Note

Asymmetric Directed Graph Coloring Games

Stephan Dominique Andres

Zentrum für angewandte Informatik Köln,
University of Cologne,
Weyertal 80, 50931 Köln, Germany

Abstract

This note generalizes the (a, b)-coloring game and the (a, b)-marking game which were introduced by Kierstead [7] for undirected graphs to directed graphs. We prove that the (a, b)-chromatic and (a, b)-coloring number for the class of orientations of forests is $b + 2$ if $b \leq a$, and infinity otherwise. From these results we deduce upper bounds for the (a, b)-coloring number of oriented outerplanar graphs and of orientations of graphs embeddable in a surface with bounded girth.

Key words: game chromatic number, game coloring number, forest, directed graph coloring game, outerplanar graph, surface, girth
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1 Introduction

The new area of research on graph coloring games was introduced by Bodlaender [3]. In Bodlaender's original game there are two players, Alice and Bob, who are given an initially uncolored graph and a set C of colors. The players alternately take turns in coloring vertices of the graph with a color from C, so that no neighbor of a vertex to be colored with $x \in C$ has been colored with x before. Here, a move consists in coloring exactly one vertex at a time. The game ends, when no move is possible any more. If all vertices are colored at the end of the game, Alice wins, otherwise Bob wins.

Email address: andres@zpr.uni-koeln.de (Stephan Dominique Andres).
URL: http://www.zaik.uni-koeln.de/∼andres (Stephan Dominique Andres).

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Since then there have been a lot of attempts to generalize this game. We will mention only one of them. Kierstead [7] modified the game by defining two positive integers a and b and the rule that each of Alice's moves consists in coloring a vertices and each of Bob's in coloring b vertices.

In this note, we generalize Kierstead's game to digraphs. This game will be called (a, b)-*coloring game* and is played on a digraph D with a color set C. Alice begins. A *feasible* coloring of a vertex v is a color that has not yet been used for any vertex w for which there is an arc (w, v). (Such a vertex w is called an *in-neighbor* of v.) However, the out-neighbors of v may have any color. Each of Alice's moves consists in coloring a vertices of the digraph feasibly, each of Bob's in coloring b vertices feasibly. If in Alice's last move there are only $x < a$ uncolored vertices left, or in Bob's last move there are only $x < b$ uncolored vertices left, the respective player has to color only x vertices during that (incomplete) move. Alice wins if and only if every vertex is colored at the end. Obviously, for undirected graphs, which are interpreted as digraphs where each arc (v, w) has an opposite arc (w, v), the game is equal to Kierstead's game. The directed $(1, 1)$-coloring game has already been investigated in [1,2].

The (a, b)-*chromatic number* $\chi_g(D; a, b)$ of the digraph D is the smallest integer n for which Alice has a winning strategy for the (a, b)-coloring game with $\#C = n$ colors. We further define for a nonempty class $\mathcal{C}$ of digraphs

$$\chi_g(\mathcal{C}; a, b) = \sup_{D \in \mathcal{C}} \chi_g(D; a, b).$$

A lot of results for graph coloring games can be obtained by considering the associated marking games, which was first observed by Zhu [8] for undirected graphs. So we introduce the (a, b)-*marking game* which is played on a digraph D with a score n. Alice begins, and Alice marks a vertices in a turn, Bob b vertices. A vertex which is marked may have $n-1$ marked in-neighbors at most. The last move may be incomplete in the same way as for the (a, b)-coloring game. The game ends when no move is possible any more. Alice wins if every vertex is marked at the end of the game, otherwise Bob wins. The lowest score for which Alice has a winning strategy is called (a, b)-*coloring number* $\mathrm{col}_g(D; a, b)$ of D. For a nonempty class $\mathcal{C}$ of digraphs let

$$\mathrm{col}_g(\mathcal{C}; a, b) = \sup_{D \in \mathcal{C}} \mathrm{col}_g(D; a, b).$$

We observe the fundamental inequality

$$\chi_g(D; a, b) \leq \mathrm{col}_g(D; a, b), \tag{1}$$

which holds for every digraph D, cf. [8].

Throughout this paper, let $\mathcal{F}$ be the class of undirected forests and $\vec{\mathcal{F}}$ be the class of orientations of forests. $\chi_g(\mathcal{F};1,1) = \mathrm{col}_g(\mathcal{F};1,1) = 4$ holds by a result of Faigle et al. [4], whereas the author [1,2] proved that $\chi_g(\vec{\mathcal{F}};1,1) = \mathrm{col}_g(\vec{\mathcal{F}};1,1) = 3$. Kierstead [7] determined $\chi_g(\mathcal{F};a,b)$ and $\mathrm{col}_g(\mathcal{F};a,b)$ for all positive integers a and b. The main aim of this note is to have an analogous result for oriented forests, which is the following theorem.

Theorem 1 *Let a and b be positive integers. Then,*

(a) *for $b \le a$: $\chi_g(\vec{\mathcal{F}};a,b) = \mathrm{col}_g(\vec{\mathcal{F}};a,b) = b+2$,*
(b) *for $a < b$: $\chi_g(\vec{\mathcal{F}};a,b) = \mathrm{col}_g(\vec{\mathcal{F}};a,b) = \infty$.*

Surprisingly, there are fewer case distinctions than in Kierstead's result [7], although the class of oriented forests seems to be more complicated than the class of undirected forests.

For the class $\mathcal{O}$ of undirected outerplanar graphs Guan and Zhu [5] determined the upper bound $\mathrm{col}_g(\mathcal{O};1,1) \le 7$. In Section 4, from our results for forests we will derive an upper bound for the (a,b)-coloring number of orientations of outerplanar graphs as well as of graphs embeddable in a surface with bounded girth.

2 Two lower bounds

Lemma 2 *For all $a, b \in \mathbb{N}$, $b+2 \le \chi_g(\vec{\mathcal{F}};a,b)$.*

PROOF. Let D be the oriented forest with vertices u_i, $v_{i,j}$ and $w_{i,j,k}$, $i = 1,\ldots,a+b$, $j = 1,\ldots,2a+1$, $k = 1,\ldots,b$, and arcs $(u_i, v_{i,j})$ and $(w_{i,j,k}, v_{i,j})$. We shall prove a winning strategy for Bob with $b'+1 \le b+1$ colors for the (a,b)-coloring game on D. In his first move, Bob colors vertices u_i, no matter what Alice has done before. After Alice's second move she will have left a subtree with vertex set $\{u_{i_0}, v_{i_0,j_0}, w_{i_0,j_0,k} \,|\, k = 1,\ldots,b\}$ in which u_{i_0} is the only colored vertex. Then Bob colors $w_{i_0,j_0,k}$ for $k = 1,\ldots,b'$ with distinct colors which are different from the color of u_{i_0}, and wins, since v_{i_0,j_0} cannot be colored any more. $\square$

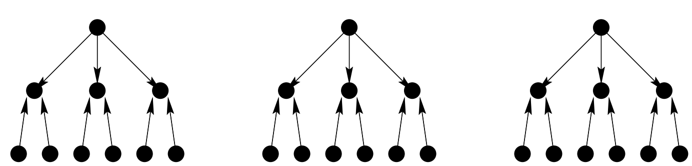

Fig. 1. The graph of Lemma 2 for $(a,b) = (1,2)$

Lemma 3 *If $b > a$, then $\chi_g(\vec{\mathcal{F}}; a, b) = \infty$.*

PROOF. Let t be a positive integer. Let D be the digraph with vertex set $\{v_i, w_{i,j} \,|\, i = 1, \ldots, b^t,\ j = 1, \ldots, t\}$ and arc set $\{(w_{i,j}, v_i) \,|\, i = 1, \ldots, b^t,\ j = 1, \ldots, t\}$. So D is a forest of b^t oriented stars. We will prove that Bob has a winning strategy with $t' \leq t$ colors for the (a, b)-coloring game on D. We divide the game into several rounds consisting of several moves. During the k-th round ($k = 1, 2, \ldots, t'$) Bob colors, with color k, b^{t-k+1} vertices $w_{i,j}$ in stars where Alice has not colored any vertex in previous rounds and Bob has colored only vertices $w_{i,j}$ with colors $1, 2, \ldots, k-1$. This is indeed possible since during the k-th round Alice may color at most $b^{t-k}a \leq b^{t-k}(b-1) = b^{t-k+1} - b^{t-k}$ of the stars Bob has colored. So there are at least $b^{t-(k+1)+1}$ such stars left for the next round. In the t'-th round there will be left at least one star the center of which is not colored but t' leaves of which are colored with t' distinct colors. As the center cannot be colored any more, Bob wins. □

3 An upper bound

When playing the marking game on a forest the players divide the forest into smaller components. Each time they mark a vertex v, the actual component of v is broken into pieces at the point v. Equivalently one may delete arcs (w, v) (since v is no danger for w), arcs (v, w) are left in their component. If necessary, i.e. if there are several arcs (v, w), the vertex v is multiplied, so that there is a copy of vertex v for each arc (v, w). These arcs (v, w) are considered to be in different components after v has been colored. These components are called *independent subtrees*.

Lemma 4 *If $b \leq a$, then $\mathrm{col}_g(\vec{\mathcal{F}}; a, b) \leq b + 2$.*

PROOF. Let F be any oriented forest. We will prove that Alice has a winning strategy for the (a, b)-marking game with score $b + 2$. As proven in [1,2] Alice has the following winning strategy with score 3 for the $(1, 1)$-marking game: She guarantees that after each of her moves every independent subtree has at most one marked vertex. This strategy also works if Bob is allowed to pass one or several moves.

We will adapt this strategy for the (a, b)-marking game with score $b + 2$. Alice still guarantees that after her moves every independent subtree has at most one marked vertex. Then after Bob's next move every independent subtree has at most $b + 1$ marked vertices. Assume that Bob has marked $v_1, v_2, \ldots, v_b$. Alice imagines that Bob has only marked v_1, and answers by marking a vertex according to her winning strategy for the $(1,1)$-marking game unless the vertex to be marked is v_i for some i. In the latter case she imagines she would have marked v_i and continues. In the next step, Alice imagines that Bob has only marked v_1 and v_2 and reacts according to her strategy for the $(1,1)$-marking game. In general, in the k-th step she imagines Bob has marked $v_1, \ldots, v_k$, always replacing those v_i she imagines to have marked herself by subsequent v_j. After $x \leq b$ steps she will have reinstalled the invariant, and there will be at most $b+1$ marked in-neighbors of a vertex in the meantime. For the next $a-x$ steps Alice plays as if Bob was passing. So at the end of her move, every independent subtree has at most one marked vertex. □

This lemma, together with the lemmata of Section 2 and the inequality (1), completes the proof of Theorem 1. Note that the argument of simulation in the proof of Lemma 4 is just the same as in the corresponding proof of Lemma 3 in Kierstead [7].

Remark. The bound of Lemma 4 still holds for a game where Bob is allowed to have the first move and where he is allowed to miss one or several turns. On the other hand the lower bounds from Section 2 are still true for a game where Alice is allowed to have the first move and is allowed to pass. So Theorem 1 still holds for all these variants of the games.

4 Outerplanar and topological digraphs

Let $D = (V, E)$, $D_1 = (V, E_1)$, and $D_2 = (V, E_2)$ be digraphs with the same vertex set. $D_1|D_2$ is an *arc partition* of D if $E = E_1 \dot\cup E_2$. By $\Delta^+(D)$ we mean the maximum in-degree of D. The following obvious observation is a generalization of an observation in [2] resp. of Theorem 2 in [5].

Observation 1 *If a digraph D has an arc partition $D_1|D_2$, then*

$$\mathrm{col}_g(D; a, b) \leq \mathrm{col}_g(D_1; a, b) + \Delta^+(D_2).$$

Let $\vec{O}$ be an orientation of an outerplanar graph. As proven in [5] for undirected graphs, $\vec{O}$ has an arc partition $D_1|D_2$, so that D_1 is a forest and D_2 has maximum (in-)degree at most 3. By Observation 1 and Theorem 1 we conclude:

Corollary 5 *Let $\vec{O}$ be an orientation of an outerplanar graph, and $a \geq b$. Then*

$$\mathrm{col}_g(\vec{O}; a, b) \leq b + 5.$$

The *lightness* of an arc (v, w) is the maximum of the in-degrees of v and w. The *lightness* $L(D)$ of a digraph D is its minimum lightness of an arc. In [6] it is shown that if every subgraph H of an undirected graph G has minimum degree $\delta \leq 1$ or $L(H) \leq k$, then the edges of G can be partitioned into a forest and a graph with maximum degree $\Delta \leq k-1$. So by Observation 1 and Theorem 1 we may also state

Corollary 6 *For $b \leq a$ and an orientation D of a graph embeddable in a surface S with girth at least g,*

$$\mathrm{col}_g(D; a, b) \leq b + u(S, g) + 1.$$

Here, $u(S, g)$ is an upper bound for the lightness of undirected graphs embeddable in the surface S with girth at least g and minimum degree $\delta \geq 2$. In [2] (and for planar graphs already in [6]) these bounds $u(S, g)$ are determined explicitly for surfaces of small orientable or nonorientable genus and sufficiently large girth.

Final remark. It is an open question for which (a, b) the (a, b)-coloring number of the class of planar (toroidal, etc.) digraphs is bounded.

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