Game-perfect graphs

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Abstract

A graph coloring game introduced by Bodlaender [3] as *coloring construction game* is the following. Two players, Alice and Bob, alternately color vertices of a given graph G with a color from a given color set C, so that adjacent vertices receive distinct colors. Alice has the first move. The game ends if no move is possible any more. Alice wins if every vertex of G is colored at the end, otherwise Bob wins. We consider two variants of Bodlaender's graph coloring game: one (A) in which Alice has the right to have the first move and to miss a turn, the other (B) in which Bob has these rights.

These games define the A-game chromatic number resp. the B-game chromatic number of a graph. For such a variant g, a graph G is g-perfect if, for every induced subgraph H of G, the clique number of H equals the g-game chromatic number of H.

We determine those graphs for which the game chromatic numbers are 2 and prove that the triangle-free B-perfect graphs are exactly the forests of stars, and the triangle-free A-perfect graphs are exactly the graphs each component of which is a complete bipartite graph or a complete bipartite graph minus one edge or a singleton. From these results we may easily derive the set of triangle-free game-perfect graphs with respect to Bodlaender's original game. We also determine the B-perfect graphs with clique number 3.

As a general result we prove that complements of bipartite graphs are A-perfect.

Key words: game-perfect, game chromatic number, perfect graphs, A-perfect, B-perfect, bipartite graphs
2000 MSC: 05C17, 05C15, 91A43

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1 Introduction

Consider the following game, which is played with a graph G and a set C of colors by two players, Alice and Bob. The players alternately color a vertex of G with a color from C in such a way that adjacent vertices receive different colors. The game ends if either an uncolored vertex is adjacent to vertices of all colors, in which case Bob wins, or every vertex is colored, in which case Alice wins. Alice can never win if $|C| < \chi(G)$, where $\chi(G)$ is the chromatic number of G, i.e. the number of colors needed to color G without having a malicious adversary Bob. Bob can never win if $|C| \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. This type of game was first considered by Bodlaender [3] and Faigle et al. [10]. In the original version Alice has the first move and missing a turn is not allowed, we, however, will deal with variants where the latter is permitted. The first variant of the game, which we call A, consists in giving Alice the right to have the first move and the right to miss one or several turns. In the second variant, B, Bob has these rights. For any variant g, the *g-game chromatic number* $\chi_g(G)$ of G is the smallest cardinality of the color set C for which Alice has a winning strategy for the variant g played on G.

For several variants of the game, determining upper bounds for the game chromatic number of different classes of graphs has recently received attention. For Bodlaender's original variant, e.g. the classes of trees [10], outerplanar graphs [12], graphs embeddable in a fixed surface [14] including planar graphs [18], line graphs of k-degenerate graphs [4] including line graphs of forests of maximum degree $\Delta \neq 4$ [1], and (a, b)-pseudo partial k-trees [17] have been examined. A new trend are results for relaxed coloring variants of the game [5,13,7–9].

A graph G is called *perfect* if, for each induced subgraph H of G, the chromatic number $\chi(H)$ equals the clique number $\omega(H)$, i.e. the size of the largest clique of H. Motivated by an application in coding theory, Berge was the first to examine the structure of perfect graphs, cf. [2]. Since then, there have been several hundreds of contributions to the theory of perfect graphs. By the famous Strong Perfect Graph Theorem [6] a graph is perfect if and only if it contains neither induced cycles of odd length $l \geq 5$ nor their complements.

We call a graph G *g-nice* if its g-game chromatic number equals its clique number, i.e., $\chi_g(G) = \omega(G)$. A graph G is *g-perfect* if every induced subgraph H of G is g-nice. We have

$$\omega(G) \leq \chi(G) \leq \chi_A(G) \leq \chi_B(G).$$

The first two inequalities are obvious, the third was first proven in [1]. Thus,

B-perfect graphs are A-perfect, and A-perfect graphs are perfect.

In this paper we completely determine the A-nice and B-nice graphs with clique number 2 and thus the A-perfect and B-perfect graphs with clique number 2 (Corollary 16 and Corollary 4). As a corollary we obtain the game-perfect graphs with clique number 2 for Bodlaender's original version of the game and for its dual (Corollary 19 and Corollary 20). For any variant g of the game, all g-nice and g-perfect graphs with clique number 2 are in particular bipartite. A graph G is bipartite if $\chi(G) \leq 2$. We also determine the B-perfect graphs with clique number 3 (Theorem 26). The only class of perfect graphs which could be recognized as A-perfect in general are complements of bipartite graphs, see Section 5.

These results are first steps towards Strong Perfect Graph Theorems for game-perfectness. However, the sets of forbidden induced subgraphs occurring in Sections 2, 3, and 6 are far from being complete. Unlike perfectness, game-perfectness does not have the Weak Perfect Graph Theorem's property, i.e. a graph is perfect if and only if its complement is perfect [15], but there are A-perfect (B-perfect) graphs the complements of which are not A-perfect (B-perfect), cf. Section 6.

We denote by P_n the path with n vertices, by C_n the cycle with n vertices, by K_n the complete graph with n vertices, and by $K_{m,n}$ the complete bipartite graph with partite sets of cardinalities m resp. n. By $K_{m,n} - M_k$ we denote the graph which is obtained from $K_{m,n}$ by deleting a matching of cardinality $k \geq 0$. For a graph G and a subgraph H of G, $V(G)$ denotes the vertex set of G, and $G \setminus H$ is the graph obtained from G by deleting all vertices of H and incident edges. The *length* of the path P_n is $n-1$. The *distance* $d(v_1, v_2)$ of two vertices v_1 and v_2 of a graph is the length of the shortest path containing v_1 and v_2, or ∞ if there is no such path. The *diameter* of a graph G is defined as

$$\operatorname{diam}(G) = \sup_{(v_1,v_2)\in V\times V} d(v_1, v_2).$$

A graph is called *trivially perfect* if it neither contains a P_4 nor a C_4 as an induced subgraph. Trivially perfect graphs were introduced by Golumbic [11], but examined even earlier, e.g. by Wolk [16]. A *universal vertex* of a graph $G = (V, E)$ is a vertex $v \in V$ that is adjacent to every vertex $w \in V$, $w \neq v$. A *star* is a tree that contains a universal vertex.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *join* graph $G_1 \vee G_2$ is defined as the graph consisting of disjoint copies G_1' of G_1 and G_2' of G_2 and additional edges in the form that every vertex of G_1' is adjacent to every vertex of G_2'. The *union* graph $G_1 \cup G_2$ is defined as the graph consisting of disjoint copies G_1' of G_1 and G_2' of G_2. For a graph G and $n \geq 0$, the graph

nG is the graph

$$\underbrace{G \cup G \cup \ldots \cup G}_{n}.$$

In case $n = 0$ it is the empty graph. The complement of a graph G is denoted by $\overline{G}$.

2 B-perfect graphs with clique number 2

The classification of B-nice (and B-perfect) graphs with clique number 2 is given in this section. First we prove that the two configurations of Fig. 1 are forbidden in B-perfect graphs.

Lemma 1 *Let G be a graph with $\omega(G) = 2$ containing an induced C_4. Then G is not B-nice.*

PROOF. Let $v_1v_2v_3v_4$ be an induced C_4. Bob has the following winning strategy with 2 colors: He misses his turns until Alice either colors a vertex v_i of the C_4 or one of its neighbors w with – say – color 1. In case Alice has colored v_i he replies by coloring v_{i+2} (index mod 4) with color 2. This is possible since neither v_{i+1}, v_{i+2}, v_{i+3} nor any other neighbor of v_{i+2} has been colored before. Bob wins, since v_{i+1} cannot be colored any more. In case Alice has colored a neighbor w of v_i outside the C_4, Bob answers by coloring v_{i+1} with color 2. This is possible, since no neighbor of v_{i+1} has been colored. Note that w is not a neighbor of v_{i+1}, otherwise there would be a triangle wv_iv_{i+1} contradicting $\omega(G) = 2$. Here again, since v_i cannot be colored any more, Bob wins. □

Lemma 2 *Let G be a graph with $\omega(G) = 2$ containing an induced P_4. Then G is not B-nice.*

PROOF. Let $v_1v_2v_3v_4$ be an induced P_4. Bob has the following winning strategy with 2 colors: He misses his turns until Alice either colors a vertex v_i of the P_4 or one of its neighbors w with – say – color 1. Then Bob colors a vertex x of the P_4 at distance 2 from v_i resp. w with color 2. In case Bob has colored w

The cycle C_4 The path P_4

Fig. 1. Two forbidden configurations for B-perfect graphs with clique number 2

there is such a vertex, otherwise there would be a triangle in G as in the proof of the previous lemma. If Bob has colored v_i there is obviously a vertex of distance 2 in the P_4. This vertex can be colored with color 2 since none of its neighbors has been colored. After that, there is a vertex y which is adjacent to v_i resp. w and x. The vertex y cannot be colored any more, so Bob wins. □

Theorem 3 *A graph G with $\omega(G) \leq 2$ is B-nice if, and only if, it is trivially perfect. This is the case if, and only if, G is a forest of stars.*

PROOF. Trivially perfect graphs with clique number of at most 2 are the graphs without induced C_3, C_4 and P_4. These are obviously forests whose components have diameter 2 at most, i.e. forests of stars. If G contains a C_4 or P_4, by Lemma 1 resp. 2, G is not B-nice. We are left to prove a winning strategy for Alice with 2 colors in case the graph is a forest of stars: If Bob colors a leaf vertex of a star whose center v_0 is uncolored, Alice colors v_0 in order to fix the coloring of the star. Otherwise Alice colors a center of a star if there is still an uncolored center of a star. In case the centers of all stars are colored, the coloring is fixed and Alice may color any uncolored vertex. In case the star is a K_2 we consider one of its two vertices as a center. By this strategy, when Alice or Bob color a center of a star, at most one leaf vertex of the star is colored, therefore 2 colors are sufficient. □

Since every induced subgraph of a forest of stars is a forest of stars we obtain

Corollary 4 *A graph G with $\omega(G) \leq 2$ is B-perfect if, and only if, it is trivially perfect. This is the case if, and only if, G is a forest of stars.*

3 A-perfect graphs with clique number 2

It is very easy to decide whether a connected graph with clique number 2 and diameter $\neq 3$ has A-game chromatic number 2, see the following propositions. But there are graphs with diameter 3 and clique number 2 as well with A-game chromatic number 2 as with larger A-game chromatic number. For example, consider the two graphs C_6 and Π of Fig. 2 which have diameter 3 and clique number 2. It is easy to see that $\chi_A(C_6) = 2$, but $\chi_A(\Pi) = 3$. Alice has the following winning strategy with two colors for C_6: she misses her first turn and, after Bob has colored a vertex v with the first color, she colors a vertex at distance 3 from v with the second color. Then the colors of the remaining vertices are fixed. Alice will also use such a fixing strategy in the case of other bipartite graphs. This strategy fails for Π if Bob colors a vertex of degree 1 in his first move.

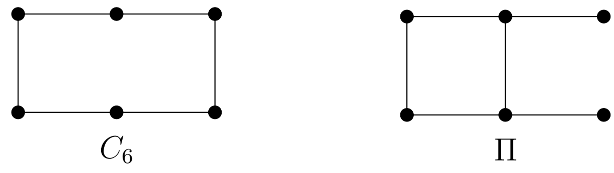

Fig. 2. Two graphs with diameter 3

This section contains a classification of A-nice graphs with clique number 2 which begins with Lemma 7 and does not make use of the notion diameter.

Proposition 5 *Let G be bipartite and diam$(G_i) \leq 2$ for each component G_i of G. Then G is A-nice.*

PROOF. We prove a winning strategy with 2 colors for Alice. Alice misses her turns until Bob colors a first vertex x in some component G_i. If x is a universal vertex of G_i, Alice continues passing since the coloring of G_i is fixed by the color of x. On the other hand, if there is a vertex y with distance $d(x, y) = 2$, then Alice colors the middle vertex z on the shortest path xzy from x to y. We state that now the coloring is fixed since every vertex of G_i has either distance at most 1 from x, or distance at most 1 from z. Assume that there is a vertex a with $d(a, x) = 2$ and $d(a, z) = 2$. Then there are paths abx and acz. In case $b = c$ we have a triangle bxz, otherwise a C_5 $abxzc$, both contradicting the fact that G is bipartite. Alice uses this strategy for each component that Bob begins to color, so there is no need for more than two colors. □

Proposition 6 *Let G be bipartite and diam$(G_i) \geq 4$ for some component G_i of G. Then G is not A-nice.*

PROOF. We prove a winning strategy for Bob with 2 colors. In G_i there is a path $v_1v_2v_3v_4v_5$ without abbreviation paths from v_1 to v_5. If Alice colors v_i or some neighbor of v_i, Bob replies by coloring a vertex at distance 2 with the remaining color, making it impossible to color the graph completely. Otherwise, Bob colors v_3. No matter what Alice does, since there are no common neighbors of v_1 and v_5, Bob may color either v_1 or v_5 different from v_3 in his next move. Again, Bob wins. □

Lemma 7 *Let $1 \leq k + 1 \leq m \leq n$. Then $K_{m,n} - M_k$ is A-nice.*

PROOF. Let P resp. Q be the partite sets with m resp. n vertices. Since $k \leq m - 1 \leq n - 1$, P and Q contain each a vertex p resp. q that is adjacent to all vertices of the other side. Alice's winning strategy with two colors is the following. In her first move she uses her right to miss a turn. W.l.o.g. Bob colors a vertex of P. If he colors a vertex that is adjacent to every vertex of Q, Alice colors q with the second color. If Bob colors a vertex that is adjacent with every vertex of Q except one vertex q', Alice colors q' with the second color. In both cases the coloring is fixed after Alice's move. $\square$

Lemma 8 *For $m \geq 2$, $K_{m,m} - M_m$ is A-nice.*

PROOF. This is the same as the second case in the proof of the previous lemma. $\square$

Lemma 9 *For $2 \leq m < n$, $K_{m,n} - M_m$ is not A-nice.*

PROOF. Let M resp. N be the partite sets with m resp. n vertices. Let $W \subseteq N$ be the vertices which are adjacent to all vertices of M. As $m < n$, $W \neq \emptyset$. We prove a winning strategy for Bob with two colors in the game A.

If Alice uses her first move to color a vertex with the first color, then Bob colors a vertex of the same partite set with the second color. As the graph is connected, Bob will win.

If Alice misses her turn, Bob colors a vertex of W with the first color. Alice now colors a vertex of M or N, or she misses her turn. If she colors a vertex x of M she has to use the second color. Bob then colors the unique vertex y of N that is not adjacent to x with the second color. If she colors a vertex of N or misses her turn, since $m \geq 2$, she leaves at least one vertex y' of $N \setminus W$ uncolored. Bob then colors y' with the second color. In any case, since the graph is connected, Bob will win. $\square$

Lemma 10 *Let G be a graph with $\omega(G) = 2$ and let H be an induced subgraph of G which does not contain any isolated vertices. Assume that, in her first move of the game A played on G with 2 colors, Alice colors a neighbor $v \in V(G \setminus H)$ of $w \in V(H)$. Then Bob wins.*

PROOF. Since there are no isolated vertices in H, there is an edge $wz \in E(H)$. There is no edge $vz \in V(G)$, otherwise there would be a triangle vwz, contradicting $\omega(G) = 2$. So Bob may color z different from v. During the game, w cannot be colored feasibly any more, i.e., Bob wins. $\square$

The proof of the following lemma is obvious.

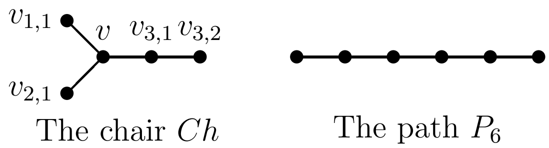

Fig. 3. Two forbidden configurations for A-perfect graphs with clique number 2

Lemma 11 *Let G be a graph with $\omega(G) = 2$ and let H be an induced subgraph of G, so that every vertex of H lies on an induced $P_4 \subseteq H$. Assume that, in her first move of the game A played on G with 2 colors, Alice colors a vertex of H. Then Bob wins.*

A *chair* is the graph on the left-hand side of Fig. 3. It will be denoted by Ch. In the Lemmas 12 and 13 we will prove that Ch and P_6 are forbidden configurations for A-perfectness. In the proof of Lemma 12 we will use the notation from Fig. 3.

Lemma 12 *Let G be a graph with $\omega(G) = 2$ that contains an induced chair Ch. Then G is not A-nice.*

PROOF. We prove a winning strategy for Bob with 2 colors. If Alice colors a neighbor of v, $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ or $v_{3,2}$ in her first move, Bob will win by Lemma 10. If Alice colors v, $v_{1,1}$, $v_{2,1}$, $v_{3,1}$ or $v_{3,2}$, then Bob will also win, by Lemma 11. So we are restricted to the case that Alice passes or colors a vertex which is neither one of Ch nor one of its neighbor vertices. In this case Bob colors $v_{1,1}$ with the first color. Now, Alice may neither color $v_{3,2}$ with the first color nor $v_{2,1}$ or $v_{3,1}$ with the second color, otherwise she will loose. If Alice colors a vertex with the first color, Bob answers by coloring either $v_{2,1}$ or $v_{3,1}$ with the second color. On the other hand, if Alice colors a vertex with the second color or if she misses her second turn, Bob colors either $v_{3,2}$ with the first color or $v_{2,1}$ with the second color. In either case, Bob wins. □

Lemma 13 *Let G be a graph with $\omega(G) = 2$ that contains an induced path P_6. Then G is not A-nice.*

PROOF. Let $v_1v_2v_3v_4v_5v_6$ be an induced P_6 in G. We prove a winning strategy for Bob with 2 colors. If Alice, in her first move, colors a neighbor of some v_i, Bob has a winning strategy by Lemma 10, if she colors some v_i, Bob wins by Lemma 11. We are left with the case that Alice colors some other vertex or misses her turn. In this case, Bob will respond by coloring v_3 with the first color. Then, if Alice colors some vertex with the first color or if she passes, Bob may color either v_1 or v_5 with the second color in order to win. In case Alice colors some vertex $v \neq v_6$ with the second color, Bob colors v_6 with the first color and wins. If Alice colors v_6 with the second color, Bob wins by coloring v_1 with the second color. □

Lemma 14 *Let G be a connected bipartite graph that does neither contain an induced chair Ch nor an induced P_6. Then G is a $K_{m,n} - M_k$, where $k \leq \min\{m, n\}$.*

PROOF. Since G is bipartite, there exist integers m and n, so that G is subgraph of $K_{m,n}$. Assume in G there are two different vertices v and w on the same side and a vertex z on the other side with the property that edges vz and wz do not exist in G. As G is connected, there are shortest paths $v \dots z$ and $w \dots z$ in G. Assume that one of them has length ≥ 5. Then this path contains an induced P_6, contrary to the precondition. So both paths have length exactly 3. Let $vxyz$ and $wabz$ be these paths which w.l.o.g. have a maximal number of common edges among all such pairs of paths. Let z_0 be the first common vertex of both paths (beginning each path from v and w, respectively). So the vertices after z_0 are equal in both paths.

Case 1: $z_0 = z$
Then $vxyzbaw$ is an induced P_7, which contradicts the precondition. Note that there is no edge ya, otherwise y would be first common vertex of the paths. By the same argument, va, wx, and bx do not exist in G, since otherwise a, x, or b would be first common vertex, respectively.

Case 2: $z_0 = y = b$
There is no edge va (resp. wx), because otherwise a (resp. x) would be the first common vertex of the paths. Thus $vxyaz$ is an induced $S_{1,1,2}$, contrary to the precondition.

Case 3: $z_0 = x = a$ (which implies $y = b$)
In this case $vwxyz$ is an induced $S_{1,1,2}$.

In every case, we obtain a contradiction, hence our assumption was wrong, and in $K_{m,n}$ misses a matching at most. □

Theorem 15 *Let G be a graph with $\omega(G) = 2$. Then G is A-nice if, and only if, each component H of G is K_1 or $K_{m,n} - M_k$, for $k < m \leq n$, or $K_{m,m} - M_m$.*

PROOF. If every component H of G is K_1 or $K_{m,n} - M_k$, for $k < m \leq n$, or $K_{m,m} - M_m$, then by Lemma 7 resp. Lemma 8 Alice has a winning strategy with two colors for every component. Note that the 'local' winning strategy of every component guarantees that she may miss her first turn, see the proof of Lemmas 7 and 8. We will describe a 'global' winning strategy, i.e. we prove that Alice wins on G with two colors. Alice misses her first turn. After that, whenever Bob colors a vertex of any component H_0, Alice replies by playing in H_0 according to her winning strategy for H_0, i.e. she either colors a vertex of H_0 or misses her turn. If Bob colors the first vertex of H_0, Alice, in order to be able to apply her local winning strategy, thinks that she has missed her first turn. If, by Bob's move, H_0 is completely colored, she misses her turn. Alice interchanges her winning strategies in the same way as Bob interchanges the components in which he plays. Bob has no chance to break the local winning strategies, so Alice will win by her global strategy with two colors on G. Thus G is A-nice.

Now consider the case that a component H is different from K_1 and $K_{m,n}-M_k$, for any $k < m \leq n$, and $K_{m,m} - M_m$. Then either H is a $K_{m,n} - M_m$ with $2 \leq m < n$, in which case Bob has a winning strategy with two colors by Lemma 9 if he only plays in H, or H is not bipartite, in which case Bob obviously wins with two colors, or H is bipartite but not of the form $K_{m,n}-M_k$ for $k \leq m \leq n$. In the latter case, by Lemma 14, H contains an induced chair or an induced P_6. Thus Bob has a winning strategy with two colors by playing only in H, according to Lemma 12 resp. Lemma 13. If in the cases in which Bob only plays in H Alice colors a vertex in a different component, Bob, in order to be able to apply his winning strategy, thinks that she has missed her turn. So G is not A-nice since $\omega(G) = 2$. □

Corollary 16 *A graph G with $\omega(G) \leq 2$ is A-perfect if, and only if, every component of G is either K_1 or $K_{m,n}$ or $K_{m,n} - M_1$ for some m, n.*

PROOF. The A-nice configurations $K_{m,n} - M_k$ for $m, n \geq 3$, $k \geq 2$ are excluded from being A-perfect, since they contain an induced subgraph isomorphic to $K_{3,2} - M_2$, which is not A-nice. □

4 Bodlaender's original version

We consider two other games. The first one is Bodlaender's original game, which was called *coloring construction game* by Bodlaender [3] and which we denote by g_A, where Alice has the first move, but missing a turn is not allowed for any player. Its dual version is g_B, where Bob has the first move and missing a turn is not allowed. The other rules of these games are the same as in the variants A and B. For the associated game chromatic numbers we have by [1] for any graph G

$$\chi_A(G) \leq \chi_{g_A}(G) \leq \chi_B(G),$$

and

$$\chi_A(G) \leq \chi_{g_B}(G) \leq \chi_B(G).$$

Thus g_A-nice and g_B-nice graphs with clique number 2 are contained in the set of A-nice graphs. As in the proof of Lemma 7 resp. 8 it is easy to see that the connected g_A-nice graphs with clique number 2 are exactly nontrivial stars, thus exactly the connected B-nice graphs with clique number 2, and the connected g_B-nice graphs with clique number 2 are the connected A-nice graphs with clique number 2.

In the following, we will consider two types of components: stars and A-components. Denote by *A-component* a connected $K_{m,m} - M_m$ or a connected $K_{m,n} - M_k$ for $k < m \leq n$ that is not a star. An *odd* resp. *even* component is a component with an odd resp. even number of vertices. Then we may formulate

Theorem 17 *Let G be a graph with clique number 2. Then G is g_A-nice if, and only if, it holds either*

(i) *every component of G is a star, or*
(ii) *G consists of an odd number of odd stars and an arbitrary number of even stars and exactly one odd A-component, or*
(iii) *G consists of an odd number of odd stars and an arbitrary number of even stars and an arbitrary number of even A-components.*

Theorem 18 *Let G be a graph with clique number 2. Then G is g_B-nice if, and only if, it holds either*

(i) *every component of G is a star, or*
(ii) *G consists of an even number of odd stars and an arbitrary number of even stars and exactly one odd A-component, or*
(iii) *G consists of an even number of odd stars and an arbitrary number of even stars and an arbitrary number of even A-components.*

PROOF. (Theorems 17 and 18) In both variants, every A-component has to be colored by Bob first, otherwise Alice will loose. The possibilities for Alice to force Bob to this are given in the theorems. $\square$

Corollary 19 *A graph G with clique number 2 is g_A-perfect if, and only if, it is a forest of stars.*

PROOF. The graphs in case (i) of Theorem 17 are obviously g_A-perfect. Consider the case that G contains an odd number of odd stars as in case (ii) and (iii). Then the subgraph of G in which one of the odd stars is deleted has an even number of odd stars and at least one A-component, hence it is not g_A-nice. $\square$

Corollary 20 *A graph G with clique number 2 is g_B-perfect if, and only if, it is either a forest of stars or a single A-component of type $K_{m,n} - M_0$ or $K_{m,n} - M_1$.*

PROOF. Case (i) of Theorem 18 describes obviously g_B-perfect graphs. Now consider case (ii) and (iii) of Theorem 18. If G contains an even number ≥ 2 of odd stars, then the subgraph where one of these odd stars is deleted has an odd number of odd stars and is not g_B-nice. Also, if G contains an arbitrary number ≥ 1 of even stars, then the subgraph where one vertex of degree 1 in an even star is deleted has an odd star, thus it is not g_B-nice. So G has no stars and a single odd A-component in case (ii) or some even A-components in case (iii). In the latter case, if G has more than one even A-component then the subgraph H which is obtained by deleting a vertex of smallest degree has either a $K_{m,n} - M_m$ with $m < n$ (which is not g_B-nice) and at least one even A-component, or one odd A-component and at least one even A-component, or a single odd star and at least one even A-component, thus H is not g_B-nice. So in case (ii) and (iii) G is a single A-component which is of the form $K_{m,n} - M_0$ or $K_{m,n} - M_1$ by Corollary 16 using $\chi_{g_B}(G) \geq \chi_A(G)$. It is easy to see that such a graph is always g_B-perfect since every subgraph is of the form $K_{m',n'} - M_k$ with $k \in \{0, 1\}$. □

5 The general case

In the previous sections we have examined game-perfectness of graphs with clique number 2. Now we will focus the more general case where there is no restriction on the clique number. Since the complete solution of the general case for a variant g would be a game-theoretic analogon of the Strong Perfect Graph Theorem [6] such a characterization might be hard to obtain and cannot be given at this time. Instead, we start with observations concerning A- resp. B-nice graphs and the relation to the game chromatic numbers of their components. After that we will discuss whether certain subclasses of perfect graphs are A- resp. B-perfect in general.

Theorem 21 *A graph G is B-nice if, and only if, for each component H of G Alice has a winning strategy in the game B played on H with $\omega(G)$ colors.*

PROOF. Assume there is a component H_0 with the property that Bob has a winning strategy in the game B played on H_0 with $\omega(G)$ colors. We have to prove that Bob has a winning strategy on G with $\omega(G)$ colors. Consider the game played on G. Then Bob only plays on H_0 according to his winning

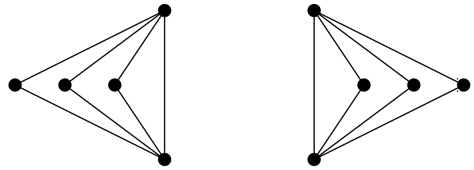

Fig. 4. The graph G_2

strategy for H_0. (If Alice plays in a component different from H_0, Bob misses his next turn.) So Bob will win globally, i.e. on G.

Now assume that Alice has a winning strategy for every component H of G in the game B with $\omega(G)$ colors. We shall prove a winning strategy for Alice on G with $\omega(G)$ colors. Alice always answers a move of Bob by playing in the same component where Bob has just colored a vertex according to her winning strategy for this component. If the component is completely colored or if Bob misses his turn, she arbitrarily chooses a component and thinks "Bob has missed his turn playing in that component." Playing like that Alice wins on G with $\omega(G)$ colors. $\square$

Theorem 22 *If a graph G is A-nice, then for each component H of G Alice has a winning strategy in the game A played on H with $\omega(G)$ colors.*

PROOF. Assume that there is a component H_0, so that Bob wins the game A on H_0 with $\omega(G)$ colors. Then Bob has a global winning strategy on G with $\omega(G)$ colors if he only plays on H_0. If Alice colors a vertex in a different component, Bob imagines that she has missed a turn. So Bob wins on G. $\square$

Remarkably, the inverse implication of Theorem 22 does not hold for $\omega(G) \geq 3$. It is easy to see that each component of the graph G_2 of Fig. 4 has A-game chromatic number $3 = \omega(G_2)$, but G_2 itself has A-game chromatic number 4.

Special classes of perfect graphs are bipartite graphs, comparability graphs and triangulated graphs, and their complements, cf. [2]. Interval graphs are special triangulated graphs. Bipartite graphs are special comparability graphs. None of these classes, except the class of complements of bipartite graphs, is contained completely in the class of B-perfect or A-perfect graphs. P_4 is a bipartite interval graph and the complement of a bipartite interval graph as it is self-complementary, but not B-perfect. An example of a bipartite interval graph is P_5, which is not A-perfect. Ch is not A-perfect but the complement of an interval and comparability graph, namely the graph in Fig. 10 (a). Some non-trivial examples for interval graphs are given in Figs. 5 and 6.

The interval graph of Fig. 5 is not B-perfect since it contains an induced P_4.

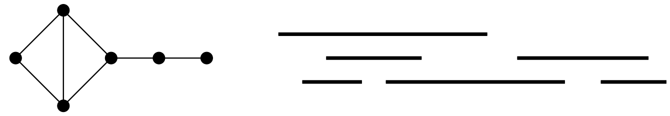

Fig. 5. An interval graph which is not B-perfect

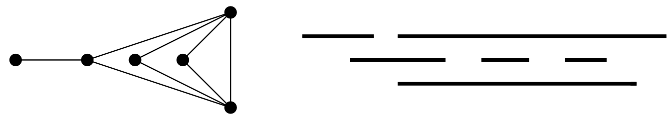

Fig. 6. An interval graph which is not A-perfect

The interval graph of Fig. 6 is not A-perfect, even not A-nice. A winning strategy for Bob with 3 colors consists in the following general idea: For one component of the graph of Fig. 4 Alice wins with 3 colors if she colors the two vertices of degree 4 in her first two moves. This strategy does not work any more for the graph of Fig. 6 because of the vertex of degree 1, as careful case distinctions show. (If Alice, in her first move, colors one of the vertices of degree 4 with color 1, Bob colors one of the vertices of degree 2 with color 2. If Alice, in her next move, leaves the other vertex v of degree 4 uncolored, she cannot avoid that Bob colors either the other vertex of degree 2 or the vertex of degree 3 with color 3, so that v cannot be colored any more. Therefore Alice must color v with color 3. Then Bob colors the vertex of degree 1 with color 2, and he wins since the vertex of degree 3 cannot be colored any more. — If Alice, in her first move, colors one of the three vertices of degree 2 or 3 with color 1, then Bob colors a second of these vertices with color 2 and wins since not both vertices of degree 4 can be colored. — If Alice, in her first move, colors the vertex of degree 1 with color 1, Bob colors a vertex of degree 2 with color 1, and not all three vertices of degree 3 and 4 can be colored (with the remaining two colors). — Finally, if Alice, in her first move, misses her turn, Bob colors the vertex of degree 3 with color 1. Since Alice can color only one vertex in her next move, she cannot avoid that Bob colors a vertex of degree 2 with either color 2 or 3 in his following move. Then one of the vertices of degree 4 cannot be colored any more. — Thus Bob wins in every case.)

Theorem 23 *Complements of bipartite graphs are A-perfect.*

PROOF. Let $G = (A \cup B, E)$ be a bipartite graph and G' be its complement. So A and B are the vertex sets of cliques in G', but not necessarily of maximum cliques. Let $\omega(G')$ be the clique number of G'. Since G' is perfect, there is a coloring $c : A \cup B \longrightarrow \{1, \ldots, \omega(G')\}$ with $\omega(G')$ colors. We call a vertex which is the only vertex in c of a certain color a *single* vertex. All other vertices are called *double* vertices as every color class in c has at most two vertices. If v is a vertex and w is a vertex of the same color in c, then w is called the *companion* of v. We will prove that for the variant A, Alice has a winning strategy with $\omega(G')$ colors.

Alice misses her first turn. If Bob colors a single vertex, Alice misses her turn. If Bob colors a double vertex v with a color not used so far and the companion w of v is uncolored, then Alice colors w with the same color as Bob has colored v. The last case is that Bob colors a double vertex v with a color already used for another vertex and the companion w of v is uncolored. In this case Alice colors w with a new color. Note that the number of colors in the partial coloring Alice and Bob produce is never greater than the number of colors in the partial coloring of c induced by the same vertices. This is so because the last case may only occur if at a certain point of the game Bob has used the color he already assigned to a single vertex for a double vertex. (And after that Bob may have used the color of a double vertex for a double vertex which is not its companion.) There are no further cases since after Alice's moves if a double vertex is colored then its compagnion is also colored. So at the end the players will have used only $\omega(G')$ colors. □

6 Towards a Strong Perfect Graph Theorem for B-perfect graphs

We define a *broken wheel* as a graph G with a universal vertex v_0 and n sets $A_1, \ldots, A_n$ of vertices and possibly an additional set B of vertices with the following properties. Between vertices of different sets A_i and A_j or B there are no edges. The subgraph induced by A_i is a complete graph. The subgraph induced by B is a complete graph without one edge $b_1 b_2$ between two vertices $b_1, b_2 \in B$. So the maximum cardinality of $A_i \cup \{v_0\}$ resp. B determines the clique number $\omega(G)$. See Fig. 7 for an example of a broken wheel with clique number 3.

Theorem 24 *A graph each component of which is a broken wheel is B-perfect.*

PROOF. First, we prove that such a graph G is B-nice. By Theorem 21 we may assume that G is a broken wheel. We give a winning strategy for Alice with $\omega(G) \geq 3$ colors. Alice has to do two things: to color the universal vertex v_0 as early as possible, but if Bob colors b_i, $i \in \{1, 2\}$, then she has to color b_{3-i} with the same color. So the universal vertex will be colored after Alice's second move (possibly with the third color) and the remaining $\omega(G)-1$ or fewer vertices of a set A_i can always be colored. The same holds for B if Alice colors b_{3-i} immediately after Bob has colored b_i. If Bob forces Alice to color the first vertex of $\{b_1, b_2\}$, then this will be at the end of the game when every vertex of B except b_1 and b_2 is colored. But then there is no danger for Alice any more to color a vertex b_i. So Alice will win in every case. For $\omega(G) \leq 2$ a broken wheel is simply a star.

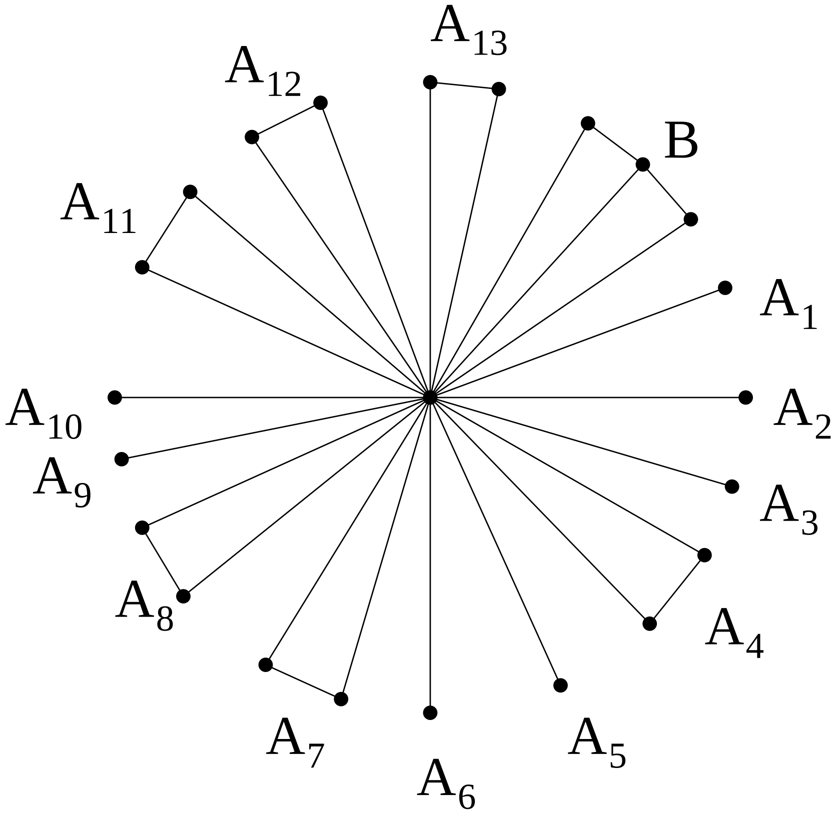

Fig. 7. A broken wheel with clique number 3

Now we have to prove that every subgraph of a graph the components of which are broken wheels is a graph all components of which are broken wheels. But this is obvious since if vertices in an set of type A_i are missing then we obtain again a clique thus a set of type A_i. On the other hand, if vertices in a set of type B are missing we either obtain a set of type A_i or of type B. Hence, in every component there is at most one set of type B. If v_0 is missing then every set A_i and B forms a component which is a trivial broken wheel. In each of these components any vertex except b_1 and b_2 is a new universal vertex of the component. We conclude that a graph of broken wheels is B-perfect. □

One may conjecture that the broken wheels are mainly all B-perfect graphs. The next theorem proves this conjecture for graphs with clique number 3. We need the following lemma concerning trivially perfect graphs which was proved by Wolk [16].

Lemma 25 (Wolk [16]) *Let G be a connected trivially perfect graph. Then G contains a universal vertex.*

Theorem 26 *A graph G with $\omega(G) = 3$ is B-perfect if and only if every component of G is a broken wheel.*

Fig. 8. Two more forbidden configurations for B-perfect graphs

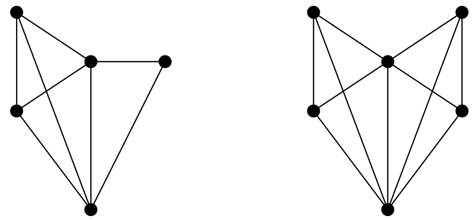

Fig. 9. Two B-perfect graphs with clique number 4

PROOF. By Theorem 24 a graph each component of which is a broken wheel is B-perfect. Now consider the case that G with $\omega(G) = 3$ is B-perfect. By the proofs of Lemmas 1 and 2, G is trivially perfect. Therefore, by Lemma 25, every component of G contains a universal vertex. Consider such a component with universal vertex v_0. Let S be a 2-connected block of this component. Since $S \setminus \{v_0\}$ does not contain induced P_4 or C_4, $S \setminus \{v_0\}$ is a star. If it is a star with three or more leaves, then Bob has a winning strategy with three colors: in his first move he colors a leaf, in his second move a leaf with a different color. So $S \setminus \{v_0\}$ is either K_1, K_2 or P_3. If two different blocks without the universal vertex are P_3, then Bob has the following winning strategy: in his first move he colors the first leaf v_1 of the first P_3 with the first color. If Alice colors one of the neighbors v_2 or v_0 of v_1 with the second color or a vertex of the second P_3 or of another block with an arbitrary color, then Bob colors the second leaf of the first P_3 with the third color, so that eventually either v_2 or v_0 will be surrounded by all three colors. So the only possibility for Alice to play safely is to color the second leaf of the first P_3 with the first color. But then Bob colors the first leaf of the second P_3 with the second color. By the same argument as above, the only chance for Alice to play safely is to color the second leaf of the second P_3 with the second color. However, then Bob colors v_2, the third vertex of the first P_3, with the third color, and Alice has lost as v_0 cannot be colored any more. Thus the respective component of G is a broken wheel. $\square$

The graphs depicted in Fig. 8 have clique number 3 but are no broken wheels, therefore they are not B-perfect by Theorem 26. There are connected graphs with clique number ≥ 4 which are B-perfect but no broken wheels, e.g. the graphs in Fig. 9. Alice has the following winning strategy with 4 colors for the graphs of Fig. 9. In her first two moves she ensures that the two vertices of degree 4 (in the left-hand graph) resp. of degree 5 (in the right-hand graph) are colored. This is possible since, if Bob uses two different colors in his moves, Alice can use the other two colors in her two first moves. The remaining vertices can be colored in any case since they have degree of at most 3.

In general, the definition of broken wheels has to be refined.

Conjecture. *A graph G with $\omega(G) = 4$ is B-perfect if, and only if, G is of the form*

$$
\begin{aligned}
&K_1 \vee (n_1K_1 \cup n_2K_2 \cup n_3P_3 \\
&\qquad \cup n_4K_3 \cup n_5(K_1 \vee 2K_2) \cup n_6(K_1 \vee (K_1 \cup K_2)) \cup n_0(K_2 \vee \overline{K_2}))
\end{aligned}
$$

with $n_i \geq 0$, $n_0 \in \{0, 1\}$, and $n_4 + n_5 + n_6 + n_0 \geq 1$.

For example, if we set $n_6 = 1$ and $n_1 = n_2 = n_3 = n_4 = n_5 = n_0 = 0$ in the conjecture, we obtain the left-hand graph of Fig. 9. The right-hand graph is obtained if we set $n_5 = 1$ and $n_1 = n_2 = n_3 = n_4 = n_6 = n_0 = 0$. A broken wheel is obtained if we set $n_5 = n_6 = 0$.

Final remark. The Weak Perfect Graph Theorem [15] states that a graph is perfect if, and only if, its complement is perfect. There is no game theoretic analogon of this theorem. An example with 5 vertices is the graph in Fig. 10 (b) which is B-perfect but its complement $C_4 \cup K_1$ is not B-perfect. The complement of the A-perfect graph in Fig. 10 (a) is the chair Ch which is not A-perfect.

Fig. 10. An (a) A-perfect resp. (b) B-perfect graph the complement of which is not A-perfect resp. B-perfect

Open question. Find an analogon to the Strong Perfect Graph Theorem for game-perfectness.

Acknowledgement

Thanks to Maria Chudnovsky. This work is a partial answer to a question she once asked me. I acknowledge several valuable suggestions of the anonymous referees, in particular the hint concerning the reference of Lemma 25.

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