (2), $\chi_g(D) \leq u(S,k)+2$. In general, these bounds can be found by decreasing by 1 the bounds in Table 5 resp. Table 6. However, the argument does not apply to the bounds in brackets which result from [12,22,24].

7 Final remarks

In Section 6 we determined upper bounds for directed coloring numbers of a simple digraph by using our result concerning the lightness of a graph. The conjecture that these bounds can be improved by applying the results concerning the positive lightness of a simple digraph seems to suggest itself. A first step towards this conjecture is the following theorem which makes use of a refined definition of i-hereditary. A digraph D is called i-$^+$*hereditary* if, for every subgraph H of D, $\delta^+(H) = 0$ or $L^+(H) \leq i$.

Theorem 17 *Let $i \geq 0$. An i-$^+$hereditary digraph D has an arc partition $D_1|D_2$, so that D_1 is acyclic, i.e., does not contain a directed cycle, and $\Delta^+(D_2) \leq i$.*

Proof. We proceed by induction on the number of arcs. If there is no arc the statement is trivial. If $\delta^+(D) = 0$, there is an arc (v,w) with $d^+(v) = 0$, and by induction hypothesis an arc partition $D_1'|D_2'$ of $D' = D - (v,w)$ exists with the desired properties for D'. Set $D_2 = D_2'$ and $D_1 = D_1' + (v,w)$. D_1 is acyclic since D_1' contains no directed cycle and $d^+(v) = 0$. On the other hand, in case $\delta^+(D) > 0$, there is an arc $e = (v,w)$ with $L^+(e) \leq i$, and by induction hypothesis an arc partition $D_1'|D_2'$ of $D' = D - e$ with the desired properties for D'. Let $D_2 = D_2' + e$ and $D_1 = D_1'$. We have $d^+_{D_2}(v) \leq d^+_D(v) \leq i$, and $d^+_{D_2}(w) \leq i$. So $\Delta^+(D_2) \leq i$. In both cases, $D_1|D_2$ is the required edge partition. □

In order to apply Observation 12, however, we have to determine the directed coloring number of acyclic digraphs embeddable in a given surface with a given girth. Maybe, this problem is as difficult as the general (not necessarily acyclic) case.

Lightness and weight. The relation (1) between lightness and weight of a digraph D motivates us to consider the following residue parameters

$$R_1(D) = 2L^+(D) - w(D),$$
$$R_2(D) = w(D) - L^+(D) - \delta^+(D).$$

Obviously, $R_1(D) = R_2(D) = 0$ for regular digraphs, i.e., digraphs where each vertex has the same in-degree. But there are also non-regular digraphs with

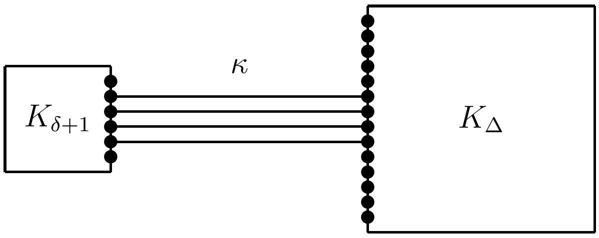

Fig. 3. A non-regular graph for which (1) is trivial

arbitrarily large maximum in-degree (or arbitrarily large clique number) Δ, arbitrarily large minimum in-degree $\delta < \Delta$, and arbitrarily large connectivity $\kappa < \delta$, with the same property. (E.g., consider the graph built by $K_{\delta+1}$ and K_Δ which are glued together by a matching of cardinality κ as in Figure 3.) A general criterion to recognize those digraphs for which lightness and weight describes the same phenomenon is given by the following

Proposition 18 *Let $D = (V, E)$ be a digraph with $E \neq \emptyset$. Then the following statements are equivalent:*

(i) $R_1(D) = R_2(D) = 0$
(ii) $L^+(D) = \delta^+(D)$
(iii) D contains an arc (v, w) with $d^+(v) = d^+(w) = \delta^+(D)$
(iv) $w(D) = 2\delta^+(D)$

Proof. The system (i) is equivalent to $L^+(D) = \delta^+(D)$ and $w(D) = 2\delta^+(D)$, thus (ii) follows from (i). On the other hand, one of the conditions (ii) and (iv) is redundant. Assume that $w(D) = 2\delta^+(D)$. Then we have

$$\begin{aligned} 0 \le R_1(D) &= 2L^+(D) - w(D) = 2L^+(D) - 2\delta^+(D), \text{ and} \\ 0 \le R_2(D) &= w(D) - L^+(D) - \delta^+(D) = \delta^+(D) - L^+(D), \end{aligned}$$

hence $\delta^+(D) = L^+(D)$. As a consequence, (iv) implies (i). Note that, if (iii) is not true, then, since $E \neq \emptyset$, each arc e has at least one end vertex v with $d^+(v) > \delta^+(D)$, and $L^+(D) = \min_e L(e) > \delta^+(D)$. This proves the implication (ii)$\Rightarrow$(iii). (iii)$\Rightarrow$(iv) follows from the definition of weight. $\square$

Remark. In general, R_1 and R_2 may not be bounded, even when restricted to (undirected) trees. To see this, for given integers $n_1 \geq 1$ and $n_2 \geq 1$, we construct a rooted tree T. Its root v has $n_1 + n_2$ descendants, each one of which has $n_1 + 2n_2$ descendants again. To form the tree $T_{(n_1,n_2)}$ take T and a copy T' of T with root v' and connect v and v' by an edge. One can easily check that $L(T_{(n_1,n_2)}) = n_1 + n_2 + 1$, and $w(T_{(n_1,n_2)}) = n_1 + 2n_2 + 2$, therefore $R_1(T_{(n_1,n_2)}) = n_1$, and $R_2(T_{(n_1,n_2)}) = n_2$. Here, the difference between the concepts of lightness and weight is expressed by different light edges: vv' is the

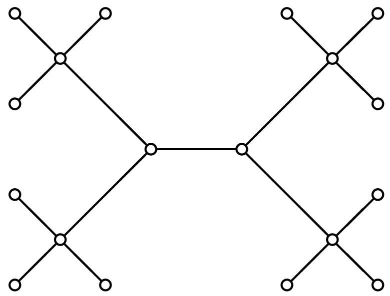

Fig. 4. The tree $T_{(1,1)}$.

only 'light edge' in the sense of lightness (since $L(vv') = L(T_{(n_1,n_2)})$,) whereas only every leaf edge e is 'light' in the sense of weight (since $w(e) = w(T_{(n_1,n_2)})$.) Figure 4 depicts $T_{(1,1)}$.

Acknowledgements

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Appendix

S / k	$s_k(0)$	$s_k(1)$	$s_k(2)$	$s_k(3)$	$s_k(4)$	$s_k(5)$
3	(8)	10	∞	∞	∞	∞
5	(5)	6	18	∞	∞	∞
7	(3)	4	7	25		
9		3	5	11		
11	(2)		4	8	30	
13		2		6	17	
15			3		13	
17				5	11	126
19					10	52
21					9	35
23				4	8	28
25						23
27					7	21
29						19
31						17
33						16
35					6	15
37			2			
39						14
43						13
47						12
51				3		
57						11
61					5	
71						10
105						9
253						8

Table 3
Upper bounds $s_k(\gamma)$ for the lightness of graphs with girth at least k and without 4-cycles and minimum degree at least 2 in the orientable surface S_γ. The bounds in brackets were already obtained by He et al. [11].

S / k	$n_k(1)$	$n_k(2)$	$n_k(3)$	$n_k(4)$	$n_k(5)$	$n_k(6)$	$n_k(7)$	$n_k(8)$
3	9	10	30	∞	∞	∞	∞	∞
5	5	6	10	30	∞	∞	∞	∞
7	3	4	5	8	20			
9		3	4	6	10	30		
11	2		3	5	7	15		
13		2		4	6	11	42	
15					5	9	24	
17				3		8	18	
19						7	15	
21			2		4		13	110
23						6	12	60
25							11	43
27							10	35
29								30
31						5	9	26
33								24
35								22
37							8	21
39								20
41					3			19
43								18
45								17
49							7	16
53								15
61				2		4		14
71								13
85							6	
87								12
121								11
221								10

Table 4
Upper bounds $n_k(\overline{\gamma})$ for the lightness of graphs with girth at least k and without 4-cycles and minimum degree at least 2 in the nonorientable surface $N_{\overline{\gamma}}$.

$k \geq$	$s_k(0)$	$s_k(1)$	$s_k(2)$	$s_k(3)$	$s_k(4)$	$s_k(5)$
3	(10)[21]	(10)[21]	(22)[12]	(24)[12]	(26)[12]	(27)[12]
5	(8)[11]	(8)[21]	21	(24)[12]	(26)[12]	(27)[12]
7	(6)[11]	(6)[21]	10			
9			8	14		
11	(5)[11]	(5)[21]	7	8		
13				9	20	
15			6		16	
17				8	14	
19					13	
21					12	
23				7	11	
25						26
27					10	24
29						22
31						20
33						19
35					9	18
37			5			
39						17
43						16
47						15
51				6		
57						14
61					8	
71						13
105						12
253						11
	$s(0)$	$s(1)$	$s(2)$	$s(3)$	$s(4)$	$s(5)$
	(17)[24]	(20)[12]	(22)[12]	(24)[12]	(26)[12]	(27)[12]

Table 5

Upper bounds $s_k(\gamma)$ for the game coloring number of graphs embeddable in the orientable surface S_γ with girth at least k and without 4-cycles, and the best-known upper bounds $s(\gamma)$ for the game coloring number of graphs embeddable in S_γ in general. Previously known bounds are in brackets. The superscript numbers refer to the bibliography. For the given surfaces, our results do not provide better bounds if the girth is augmented, without improving the lower bounds for $M(S)$.

$k \geq$	$n_k(1)$	$n_k(2)$	$n_k(3)$	$n_k(4)$	$n_k(5)$	$n_k(6)$	$n_k(7)$	$n_k(8)$
3	(10)[21]	(10)[21]	33	???	???	???	???	???
5	8	(8)[21]	13	33	???	???	???	???
7	6	(6)[21]	8	11	23	???	???	???
9			7	9	13	33	???	???
11	5	(5)[21]	6	8	10	18	???	???
13				7	9	14	45	???
15					8	12	27	???
17				6		11	21	???
19						10	18	???
21			5		7		16	113
23						9	15	63
25							14	46
27							13	38
29								33
31						8	12	29
33								27
35								25
37							11	24
39								23
41					6			22
43								21
45								20
49							10	19
53								18
61				5		7		17
71								16
85							9	
87								15
121								14
221								13
	$n(1)$	$n(2)$	$n(3)$	$n(4)$	$n(5)$	$n(6)$	$n(7)$	$n(8)$
	(19)[22]	???	???	???	???	???	???	???

Table 6
Upper bounds $n_k(\overline{\gamma})$ for the game coloring number of graphs embeddable in the nonorientable surface $N_{\overline{\gamma}}$ with girth at least k and without 4-cycles, and upper bounds $n(\overline{\gamma})$ for the game coloring number of graphs embeddable in $N_{\overline{\gamma}}$. Previously known bounds are in brackets. The superscript numbers refer to the bibliography. For the entries with question marks, it is not known whether bounds exist.