On some properties of the Lyapunov equation for damped systems

Ninoslav Truhar,
University of Osijek,
Department of Mathematics,
31000 Osijek, Croatia[1]
ntruhar@mathos.hr
Krešimir Veselić
Lehrgebiet Mathematische Physik,
Fernuniversität,
58084 Hagen, Germany
kresimir.veselic@fernuni-hagen.de

Abstract

We consider a damped linear vibrational system whose dampers depend linearly on the viscosity parameter v. We show that the trace of the corresponding Lyapunov solution can be represented as a rational function of v whose poles are the eigenvalues of a certain skew symmetric matrix. This makes it possible to derive an asymptotic expansion of the solution in the neighborhood of zero (small damping).

1 Introduction

We consider a damped linear vibrational system

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{1.1}$$

where the matrices M, C, K (mass, damping, stiffness) are symmetric, M, K are positive definite and C is positive semidefinite. If internal damping is neglected C has often small rank as it describes a few dampers built in to calm down dangerous vibrations. Often C has the form

$$C = vC_0$$

where v is a variable viscosity and C_0 describes the geometry of a damper. C_0 will have rank $1, 2$ or 3 according to whether the damper can exhibit linear, planar or spatial displacements.

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An example is the so-called *n-mass oscillator* or *oscillator ladder* (Fig. 1) where

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n)$$

$$K = \begin{bmatrix} k_1 & -k_1 & & & \\ -k_1 & k_1+k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -k_{n-2} & k_{n-2}+k_{n-1} & -k_{n-1} \\ & & & -k_{n-1} & k_{n-1}+k_n \end{bmatrix},$$

$$C = ve_1e_1^T + v(e_3 - e_2)(e_3 - e_2)^T. \tag{1.2}$$

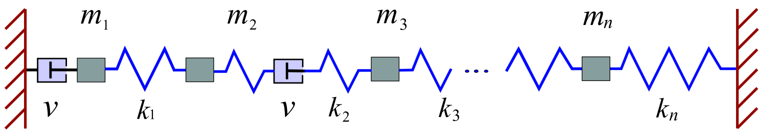

Figure 1: The n-mass oscillator with two dampers

Here $m_i > 0$ are the masses, $k_i > 0$ the spring constants or stiffnesses, e_i is the i-th canonical basis vector, and v is the viscosity of the damper applied on the i-th mass.

After the substitution

$$y_1 = \Omega\,\Phi^{-1}x\,, \quad y_2 = \Phi^{-1}\dot{x}\,, \tag{1.3}$$

where

$$K\Phi = M\Phi\Omega^2\,, \quad \Phi^T M\Phi = I \tag{1.4}$$

$$\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n), \quad \omega_1 < \ldots < \omega_n \tag{1.5}$$

is the eigenreduction of the symmetric positive definite matrix pair K, M, the system (1.1) goes over into

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}\,, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \tag{1.6}$$

$$\mathbf{A} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -D \end{bmatrix}, \quad D = \Phi^T C\Phi\,. \tag{1.7}$$

Then

$$E = \frac{1}{2}\int_0^\infty \left(\dot{x}^T M\dot{x} + x^T Kx\right) dt = \int_0^\infty \|\mathbf{y}\|^2 dt = \int_0^\infty \|e^{\mathbf{A}\,t}\,\mathbf{y}_0\|^2 dt\,, \tag{1.8}$$

where $\mathbf{y}_0$ is the initial data. Thus,

$$E \equiv E(\mathbf{y}_0) = \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 \,,$$

where

$$\mathbf{X} = \int_0^\infty e^{\mathbf{A}^T t} e^{\mathbf{A} t} dt \tag{1.9}$$

solves the Lyapunov equation

$$\mathbf{A}^T \mathbf{X} + \mathbf{X}\mathbf{A} = -\mathbf{I} \,. \tag{1.10}$$

Our penalty function is obtained by averaging E over all initial data with the equal energy, that is, we form

$$\bar{E} = \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 d\sigma$$

where $d\sigma$ is a probability measure on the unit sphere in $\mathbb{R}^{2n}$. Since by the map

$$\mathbf{X} \mapsto \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 \; d\sigma$$

is given a linear functional on the space of the symmetric matrices, by Riesz theorem there exists a symmetric matrix $\mathbf{Z}$ such that

$$\mathbf{X} \mapsto \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 \; d\sigma = Tr(\mathbf{Z}\mathbf{X}) \,, \text{ for all symmetric matrices } \mathbf{X} \,.$$

Let $\mathbf{y} \in \mathbb{R}^{2n}$ be arbitrary. Set $X = \mathbf{y}\mathbf{y}^T$. Then

$$0 \leq \int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 \; d\sigma = Tr(\mathbf{Z}\mathbf{X}) = Tr(\mathbf{Z}\mathbf{y}\mathbf{y}^T) = Tr(\mathbf{y}^T \mathbf{Z} \mathbf{y}) \,,$$

hence $\mathbf{Z}$ is always positive semi-definite.

For any given measure there is a unique positive semidefinite matrix $\mathbf{Z}$ such that

$$\bar{E} = Tr(\mathbf{Z}\mathbf{X}) \,. \tag{1.11}$$

For the measure σ generated by the Lebesgue measure (i.e. the usual surface measure) on $\mathbb{R}^{2n}$, we obtain $\mathbf{Z} = \frac{1}{2n}\mathbf{I}$. For the convenience of the reader, we give a sketch of the proof:

Recall,

$$\mathbf{Z}_{ij} = \int_S \mathbf{y}_i \mathbf{y}_j \, \sigma(d\mathbf{y}) \,.$$

One can easily see using Minkowski formula (see [2]) that

$$\mathbf{Z}_{ij} = \int_S \mathbf{y}_i \mathbf{y}_j \, \sigma(d\mathbf{y}) = \frac{1}{2\varepsilon} \lim_{\varepsilon \to 0} \int_{\mathrm{d}(\mathbf{y}, S) \leq \varepsilon} \mathbf{y}_i \mathbf{y}_j \, \sigma(d\mathbf{y}) \,,$$

here S denotes the unit sphere in $\mathbb{R}^{2n}$ and $\mathrm{d}(\mathbf{y}, S)$ is a corresponding distance.

Obviously, $\mathbf{Z}_{ij} = 0$ for $i \neq j$ and $\mathbf{Z}_{ii} = \mathbf{Z}_{jj}$, for $i, j \in \{1, 2, \ldots, 2n\}$. Since

$$\mathbf{Z}_{11} + \mathbf{Z}_{11} + \ldots \mathbf{Z}_{2n2n} = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \, vol \left(\mathbf{y} \in \mathbb{R}^{2n} : \mathrm{d}(\mathbf{y}, S) \leq \varepsilon \right) = 1 \,,$$

it follows $\mathbf{Z} = \frac{1}{2n}\mathbf{I}$. The more details about the structure of the matrix $\mathbf{Z}$ one can find in [4].

We have shown that

$$\int_{\|\mathbf{y}_0\|=1} \mathbf{y}_0^T \mathbf{X} \mathbf{y}_0 \; d\sigma = \min$$

is equivalent to

$$Tr(\mathbf{Z}\mathbf{X}) = \min \,. \tag{1.12}$$

where $\mathbf{Z}$ is a symmetric positive semidefinite matrix which may be normalized to have a unit trace.

If one is interested in damping a certain part of the spectrum of the matrix $\mathbf{A}$ (which is very important in applications) then the matrix $\mathbf{Z}$ will have a special structure. For example, let $\sigma = \sigma_1 \times \sigma_2 \times \sigma_1 \times \sigma_2$, where σ_1 is a measure on the frequency subspace determined by $\omega \leq \omega_{\max} \equiv \omega_s$ generated by Lebesgue measure, that is σ_1 is a measure on the frequency subspace which corresponds to the eigenfrequencies (defined by (1.5)) $\omega_1, \ldots \omega_s$ and σ_2 is Dirac measure on the complement. Then we obtain that the corresponding matrix Z has the form

$$\mathbf{Z}_s = \mathbf{Z} = \frac{1}{2s} \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \,, \tag{1.13}$$

where I_s is the identity matrix of the dimension s which is defined by $\omega_{\max} = \omega_s$. Here $\omega_{\max} = \omega_s$ is critical frequency with the property that the eigenfrequencies from (1.5) greater then ω_s are not dangerous. Hence, we damp first s eigenfrequencies. The construction of $\mathbf{Z}$ from (1.13) is a generalization of the simplest case $\mathbf{Z} = \frac{1}{2n}\mathbf{I}$.

In [7] a simple solution of the problem (1.11) has been presented for $\mathbf{A}$ from (1.7) and $\mathrm{rank}(C) = 1$. In particular,

$$Tr(\mathbf{Z}\mathbf{X}(v)) = const + \frac{a}{v} + bv, \quad a, b > 0 \,, \tag{1.14}$$

which made it possible to find the minimum by a simple formula explicitly. The case $\mathrm{rank}(C) > 1$ seems to be essentially more difficult to handle.

The main result of this paper is the explicit formula:

$$\mathbf{X}(v) = \frac{\mathbf{\Psi}_{-1}}{v} - \mathbf{\Psi}_0 - v\widehat{\mathbf{\Psi}}_1 + v \sum_{i=1}^{s} \frac{\lambda_i \left(\lambda_i \mathbf{\Phi}_i - v \mathbf{\Upsilon}_i\right)}{\lambda_i^2 + v^2} \,. \tag{1.15}$$

where $\mathbf{\Psi}_{-1}$, $\mathbf{\Psi}_0$, $\widehat{\mathbf{\Psi}}_1$, $\mathbf{\Phi}_i$ and $\mathbf{\Upsilon}_i$ are $m \times m$ matrices and λ_i are eigenvalues of the pencil $(\mathbb{A}_0, \mathbb{D})$ where $\mathbb{A}_0$ and $\mathbb{D}$ are matrices which correspond to the linear operators

$$\begin{aligned} \mathbf{X} &\mapsto -\mathbf{A}_0\mathbf{X} + \mathbf{X}\mathbf{A}_0\,, \\ \mathbf{X} &\mapsto \mathbf{D}\mathbf{X} + \mathbf{X}\mathbf{D}\,, \end{aligned}$$

respectively. Thus, technically, we have turned the viscosity into the spectral parameter.

Further, we have obtained a simple formula for the $\mathbf{\Psi}_{-1}$ in (1.15). This matrix is responsible for the behavior of the solution $\mathbf{X}(v)$ in the neighborhood of zero (small damping):

$$\mathbf{\Psi}_{-1} = \begin{bmatrix} \mathbf{D}_\Delta & 0 \\ 0 & \mathbf{D}_\Delta \end{bmatrix},$$

where

$$\mathbf{D}_\Delta = (\operatorname{diag}(\mathbf{D}))^{-1}\,.$$

We will use the following notation: matrices written in the simple Roman fonts, M, D or K for example will have n^2 entries. Matrices written in the mathematical bold fonts, $\mathbf{A}$, $\mathbf{B}$ will have m^2 entries, where $m = 2n$ (that is $\mathbf{A}$, $\mathbf{B}$ are matrices defined on the $2n$-dimensional phase space). Finally, matrices written in the Blackboard bold fonts $\mathbb{A}$, or $\mathbb{D}$ will have more than m^2 entries.

2 The main result

As we have said in the Introduction, our aim is to obtain the solution $\mathbf{X}$ of the Lyapunov equation

$$\mathbf{A}(v)^T\mathbf{X} + \mathbf{X}\mathbf{A}(v) = -\mathbf{I}, \tag{2.1}$$

where $\mathbf{Z}$ is defined by (1.13) and $\mathbf{I}$ is $m \times m$ identity matrix.

From (1.7) it follows that $\mathbf{A}(v)$ can be written as

$$\mathbf{A}(v) \equiv \mathbf{A}_0 - v\,\mathbf{D}\,, \text{where } \mathbf{A}_0 = \begin{bmatrix} \Omega_1 & & & \\ & \Omega_2 & & \\ & & \ddots & \\ & & & \Omega_n \end{bmatrix}, \quad \Omega_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \tag{2.2}$$

and $\mathbf{D} = \mathbf{D}_0\mathbf{D}_0^T$, where

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ d_{11} & d_{12} & \dots & d_{1r} \\ 0 & 0 & \dots & 0 \\ d_{21} & d_{22} & \dots & d_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ d_{n1} & d_{n2} & \dots & d_{nr} \end{bmatrix},$$

d_{ij} are entries of the matrix

$$\mathbf{D}_0 = \mathbf{P} \begin{bmatrix} 0 \\ L_C \end{bmatrix}, \qquad C_0 = L_C L_C^T,$$

and $\mathbf{P}$ is the "perfect shuffling" permutation.

Now, we proceed with solving equation (2.1). As it is well known, Lyapunov equation (2.1) is equivalent to ([3, Theorem 12.3.1])

$$\left(\mathbf{I} \otimes (\mathbf{A}_0 - v\,\mathbf{D})^T + (\mathbf{A}_0 - v\,\mathbf{D})^T \otimes \mathbf{I}\right) \cdot \mathrm{vec}(\mathbf{X}) = -\mathrm{vec}(\mathbf{I})\,, \tag{2.3}$$

where $\mathbf{L}\otimes\mathbf{T}$ denotes the Kronecker product of $\mathbf{L}$ and $\mathbf{T}$, and vec($\mathbf{I}$) is the vector formed by "stacking" the columns of $\mathbf{I}$ into one long vector.

Further, we will need the following two $m^2 \times m^2$ matrices defined by

$$\mathbb{A}_0 = \mathbf{I} \otimes \mathbf{A}_0^T + \mathbf{A}_0^T \otimes \mathbf{I}\,, \quad \mathbb{D} = \mathbf{I} \otimes \mathbf{D}_0\mathbf{D}_0^T + \mathbf{D}_0\mathbf{D}_0^T \otimes \mathbf{I}\,. \tag{2.4}$$

It is easy to show that $\mathbb{D} = \mathbb{D}_F\,\mathbb{D}_F^T$, where

$$\mathbb{D}_F = \begin{bmatrix} I \otimes \mathbf{D}_0 & \mathbf{D}_0 \otimes I \end{bmatrix}. \tag{2.5}$$

Now, using (2.5) and (2.4) it follows that solution vec($\mathbf{X}$) of equation (2.3) can be written as

$$\mathrm{vec}(\mathbf{X}) = -\left(\mathbb{A}_0 - v\mathbb{D}_F\mathbb{D}_F^T\right)^{-1} \mathrm{vec}(\mathbf{I})\,. \tag{2.6}$$

Obviously, there exists a unitary matrix $\mathbb{U}$ such that

$$\mathbb{U}^T \mathbb{A}_0 \mathbb{U} = \begin{bmatrix} 0 & \\ & \widehat{\mathbb{A}}_0 \end{bmatrix}, \tag{2.7}$$

where $\mathbb{A}_0$ is the skew-symmetric matrix corresponds with linear operator defined in (2.4) and $\widehat{\mathbb{A}}_0$ is a non-singular block diagonal matrix defined by

$$\widehat{\mathbb{A}}_0 = \mathrm{diag}(\Xi_1, \ldots, \Xi_{m_2}) \quad \text{where} \quad \Xi_i = \begin{bmatrix} 0 & -\mu_i \\ \mu_i & 0 \end{bmatrix} \tag{2.8}$$

where $\pm i\mu_i$, $i = 1, \ldots, m_2$ are non-zero eigenvalues of matrix $\mathbb{A}_0$, that is, $0 \neq \mu_i = (\pm\omega_i) - (\pm\omega_j)$ for $i, j = 1, \ldots, n$ (see [3, Corollary 12.2.2]). Note that $m_2 = (m^2 - m)/2$. Set

$$\begin{bmatrix} \mathbb{D}_1 \\ \mathbb{D}_2 \end{bmatrix} = \mathbb{U}^T \mathbb{D}_F\,,$$

where $\mathbb{D}_F$ is defined in (2.5).

Now,

$$\mathbb{A}_0 - v\mathbb{D}_F\mathbb{D}_F^T = \mathbb{U}\left(\begin{bmatrix} 0 & \\ & \widehat{\mathbb{A}}_0 \end{bmatrix} - v\begin{bmatrix} \mathbb{D}_1\mathbb{D}_1^T & \mathbb{D}_1\mathbb{D}_2^T \\ \mathbb{D}_2\mathbb{D}_1^T & \mathbb{D}_2\mathbb{D}_2^T \end{bmatrix}\right)\mathbb{U}^T. \tag{2.9}$$

Taking

$$\mathbb{G} = \begin{bmatrix} I & 0 \\ -\mathbb{D}_2\mathbb{D}_1^T(\mathbb{D}_1\mathbb{D}_1^T)^{-1} & I \end{bmatrix} \tag{2.10}$$

we obtain

$$\mathbb{A}_0 - v\mathbb{D}_F\mathbb{D}_F^T = \mathbb{U}\,\mathbb{G}^{-1}\begin{bmatrix}-v\mathbb{D}_1\mathbb{D}_1^T & \\ & \widetilde{\mathbb{A}}\end{bmatrix}\mathbb{G}^{-T}\mathbb{U}^T, \tag{2.11}$$

where

$$\widetilde{\mathbb{A}} = \widehat{\mathbb{A}}_0 - v\mathbb{D}_2(I - \mathbb{D}_1^T(\mathbb{D}_1\mathbb{D}_1^T)^{-1}\mathbb{D}_1)\mathbb{D}_2^T. \tag{2.12}$$

Note that we can write

$$\mathbb{D}_2(I - \mathbb{D}_1^T(\mathbb{D}_1\mathbb{D}_1^T)^{-1}\mathbb{D}_1)\mathbb{D}_2^T = \mathbb{F}\mathbb{F}^T\,.$$

Further, we have to find the inverse of $\widetilde{\mathbb{A}} = \widehat{\mathbb{A}}_0 - v\mathbb{F}\mathbb{F}^T$. This is obtained by using the *Sherman-Morrison-Woodbury formula* ([1, (2.1.4), pg.51.]), that is,

$$(\widehat{\mathbb{A}}_0 - v\,\mathbb{F}\mathbb{F}^T)^{-1} = \widehat{\mathbb{A}}_0^{-1} + v\widehat{\mathbb{A}}_0^{-1}\mathbb{F}\left(I - v\mathbb{F}^T\widehat{\mathbb{A}}_0^{-1}\mathbb{F}\right)^{-1}\mathbb{F}^T\widehat{\mathbb{A}}_0^{-1}. \tag{2.13}$$

The inverse of $I - v\mathbb{F}^T\widehat{\mathbb{A}}_0^{-1}\mathbb{F}$ remains to be found. Let

$$\Lambda = \operatorname{diag}\left(\begin{bmatrix}0 & \lambda_1\\ -\lambda_1 & 0\end{bmatrix}, \begin{bmatrix}0 & \lambda_2\\ -\lambda_2 & 0\end{bmatrix}, \ldots, \begin{bmatrix}0 & \lambda_s\\ -\lambda_s & 0\end{bmatrix}\right),$$

where $\pm i\lambda_1$, $\pm i\lambda_2$,..., $\pm i\lambda_s$ are non-vanishing finite eigenvalues of the problem

$$(\mathbb{A}_0 - \lambda\mathbb{D})\operatorname{vec}(\mathbf{Y}) = 0.$$

Since $\mathbb{F}^T\widehat{\mathbb{A}}_0^{-1}\mathbb{F}$ is skew-symmetric, then there exists an orthogonal matrix U_S of order $2(r-1)m$ such that

$$U_S^T\mathbb{F}^T\widehat{\mathbb{A}}_0^{-1}\mathbb{F}U_S = \begin{bmatrix}0 & 0\\ 0 & \Gamma\end{bmatrix}, \tag{2.14}$$

where $\Gamma = \Lambda^{-1}$.

Using (2.11), (2.13), (2.14) and (2.6) it follows

$$\operatorname{vec}(\mathbf{X}) = -\mathbb{U}\,\mathbb{G}^T\begin{bmatrix}\Delta_1 & \\ & \Delta_2\end{bmatrix}\mathbb{G}\mathbb{U}^T\operatorname{vec}(\mathbf{I})\,, \tag{2.15}$$

where

$$\Delta_1 = -\frac{(\mathbb{D}_1\mathbb{D}_1^T)^{-1}}{v}, \quad \Delta_2 = \widehat{\mathbb{A}}_0^{-1} + v\widehat{\mathbb{A}}_0^{-1}\mathbb{F}U_S\begin{bmatrix}I & 0\\ & (I - v\Gamma)^{-1}\end{bmatrix}U_s^T\mathbb{F}^T\widehat{\mathbb{A}}_0^{-1}. \tag{2.16}$$

Since Γ is block diagonal we have

$$(I - v\Gamma)^{-1} = \operatorname{diag}\left(\frac{1}{\lambda_1^2 + v^2}\begin{bmatrix}\lambda_1^2 & -v\lambda_1\\ v\lambda_1 & \lambda_1^2\end{bmatrix}, \ldots, \frac{1}{\lambda_s^2 + v^2}\begin{bmatrix}\lambda_s^2 & -v\lambda_s\\ v\lambda_s & \lambda_s^2\end{bmatrix}\right). \tag{2.17}$$

Using (2.16) and (2.15) it follows

$$\operatorname{vec}(\mathbf{X}) = \left(\frac{\mathbb{V}_{-1}}{v} - \mathbb{V}_0 - v\widehat{\mathbb{V}}_1 + v\sum_{i=1}^{s}\frac{\lambda_i\,(\lambda_i\mathbb{W}_i - v\mathbb{Z}_i)}{\lambda_i^2 + v^2}\right)\operatorname{vec}(\mathbf{I})\,, \tag{2.18}$$

where matrices $\mathbb{V}_{-1}$, $\mathbb{V}_0$, $\widehat{\mathbb{V}}_1$, $\mathbb{W}_i$ and $\mathbb{Z}_i$ are constructed using (2.10), (2.16) and (2.15).

By "reshaping" vectors in (2.18) back into $m \times m$ matrices we obtain the solution of equation (2.1)

$$\mathbf{X} = \frac{\mathbf{\Psi}_{-1}}{v} - \mathbf{\Psi}_0 - v\widehat{\mathbf{\Psi}}_1 + v\sum_{i=1}^{s} \frac{\lambda_i\left(\lambda_i \mathbf{\Phi}_i - v\mathbf{\Upsilon}_i\right)}{\lambda_i^2 + v^2}\,. \tag{2.19}$$

If one is interested in deriving an optimal damping, then according to (1.11) one has to minimize the function $f(v) = tr(\mathbf{ZX}(v))$, where

$$tr(\mathbf{ZX}(v)) = \mathrm{vec}(\mathbf{Z})^T \mathrm{vec}(\mathbf{X}) = -\mathrm{vec}(\mathbf{Z})^T \left(\mathbb{A}_0 - v\mathbb{D}_F\mathbb{D}_F^T\right)^{-1}\mathrm{vec}(\mathbf{I})\,.$$

This gives

$$Tr(\mathbf{ZX}) = \frac{X_{-1}}{v} - X_0 - v\widehat{X}_1 + v\sum_{i=1}^{s} \frac{\lambda_i\left(\lambda_i X_i - vY_i\right)}{\lambda_i^2 + v^2}\,, \tag{2.20}$$

where

$$\begin{aligned} X_{-1} &= \mathrm{vec}(\mathbf{Z})^T \mathbb{V}_{-1}\,\mathrm{vec}(\mathbf{I})\,,\\ X_0 &= \mathrm{vec}(\mathbf{Z})^T \mathbb{V}_0\,\mathrm{vec}(\mathbf{I})\,,\\ \widehat{X}_1 &= \mathrm{vec}(\mathbf{Z})^T \widehat{\mathbb{V}}_{-1}\,\mathrm{vec}(\mathbf{I})\,, \end{aligned} \tag{2.21}$$

and

$$X_i = \mathrm{vec}(\mathbf{Z})^T \mathbb{W}_i\,\mathrm{vec}(\mathbf{I})\,, \quad Y_i = \mathrm{vec}(\mathbf{Z})^T \mathbb{Z}_i\,\mathrm{vec}(\mathbf{I})\,. \tag{2.22}$$

Note that the function $Tr(\mathbf{ZX})$ from (2.20) is a generalization of the function $Tr(\mathbf{X})$ defined in (1.14).

Remark **2.1** *In the case when* $\mathbf{Z}$ *is a diagonal matrix, the function* $Tr(\mathbf{ZX})$ *from (2.20) has the following simpler form:*

$$Tr(\mathbf{ZX}) = \frac{X_{-1}}{v} - v\widehat{X}_1 + v\sum_{i=1}^{s} \frac{\lambda_i^2\, X_i}{\lambda_i^2 + v^2}\,.$$

Finally we derive an explicit formula for the matrix $\mathbf{\Psi}_{-1}$. Let $\mathbb{U}_0$ be that part of $\mathbb{U}$ corresponding to the null-space of $\mathbb{A}_0$. From (2.18) it follows that

$$\mathrm{vec}(\mathbf{\Psi}_{-1}) = \mathbb{V}_{-1}\ \mathrm{vec}(\mathbf{I}) \equiv -\mathbb{U}_0(\mathbb{D}_1\mathbb{D}_1^T)^{-1}\mathbb{U}_0^T\ \mathrm{vec}(\mathbf{I})\,. \tag{2.23}$$

It is easily seen that i-th column of the matrix $\mathbb{U}_0$ can be written as $\mathrm{vec}(\mathbf{O}_i)$, where $\mathbf{O}_i$ is a block-diagonal matrix and

$$\begin{aligned} \mathbf{O}_{2i-1} &= \mathrm{diag}(0,\ldots,0,O_{ii},0,\ldots,0)\\ \mathbf{O}_{2i} &= \mathrm{diag}(0,\ldots,0,\widehat{O}_{ii},0,\ldots,0) \end{aligned}$$

where O_{ii} and $\widehat{O}_{ii}$ are "orthogonal" solutions of

$$-\Omega_i O_{ij} + O_{ij}\Omega_j = 0 \qquad i,j = 1,\ldots,n\,,$$

that is

$$O_{ii} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}\,, \text{ and } \widehat{O}_{ii} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}\,.$$

Let $\langle A, B\rangle = Tr(A^T B)$ be the usual Frobenius scalar product. Then, we can write the (p,q)-th element of $\mathbb{U}_0^T \mathbb{D}_F \mathbb{D}_F^T \mathbb{U}_0$ as

$$(\mathbb{U}_0)^T_{(:,p)} \mathbb{D}_F \mathbb{D}_F^T (\mathbb{U}_0)_{(:,q)} = \langle \mathbf{D}_{pq}\,\mathbf{O}_p + \mathbf{O}_p\,\mathbf{D}_{pq}, \mathbf{O}_q\rangle \qquad p,q = 1,\ldots,m\,,$$

where $\mathbf{D}_{pq} = (D_{pq})$ and

$$D_{pq} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{D}_{(p,:)}\mathbf{D}^T_{(q,:)} \end{bmatrix}\,.$$

The orthonormality property

$$\langle \mathbf{O}_p, \mathbf{O}_q\rangle = \delta_{pq}\,,$$

implies

$$\left(\mathbb{U}_0^T \mathbb{D}_F \mathbb{D}_F^T \mathbb{U}_0\right)^{-1} = \mathrm{diag}(1/(\mathbf{D})_{22}, 1/(\mathbf{D})_{22}, \ldots, 1/(\mathbf{D})_{mm}, 1/(\mathbf{D})_{mm})\,.$$

Using the fact that $\mathbb{U}_0 \mathbb{U}_0^T\ \mathrm{vec}(\mathbf{I}) = \mathrm{vec}(\mathbf{I})$, from (2.23) it follows that

$$\mathbf{\Psi}_{-1} = \mathrm{diag}\left(\frac{1}{\mathbf{D}_{22}}, \frac{1}{\mathbf{D}_{22}}, \ldots, \frac{1}{\mathbf{D}_{mm}}, \frac{1}{\mathbf{D}_{mm}}\right)\,. \qquad (2.24)$$

After applying a perfect shuffle permutation we have

$$\mathbf{\Psi}_{-1} = \begin{bmatrix} \mathbf{D}_\Delta & 0 \\ 0 & \mathbf{D}_\Delta \end{bmatrix}\,, \qquad (2.25)$$

where

$$\mathbf{D}_\Delta = (\mathrm{diag}\,(\mathbf{D}))^{-1}\,.$$

The explicitness of the obtained formulas is attractive for possible numerical computation. Our first attempts to perform this task did not succeed due to unexpected complexity problems. We will come back to this issue in our future research.

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