Abstract

What can numerical linear algebra learn from singular point computations?

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During the past 20 years several stable and efficient algorithms for computing singular points y^* of nonlinear equations $F(y) = F(x,\lambda) = 0$ where $F : \mathbb{R}^{n+1} \to \mathbb{R}^n$ have been developed by, e.g., ALLGOWER, GRIEWANK, KELLER, PÖNISCH, REDDIEN, RHEINBOLDT, SCHWETLICK and others. Most of these algorithms are based on so-called *minimally extended systems* $H(y,\mu) = 0$ for characterizing the special singular point in such a way that $H(y^*,\mu^*) = 0$ and, moreover, $\partial H(y^*,\mu^*)$ is nonsingular so that locally and superlinearly convergent Newton-type methods can be used. Here μ denotes a vector of some artificial variables, typically unfolding parameters.

It turns out that some of these algorithms in a natural way lead to certain algorithms of invers iteration type for approximating eigenvalues or singular values of matrices, resp. These algorithms work with nonsingular bordered matrices where the bordering blocks are automatically chosen as "almost optimal" with respect to the condition number.

An example of such a method for improving approximations λ_k to an algebraically simple eigenvalue of a non-normal matrix has recently been published by SCHWETLICK/LÖSCHE. There an bifurcation point algorithm introduced by GRIEWANK/REDDIEN and modified by ALLGOWER/SCHWETLICK has been applied to $F(x,\lambda) := Ax - \lambda x$ without any normalising condition starting from the trivial branch $x = 0$. It leads to an "alternating " generalized Rayleigh-Quotient iteration where two linear systems with nonsingular matrices

$$C_k := \begin{pmatrix} A - \lambda_k I & y^k \\ (z^k)^T & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

and C_k^T have to be solved per step. Here y^k, z^k with $\|y^k\| = \|z^k\| = 1$ are approximations to the corresponding left and right eigenvector, resp., which can be updated by quantities available from the algorithm. These updates give $\lambda_{k+1} := [(y^k)^T A z^k]/[(y^k)^T z^k]$ as generalized Rayleigh quotient. This method is related to a new RQ iteration of O'LEARY/STEWART. Using results of nonlinear analysis, the Q-quadratic convergence of the $\{\lambda_k\}$ and the R-quadratic of the $\{y^k\}$ and $\{z^k\}$ could be shown provided that λ_0 is sufficiently good.

As an example of a related method for singular value computations, a new generalized invers subspace iteration for computing the $q \ll n$ smalles singular values and the corresponding singular vectors of a matrix $A \in \mathbb{R}^{n\times n}$ will be described in the talk (joint work with Uwe Schnabel). That method requires per step to solve alternatingly linear systems with bordered nonsingular matrices $B_{2\ell} = B(Y_{2\ell}, Z_{2\ell-1}, \Omega_{2\ell})$ and $B_{2\ell+1}^T = B(Y_{2\ell}, Z_{2\ell+1}, \Omega_{2\ell+1})^T$, resp., $\ell \geq 0$, where

$$B(Y,Z,\Omega) := \begin{pmatrix} A & Y \\ Z^T & \Omega \end{pmatrix} \in \mathbb{R}^{(n+q)\times(n+q)} \quad \text{with} \quad Y^TY = Z^TZ = I_q.$$

These matrices have decreasing, asymptotically optimal condition numbers, and under weak assumptions the approximations to the right and left singular subspaces belonging to $\sigma_{n-q+1}, \ldots, \sigma_n$ which are defined by the X and Y blocks of B_k, resp., converge linearly (measured by the tangent of the angle) with factor $\kappa := \sigma_{n-q+1}/\sigma_{n-q}$ provided that $\sigma_{n-q+1} < \sigma_{n-q}$ whereas certain singular value approximations extracted from these subspaces by a Rayleigh-Ritz-like process that requires computing the SVD of the $(q \times q)$-matrix Y^TAX have accuracy κ^{2k}. Extensions to rectangular matrices are possible.