A continuation method for a right definite two-parameter eigenvalue problem

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We present a new numerical method for the right definite two-parameter eigenvalue problem. The method is based on the continuation method which has been successfully applied to one-parameter eigenvalue problems. As a corrector in the continuation method we use the Tensor Rayleigh Quotient Iteration (TRQI) which is a two-parameter generalization of the Rayleigh quotient iteration. We show its convergence and compare it with Newton's method.

Multiparameter eigenvalue problems arise in a variety of applications, particularly in mathematical physics when the method of separation of variables is used to solve boundary value problems. We consider a right definite two-parameter eigenvalue problem $\boldsymbol{W}$

$$\begin{aligned} A_1 x &= \lambda B_1 x + \mu C_1 x, \\ A_2 y &= \lambda B_2 y + \mu C_2 y, \end{aligned}$$

where A_i, B_i, C_i are real symmetric $n_i \times n_i$ matrices, $i = 1, 2$, and where exists $\delta > 0$ such that

$$\begin{vmatrix} x^T B_1 x & x^T C_1 x \\ y^T B_2 y & y^T C_2 y \end{vmatrix} \geq \delta$$

for all vectors $\|x\| = \|y\| = 1$. We say that (λ, μ) is an eigenvalue of $\boldsymbol{W}$ if $\det(A_i - \lambda B_i - \mu C_i) = 0$ for $i = 1, 2$.

On the tensor product space $V := \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$ we define operator determinants

$$\Delta_0 = \begin{vmatrix} B_1^\dagger & C_1^\dagger \\ B_2^\dagger & C_2^\dagger \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} A_1^\dagger & C_1^\dagger \\ A_2^\dagger & C_2^\dagger \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} B_1^\dagger & A_1^\dagger \\ B_2^\dagger & A_2^\dagger \end{vmatrix},$$

where $A_1^\dagger, A_2^\dagger, B_1^\dagger, B_2^\dagger, C_1^\dagger, C_2^\dagger$ are the induced linear transformations on V. A two-parameter system $\boldsymbol{W}$ is called nonsingular if the corresponding operator determinant Δ_0 is invertible. In this case problem $\boldsymbol{W}$ is equivalent to the simultaneous problem $\boldsymbol{\Delta}$

$$\begin{aligned} \Delta_1 z &= \lambda \Delta_0 z, \\ \Delta_2 z &= \mu \Delta_0 z \end{aligned}$$

for decomposable tensors $z \in V$. Right definiteness is equivalent to the condition that Δ_0 is positive definite.