Relative Perturbation Theory for the Hyperbolic Singular Value Problem

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Abstract

Let $\mathbf{G}$ be $\mathbf{m} \times \mathbf{n}$ **full rank** matrix and let $\mathbf{J}$ be a $\mathbf{n} \times \mathbf{n}$ diagonal matrix of signs $\mathbf{J_{ii}} \in \{-1, 1\}$. The HSVD for the pair $(\mathbf{G}, \mathbf{J})$ is a factorization

$$\mathbf{G} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{-1}.$$

where $\mathbf{U}$ is unitary, Σ is nonnegative diagonal, and $\mathbf{V}$ is $\mathbf{J}$-unitary. The latter is equivalent to

$$\mathbf{V}^*\mathbf{J}\mathbf{V} = \mathbf{J} \qquad \text{or} \qquad \mathbf{V}^{-1} = \mathbf{J}\mathbf{V}^*\mathbf{J}.$$

This is a difference between the classical SVD where $\mathbf{V}$ is also unitary. As the classical SVD, the HSVD can be written in economical form depending upon dimension. Similarly to the classical SVD, the HSVD is closely related to two eigenvalue problems: the Hermitian eigenvalue problem

$$\mathbf{G}\mathbf{J}\mathbf{G}^* = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^* \qquad \text{where} \qquad \boldsymbol{\Lambda} = \mathbf{diag} = \boldsymbol{\Sigma}\mathbf{J}\boldsymbol{\Sigma}^*,$$

and the $\mathbf{J}$-Hermitian or **hyperbolic** eigenvalue problem for the pair $(\mathbf{G}^*\mathbf{G}, \mathbf{J})$,

$$\mathbf{V}^*\mathbf{G}^*\mathbf{G}\mathbf{V} = \mathbf{diag} = \boldsymbol{\Sigma}^*\boldsymbol{\Sigma}, \qquad \mathbf{V}^*\mathbf{J}\mathbf{V} = \mathbf{J}.$$

Note that if $\mathbf{G}$ has full row rank we must additionaly require that $\mathbf{G}\mathbf{J}\mathbf{G}^*$ is non-singular – the counter example is

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{or the pair} \qquad \left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \right)$$

The HSVD appears in various applications of the **downdating problem**: the Hermitian eigenvalue problem for the matrix

$$\mathbf{A}\mathbf{A}^* - \mathbf{B}\mathbf{B}^*$$

can be solved as the Hemitian eigenvalue problem for the matrix $\mathbf{G}\mathbf{J}\mathbf{G}^*$ where

$$\mathbf{G} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} \qquad \text{and} \qquad \mathbf{J} = \begin{bmatrix} \boldsymbol{I} & \\ & -\boldsymbol{I} \end{bmatrix}$$

with the blocks of appropriate dimensions. We consider:

- $\mathbf{G}$ with full column rank; $\mathbf{G}$ with full row rank (here $\mathbf{G}\mathbf{J}\mathbf{G}^*$ is non-singular); non-singular $\mathbf{G}$ is included in both cases;
- relative element-wise perturbations $|\boldsymbol{\delta}\mathbf{G}| \leq \varepsilon|\mathbf{G}|$;
- right or left scalings, $\mathbf{G} = \mathbf{B}\mathbf{D}$ or $\mathbf{G} = \bar{\mathbf{D}}\bar{\mathbf{B}}$, respectively;
- relative perturbation bounds for singular values $\dfrac{\boldsymbol{\delta\sigma}}{\boldsymbol{\sigma}}$ (Weyl and Wielandt–Hoffman type);
- perturbation bounds for singular vectors ($\mathbf{sin}\boldsymbol{\Theta}$ theorems for columns of $\mathbf{U}$ and $\mathbf{V}$).

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