OPTIMIZING THE SOLUTION OF THE LJAPUNOV EQUATION

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Abstract We consider a second order damped-vibration equation $M\ddot{x}+C\dot{x}+Kx = 0$, where M, C, K (called mass, damping, stiffness matrix, respectively) are real, symmetric matrices of order n with M, K positive definite and C positive semidefinite with $\text{rank}(\mathbf{C}) = r$.

We optimize the solution $\mathbf{X}$ of the Ljapunov equation $\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} = -\mathbf{B}$, (where $\mathbf{A} = \mathbf{A}(\varepsilon)$ is obtained from M, C, K and $\mathbf{B}$ is positive definite) in the following sense. Parameter ε is called optimal if $Tr(\mathbf{ZX}) = \min$, where $\mathbf{Z}$ is a suitable chosen positive semidefinite matrix.

We obtain an explicit formula which is easily minimized. This optimization is sensibly more efficient from than standard methods (Bartels–Stewart), if r is small.

Keywords: Damped-vibration, Ljapunov equation, optimization of viscosities of dampers.

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1. Introduction

We consider a damped linear vibrational system described by the differential equation

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{1.1}$$

where M, C, K (called mass, damping, stiffness matrix, respectively) are real, symmetric matrices of order n with M, K positive definite and C positive semidefinite. Often the matrix C has a small rank. An example is the so-called *n-mass oscillator* or *oscillator ladder* (Fig. 1) where

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n)$$

$$K = \begin{bmatrix} k_0 + k_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -k_{n-1} \\ & & & -k_{n-1} & k_{n-1} + k_n \end{bmatrix},$$

$$C = \sum_k c_k e_k e_k^T\,, \tag{1.2}$$

where e_k is the k-th canonical basis vector.

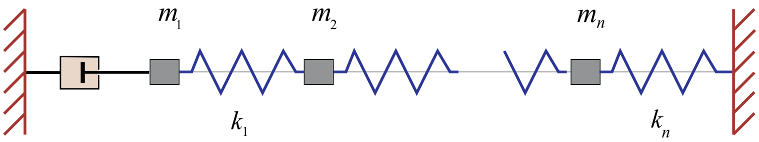

Figure 1. The n-mass oscillator with one damper

Here $m_i > 0$ are the masses, $k_i > 0$ the spring constants or stiffnesses and $c_i > 0$ are the damping constants of the damper applied to the mass m_i (in Fig. 1 $k_0 = 0$).

Using the eigenvalue decomposition

$$\Phi^T K \Phi = \Omega^2\,, \quad \Phi^T M \Phi = I\,,$$

where $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n)$ and setting

$$y_1 = \Omega\,\Phi^T x \quad y_2 = \Phi^T \dot{x}\,,$$

(1.1) can be written as

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y},, \tag{1.3}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T C \Phi \end{bmatrix}, \tag{1.4}$$

with the solution

$$\mathbf{y} = e^{\mathbf{A}t}\,\mathbf{y}_0\,, \quad \text{where } \mathbf{y}_0 \text{ is the initial data.}$$

Moreover,

$$\mathbf{y}^T\mathbf{y} = \|\mathbf{y}\|^2 = x^T K x + \dot{x}^T M \dot{x} = 2E(t)\,.$$

In other words, the Euclidian norm of this phase-space representation equals twice the total energy of the system.

In this paper we study the matrix

$$\mathbf{X} = \int_0^\infty e^{\mathbf{A}^T t}\,\mathbf{B}\,e^{\mathbf{A}t} dt$$

which solves the Ljapunov equation

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} = -\mathbf{B}, \tag{1.5}$$

where $\mathbf{Z}$ is a symmetric positive semidefinite matrix which may be normalized to have a unit trace.

Throughout this paper the matrix $\mathbf{B}$ (and therefore the solution $\mathbf{X}$) will be symmetric and positive definite. Our aim is, in fact, to optimize the solution $\mathbf{X}$ of (1.5) in the following sense. Let $\mathbf{A}$ be written as

$$\mathbf{A} \equiv \mathbf{A}_\varepsilon = \mathbf{A}_0 - \varepsilon\mathbf{D}\,, \tag{1.6}$$

where $\mathbf{A}_0$ is skew-symmetric, $\mathbf{D}$ is positive semidefinite, $\varepsilon > 0$. Parameter ε is called optimal if

$$Tr(\mathbf{Z}\mathbf{X}) = \min \tag{1.7}$$

For the physical background for this penalty function see [5], [6], [7].

In [7] a simple solution of the problem (1.7) has been presented for $\mathbf{A}$ from (1.6) and $\mathrm{rank}(C) = 1$. In particular,

$$Tr(\mathbf{Z}\mathbf{X}) = const + \frac{a}{\varepsilon} + b\varepsilon, \quad a, b > 0\,, \tag{1.8}$$

which made possible to find the minimum explicitly by a simple formula. The case $\mathrm{rank}(C) > 1$ seems to be essentially more difficult to handle.

Our approach here consists of deriving the formula for the solution $\mathbf{X}$ as the function of ε which will further allow us to discuss some properties of the solution and to find the graph of the meromorphic function $\varepsilon \rightarrowtail Tr(\mathbf{ZX})$ by finding its poles and performing a corresponding partial fraction decomposition. We obtain the formula for the solution $\mathbf{X}$:

$$\mathbf{X} = \frac{\mathbf{\Psi}_{-1}}{\varepsilon} - \mathbf{\Psi}_0 - \varepsilon \widehat{\mathbf{\Psi}}_1 + \varepsilon \sum_{i=1}^{s} \frac{\lambda_i\left(\lambda_i \mathbf{\Phi}_i - \varepsilon \mathbf{\Upsilon}_i\right)}{\lambda_i^2 + \varepsilon^2}\,. \tag{1.9}$$

where $\mathbf{\Psi}_{-1}$, $\mathbf{\Psi}_0$, $\widehat{\mathbf{\Psi}}_1$, $\mathbf{\Phi}_i$ and $\mathbf{\Upsilon}_i$ are $m \times m$ matrices and λ_i are eigenvalues of the pencil $(\mathbb{A}_0, \mathbb{D})$ where $\mathbb{A}_0$ and $\mathbb{D}$ are matrices which correspond to the linear operators

$$\begin{aligned} \mathbf{X} &\mapsto -\mathbf{A}_0\mathbf{X} + \mathbf{X}\mathbf{A}_0\,, \\ \mathbf{X} &\mapsto \mathbf{D}\mathbf{X} + \mathbf{X}\mathbf{D}\,, \end{aligned}$$

respectively.

Further, we have obtained a simple formula for the matrix $\mathbf{\Psi}_{-1}$. This matrix is responsible for the behavior of the solution in the neighborhood of zero (small damping):

$$\mathbf{\Psi}_{-1} = \left[\frac{b_1}{\mathbf{D}_{22}}\, u_1\,, \frac{b_2}{\mathbf{D}_{22}}\, u_2\,, \ldots, \frac{b_{m-1}}{\mathbf{D}_{mm}}\, u_{m-1}\,, \frac{b_m}{\mathbf{D}_{mm}}\, u_m\right]\,,$$

where $\mathbb{U}_0 = \left[u_1, u_2, \ldots, u_m\right]$ is a matrix whose columns span the null-subspace of the matrix $\mathbb{A}_0$, and $b_i = u_i^T \mathrm{vec}(\mathbf{B}) \in \mathbb{R}$, where $\mathrm{vec}(\mathbf{B})$ is the vector formed by "stacking" the columns of $\mathbf{B}$ into one long vector.

Moreover, we present an algorithm to compute the necessary coefficients, in the formula (1.9). Our formulae are heuristically obtained by analyzing dimensions $n = 2, 3, 4, 5$ by computer algebra and then guessing for general n. There is no proof (there hardly will ever be) but our results numerically coincide with those obtained by standard methods.

2. The main result

As we have said in the Introduction, our aim is to find the solution of the Ljapunov equation

$$\widehat{\mathbf{A}}^T \widehat{\mathbf{X}} + \widehat{\mathbf{X}} \widehat{\mathbf{A}} = -\widehat{\mathbf{B}}, \tag{2.1}$$

where $\widehat{\mathbf{Z}}$ and $\widehat{\mathbf{B}}$ are symmetric positive semidefinite matrices. From (1.4) it follows that we can write $\widehat{\mathbf{A}}$ from (2.1) as

$$\widehat{\mathbf{A}} \equiv \widehat{\mathbf{A}}_0 - \varepsilon\, \widehat{\mathbf{D}} = \begin{bmatrix} 0 & L_1^T L_2^{-T} \\ -L_2^{-1} L_1 & 0 \end{bmatrix} - \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & L_2^{-1} C_0 C_0^T L_2^{-T} \end{bmatrix}\,,$$

where $C = \varepsilon\, C_0 C_0^T$. Let $C_0 \in \mathbb{R}^{n\times r}$ have a full column rank, that is

$$C_0 = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1r} \\ c_{21} & c_{22} & \dots & c_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} & \dots & c_{nr} \end{bmatrix}.$$

Further, let $L_2^{-1}L_1 = V\Omega U^T$ be the singular value decomposition of $L_2^{-1}L_1$, where $\Omega = \mathrm{diag}(\omega_1,\dots,\omega_n) > 0$ and U and V are unitary matrices. Pre- and post-multiplying (2.1) by unitary matrices $\mathbf{W} = \mathrm{diag}(U,V)$ and $\mathbf{W}^T$, and by $\mathbf{P}$ and $\mathbf{P}^T$, respectively, where $\mathbf{P}$ is a perfect shuffle permutation, the Ljapunov equation (2.1) has the following form

$$(\mathbf{A}_0 - \varepsilon\mathbf{D})^T\mathbf{X} + \mathbf{X}(\mathbf{A}_0 - \varepsilon\mathbf{D}) = -\mathbf{B}. \tag{2.2}$$

In (2.2)

$$\mathbf{A}_0 = \begin{bmatrix} \Omega_1 & & & \\ & \Omega_2 & & \\ & & \ddots & \\ & & & \Omega_n \end{bmatrix},$$

$\Omega_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}$ and $\mathbf{D} = \mathbf{D}_0\mathbf{D}_0^T$, where

$$\mathbf{D}_0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ d_{11} & d_{12} & \dots & d_{1r} \\ 0 & 0 & \dots & 0 \\ d_{21} & d_{22} & \dots & d_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \\ d_{n1} & d_{n2} & \dots & d_{nr} \end{bmatrix},$$

and d_{ij} are entries of the matrix $D_0 = V^T L_2^{-1} C_0$. The solution $\mathbf{X}$ of (2.2) can be written as $\mathbf{X} = \mathbf{P}\mathbf{W}\widehat{\mathbf{X}}\mathbf{W}^T\mathbf{P}^T$. Similarly, $\mathbf{Z}$ can be written as $\mathbf{Z} = \mathbf{P}\mathbf{W}\widehat{\mathbf{Z}}\mathbf{W}^T\mathbf{P}^T$ (note that $Tr(\mathbf{Z}\mathbf{X}) = Tr(\widehat{\mathbf{Z}}\widehat{\mathbf{X}})$).

Thus, let

$$\mathbb{A} \equiv \mathbb{A}_\varepsilon = \mathbb{A}_0 - \varepsilon\mathbb{D}, \quad \varepsilon > 0$$

be the matrix of the linear operator defined on the space of square matrices by

$$\begin{aligned} \mathbf{X} &\mapsto -\mathbf{A}_0\mathbf{X} + \mathbf{X}\mathbf{A}_0 \\ \mathbf{X} &\mapsto \mathbf{D}\mathbf{X} + \mathbf{X}\mathbf{D}\,. \end{aligned}$$

Then we can write

$$\mathrm{vec}(\mathbf{X}) \equiv \mathrm{vec}(\mathbf{X})_\varepsilon = -\mathbb{A}^{-1}\,\mathrm{vec}(\mathbf{B})\,. \tag{2.3}$$

Using the Kronecker product of matrices we can write ([3, Theorem 12.3.1])

$$\begin{aligned} \mathbb{A}_0 &= -\mathbf{I}\otimes\mathbf{A}_0 + \mathbf{A}_0\otimes\mathbf{I} \\ \mathbb{D} &= \mathbf{I}\otimes\mathbf{D} + \mathbf{D}\otimes\mathbf{I}\,. \end{aligned} \tag{2.4}$$

Note that from (2.3) it follows that

$$\mathbf{X} = \left[\mathbf{x}_1,\ldots,\mathbf{x}_m\right]\,, \quad \text{where} \quad \mathbf{x}_i = -\mathbb{A}^{-1}_{((i-1)m+1:im,:)}\,\mathrm{vec}(\mathbf{B})\,. \tag{2.5}$$

Using the fact that $Tr(\mathbf{ZX}) = \mathrm{vec}(\mathbf{Z})^T\,\mathrm{vec}(\mathbf{X})$ it follows

$$Tr(\mathbf{ZX}) = -\,\mathrm{vec}(\mathbf{Z})^T\,(\mathbb{A}_0 - \varepsilon\mathbb{D})^{-1}\,\mathrm{vec}(\mathbf{B})\,. \tag{2.6}$$

Before we continue, we collect here some simple facts about the matrix pair $(\mathbb{A}_0, \mathbb{D})$. Their proofs are straightforward.

(i) $\mathbb{A}_0$ is skew-symmetric, $\mathbb{D}$ is symmetric and positive semidefinite.[1]

(ii) $\mathbb{A}_0 - \varepsilon\mathbb{D}$ is non-singular and dissipative for any $\varepsilon > 0$. Both $\mathbb{A}_0$ and $\mathbb{D}$ are singular but their null subspaces have zero intersection.

(iii) The eigenvalues of the pair $(\mathbb{A}_0, \mathbb{D})$ (including zero and infinity) are purely imaginary.

(iv) The orthogonal decomposition of the space of all matrices into subspaces of the symmetric and skew-symmetric matrices reduces the family $(\mathbb{A}_0 - \lambda\mathbb{D})$. Any of (i), (ii) and (iii) holds for the restriction of $(\mathbb{A}_0 - \lambda\mathbb{D})$ to any of these spaces.

(v) The operator

$$\mathbf{X} \;\mapsto\; (-\mathbf{A}_0 - \varepsilon\mathbf{D})\mathbf{X} + \mathbf{X}(\mathbf{A}_0 - \varepsilon\mathbf{D})$$

is positive in the sense that it maps the cone of positive definite matrices into itself.

We continue by deriving an explicit form of $(\mathbb{A}_0 - \varepsilon\mathbb{D})^{-1}$, that is, we describe how one can find an explicit form of the solution $\mathbf{X}$ as the rational function depends on ε. From now on we assume that $0 < \omega_1 < \omega_2 < \ldots < \omega_n$.

First, we determine the unitary matrix $\mathbb{U}$ such that

$$\mathbb{U}^T \mathbb{A}_0 \mathbb{U} = \begin{bmatrix} 0 & \\ & \widehat{\mathbb{A}}_0 \end{bmatrix}, \tag{2.7}$$

where $\mathbb{A}_0$ is a skew-symmetric matrix corresponds with linear operator

$$\mathbf{X} \mapsto -\mathbf{A}_0\mathbf{X} + \mathbf{X}\mathbf{A}_0\,. \tag{2.8}$$

and $\widehat{\mathbb{A}}_0$ is a non-singular block diagonal matrix defined by

$$\widehat{\mathbb{A}}_0 = \operatorname{diag}(\Xi_1, \ldots, \Xi_{m_2}) \quad \text{where} \quad \Xi_i = \begin{bmatrix} 0 & -\mu_i \\ \mu_i & 0 \end{bmatrix} \tag{2.9}$$

where $\pm i\mu_i$, $i = 1, \ldots, m_2$ are non-zero eigenvalues of matrix $\mathbb{A}_0$, that is, $0 \neq \mu_i = (\pm\omega_i) - (\pm\omega_j)$ for $i, j = 1, \ldots, n$ (see [3, Corollary 12.2.2]). Note that $m_2 = (m^2 - m)/2$.

Further, we write $\mathbb{D} = \mathbb{D}_F\, \mathbb{D}_F^T$, where $\mathbb{D}$ is defined by (2.4). Using [3, Proposition 12.1.2] it follows that

$$\mathbb{D}_F = \begin{bmatrix} I_m \otimes \mathbf{D}_0 & \mathbf{D}_0 \otimes I_m \end{bmatrix}. \tag{2.10}$$

Indeed

$$\begin{aligned} \mathbb{D}_F \mathbb{D}_F^T &= (I_m \otimes \mathbf{D}_0)\,(I_m \otimes \mathbf{D}_0^T) + (\mathbf{D}_0 \otimes I_m)\,(\mathbf{D}_0^T \otimes I_m) \\ &= I_m \otimes \mathbf{D}_0\,\mathbf{D}_0^T + \mathbf{D}_0\,\mathbf{D}_0^T \otimes I_m\,. \end{aligned}$$

Note that factor $\mathbb{D}_F$ is an $m^2 \times 2rm$ matrix.

Now, if we write $\mathbb{U} \equiv \begin{bmatrix} \mathbb{U}_0 & \mathbb{U}_0^{\perp} \end{bmatrix}$, then

$$\mathbb{A}_0 - \varepsilon \mathbb{D}_F \mathbb{D}_F^T = \begin{bmatrix} \mathbb{U}_0 & \mathbb{U}_0^{\perp} \end{bmatrix} \left(\begin{bmatrix} 0 & \\ & \widehat{\mathbb{A}}_0 \end{bmatrix} - \varepsilon \begin{bmatrix} \mathbb{D}_1 \mathbb{D}_1^T & \mathbb{D}_1 \mathbb{D}_2^T \\ \mathbb{D}_2 \mathbb{D}_1^T & \mathbb{D}_2 \mathbb{D}_2^T \end{bmatrix} \right) \begin{bmatrix} \mathbb{U}_0^T \\ (\mathbb{U}_0^{\perp})^T \end{bmatrix}, \tag{2.11}$$

where

$$\mathbb{D}_1 = \mathbb{U}_0^T \mathbb{D}_F \in \mathbb{R}^{m \times 2rm} \quad \text{and} \quad \mathbb{D}_2 = (\mathbb{U}_0^{\perp})^T \mathbb{D}_F \in \mathbb{R}^{(m^2 - m) \times 2rm}\,. \tag{2.12}$$

Taking

$$\mathbb{G} = \begin{bmatrix} I & 0 \\ -\mathbb{D}_2 \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} & I \end{bmatrix} \tag{2.13}$$

we obtain

$$\mathbb{A}_0 - \varepsilon \mathbb{D}_F \mathbb{D}_F^T = \mathbb{U}\mathbb{G}^{-1} \begin{bmatrix} -\varepsilon \mathbb{D}_1 \mathbb{D}_1^T & \\ & \widetilde{\mathbb{A}} \end{bmatrix} \mathbb{G}^{-T} \mathbb{U}^T, \tag{2.14}$$

where
$$\widetilde{\mathbb{A}} = \widehat{\mathbb{A}}_0 - \varepsilon \mathbb{D}_2 (I - \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} \mathbb{D}_1) \mathbb{D}_2^T . \tag{2.15}$$
Now, we compute matrix $\mathbb{F} \in \mathbb{R}^{(m^2-m)\times 2(r-1)m}$, such that
$$\mathbb{D}_2 (I - \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} \mathbb{D}_1) \mathbb{D}_2^T = \mathbb{F}\mathbb{F}^T ,$$
as follows. Let
$$\mathbb{D}_1^T = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} \widetilde{R} \\ 0 \end{bmatrix}$$
be the QR decomposition of $\mathbb{D}_1^T$.Then the orthogonality of $\begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ can be expressed as
$$Q_1 Q_1^T + Q_2 Q_2^T = I ,$$
which further implies
$$I - \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} \mathbb{D}_1 = Q_2 Q_2^T ,$$
Then, matrix $\mathbb{F}$ is defined by
$$\mathbb{F} = \mathbb{D}_2 Q_2 . \tag{2.16}$$

Further, we have to find the inverse of $\widetilde{\mathbb{A}} = \widehat{\mathbb{A}}_0 - \varepsilon \mathbb{F}\mathbb{F}^T$. This is obtained by using the *Sherman-Morrison-Woodbury formula* ([2, (2.1.4),pg.51.]), that is,
$$(\widehat{\mathbb{A}}_0 - \varepsilon\, \mathbb{F}\mathbb{F}^T)^{-1} = \widehat{\mathbb{A}}_0^{-1} + \varepsilon \widehat{\mathbb{A}}_0^{-1} \mathbb{F} \left(I - \varepsilon \mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F} \right)^{-1} \mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} . \tag{2.17}$$
The inverse of $I - \varepsilon \mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F}$ remains to be found. Let
$$\Lambda = \operatorname{diag} \left(\begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{bmatrix}, \ldots., \begin{bmatrix} 0 & \lambda_s \\ -\lambda_s & 0 \end{bmatrix} \right) ,$$
where $\pm i\lambda_1$, $\pm i\lambda_2$,..., $\pm i\lambda_s$ are non-vanishing finite eigenvalues of the problem
$$(\mathbb{A}_0 - \lambda \mathbb{D}) \operatorname{vec}(\mathbf{Y}) = 0.$$
Since $\mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F}$ is skew-symmetric, then there exists an orthogonal matrix U_S of order $2(r-1)m$ such that
$$U_S^T \mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F} U_S = \begin{bmatrix} 0 & 0 \\ 0 & \Gamma \end{bmatrix} , \tag{2.18}$$
where $\Gamma = \Lambda^{-1}$.

Using (2.14), (2.17), (2.18) and (2.3) it follows

$$\operatorname{vec}(\mathbf{X}) = -\mathbb{U}\mathbb{G}^T \begin{bmatrix} \Delta_1 & \\ & \Delta_2 \end{bmatrix} \mathbb{G}\mathbb{U}^T \operatorname{vec}(\mathbf{B}) , \tag{2.19}$$

where

$$\Delta_1 = -\frac{(\mathbb{D}_1\mathbb{D}_1^T)^{-1}}{\varepsilon}, \quad \Delta_2 = \widehat{\mathbb{A}}_0^{-1} + \varepsilon\widehat{\mathbb{A}}_0^{-1}\mathbb{F}U_S \begin{bmatrix} I & 0 \\ & (I-\varepsilon\Gamma)^{-1} \end{bmatrix} U_s^T\mathbb{F}^T\widehat{\mathbb{A}}_0^{-1}.$$

Let

$$\mathbb{T} \equiv \begin{bmatrix} \mathbb{T}_1 & \mathbb{T}_2 \end{bmatrix} = \widehat{\mathbb{A}}_0^{-1}\mathbb{F}U_S , \tag{2.20}$$

be an $(m^2-m)\times(2r-1)m$ matrix, decomposed so that $\mathbb{T}_1$ has s_0 and $\mathbb{T}_2$ has $2s$ columns (Γ is a $2s\times 2s$ matrix, and $2s+s_0=(2r-1)m$). Also let

$$\mathbb{U}\mathbb{G}^T \equiv \begin{bmatrix} \mathbb{U}_0 & \widehat{\mathbb{U}} \end{bmatrix}, \quad \text{where} \quad \widehat{\mathbb{U}} = -\mathbb{U}_0(\mathbb{D}_1\mathbb{D}_1^T)^{-T}\mathbb{D}_1\mathbb{D}_2^T + \mathbb{U}_0^{\perp} . \tag{2.21}$$

Then (2.19), (2.20) and (2.21) imply

$$\operatorname{vec}(\mathbf{X}) = \left(\frac{\mathbb{V}_{-1}}{\varepsilon} - \mathbb{V}_0 - \varepsilon\,\mathbb{T}_1\mathbb{T}_1^T + \varepsilon\left(\mathbb{T}_2(I-\varepsilon\Gamma)^{-1}\mathbb{T}_2^T\right)\right)\operatorname{vec}(B) , \tag{2.22}$$

where

$$\mathbb{V}_0 = \widehat{\mathbb{U}}\widehat{\mathbb{A}}_0^{-1}\widehat{\mathbb{U}}^T \quad \text{and} \quad \mathbb{V}_{-1} = -\mathbb{U}_0(\mathbb{D}_1\mathbb{D}_1^T)^{-1}\mathbb{U}_0^T . \tag{2.23}$$

Further, note that

$$(I-\varepsilon\Gamma)^{-1} = \operatorname{diag}\left(\frac{1}{\lambda_1^2+\varepsilon^2}\begin{bmatrix} \lambda_1^2 & -\varepsilon\lambda_1 \\ \varepsilon\lambda_1 & \lambda_1^2 \end{bmatrix}, ..., \frac{1}{\lambda_s^2+\varepsilon^2}\begin{bmatrix} \lambda_s^2 & -\varepsilon\lambda_s \\ \varepsilon\lambda_s & \lambda_s^2 \end{bmatrix}\right) . \tag{2.24}$$

Finally, if we write

$$\mathbb{T}_2 = \begin{bmatrix} \mathbf{t}_1, ..., \mathbf{t}_s \end{bmatrix} , \tag{2.25}$$

where $\mathbf{t}_i = \begin{bmatrix} \mathbf{t}_i^1 & \mathbf{t}_i^2 \end{bmatrix}$ contains two columns of $\mathbb{T}_2$, then from (2.22) follows

$$\operatorname{vec}(\mathbf{X}) = \left(\frac{\mathbb{V}_{-1}}{\varepsilon} - \mathbb{V}_0 - \varepsilon\widehat{\mathbb{V}}_1 + \varepsilon\sum_{i=1}^{s}\frac{\lambda_i\left(\lambda_i\mathbb{W}_i - \varepsilon\mathbb{Z}_i\right)}{\lambda_i^2+\varepsilon^2}\right)\operatorname{vec}(\mathbf{B}) , \tag{2.26}$$

where

$$\mathbb{W}_i = \mathbf{t}_i^1(\mathbf{t}_i^1)^T + \mathbf{t}_i^2(\mathbf{t}_i^2)^T, \quad \mathbb{Z}_i = \mathbf{t}_i^1(\mathbf{t}_i^2)^T - \mathbf{t}_i^2(\mathbf{t}_i^1)^T , \quad \widehat{\mathbb{V}}_1 = \mathbb{T}_1\mathbb{T}_1^T .$$

By "reshaping" vectors in (2.26) into $m\times m$ matrices we obtain the solution of equation (2.2)

$$\mathbf{X} = \frac{\mathbf{\Psi}_{-1}}{\varepsilon} - \mathbf{\Psi}_0 - \varepsilon\widehat{\mathbf{\Psi}}_1 + \varepsilon\sum_{i=1}^{s}\frac{\lambda_i\left(\lambda_i\mathbf{\Phi}_i - \varepsilon\mathbf{\Upsilon}_i\right)}{\lambda_i^2+\varepsilon^2} . \tag{2.27}$$

On the other hand, from (2.6) it follows

$$Tr(\mathbf{ZX}) = \frac{X_{-1}}{\varepsilon} - X_0 - \varepsilon \widehat{X}_1 + \varepsilon \sum_{i=1}^{s} \frac{\lambda_i\,(\lambda_i X_i - \varepsilon Y_i)}{\lambda_i^2 + \varepsilon^2}\,, \tag{2.28}$$

where

$$\begin{aligned} X_{-1} &= \mathrm{vec}(\mathbf{Z})^T \mathbb{V}_{-1}\,\mathrm{vec}(\mathbf{B})\,, \\ X_0 &= \mathrm{vec}(\mathbf{Z})^T \mathbb{V}_0\,\mathrm{vec}(\mathbf{B})\,, \\ \widehat{X}_1 &= \mathrm{vec}(\mathbf{Z})^T \widehat{\mathbb{V}}_{-1}\,\mathrm{vec}(\mathbf{B})\,, \end{aligned} \tag{2.29}$$

and

$$X_i = \mathrm{vec}(\mathbf{Z})^T \mathbb{W}_i\,\mathrm{vec}(\mathbf{B})\,, \quad Y_i = \mathrm{vec}(\mathbf{Z})^T \mathbb{Z}_i\,\mathrm{vec}(\mathbf{B})\,. \tag{2.30}$$

Note that the function $Tr(\mathbf{ZX})$ from (2.28) is a generalization of function $Tr(\mathbf{X})$ presented in [7], and it is easily minimized.

As we will see in the following section the coefficients X_i and Y_i in (2.28) cannot be easily derived. Since most of matrices which appear in the construction of coefficients X_i and Y_i have $O(m^4)$ or $O(m^3)$ entries, we have to use the structure of our original problem to avoid forming of any matrix which has more than $O(m^2)$ entries and to make all calculations (mostly matrix multiplications) in $O(m^3)$ flops.

REMARK **2.1** *Here we derive an explicit formula for matrix $\mathbf{\Psi}_{-1}$. From (2.26) and (2.23) it follows that*

$$\mathrm{vec}(\mathbf{\Psi}_{-1}) = \mathbb{V}_{-1}\ \mathrm{vec}(\mathbf{B}) \equiv -\mathbb{U}_0\,(\mathbb{D}_1 \mathbb{D}_1^T)^{-1} \mathbb{U}_0^T\ \mathrm{vec}(\mathbf{B}) \tag{2.31}$$

As we will see in STEP 1 *from the Section 3 i-th column of the matrix $\mathbb{U}_0$ can be written as $\mathrm{vec}(\mathbf{O}_i)$, where $\mathbf{O_i}$ is a block-diagonal matrix and*

$$\begin{aligned} \mathbf{O}_{2i-1} &= \mathrm{diag}(0,\ldots,0,O_{ii},0,\ldots,0) \\ \mathbf{O}_{2i} &= \mathrm{diag}(0,\ldots,0,\widehat{O}_{ii},0,\ldots,0) \end{aligned}$$

where O_{ii} and $\widehat{O}_{ii}$ are "orthogonal" solutions of

$$-\Omega_i O_{ij} + O_{ij}\Omega_j = 0 \qquad i,j = 1,\ldots,n.$$

Let $\star$ denote the matrix operation between two $2{\times}2$ matrices defined by the "point product" (. in MATLAB) and then sum of all four elements. Now, we can write the (p,q)-th element of $\mathbb{U}_0^T \mathbb{D}_F \mathbb{D}_F^T \mathbb{U}_0$ as*

$$(\mathbb{U}_0)_{(:,p)}^T \mathbb{D}_F \mathbb{D}_F^T (\mathbb{U}_0)_{(:,q)} = (\mathbf{D_{pq}}\,\mathbf{O}_p + \mathbf{O}_p\,\mathbf{D_{pq}}) \star \mathbf{O}_q \quad p,q = 1,\ldots,m\,,$$

where $\mathbf{D_{pq}} = (D_{pq})$ *and*

$$D_{pq} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{D}_{(p,:)}\mathbf{D}^T_{(q,:)} \end{bmatrix} .$$

Now from the fact that

$$\mathbf{O}_p \star \mathbf{O}_q = \delta_{pq} ,$$

(see STEP 1 *from the Section 3) where* δ_{pq} *is Kronecker, it follows*

$$\left(\mathbb{U}_0^T \mathbb{D}_F \mathbb{D}_F^T \mathbb{U}_0\right)^{-1} = \mathrm{diag}(1/(\mathbf{D})_{22}, 1/(\mathbf{D})_{22}, \ldots, 1/(\mathbf{D})_{mm}, 1/(\mathbf{D})_{mm}) .$$

This together with (2.31) implies

$$\mathrm{vec}(\boldsymbol{\Psi}_{-1}) = -\mathbb{U}_0 \left[\frac{b_1}{\mathbf{D}_{22}}\ \frac{b_2}{\mathbf{D}_{22}}\ \ldots, \frac{b_{m-1}}{\mathbf{D}_{mm}}\ \frac{b_m}{\mathbf{D}_{mm}}\right]^T$$

and

$$\boldsymbol{\Psi}_{-1} = \left[\frac{b_1}{\mathbf{D}_{22}}\, u_1\, , \frac{b_2}{\mathbf{D}_{22}}\, u_2\, , \ldots, \frac{b_{m-1}}{\mathbf{D}_{mm}}\, u_{m-1}\, , \frac{b_m}{\mathbf{D}_{mm}}\, u_m\right] .$$

where $\mathbb{U}_0 = \left[u_1, u_2, \ldots, u_m\right]$.

3. The algorithm

The whole procedure of constructing coefficients X_0, X_1 and $\widehat{X}_1$ defined by (2.29), and X_i and Y_i defined by (2.30) is long and tedious. Thus, we are going to describe only the most important parts of the whole procedure for the construction of the function $Tr(\mathbf{ZX})$ from (2.28).

STEP 1. As it has been shown in (2.7), first we have to determine the unitary matrix $\mathbb{U}$ such that

$$\mathbb{U}^T \mathbb{A}_0 \mathbb{U} = \begin{bmatrix} 0 & \\ & \widehat{\mathbb{A}}_0 \end{bmatrix} .$$

Our algorithm for deriving a zero-subspace of matrix $\mathbb{A}_0$ in Matlab notation has the following form:

```
for k=2:2:m
U0(m*(k-2)+(k-1),(k-1))=1/sqrt(2);
U0(m*(k-1)+k,(k-1))=1/sqrt(2);
U0(m*(k-2)+k,k)=-1/sqrt(2);
U0(m*(k-1)+(k-1),k)=1/sqrt(2);
 end
```

where the columns of `U0` span a zero-subspace of matrix $\mathbb{A}_0$.

The rest of matrix $\mathbb{U}$ can be constructed using an algorithm similar to the one for $\mathbb{U}_0$. The number of flops in STEP 2 is $O(m^2)$.

STEP 2. Let

$$y_L \equiv \begin{bmatrix} y_L^1 \\ y_L^2 \end{bmatrix} = \mathbb{G}\mathbb{U}^T \operatorname{vec}(\mathbf{Z}) \quad \text{and} \quad y_R \equiv \begin{bmatrix} y_R^1 \\ y_R^2 \end{bmatrix} = \mathbb{G}\mathbb{U}^T \operatorname{vec}(\mathbf{B}), \tag{3.1}$$

be vectors needed in the construction of coefficients from (2.29) and (2.30). In (3.1) y_L^1 (y_R^1) is an m and y_L^2 (y_R^2) is an $m^2 - m$ dimensional vector. Let us define

$$u_L = \mathbb{U}^T \operatorname{vec}(\mathbf{Z}) \quad \text{and} \quad u_R = \mathbb{U}^T \operatorname{vec}(\mathbf{B})$$

then by using (2.13) and (3.1) we obtain

$$y_L^1 = u_{L(:,1:m)} \quad \text{and} \quad y_R^1 = u_{R(:,1:m)} \tag{3.2}$$

and

$$\begin{aligned} y_L^2 &= -\mathbb{D}_2\mathbb{D}_1^T(\mathbb{D}_1\mathbb{D}_1^T)^{-1}u_{L(:,1:m)} + u_{L(:,m+1:m^2)} \\ y_R^2 &= -\mathbb{D}_2\mathbb{D}_1^T(\mathbb{D}_1\mathbb{D}_1^T)^{-1}u_{R(:,1:m)} + u_{R(:,m+1:m^2)}. \end{aligned} \tag{3.3}$$

In this step we calculate the vectors u_L and u_R. Using the structure of matrix $\mathbb{U}$ described in STEP 1 we can derive vectors u_L (u_R) in $O(m^2)$ operations.

STEP 3. We proceed with a description of the algorithm which calculates the matrix $\mathbb{D}_1$ defined in (2.12). Note that, in definition of $\mathbb{D}_1 = \mathbb{U}_0^T\mathbb{D}_F$ both matrices $\mathbb{U}_0^T$ and $\mathbb{D}_F$ have $O(m^3)$ entries. Thus, we have to avoid constructing any of them.

Here we have to handle the problem of how to multiply two matrices, without constructing any of them, and without useing a "sparse" procedure from MATLAB or any other known program package for "sparse" matrices (unfortunately, it appears that "sparse" procedures are not satisfactory here).

This problem (and other similar forthcoming problems of the product of two other more "structured" matrices) have been solved by using computer algebra. As we have already mentioned in Section 1, we do not have proofs for our formulae which are obtained heuristically by analyzing dimensions $n = 2, 3, 4, 5$ and then guessing for general n. Unfortunately, these programs for multiplication of the "structured" matrices (apart from a few which will be presented here) are too big and clumsy to be handled, thus they will not be presented here. However, the whole MATLAB code is available from the first author upon request.

According to the upper considerations, we have to point out that only proof which we have for this part of our program (multiplications of two sparse matrices) is purely heuristic.

The following algorithm calculates $\mathbb{D}_1$:

```
for i=1:n
dda1(2*i-1, (2*i-1)*r+(1:r)) =(1/sqrt(2))*c(i,:);
dda1(2*i,2*(i-1)*r+(1:r)) =(-1/sqrt(2))*c(i,:);
end

for i=1:n
for j=1:r
dda2(2*i,(j-1)*m + (2*i-1)) =(1/sqrt(2))*c(i,j);
dda2(2*i-1, ((j-1)*m)+ 2*i ) =(1/sqrt(2))*c(i,j);
end
end
d1=[dda1, dda2];
```

where `d1`$=\mathbb{D}_1$ and `c(i,j)` $= c_{ij}$. From the above algorithm follows that for $\mathbb{D}_1$ we need $O(rm)$ operations.

STEP 4. In this step we derive a skew-symmetric matrix $\mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F}$, where $\mathbb{F}$ and $\widehat{\mathbb{A}}_0^{-1}$ are defined in (2.16) and (2.9), respectively.

This algorithm is not as simple as the one from the above steps, thus we will not write it here.

For this step we need $\underline{O(r^3m^3)}$ operations.

STEP 5. In this step we calculate the eigenvalue decomposition of a skew-symmetric matrix $\mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F}$ defined by (2.18).

For this purpose, we first reduce $\mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F}$ to the tridiagonal form using Hausholder tridiagonalization ([2, Algorithm 7.7.3]) which costs us $\underline{8/3(2r-1)^3m^3}$ operations. Further, let

$$W^T \mathbb{F}^T \widehat{\mathbb{A}}_0^{-1} \mathbb{F} W = H_s \,,$$

where H_s is a tridiagonal matrix with zeros on the main diagonal. We make permutation so that

$$P_S^T H_S P_S = \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix} ,$$

where B is bidiagonal.

Finally, we calculate SVD of matrix B, which additionally costs us $O((2r-1)^2m^2)$ operations. If we write

$$B = U_b \Sigma_B V_B^T ,$$

where U_B and V_B are unitary and Σ_B is diagonal, then (2.18) holds with

$$U_S = W P_S \begin{bmatrix} U_B & 0 \\ 0 & V_B \end{bmatrix} P_2^T ,$$

where P_2 is a proper permutation. The last matrix multiplications cost us a $\underline{2(2r-1)^3m^3}$ operations.

STEP 6. We proceed with forming vectors y_L and y_R from (3.1).

As we can see from (3.2), in definition of vectors y_L^2 and y_R^2 there exists matrix $\mathbb{D}_2$, whose implicit construction has to be avoided. Thus, again we have to find the way to derive vectors y_L^2 and y_R^2 "directly". For this purpose consider y_L^2 (y_R^2 is obtained similarly). Note that

$$y_L^2 = -(\mathbb{U}_0^\perp)^T \mathbb{D}_F \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} u_{L(:,1:m)} + u_{L(:,m+1:m^2)}.$$

If we write $d_{FL} = \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} u_{L(:,1:m)}$ by using (2.10) it is easy to obtain the algorithm which derives $\widehat{d}_{FL} = \mathbb{D}_F \mathbb{D}_1^T (\mathbb{D}_1 \mathbb{D}_1^T)^{-1} u_{L(:,1:m)}$. Indeed, let `dFL` $= d_{FL}$ and `dQ` $= \widehat{d}_{FL}$, then the following algorithm calculates $\widehat{d}_{FL}$:

```
j=1:n;
k=1:r;
for i=1:m
C1((i-1)*m+2*j,:)=c(j,k)*dFL((i-1)*r+k,:);
end
i1=1:n;
for j1=1:m
C2((2*i1-1)*m+j1,:)=c(i1,k)*dFL(r*m+(k-1)*m+j1,:);
end
dQ=C1+C2;
```

The number of operation in the above algorithm does not exceed $O(m^2)$. Further, we proceed with the computation of vector $(\mathbb{U}_0^\perp)^T \widehat{d}_{FL}$. This can be done using a part of the algorithm described in STEP 2. This algorithm need under the $O(m^2)$ operations.

STEP 7. In this step we construct coefficients X_0, X_1 and $\widehat{X}_1$ defined by (2.29), and X_i and Y_i defined by (2.30).

Whole number of operations needed in this step is $O(r^2m^2)$.

Now we have all necessary elements for construction of $Tr(\mathbf{ZX})$ from (2.28). The number of operations needed for construction of function $Tr(\mathbf{ZX})$ is $O(r^3m^3)$.

The function $Tr(\mathbf{ZX})$ in general can be very complicated. The following example shows that it is not always convex (this was expected from some simple known examples).

Example 3.1 *Consider*

$$M\ddot{x} + C\dot{x} + Kx = 0$$

where

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}, \quad C = C_0C_0^*,$$
$$\text{and} \;\; C_0 = \begin{bmatrix} -0.021663 & 0.52927 \\ 0.017933 & 0.63936 \end{bmatrix}.$$

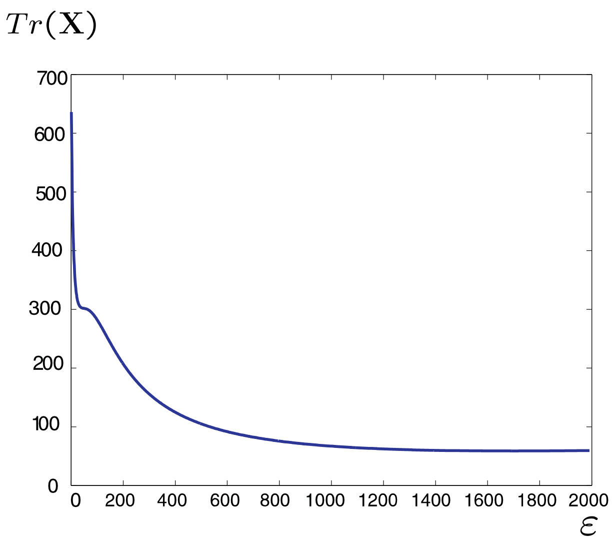

Figure 2. Non-convex function

Notes

1. We assume the space of all real matrices to be equipped with the usual Euclidian scalar product.

References

[1] R.H. Bartels and G.W. Stewart. A solution of the matrix equation AX + XB = C. *Comm. ACM* 15:820–826, (1972).

[2] G.H. Golub and Ch.F. van Loan. *Matrix computations.* J. Hopkins University Press, Baltimore, 1989.

[3] P. Lancaster and M. Tismenetsky. *The Theory of Matrices.* Academic Press, New York 1985.

[4] P.C. Müller and W.O. Schiehlen. *Lineare Schwingungen.* Akademische Verlagsgesellschaft Wiesbaden 1976.

[5] K. Veselić, K. Brabender and K. Delinić. Passive control of linear systems. *Applied Mathematics and Computation, M. Rogina et al. Eds. Dept. of Math. Univ. Zagreb.* 39-68, 2001.

[6] K. Veselić. On linear vibrational systems with one dimensional damping. *Appl. Anal.* 29:1–18, 1988.

[7] K. Veselić. On linear vibrational systems with one dimensional damping II. *Integral Eq. Operator Th.* 13:883-897, 1990.