Two disjoint negative cycles in a signed graph

Winfried Hochstättler, Robert Nickel

FernUniversität in Hagen, Diskrete Mathematik und Optimierung, D–58084 Hagen, Germany

Britta Peis

Zentrum für Angewandte Informatik (ZAIK), Universität zu Köln, D–50931 Köln, Germany

1 Motivation

Signed graphs have been studied a lot during the last two decades. We address the vertex disjoint negative cycle packing problem posed by Zaslavsky [7, Problem II.A.1a]. Our aim is to find two vertex disjoint negative cycles in reasonable time which generalizes the problem of finding two vertex disjoint odd cycles in a graph.

In the first stage we show that the problem is solvable in polynomial time as it reduces to testing a given matrix for total unimodularity. Such an algorithm operates on the incidence matrix of the signed graph and does not make use of the underlying graphic structure.

There are at least two ways to accelerate the procedure. The first is to disassemble the total unimularity test and to consider minimum non-totally unimodular submatrices resulting in a complexity of at least one total unimodularity test which is still not very efficient.

Instead, we try to find an algorithm that acts on the graph itself, investigate structural properties of the problem and present instances where straightforward approaches fail.

Email addresses: `winfried.hochstaettler@fernuni-hagen.de` (Winfried Hochstättler,), `robert.nickel@fernuni-hagen.de` (Robert Nickel), `peis@zpr.uni-koeln.de` (Britta Peis).

Signed Graphs

A *signed graph* $G(V, E, \Sigma)$ is a (multi-)graph $G(V, E)$ together with a partition $E = \Sigma \dot{\cup} (E \setminus \Sigma)$ of its edges into *negative* and *positive* edges. A subset of edges is called *negative* if it contains an odd number of negative edges and *even*, otherwise. A *cycle* of a signed graph is a cycle of $G(V, E)$.

We consider the problem of finding two vertex disjoint negative cycles in $G(V, E, \Sigma)$. If $\Sigma = E$ this problem is equivalent to finding two vertex disjoint cycles of odd length.

Proving Polynomiality

The Bias Matroid of a Signed Graph

The node-edge incidence matrix $A(G)$ of a signed graph is the matrix with ± 1 in component $a_{v,e}$ when $e \in E$ is a non-loop edge incident with $v \in V$. The signs in a column corresponding to a non-loop edge e alternate/correspond when e is positive respectively negative. Positive loops correspond to zero-columns and negative loops correspond to columns that have a 2 in the row corresponding to the incident vertex.

The *circuits* of a signed graph G are the edge sets corresponding to minimal linear dependent columns of $A(G)$ and form the set of circuits of a matroid (see Oxley [2]) which is called the *bias matroid* of G. Zaslawski [8] has shown that these are exactly the edge sets corresponding to

· positive cycles of G
· pairs of negative cycles intersecting in a single vertex
· pairs of vertex disjoint negative cycles connected with a (non-trivial) path.

The last type is called a *handcuff*. Our problem now reduces to searching a connected signed graph for a handcuff.

If $G(V, E)$ is not 2-connected one can try and find in any block of G a negative cycle in linear time (at least one has to be careful that two such cycles do not intersect in a cut vertex). Thus, it suffices to consider the 2-connected case.

Proposition 1.1 (Tutte [6]) *A matroid is binary if and only if it does not contain a 4-element minor whose circuits are* $\{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.

The minor in the above theorem is the uniform matroid $U_{2,4}$ of rank two with four elements, the four point line. One easily checks that $U_{2,4}$ is isomorphic to the bias matroid of the following signed graph $\pm K_2^o$ (negative edges are bold):

By the 2-connectedness and Menger's Theorem two disjoint negative cycles can be connected by two vertex disjoint paths. These paths and cycles together contract to $\pm K_2^o$. Hence we get the following result of Pagano [3]:

Proposition 1.2 *The bias matroid of a 2-connected signed graph G is not binary if and only if $\hat{G}$ contains two disjoint negative cycles.*

Here, $\hat{G}$ is the signed graph obtained from G by contracting all blocks that do not contain a negative cycle.

With its rational representation matrix $A(G)$ the bias matroid of a signed graph is orientable and, therefore a 2-connected signed graph (resp. its bias matroid) is regular if and only if it has no $\pm K_2^o$ minor.

A matroid representable over $\mathbb{Q}$ is regular if and only if it has a representation by a totally unimodular matrix. Total unimodularity testing is possible in polynomial time (see Cunningham and Edmonds [1]). As regular matroids are uniquely representable it suffices to test $A(G)$ for total unimodularity.

Proposition 1.3 *Let $\mathcal{M}(G)$ be a minimal non-regular bias matroid of a signed graph (i. e. the deletion of any edge makes $\mathcal{M}(G)$ into a regular matroid). Then $\mathcal{M}(G)$ has corank 2, i. e. $\mathcal{M}(G) \setminus e$ contains exactly one circuit for any edge e.*

An outline of a polynomial algorithm is as follows: Delete edges as long as the incidence matrix remains non-totally unimodular. Then find a handcuff in the corank 2 minor of G' of G by deleting each edge once.

Structural Considerations

The fact that the family of odd cycles does not have the Erdös-Pósa property constitutes that the problem at least is not trivial:

Proposition 1.4 (Rautenbach and Reed [4]) *There is no integer k such that every graph G either contains 2 vertex disjoint odd cycles or a set K of k vertices such that $G \setminus K$ is bipartite.*

The graph instances that require the deletion of $o(|V|)$ vertices are the *Escher walls* (see [4]). Hence, the existence of two vertex disjoint odd cycles cannot be checked by deleting a subset of vertices of fixed size.

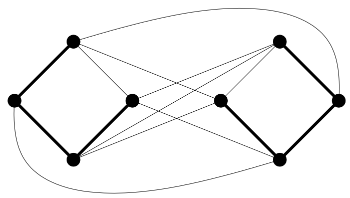
Switching a vertex v, i. e. making positive edges incident with v negative and vice versa, leaves the circuit structure invariant. The goal is to find a switching so that there are two disjoint negative cycles each having only one negative edge. Then we have

Theorem 1.5 *If G contains two vertex disjoint negative cycles C_1, C_2 satisfying $C_1 \cap \Sigma = C_2 \cap \Sigma = 1$ then these can be found in polynomial time.*

The proof of this is very easy. Consider any pair of two nonadjacent negative edges $s_1 t_1$ and $s_2 t_2$ and find in $G \setminus \Sigma$ two vertex disjoint paths from s_i to t_i for $i \in \{1, 2\}$. This is possible in $\mathcal{O}(|V| \cdot |E|)$ time (see [5]).

The above argument also holds for a fixed number of p vertex disjoint negative circles but the corresponding polynomial algorithms are hardly practical for $p > 2$ ([5]). p vertex disjoint paths can be found in $\mathcal{O}(|V|^2|E|)$ but with a large constant factor.

The problem is to find a switching so that two such cycles exist. Take any spanning tree T of G. Then there is an $S \subseteq V$ so that all edges of T are positive. Obviously, there are trees so that the assumption of the above theorem holds. Unfortunately, we can construct counterexamples so that spanning trees with nice structural properties such as DFS trees or Hamiltonian paths leave one pair of two disjoint negative cycles that both have more than one negative edge:



A signed graph with a unique pair of two negative disjoint cycles spanned by a positive path

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