ON WEAKLY FORMULATED SYLVESTER EQUATIONS AND APPLICATIONS

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ABSTRACT. We use a "weakly formulated" Sylvester equation

$$\mathbf{H}^{1/2}T\mathbf{M}^{-1/2} - \mathbf{H}^{-1/2}T\mathbf{M}^{1/2} = F$$

to obtain new bounds for the rotation of spectral subspaces of a nonnegative selfadjoint operator in a Hilbert space. Our bound extends the known results of Davis and Kahan. Another application is a bound for the square root of a positive selfadjoint operator which extends the known rule: "The relative error in the square root is bounded by the one half of the relative error in the radicand". Both bounds are illustrated on differential operators which are defined via quadratic forms.

1. PRELIMINARIES

In this work we will study properties of nonnegative selfadjoint operators in a Hilbert space which are close in the sense of the inequality

$$|h(\phi,\psi) - m(\phi,\psi)| \le \eta\sqrt{h[\phi]m[\psi]} \tag{1.1}$$

where the sesquilinear forms h, m belong to the operators $\mathbf{H}, \mathbf{M}$ respectively and $m[\psi] = m(\psi,\psi)$, $h[\phi] = h(\phi,\phi)$. By $\mathcal{Q}(h)$ we denote the *domain space* of a sesquilinear form h and in (1.1) we assume that $\mathcal{Q}(h) = \mathcal{Q}(m)$.

In the first part of the paper we show that (1.1) implies an estimate of the separation between "matching" eigensubspaces of $\mathbf{H}$ and $\mathbf{A}$. To be more precise one of the typical situations is: Let

$$0 \le \lambda_1(\mathbf{H}) \le \lambda_2(\mathbf{H}) \le \cdots \le \lambda_n(\mathbf{H}) < D < \lambda_{n+1}(\mathbf{H}) \le \cdots \tag{1.2}$$

$$0 \le \lambda_1(\mathbf{M}) \le \lambda_2(\mathbf{M}) \le \cdots \le \lambda_n(\mathbf{M}) < D < \lambda_{n+1}(\mathbf{M}) \le \cdots \tag{1.3}$$

be the eigenvalues of the operators $\mathbf{H}$ and $\mathbf{M}$ which satisfy (1.1) then

$$\|E_{\mathbf{H}}(D) - E_{\mathbf{M}}(D)\| \le \min\left\{\frac{\sqrt{D\lambda_n(\mathbf{H})}}{D-\lambda_n(\mathbf{H})}, \frac{\sqrt{D\lambda_n(\mathbf{M})}}{D-\lambda_n(\mathbf{M})}\right\}\eta.$$

Such an estimate was implicit in [7]. We then generalize this inequality to hold both for the operator norm $\|\cdot\|$ and the Hilbert–Schmidt norm $\|\cdot\|_{HS}$. We also allow that $E_{\mathbf{H}}(D)$ and $E_{\mathbf{M}}(D)$ be possibly infinite dimensional. For recent estimates of the separation between eigensubspaces see [10].

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In the second part of the paper we establish estimates for a perturbation of the square root of a positive operator. It will be shown that the inequality (1.1) implies

$$|h_2(\phi,\psi) - m_2(\phi,\psi)| \leq \frac{\eta}{2}\sqrt{h_2[\phi]m_2[\psi]},$$

where the sesquilinear forms h_2, m_2 belong to the operators $\mathbf{H}^{1/2}, \mathbf{M}^{1/2}$, respectively. This will show that it is meaningful to consider weakly formulated Sylvester equations where all the coefficient operators are unbounded, cf. (1.4).

Both of this problems will be solved through a study of the weak Sylvester equation, which reads formally

$$\mathbf{H}T - T\mathbf{M} = \mathbf{H}^{1/2}F\mathbf{M}^{1/2}. \tag{1.4}$$

These two case studies represent two different classes of additional assumptions which have to be imposed on the coefficient operators $\mathbf{H}$, $\mathbf{M}$ and F in order that (1.4) defines a meaningful operator T.

The main novelty (and contribution) of this work is that we present an abstract study of the operator equation (1.4) in the case when only F is a *bona fide* operator. The expression $\mathbf{H}^{1/2}F\mathbf{M}^{1/2}$ need not possess an operator representation. In comparison, $\mathbf{H}^{1/2}F\mathbf{M}^{1/2}$ was always a bounded operator for the Sylvester equations which were studied in [1, 2, 12]. Most recent and most general result of this type in *the case of matrix coefficients* is [12, Theorem 1] which reads

> *Let M and H be positive semi definite (finite) matrices such that the intersection of their spectra is empty. Then the solution T of (1.4) satisfies*
>
> $$\|\!| T \|\!| \leq \frac{\pi}{2} \frac{\|\!| F \|\!|}{\min\{|\ln \lambda/\mu| \ : \ \lambda \in \sigma(H), \mu \in \sigma(M)\}}$$
>
> *Here $\sigma(H)$ and $\sigma(M)$ denote the spectra of H and M and $\|\!| \,.\, \|\!|$ is any unitary invariant matrix norm.*

We consider a very general class of (unbounded) operator coefficients for the weak Sylvester equation. In order to regularize the problem we need to impose more stringent conditions (as compared with those in the result we have just stated) on the location of $\sigma(\mathbf{H})$ and $\sigma(\mathbf{M})$ or on the unitary invariant norm $\|\!| \cdot \|\!|$, see Theorems 2.1, 2.4, 2.7, 2.8 and 5.1 below. It should be noted that in the matrix case and in the situation in which all of these results apply their numerical performance is comparable, cf. [12].

A main technique which led to [12, Theorem 1] is the inequality

$$\|\!| \ln(\mathbf{H})T - T\ln(\mathbf{M}) \|\!| \leq \|\!| \mathbf{H}^{1/2}T\mathbf{M}^{-1/2} - \mathbf{H}^{-1/2}T\mathbf{M}^{1/2} \|\!| = \|\!| F \|\!| \tag{1.5}$$

This deep result from [9] can unfortunately only be assumed as formally correct in our setting since the products $\ln(\mathbf{H})T$ and $T\ln(\mathbf{M})$ do not have to be bona fide operators. To some extent it could be said that the main novelty in this work is a form theoretic approach to the problem of regularizing the equation (1.4).

More specifically, our first main result—contained in Theorem 2.1 below—extends our previous result from [7] in various ways. In particular, we allow the perturbed projection to be infinite dimensional. In the proof we also overcome a technical error contained in [7]. We then extend this result to the case of other unitary invariant operator norms[1]. Particular attention is paid to the Hilbert–Schmidt norm because of its possible importance in applications. This special case is handled by another technique which allows an arbitrary interlacing of the involved spectra.

1.1. **Notation and Lemmata.** The main object in this work shall be a closed nonnegative symmetric form in a Hilbert space. When dealing with symmetric forms in a Hilbert space, we shall follow the terminology of Kato, cf. [8]. For reader's convenience we now give definitions of some terms that will frequently be used, cf. [3, 8].

Definition 1.1. Let h be a positive definite symmetric form in $\mathcal{H}$. A sesquilinear form a, which need not be closed, is said to be *h-bounded*, if $\mathcal{Q}(h) \subset \mathcal{Q}(a)$ and there exists $\eta \geq 0$

$$|a[u]| \leq \eta h[u] \qquad u \in \mathcal{Q}(h).$$

If h is positive definite the space $(\mathcal{Q}(h), h)$ can be considered as a Hilbert space and h-bounded form a defines a bounded operator on $(\mathcal{Q}(h), h)$.

Definition 1.2. A bounded operator $A : \mathcal{H} \to \mathcal{U}$ is called *degenerate* if its *range space* $\mathsf{R}(A) := \{Au \ : \ u \in \mathcal{H}\}$ is finite dimensional.

Definition 1.3. If $\mathbf{H}$ is a selfadjoint operator and P a projection, to say that P commutes with $\mathbf{H}$ means that $u \in \mathcal{D}(\mathbf{H})$ implies $Pu \in \mathcal{D}(\mathbf{H})$ and

$$\mathbf{H}Pu = P\mathbf{H}u, \qquad u \in \mathcal{D}(\mathbf{H}).$$

Definition 1.4. Let $\mathbf{H}$ and $\mathbf{M}$ be nonnegative operators. We define the *order relation* $\leq$ between the nonnegative operators by saying that $\mathbf{M} \leq \mathbf{H}$ if and only if $\mathcal{D}(\mathbf{H}^{1/2}) \subset \mathcal{D}(\mathbf{M}^{1/2})$ and

$$\|\mathbf{M}^{1/2}u\| \leq \|\mathbf{H}^{1/2}u\|, \qquad u \in \mathcal{D}(\mathbf{H}^{1/2}),$$

or equivalently $m[u] \leq h[u]$, $u \in \mathcal{Q}(h) := \mathcal{D}(\mathbf{H}^{1/2})$, when m and h are nonnegative forms defined by the operators $\mathbf{M}$ and $\mathbf{H}$ and $\mathbf{M} \leq \mathbf{H}$.

As a notational convention we use normal math-script letters (e.g. M) to denote bounded operators and matrices and boldface math-script letters (e.g. $\mathbf{H}$) to denote unbounded operators.

A main principle we shall use to develop the perturbation theory will be the *monotonicity of the spectrum* with regard to the order relation between nonnegative operators. This principle can be expressed in many ways. The relevant results, which are scattered over the monographs [3, 8], are summed up in the following theorem, see also [11, Corollary A.1].

Theorem 1.5. *Let $\mathbf{M} = \int \lambda \ dE_{\mathbf{M}}(\lambda)$ and $\mathbf{H} = \int \lambda \ dE_{\mathbf{H}}(\lambda)$ be nonnegative operators in $\mathcal{H}$ and let $\mathbf{M} \leq \mathbf{H}$. Let the eigenvalues of $\mathbf{H}$ and $\mathbf{M}$ be as in (1.2) and (1.3) then*

(1) $\lambda_e(\mathbf{M}) \leq \lambda_e(\mathbf{H})$
(2) $\mathsf{dim}\ E_{\mathbf{H}}(\gamma) \leq \mathsf{dim}\ E_{\mathbf{M}}(\gamma)$, *for every* $\gamma \in \mathbb{R}$
(3) $\lambda_k(\mathbf{M}) \leq \lambda_k(\mathbf{H})$, $\qquad k = 1, 2, \cdots$.

The infimum of the essential spectrum of some operator $\mathbf{H}$ is denoted by $\lambda_e(\mathbf{H})$.

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[1] Also called "cross-norms" in the terminology of [8] or "symmetric norms" in the terminology of [4, 17].

With this theorem in hand we review spectral properties of operators $\mathbf{H}$ and $\mathbf{M}$, for which there exists $0 \leq \varepsilon < 1$ such that

$$(1 - \varepsilon)m[u] \leq h[u] \leq (1 + \varepsilon)m[u], \qquad u \in \mathcal{Q} := \mathcal{Q}(h) = \mathcal{Q}(m). \tag{1.6}$$

Let us assume $h[u] > 0$ then $m[u] > 0$ and

$$(1 - \frac{\varepsilon}{1-\varepsilon})h[u] \leq m[u] \leq (1 + \frac{\varepsilon}{1-\varepsilon})h[u]. \tag{1.7}$$

Inequality (1.6) implies that $\mathsf{N}(\mathbf{H}) = \mathsf{N}(\mathbf{M})$, so (1.7) holds for all $u \in \mathcal{Q}$. By $\mathsf{N}(\mathbf{H})$ we denote the *null space* of some operator $\mathbf{H}$.

Lemma 1.6. *Let m and h be nonnegative forms such that $\lambda_e(\mathbf{M}) > 0$ and $\lambda_e(\mathbf{H}) > 0$ and let (1.6) hold. Then*

$$|\lambda_i(\mathbf{H}) - \lambda_i(\mathbf{M})| \leq \varepsilon \lambda_i(\mathbf{M}) \tag{1.8}$$

$$|\lambda_i(\mathbf{H}) - \lambda_i(\mathbf{M})| \leq \frac{\varepsilon}{1-\varepsilon}\lambda_i(\mathbf{H}) \tag{1.9}$$

$\lambda_i(\mathbf{H})$ and $\lambda_i(\mathbf{M})$ are as in (1.2) and (1.3). Assume that $\lambda_{i-1}(\mathbf{H}) < \lambda_i(\mathbf{H}) < \lambda_{i+1}(\mathbf{H})$ and

$$\frac{\varepsilon}{1-\varepsilon} < \max\Big\{\frac{\lambda_{i+1}(\mathbf{H}) - \lambda_i(\mathbf{H})}{\lambda_{i+1}(\mathbf{H}) + \lambda_i(\mathbf{H})}, \frac{\lambda_i(\mathbf{H}) - \lambda_{i-1}(\mathbf{H})}{\lambda_i(\mathbf{H}) + \lambda_{i-1}(\mathbf{H})}, 1\Big\} \tag{1.10}$$

then

$$\min_{\lambda_j(\mathbf{M})} \frac{|\lambda_i(\mathbf{H}) - \lambda_j(\mathbf{M})|}{\lambda_i(\mathbf{H})} = \frac{|\lambda_i(\mathbf{H}) - \lambda_i(\mathbf{M})|}{\lambda_i(\mathbf{H})} < 1. \tag{1.11}$$

If $\lambda_{i-1}(\mathbf{H}) < \lambda_i(\mathbf{H}) = \cdots = \lambda_{i+n-1}(\mathbf{H}) < \lambda_{i+n}(\mathbf{H})$ and

$$\frac{\varepsilon}{1-\varepsilon} < \max\Big\{\frac{\lambda_{i+n}(\mathbf{H}) - \lambda_i(\mathbf{H})}{\lambda_{i+n}(\mathbf{H}) + \lambda_i(\mathbf{H})}, \frac{\lambda_i(\mathbf{H}) - \lambda_{i-1}(\mathbf{H})}{\lambda_i(\mathbf{H}) + \lambda_{i-1}(\mathbf{H})}, 1\Big\} \tag{1.12}$$

then

$$\operatorname*{argmin}_{j\in\mathbb{N}} \frac{|\lambda_{i-1}(\mathbf{H}) - \lambda_j(\mathbf{M})|}{\lambda_{i-1}(\mathbf{H})} \leq i - 1 \tag{1.13}$$

$$\operatorname*{argmin}_{j\in\mathbb{N}} \frac{|\lambda_{i+n}(\mathbf{H}) - \lambda_j(\mathbf{M})|}{\lambda_{i+n}(\mathbf{H})} \geq i + n. \tag{1.14}$$

Proof. Estimates (1.8)–(1.9) are a consequence of (1.6)–(1.7) and Theorem 1.5. The rest of the theorem follows from a proof which analogous to the proof of [5, Theorem 4.16]. We repeat the argument in this new setting.

Let $i \neq j$ then

$$\begin{aligned}
\frac{|\lambda_i(\mathbf{H}) - \lambda_j(\mathbf{M})|}{\lambda_i(\mathbf{H})} &\geq \frac{|\lambda_i(\mathbf{H}) - \lambda_j(\mathbf{H})|}{\lambda_i(\mathbf{H}) + \lambda_j(\mathbf{H})}\frac{\lambda_i(\mathbf{H}) + \lambda_j(\mathbf{H})}{\lambda_i(\mathbf{H})} - \frac{|\lambda_j(\mathbf{H}) - \lambda_j(\mathbf{M})|}{\lambda_j(\mathbf{H})}\frac{\lambda_j(\mathbf{H})}{\lambda_i(\mathbf{H})} \\
&\geq \gamma\Big(1 + \frac{\lambda_j(\mathbf{H})}{\lambda_i(\mathbf{H})}\Big) - \frac{\varepsilon}{1-\varepsilon}\frac{\lambda_j(\mathbf{H})}{\lambda_i(\mathbf{H})} > \gamma \\
&> \frac{|\lambda_i(\mathbf{H}) - \lambda_i(\mathbf{M})|}{\lambda_i(\mathbf{H})}.
\end{aligned}$$

With this we have established (1.11). (1.13)–(1.14) are a way to state (1.11) in a presence of a multiple eigenvalue $\lambda_i(\mathbf{H})$. The proof follows by a repetition of the previous argument for $j \geq i$ and $j \leq i+n-1$. For instance, we establish (1.13) by proving

$$\frac{|\lambda_{i-1}(\mathbf{H}) - \lambda_j(\mathbf{M})|}{\lambda_{i-1}(\mathbf{H})} > \frac{|\lambda_{i-1}(\mathbf{H}) - \lambda_{i-1}(\mathbf{M})|}{\lambda_{i-1}(\mathbf{H})}$$

for all $j \geq i$. □

Remark 1.7. The significance of this lemma is that it detects which spectral subspaces should be compared. When we were comparing discrete eigenvalues, the order relation between the real numbers (eigenvalues) solved this problem automatically. For spectral subspaces we need to assume more than (1.6) in order to be able to construct meaningful estimates. Assumptions (1.10) and (1.12) show how much more we (will) assume.

Next we show that (1.6) implies (1.1) with $\eta = \varepsilon(1-\varepsilon)^{-1/2}$. To establish this claim we need a notion of a pseudo inverse of a closed operator. A definition from [18] will be used. The *pseudo inverse* of a selfadjoint operator $\mathbf{H}$ is the selfadjoint operator $\mathbf{H}^\dagger$ defined by

$$\begin{aligned} \mathcal{D}(\mathbf{H}^\dagger) &= \mathsf{R}(\mathbf{H}) \oplus \mathcal{D}(\mathbf{H})^\perp, \\ \mathbf{H}^\dagger(u+v) &= \mathbf{H}^{-1}u, \qquad u \in \mathsf{R}(\mathbf{H}),\ g \in \mathcal{D}(\mathbf{H})^\perp. \end{aligned}$$

It follows that $\mathbf{H}^\dagger = \mathbf{H}^{-1}$ in $\overline{\mathsf{R}(\mathbf{H})}$. Note that we did not assume $\mathbf{H}^\dagger$ to be bounded or densely defined. The operator $\mathbf{H}^\dagger$ will be bounded if and only if $\mathsf{R}(\mathbf{H})$ is closed in $\mathcal{H}$, see [15]. The operator $\mathbf{H}^\dagger$ could have also been defined by the spectral calculus, since

$$\mathbf{H}^\dagger = f(\mathbf{H}), \qquad f(\lambda) = \begin{cases} 0, & \lambda = 0, \\ \frac{1}{\lambda}, & \lambda \neq 0. \end{cases}$$

In [18] Weidmann has given a short survey of the properties of the pseudo inverse of a nondensely defined operator $\mathbf{H}$. In particular, let $\mathbf{H}_1$ and $\mathbf{H}_2$ be two nonnegative operators in $\overline{\mathcal{D}(\mathbf{H}_1)}$ and $\overline{\mathcal{D}(\mathbf{H}_2)}$ respectively then

$$\|\mathbf{H}_1^{1/2}u\| \leq \|\mathbf{H}_2^{1/2}u\| \Leftrightarrow \|\mathbf{H}_2^{1/2\dagger}u\| \leq \|\mathbf{H}_1^{1/2\dagger}u\|. \tag{1.15}$$

Analogously, let h_1 and h_2 be two closed, not necessarily densely defined, positive definite forms and let $\mathbf{H}_1$ and $\mathbf{H}_2$ be the selfadjoint operators defined by h_1 and h_2 in $\overline{\mathcal{Q}(h_1)}$ and $\overline{\mathcal{Q}(h_2)}$. We say $h_1 \leq h_2$ when $\mathcal{Q}(h_2) \subset \mathcal{Q}(h_1)$ and

$$h_1[u] = \|\mathbf{H}_1^{1/2}u\|^2 \leq h_2[u] = \|\mathbf{H}_2^{1/2}u\|^2, \qquad u \in \mathcal{Q}(h_2). \tag{1.16}$$

Equivalently, we write $\mathbf{H}_1 \leq \mathbf{H}_2$ when $h_1 \leq h_2$. Now, we can write the fact (1.15) as

$$\mathbf{H}_1 \leq \mathbf{H}_2 \Longleftrightarrow \mathbf{H}_2^\dagger \leq \mathbf{H}_1^\dagger. \tag{1.17}$$

In one point we will depart from the conventions in [8].

Definition 1.8. A nonnegative form

$$h(u,v) = (\mathbf{H}^{1/2}u, \mathbf{H}^{1/2}v)$$

will be called *nonnegative definite* when $\mathbf{H}^\dagger$ is bounded. Analogously, the nonnegative operator $\mathbf{H}$ such that $\mathbf{H}^\dagger$ is bounded will also be called *nonnegative definite.*

In the sequel we establish a connection between (1.6) and (1.1) when h and m are nonnegative definite forms.

Lemma 1.9. *Let* $\mathbf{H}$ *and* $\mathbf{M}$ *be nonnegative definite operators in a Hilbert space* $\mathcal{H}$ *such that (1.6) holds for* $0 \le \varepsilon < 1$. *Let*

$$S = \mathbf{H}^{1/2}\mathbf{M}^{\dagger 1/2} - \overline{\mathbf{H}^{\dagger 1/2}\mathbf{M}^{1/2}} \tag{1.18}$$

then S *is bounded and*

$$|(\psi, S\phi)| \le \frac{\varepsilon}{\sqrt{1-\varepsilon}}\|\psi\|\|\phi\|. \tag{1.19}$$

Proof. The closed graph theorem implies that the operator

$$S = \mathbf{H}^{1/2}\mathbf{M}^{\dagger 1/2} - \overline{\mathbf{H}^{\dagger 1/2}\mathbf{M}^{1/2}}$$

is bounded. Also, $\mathsf{N}(\mathbf{H}) = \mathsf{N}(\mathbf{M}) = \mathsf{N}(S)$ and $P_{\mathsf{N}(S)}$ commutes with S. It is sufficient to prove the estimate for $x, y \in \mathsf{R}(\mathbf{H})$. The assumption (1.6) gives

$$|(h-m)(\mathbf{H}^{\dagger 1/2}x, \mathbf{M}^{\dagger 1/2}y)| \le \varepsilon\|y\|\; m[\mathbf{H}^{\dagger 1/2}x]^{1/2}.$$

Analogously, (1.6) implies

$$\|\mathbf{M}^{1/2}\mathbf{H}^{\dagger 1/2}\| \le \frac{1}{\sqrt{1-\varepsilon}}\ . \tag{1.20}$$

Altogether, the estimate (1.19) follows. □

Now, we rewrite the conclusion of this lemma in the symmetric form setting. The result is given in the form of a proposition which we present without proof.

Proposition 1.10. *Let* m *and* h *be nonnegative definite forms and let there exist* $0 \le \varepsilon < 1$ *such that (1.6) holds then* $\mathsf{N}(\mathbf{H}) = \mathsf{N}(\mathbf{H})$ *and*

$$|h(u,v) - m(u,v)| \le \frac{\varepsilon}{\sqrt{1-\varepsilon}}\sqrt{h[u]m[v]}.$$

When we only know that h and m satisfy (1.1) then we can establish a similar result about $\mathsf{N}(\mathbf{H})$ and $\mathsf{N}(\mathbf{M})$.

Proposition 1.11. *Let* m *and* h *be nonnegative definite forms such that (1.1) holds then*

$$S = \mathbf{H}^{1/2}\mathbf{M}^{\dagger 1/2} - \overline{\mathbf{H}^{\dagger 1/2}\mathbf{M}^{1/2}}$$
$$S^* = \overline{\mathbf{M}^{\dagger 1/2}\mathbf{H}^{1/2}} - \mathbf{M}^{1/2}\mathbf{H}^{\dagger 1/2}$$

are bounded operators and $\|S^*\| = \|S\| \le \eta$. *Furthermore,* $\mathsf{N}(\mathbf{H}) = \mathsf{N}(\mathbf{M})$ *and a fortiori* $\mathsf{R}(\mathbf{H}) = \mathsf{R}(\mathbf{M})$.

The operator S has a special structure. Assume $\mathbf{M}u = \mu u$ and $\mathbf{H}v = \lambda v$, then

$$\begin{aligned}(v, Su) &= \lambda^{1/2}(v,u)\mu^{1/2} - \lambda^{-1/2}(v,u)\mu^{1/2} \\ &= \frac{\lambda-\mu}{\sqrt{\lambda\mu}}(v,u)\ . \end{aligned} \tag{1.21}$$

The equation (1.21) suggests the distance function

$$\frac{|\lambda-\mu|}{\sqrt{\lambda\mu}}$$

which measures the distance between the eigenvalues of operators $\mathbf{H}$ and $\mathbf{M}$. We state this result as the following corollary..

Corollary 1.12. *Let* $\mathbf{M}u = \mu u$, $\|u\| = 1$ *and* $\mathbf{H}v = \lambda v$, $\|v\| = 1$ *and let* S *be as in Proposition 1.11 then*

$$\frac{|\lambda - \mu|}{\sqrt{\lambda\mu}} \le \frac{\eta}{|(u, v)|}.$$

Our theory is designed to be directly applicable to differential operators given in a weak form. This will enable us to obtain estimates for the difference between the spectral projections of the operators to which the theory of [1, 2] does not apply, see Example 3.4 below.

2. Weak Sylvester equation

Let us outline the general picture. We have an unbounded positive definite operator $\mathbf{A}$ and a bounded positive definite operator M. They are defined in, possibly, different subspaces of the environment Hilbert space $\mathcal{H}$. Thus, $\mathcal{H}_M = \mathsf{R}(M)$ is (of necessity) a closed subspace of $\mathcal{H}$ and likewise

$$\overline{\mathcal{D}(\mathbf{A}^{1/2})}^{\mathcal{H}} = \mathsf{R}(\mathbf{A}^{1/2}) = \mathcal{H}_{\mathbf{A}}.$$

Let the bounded operator $F : \mathcal{H}_M \to \mathcal{H}_{\mathbf{A}}$ be given, then we are looking for the bounded operator $T : \mathcal{H}_M \to \mathcal{H}_{\mathbf{A}}$ such that

(2.1)
$$(\mathbf{A}^{1/2}v, TM^{-1/2}u) - (\mathbf{A}^{-1/2}v, TM^{1/2}u) = (v, Fu)\ , \qquad v \in \mathcal{D}(\mathbf{A}^{1/2}),\ u \in \mathcal{H}_M.$$

Formally, we say that T solves the equation

$$\mathbf{A}T - TM = \mathbf{A}^{1/2}FM^{1/2}. \tag{2.2}$$

Here $G = \mathbf{A}^{1/2}FM^{1/2}$ is naturally only a formal expression and does not represent a *bona fide* operator. In the case in which G be a *bona fide* operator equation (2.2) becomes the rigorous equation

$$\mathbf{A}T - TM = G,$$

called the (standard) *Sylvester equation*, cf. [1, 2]. The case when $\mathbf{A}$ and M are finite matrices has been considered in [12] where (2.2) was called the structured Sylvester equation.

We call the relation (2.1) the *weak Sylvester equation.* This equation has the same form as (1.4), but its coefficients are less general since we assume M to be a bounded operator. On the other hand, this "special" Sylvester equation allows us to tackle the perturbation problem for $E_{\mathbf{H}}(D)$ and $E_{\mathbf{M}}(D)$ in full generality (e.g. take $\mathbf{A} : \mathcal{H}_{\mathbf{A}} \to \mathcal{H}_{\mathbf{A}}$ as the compression of $\mathbf{H}$ on the subspace $\mathcal{H}_{\mathbf{A}} := \mathsf{R}(E_{\mathbf{H}}(D))^{\perp}$ and $M : \mathcal{H}_M \to \mathcal{H}_M$ as the compression of $\mathbf{M}$ on the subspace $\mathcal{H}_M := \mathsf{R}(E_{\mathbf{M}}(D))$, for details see Section 3). We have adapted the notation to reflect this structural fact.

The weak Sylvester equation represents a generalization of the concept of the structured Sylvester equation (2.2) from finite matrix setting to unbounded operator setting. The following theorem slightly generalizes the corresponding result from the joint paper [7] and corrects a technical glitch in one of the proofs.

Theorem 2.1. *Let* $\mathbf{A}$ *and* M *be positive definite operators in* $\mathcal{H}_{\mathbf{A}}$ *and* $\mathcal{H}_M$*, respectively and let* F *be a bounded operator from* $\mathcal{H}_M$ *into* $\mathsf{R}(\mathbf{A}^{1/2}) = \mathcal{H}_{\mathbf{A}}$*. If* M *is bounded and*

$$\|M\| < \frac{1}{\|\mathbf{A}^{-1}\|} \tag{2.3}$$

then the weakly formulated Sylvester equation

$$\left(\mathbf{A}^{1/2}v, TM^{-1/2}u\right) - \left(v, \mathbf{A}^{-1/2}TM^{1/2}u\right) = (v, Fu) \tag{2.4}$$

has a unique solution T, given by $\tau(v,u) = (v, Tu)$ and

$$\tau(v,u) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}(\mathbf{A}^{1/2}v, (\mathbf{A} - \mathrm{i}\zeta - d)^{-1}F(M - \mathrm{i}\zeta - d)^{-1}M^{1/2}u)d\zeta, \tag{2.5}$$

where d is any number satisfying

$$\|M\| < d < \frac{1}{\|\mathbf{A}^{-1}\|}\ . \tag{2.6}$$

Proof. The uniqueness means that

$$\left(\mathbf{A}^{1/2}v, WM^{-1/2}u\right) - \left(v, \mathbf{A}^{-1/2}WM^{1/2}u\right) = 0, \tag{2.7}$$

for $u \in \mathcal{H}_M$, $v \in \mathcal{D}(\mathbf{A}^{1/2})$, has the only bounded solution $W = 0$. Let

$$E_n = \int_0^n d\ E_{\mathbf{A}^{1/2}}(\lambda),$$

then in particular

$$\left(\mathbf{A}^{1/2}v, E_nWM^{-1/2}u\right) - \left(v, \mathbf{A}^{-1/2}E_nWM^{1/2}u\right) = 0,$$

for $u \in \mathcal{H}_M$, $v \in \mathcal{D}(\mathbf{A}^{1/2}) \cap E_n\mathcal{H}$. Define the cut–off function

$$f_n(x) = \begin{cases} x, & D \le x \le n \\ n, & n \le x \end{cases}$$

with $D = 1/\|\mathbf{A}^{-1}\|$. The operator $f_n(\mathbf{A}^{1/2})$ is bicontinuous and

$$f_n(\mathbf{A}^{1/2})E_nWM^{-1/2} - f_n(\mathbf{A}^{1/2})^{-1}E_nWM^{1/2} = 0. \tag{2.8}$$

Since $f_n(\mathbf{A}^{1/2})$ and $M^{1/2}$ are bounded and positive definite operators, the standard Sylvester equation (2.8) has the unique solution

$$E_nW = 0, \qquad n \in \mathbb{N}\ . \tag{2.9}$$

This is a consequence of the standard theory of the Sylvester equation with bounded coefficients, see [1, 2]. The statement (2.9) implies $W = 0$.

Now for the existence. We use the spectral integral $\mathbf{A} = \int \lambda\ dE(\lambda)$ to compute

$$\begin{aligned}\int_{-\infty}^{\infty}\|(\mathbf{A} + i\zeta - d)^{-1}\mathbf{A}^{1/2}v\|^2\ d\zeta &= \int_{-\infty}^{\infty}(\mathbf{A}^{1/2}v, \left|\mathbf{A} - i\zeta - d\right|^{-2}\mathbf{A}v)\ d\zeta \\ &= \int_{-\infty}^{\infty} d\zeta \int_D^{\infty}\frac{\lambda\ d(E(\lambda)\mathbf{A}^{1/2}v, \mathbf{A}^{1/2}v)}{(\lambda - d)^2 + \zeta^2} \\ &= \int_D^{\infty}\lambda\ d(E(\lambda)\mathbf{A}^{1/2}v, \mathbf{A}^{1/2}v)\int_{-\infty}^{\infty}\frac{d\zeta}{(\lambda - d)^2 + \zeta^2} \\ &= \int_D^{\infty}\frac{\pi\lambda\ d(E(\lambda)\mathbf{A}^{1/2}v, \mathbf{A}^{1/2}v)}{\lambda - d} \\ &= \pi(\mathbf{A}(\mathbf{A} - d)^{-1}v, v). \end{aligned} \tag{2.10}$$

Analogously, one establishes

$$\int_{-\infty}^{\infty}\|(M - i\zeta - d)^{-1}M^{1/2}u\|^2\ d\zeta = \pi(M(d - M)^{-1}u, u). \tag{2.11}$$

The convergence of these integrals justifies the following computation. Set

$$\tau(v,u) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} (\mathbf{A}^{1/2}v, (\mathbf{A}-\mathrm{i}\zeta-d)^{-1}F(M-\mathrm{i}\zeta-d)^{-1}M^{1/2}u)d\zeta$$

and then compute using (2.10) and (2.11)

$$\begin{aligned}|\tau(v,u)|^2 &= \frac{1}{(2\pi)^2}\Big[\int_{-\infty}^{\infty}((\mathbf{A}+\mathrm{i}\zeta-d)^{-1}\mathbf{A}^{1/2}v, F(M-\mathrm{i}\zeta-d)^{-1}M^{1/2}u)d\zeta\Big]^2\\
&\le \frac{\|F\|^2}{(2\pi)^2}\Big[\int_{-\infty}^{\infty}\|(\mathbf{A}+i\zeta-d)^{-1}\mathbf{A}^{1/2}v\|\;\|(M-i\zeta-d)^{-1}M^{1/2}u\|d\zeta\Big]^2\\
&\le \frac{\|F\|^2}{4}(\mathbf{A}(\mathbf{A}-d)^{-1}v,v)(M(d-M)^{-1}u,u). \qquad (2.12)\end{aligned}$$

This in turn implies that the operator

$$\tau(v,u) = (v,Tu)$$

is a bounded operator and also gives the meaning to the formula (2.5).

Now we will prove that this T satisfies the equation (2.4). Note that

$$\mathbf{A}(\mathbf{A}-\rho-d)^{-1} = \mathbf{I} + (\rho+d)(\mathbf{A}-\rho-d)^{-1}, \qquad \rho\notin\sigma(\mathbf{A})$$

and then take $v\in\mathcal{D}(\mathbf{A})$ to compute

$$\begin{aligned}&(\mathbf{A}^{1/2}v, TM^{-1/2}u) - (\mathbf{A}^{-1/2}v, TM^{1/2}u) =\\
&= -\frac{1}{2\pi}\Big[\int_{-\infty}^{\infty}(\mathbf{A}v, (\mathbf{A}-\mathrm{i}\zeta-d)^{-1}F(M-\mathrm{i}\zeta-d)^{-1}u)\;d\zeta\\
&\quad - \int_{-\infty}^{\infty}(v, (\mathbf{A}-\mathrm{i}\zeta-d)^{-1}F(M-\mathrm{i}\zeta-d)^{-1}Mu)\;d\zeta\Big]\\
&= -\frac{1}{2\pi}\Big[\mathrm{v.p.}\int_{-\infty}^{\infty}(v, F(M-i\zeta-d)^{-1}u)\;d\zeta\\
&\quad + \int_{-\infty}^{\infty}(i\zeta+d)((\mathbf{A}-i\zeta-d)^{-1}v, F(M-i\zeta-d)^{-1}u)\;d\zeta\\
&\quad - \int_{-\infty}^{\infty}(i\zeta+d)((\mathbf{A}-i\zeta-d)^{-1}v, F(M-i\zeta-d)^{-1}u)\;d\zeta\\
&\quad - \mathrm{v.p.}\int_{-\infty}^{\infty}((\mathbf{A}-i\zeta-d)^{-1}v, Fu)\;d\zeta\Big]\\
&= (v,Fu).\end{aligned}$$

By a usual density argument we conclude that the operator T satisfies (2.4). □

Theorem 2.2. *Let* $\mathbf{A}$*,* M *and* F *be as in Theorem 2.1 then*

$$\|T\| \le \sqrt{\frac{D\|M\|}{(D-d)(d-\|M\|)}}\;\frac{\|F\|}{2}$$

for any $\|M\| < d < D$*. The optimal* d *is* $d = (\|M\|+D)/2$ *and then we obtain*

$$\|T\| \le \frac{\sqrt{D\|M\|}}{(D-\|M\|)}\;\|F\| \qquad (2.13)$$

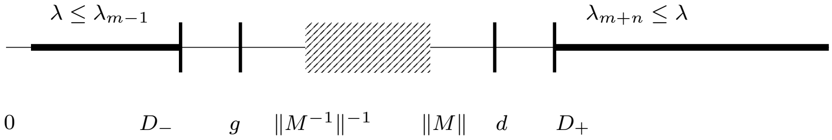

FIGURE 1. The spectral gaps

Proof. Estimate (2.12) yields

$$\|T\| \le \frac{\|F\|}{2}\|\mathbf{A}(\mathbf{A}-d)^{-1}\|\;\|M(d-M)^{-1}\| \le \frac{\|F\|}{2}\sqrt{\frac{D\|M\|}{(D-d)(d-\|M\|)}}$$

This in turn implies the desired estimate. The optimality of the $d = (\|M\| + D)/2$ can now be checked by a direct computation. □

Remark 2.3. In fact, we will see that the estimate of Theorem 2.2 is optimal in the following sense. Let us consider the equation (2.4) in another light. Theorem 2.1 gives a set of conditions when the equation (2.1) has a unique solution. Theorem 2.2 then provides us with an estimate of this solution.

Since for given F, under the conditions of Theorem 2.1, there exists the unique T such that (2.4) holds, we can define the so called "Sylvester operator" which associates the solution T to every operator F. The estimate (2.13) is then an estimate of the norm of the inverse of such an operator.

The bound (2.13) is sharp in this sense as shows the following example. Let M and $\mathbf{A}$ be such that

$$Mq = \|M\|q, \qquad \mathbf{A}p = Dp,$$

for p and q one dimensional projections and let $F = pq$.

Then (2.4) is obviously satisfied by

$$T = \frac{\sqrt{D\|M\|}}{D-\|M\|}pq.$$

2.1. **Allowing for a more general relation between $\sigma(M)$ and $\sigma(\mathbf{A})$.** An analogue of Theorem 2.1 holds, if the assumption (2.6) is replaced by a more general one, namely that the interval

$$\left[\|M^{-1}\|^{-1}, \|M\|\right]$$

be contained in the resolvent set of the operator $\mathbf{A}$. We omit the proof of the following result.

Theorem 2.4. *Let the operators $\mathbf{A}$, M and F be as in Theorem 2.1, and let their spectra be arranged as on Figure 1, then (in the sense of (2.5))*

$$\begin{aligned} T \;&=\; -\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{A}^{1/2}(\mathbf{A}-\mathrm{i}\zeta-d)^{-1}F(M-\mathrm{i}\zeta-d)^{-1}M^{1/2}d\zeta \\ &\quad +\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{A}^{1/2}(\mathbf{A}-\mathrm{i}\zeta-g)^{-1}F(M-\mathrm{i}\zeta-g)^{-1}M^{1/2}d\zeta, \end{aligned}$$

where d, g are chosen from the right and left spectral gap, see Figure 1, is the solution of the weak Sylvester equation (2.4). We also have the estimate

$$\|T\| \leq \Big(\frac{\sqrt{\|M^{-1}\|^{-1}D_-}}{\|M^{-1}\|^{-1} - D_-} + \frac{\sqrt{D_+\|M\|}}{D_+ - \|M\|}\Big)\|F\|.$$

2.2. **Estimates in the Hilbert–Schmidt norm.** A bounded operator $H : \mathcal{H} \to \mathcal{H}$ is a Hilbert–Schmidt operator if H^*H is trace class and then, cf. [8, Ch. X.1.3],

$$\| H \|_{HS} := \mathsf{Tr}\sqrt{H^*H}. \tag{2.14}$$

Let $\mathbf{A}$ and $\mathbf{M}$ be positive definite operators in $\mathcal{H}_\mathbf{A} \subset \mathcal{H}$ and $\mathcal{H}_\mathbf{M} \subset \mathcal{H}$, respectively. We will analyze the weakly formulated Sylvester equation under the assumption that $\| F \|_{HS} < \infty$ and

$$\mathsf{gap}(\sigma(\mathbf{M}), \sigma(\mathbf{H})) := \inf_{\substack{\mu\in\sigma(\mathbf{M}),\\ \lambda\in\sigma(\mathbf{A})}} \frac{|\mu - \lambda|}{\sqrt{\mu\lambda}} > 0. \tag{2.15}$$

To prove our result, we will need a basic result on the spectral representation of selfadjoint operators, see [19, Satz 8.17].

Theorem 2.5 (Spectral representation)**.** *For every selfadjoint operator $\mathbf{H}$ in a sparable Hilbert space $\mathcal{H}$ there exists a σ-finite measure space $(\mathcal{M}, \mu)$, a μ-measurable function $h : \mathcal{M} \to \mathbb{R}$ and a unitary operator $V : \mathcal{H} \to L^2(\mathcal{M}, \mu)$ such that*

$$\mathbf{H} = V^{-1}\widetilde{\mathbf{H}}V.$$

Here $\widetilde{\mathbf{H}} : L^2(\mathcal{M}, \mu) \to L^2(\mathcal{M}, \mu)$ is the multiplication operator which is defined by the function h.

We will also need the following theorem on the integral representation of Hilbert–Schmidt operators. For the proof see [19, Satz 3.19].

Theorem 2.6. *A bounded operator $T : L^2(\mathcal{M}_1, \mu) \to L^2(\mathcal{M}_2, \nu)$ is a Hilbert–Schmidt operator if and only if there exists a function $t \in L^2(\mathcal{M}_1 \times \mathcal{M}_2, \mu \times \nu)$ such that*

$$(Tg)(y) = \int_{\mathcal{M}_1} t(x, y)g(x)d\mu \qquad \nu\text{-almost everywhere,} \qquad g \in L^2(\mathcal{M}_1, \mu).$$

Furthermore, we have

$$\| T \|_{HS} = \|t\|_{L^2(\mathcal{M}_1\times\mathcal{M}_2,\mu\times\nu)}.$$

We now prove a "Hilbert–Schmidt" version of Theorem 2.1. We will assume that $\| F \|_{HS} < \infty$ and that $\mathcal{H}$ be separable. On the other hand, the spectra of $\mathbf{A}$ and $\mathbf{M}$ may be arbitrarily interlaced.

Theorem 2.7. *Let $\mathbf{A}$ and $\mathbf{M}$ be positive definite operators in $\mathcal{H}_\mathbf{A}$ and $\mathcal{H}_\mathbf{M}$, respectively and let $F : \mathcal{H}_\mathbf{M} \to \mathcal{H}_\mathbf{A}$ be a bounded operator. Assume further that $\| F \|_{HS} < \infty$ and $\mathsf{gap}(\sigma(\mathbf{M}), \sigma(\mathbf{H})) > 0$ then there exists a unique Hilbert–Schmidt operator T such that*

$$\Big(\mathbf{A}^{1/2}v, T\mathbf{M}^{-1/2}u\Big) - \Big(v, \mathbf{A}^{-1/2}T\mathbf{M}^{1/2}u\Big) = (v, Fu) \tag{2.16}$$

and

$$\| T \|_{HS} \leq \frac{\| F \|_{HS}}{\mathsf{gap}(\sigma(\mathbf{M}), \sigma(\mathbf{H}))}. \tag{2.17}$$

Proof. The uniqueness of the bounded solution of the equation (2.16) follows by a double cut-off argument analogous to the one used in (2.8)–(2.9). We leave out the details.

By Theorem 2.6 there exist measure spaces $(\mathcal{M}_{\mathbf{M}}, \mu)$ and $(\mathcal{M}_{\mathbf{A}}, \mu)$, measurable functions $m : \mathcal{M}_{\mathbf{M}} \to \mathbb{R}$ and $a : \mathcal{M}_{\mathbf{A}} \to \mathbb{R}$ and unitary operators $U : \mathcal{H} \to L^2(\mathcal{M}_{\mathbf{M}}, \mu)$ and $V : \mathcal{H} \to L^2(\mathcal{M}_{\mathbf{A}}, \mu)$ such that

$$\mathbf{A} = V^{-1}\widetilde{\mathbf{A}}V$$
$$\mathbf{M} = U^{-1}\widetilde{\mathbf{M}}U.$$

Here we have taken $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{M}}$ to be the multiplication operators which were defined by the functions a and m respectively. Since $\| F \|_{HS} < \infty$, the operator $VFU : L^2(\mathcal{M}_{\mathbf{M}}, \mu) \to L^2(\mathcal{M}_{\mathbf{A}}, \mu)$ is obviously a Hilbert–Schmidt operator and $\| VFU \|_{HS} = \| F \|_{HS}$. We can therefore assume, without loosing generality, that we work with $\mathcal{H}_{\mathbf{M}} = L^2(\mathcal{M}_{\mathbf{M}}, \mu)$, $\mathcal{H}_{\mathbf{A}} = L^2(\mathcal{M}_{\mathbf{H}}, \nu)$ and that $\mathbf{A} = \widetilde{\mathbf{A}}$, $\mathbf{M} = \widetilde{\mathbf{M}}$ and $F = VFU$.

Theorem 2.6 implies that there exists a function $f \in L^2(\mathcal{M}_{\mathbf{M}} \times \mathcal{M}_{\mathbf{A}}, \mu \times \nu)$ such that

$$(Fg)(y) = \int_{\mathcal{M}_{\mathbf{M}}} f(x,y)g(x)d\mu \qquad \nu\text{-almost everywhere}, \quad g \in L^2(\mathcal{M}_{\mathbf{M}}, \mu).$$

Set

$$t(x,y) = \frac{f(x,y)}{\frac{a(y)^{1/2}}{m(x)^{1/2}} - \frac{m(x)^{1/2}}{a(y)^{1/2}}}, \qquad \mu \times \nu\text{-almost everywhere.} \tag{2.18}$$

Relation (2.15) and the positive definiteness of $\mathbf{A}$ and $\mathbf{M}$ imply that

$$\| \frac{a(\cdot)^{1/2} m(\cdot\cdot)^{1/2}}{a(\cdot) - m(\cdot\cdot)} \|_{L^\infty(\mathcal{M}_{\mathbf{M}} \times \mathcal{M}_{\mathbf{A}}, \mu \times \nu)} \leq \frac{1}{\mathsf{gap}(\sigma(\mathbf{M}), \sigma(\mathbf{A}))}$$

thus $t \in L^2(\mathcal{M}_{\mathbf{M}} \times \mathcal{M}_{\mathbf{A}}, \mu \times \nu)$ and

$$\|t\|_{L^2(\mathcal{M}_{\mathbf{M}} \times \mathcal{M}_{\mathbf{A}}, \mu \times \nu)} \leq \frac{1}{\mathsf{gap}(\sigma(\mathbf{M}), \sigma(\mathbf{A}))} \|f\|_{L^2(\mathcal{M}_{\mathbf{M}} \times \mathcal{M}_{\mathbf{A}}, \mu \times \nu)}. \tag{2.19}$$

Now (2.18) can be rewritten as

$$a(y)^{1/2} t(x,y) m(x)^{-1/2} - a(y)^{-1/2} t(x,y) m(x)^{1/2} = f(x,y) \tag{2.20}$$

The kernel t defines a Hilbert–Schmidt operator T with

$$(v, Tu) = \int \overline{v(y)} t(x,y) u(x) d\mu \ d\nu.$$

By taking integrals for $v \in \mathcal{D}(\mathbf{A}^{1/2})$ and $u \in \mathcal{D}(\mathbf{M}^{1/2})$ we establish that the equation (2.20) is equivalent to (2.16) and the estimate (2.19) implies (2.17). □

2.3. **Estimates by other unitary invariant operator norms.** Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded operators on the Hilbert space $\mathcal{H}$. We will consider *symmetric norms* $\| \cdot \|$ on a subspace $\mathcal{S}$ of $\mathcal{L}(\mathcal{H})$. To say that the norm is symmetric on $\mathcal{S} \subset \mathcal{L}(\mathcal{H})$ means that, beside the usual properties of any norm, it additionally satisfies:

(i): If $B \in \mathcal{S}$, $A, C \in \mathcal{L}(\mathcal{H})$ then $ABC \in \mathcal{S}$ and

$$\| ABC \| \leq \|A\| \| B \| \|C\|.$$

(ii): If A has rank 1 then $\|\!| A \|\!| = \|A\|$, where $\|\cdot\|$ always denotes the standard operator norm on $\mathcal{L}(\mathcal{H})$.
(iii): If $A \in \mathcal{S}$ and U, V are unitary on $\mathcal{H}$, then $UAV \in \mathcal{S}$ and $\|\!| UAV \|\!| = \|\!| A \|\!|$.
(iv): $\mathcal{S}$ is complete under the norm $\|\!| \cdot \|\!|$.

The subspace $\mathcal{S}$ is defined as a $\|\!| \cdot \|\!|$–closure of the set of all degenerate operators in $\mathcal{L}(\mathcal{H})$. Such $\mathcal{S}$ is an ideal in the algebra $\mathcal{L}(\mathcal{H})$, cf. [4, 17]. Symmetric norms were used in [1] in the context of subspace estimates. If we assume, additionally to the assumptions of Theorem 3.2 that $\|\!| F \|\!| < \infty$ then there exists a unique bounded solution T of the weak Sylvester equation and

$$\|\!| T \|\!| \leq \frac{\sqrt{D\|M\|}}{D - \|M\|} \|\!| F \|\!| .$$

We now prove this fact.

Theorem 2.8. *Let* $\mathbf{A}$ *and* M *be the selfadjoint operators which satisfy the assumptions of Theorem 2.1 and let the symmetric norm* $\|\!| \cdot \|\!|$ *have the property*

($\mathcal{P}$) If $\sup \|\!| A_n \|\!| < \infty$ *and* $A = \text{w-lim}_n A_n$ *then* $A \in \mathcal{S}$ *and*

$$\|\!| A \|\!| \leq \sup \|\!| A_n \|\!| .$$

If $\|\!| F \|\!| < \infty$ *then there exists a unique bounded operator* T *such that*

$$\Big(\mathbf{A}^{1/2}v, TM^{-1/2}u\Big) - \Big(v, \mathbf{A}^{-1/2}TM^{1/2}u\Big) = (v, Fu)$$

and

$$\|\!| T \|\!| \leq \frac{\sqrt{D\|M\|}}{D - \|M\|} \|\!| F \|\!| .$$

Proof. The proof follows by a cut-off argument. We (re)use the construction which was used in (2.8). Let $f_n(\mathbf{A}^{1/2})$ and E_n be as in (2.8). The equation

$$\Big(f_n(\mathbf{A}^{1/2})v, T_nM^{-1/2}u\Big) - \Big(f_n(\mathbf{A}^{1/2})^{-1}v, T_nM^{1/2}u\Big) = (v, E_nFu) \tag{2.21}$$

can now be written as the standard Sylvester equation

$$f_n(\mathbf{A}^{1/2})^2 T_n - T_n M = f_n(\mathbf{A}^{1/2})E_nFM^{1/2}$$

which has the unique bounded solution $T_n : \mathcal{H}_M \to \mathsf{R}(E_n)$ and $\|\!| T_n \|\!| < \infty$ (this follows from [2, Theorem 5.2]). The operator E_nT is bounded and satisfies the equation (2.21) therefore $T_n = E_nT$. Here we have tacitly assumed $\mathcal{L}(\mathcal{H}_\mathbf{A}) \subset \mathcal{L}(\mathcal{H})$. Furthermore,

$$\mathbf{A}^{1/2}E_nTM^{-1/2} - \mathbf{A}^{-1/2}E_nTM^{1/2} = E_nF. \tag{2.22}$$

We compute, using the property **(i)**,

$$\|\!| \mathbf{A}^{-1/2}E_nTM^{1/2} \|\!| \leq \frac{\|M^{1/2}\|}{\sqrt{\|\mathbf{A}^{-1}\|^{-1}}} \|\!| E_nT \|\!|$$

$$\|\!| \mathbf{A}^{1/2}E_nTM^{-1/2} \|\!| \geq \frac{\sqrt{\|\mathbf{A}^{-1}\|^{-1}}}{\|M^{1/2}\|} \|\!| E_nT \|\!| .$$

From these estimates and (2.22) we obtain the uniform upper bound

$$\|\!| E_nT \|\!| \leq \frac{\sqrt{D\|M\|}}{D - \|M\|} \|\!| E_nF \|\!| \leq \frac{\sqrt{D\|M\|}}{D - \|M\|} \|\!| F \|\!| . \tag{2.23}$$

Since $E_n T \to T$ in the strong operator topology, Property $(\mathcal{P})$ and the uniform bound (2.23) imply $\| T \| < \infty$ and the desired estimate follows. □

3. Perturbations of spectral subspaces

When comparing two spectral subspaces of operators $\mathbf{H}$ and $\mathbf{M}$, which satisfy (1.1), we have to make an additional assumption on the location of the spectra. Namely we assume that there exist $D_1 < D_2$ such that the interval $[D_1, D_2] \subset \mathbb{R}$ is contained in the resolvent sets of both $\mathbf{H}$ and $\mathbf{M}$. Let $Q = E_{\mathbf{H}}(D_1)$ and $P = E_{\mathbf{M}}(D_1)$. We want to estimate the norm of $P - Q$. The following description of a relation between a pair of orthogonal projections in a Hilbert space will be sufficient for our considerations. For the proof see [8].

Theorem 3.1 (Kato). *Let P and Q be two orthogonal projections such that*

$$\|P(\mathbf{I} - Q)\| < 1.$$

Then we have the following alternative. Either

(1) $\mathsf{R}(P)$ *and* $\mathsf{R}(Q)$ *are isomorphic and*

$$\|P(\mathbf{I} - Q)\| = \|Q(\mathbf{I} - P)\| = \|P - Q\| \qquad \textit{or}$$

(2) $\mathsf{R}(P)$ *is isomorphic to true subspace of* $\mathsf{R}(Q)$ *and*

$$\|Q(\mathbf{I} - P)\| = \|P - Q\| = 1.$$

To ease the presentation set $P_\perp = \mathbf{I} - P$ and $Q_\perp = \mathbf{I} - Q$. First, let us consider the case when h and m are positive definite. With the help of Proposition 1.11 we shall later reduce the nonnegative definite case to the positive definite one.

We define the operators

$$\mathbf{A} = Q_\perp H Q_\perp, \quad H = Q\mathbf{H}Q, \quad M = P\mathbf{M}P \quad \text{and} \quad \mathbf{W} = P_\perp \mathbf{M} P_\perp. \tag{3.1}$$

We shall not notationally distinguish the operators $\mathbf{A}$, M, $\mathbf{W}$ and H from their restrictions to the complement of their respective null spaces. Obviously,

$$\mathbf{H} = H + \mathbf{A}, \qquad \mathbf{M} = M + \mathbf{W}$$

and we compute, for S from (1.18),

$$\begin{aligned} Q_\perp S P &= (\mathbf{H}^{1/2} Q_\perp P \mathbf{M}^{-1/2} - \mathbf{H}^{-1/2} Q_\perp P \mathbf{M}^{1/2}) P \\ &= \mathbf{A}^{1/2} Q_\perp P M^{-1/2} - \mathbf{A}^{-1/2} Q_\perp P M^{1/2} \\ &= \mathbf{A}^{1/2} T M^{-1/2} - \mathbf{A}^{-1/2} T M^{1/2}. \end{aligned} \tag{3.2}$$

Here we have defined $T = Q_\perp P$. If we assume that $\mathsf{dim}(Q) = \mathsf{dim}(P) < \infty$ then Theorem 3.1 yields

$$\|P - Q\| = \|T\|.$$

The case when $\mathsf{dim}(Q) = \mathsf{dim}(P) = \infty$ will follow in a similar fashion.

The operator equation can be written in the following variational form

$$\begin{aligned} (\mathbf{A}^{1/2} v, T M^{-1/2} u) - (\mathbf{A}^{-1/2} v, T M^{1/2} u) &= (v, Su), \\ v \in \mathcal{D}(\mathbf{A}^{1/2}), \quad u &\in \mathsf{R}(P), \end{aligned} \tag{3.3}$$

which we have called the weakly formulated Sylvester equation.

Theorem 3.2. *Let the positive definite forms* m *and* h *be given such that (1.1) holds. Let there exist* $D_1 < D_2$ *such that the interval* $[D_1, D_2] \subset \mathbb{R}$ *be contained in the resolvent sets of both* $\mathbf{H}$ *and* $\mathbf{M}$. *Set* $Q = E_{\mathbf{H}}(D_1)$, $P = E_{\mathbf{M}}(D_1)$ *and assume* $\eta < (D_2 - D_1)(D_2 D_1)^{-1/2}$ *then*

$$\|P - Q\| \leq \frac{\sqrt{D_2 D_1}}{D_2 - D_1}\, \eta\ . \tag{3.4}$$

Proof. $T = Q_\perp P$ is the unique solution of the equation (3.3). Theorem 2.2 implies

$$\|T\| \quad \leq \quad \frac{\eta}{2}\sqrt{\frac{D_2 \lambda_n(\mathbf{M})}{(D_2 - d)(d - \lambda_n(\mathbf{M}))}}.$$

for any $\lambda_n(\mathbf{M}) < d < D_2$. The optimal d equals $\frac{(D_2 + \lambda_n(\mathbf{M}))}{2}$ and since $\|M\| < D_1$ we conclude

$$\|Q_\perp P\| \leq \eta\, \frac{\sqrt{D_2 \lambda_n(\mathbf{M})}}{D_2 - \lambda_n(\mathbf{M})} \leq \eta\, \frac{\sqrt{D_2 D_1}}{D_2 - D_1} < 1.$$

Analogous argumentation for $T = P_\perp Q$, with the roles of $\mathbf{H}$ and $\mathbf{M}$ in (3.3) being interchanged, yields the inequality

$$\|P_\perp Q\| \leq \eta\, \frac{\sqrt{D_2 D_1}}{D_2 - D_1} < 1.$$

Theorem (3.1) now implies that

$$\|Q_\perp P\| = \|P_\perp Q\| = \|Q - P\|.$$

This in turn establishes (3.4). □

In the case in which h is only nonnegative definite, assumption (1.1) implies that $\mathsf{N}(\mathbf{M}) = \mathsf{N}(\mathbf{H})$ and $\mathsf{R}(\mathbf{H}) = \mathsf{R}(\mathbf{M})$, since $\mathbf{H}$ and $\mathbf{M}$ are selfadjoint. This in turn allows us to conclude that $\mathcal{N} := \mathsf{R}(P) \cap \mathsf{N}(\mathbf{H}) \subset \mathsf{R}(Q)$, so

$$\widetilde{Q} = Q - P_{\mathcal{N}}, \qquad \widetilde{P} = P - P_{\mathcal{N}}$$

are orthogonal projections and

$$\|Q - P\| = \|\widetilde{Q} - \widetilde{P}\|.$$

Since $\mathsf{R}(\widetilde{P}) \subset \mathsf{R}(\mathbf{H})$ and $\mathsf{R}(\widetilde{Q}) \subset \mathsf{R}(\mathbf{H})$ we can reduce the problem to the positive definite case.

Theorem 3.3. *Let the positive definite forms* m *and* h *be given such that (1.1) holds. Let there exist* $0 < L_1 < L_2 < D_1 < D_2$ *such that the intervals* $[L_1, L_2] \subset \mathbb{R}$ *and* $[D_1, D_2] \subset \mathbb{R}$ *be contained in the resolvent sets of both* $\mathbf{H}$ *and* $\mathbf{M}$. *Set* $Q = E_{\mathbf{H}}[L_1, L_2]$, $P = E_{\mathbf{M}}[D_1, D_2]$ *and assume*

$$\Big[\frac{\sqrt{D_2 D_1}}{D_2 - D_1} + \frac{\sqrt{L_2 L_1}}{L_2 - L_1}\Big]\, \eta < 1$$

then

$$\|P - Q\| \leq \Big[\frac{\sqrt{D_2 D_1}}{D_2 - D_1} + \frac{\sqrt{L_2 L_1}}{L_2 - L_1}\Big]\, \eta\ . \tag{3.5}$$

Proof. The assumption $L_1 > 0$ implies that we may assume, without losing any generality, that we have the positive definite forms m and h. Theorem 2.7 and the same argument as in Theorem 3.2 implies

$$\|P_\perp Q\| = \|Q_\perp P\| = \|P - Q\|.$$

This in turn allows us to conclude that

$$\|P - Q\| \le \Big[\frac{\sqrt{D_2 D_1}}{D_2 - D_1} + \frac{\sqrt{L_2 L_1}}{L_2 - L_1}\Big]\ \eta. \tag{3.6}$$

□

Numerical experiments with the Sturm–Liouville eigenvalue problem, which were performed in [7], illustrated that in some situations the results of Theorems 3.2 and 3.3 yield considerably sharper estimates of the perturbations of the spectral subspaces than the results of [1, 2]. We now show that our theorems also apply in situations in which the theory from [1, 2] does not.

Example 3.4. Take $\mathbf{H}$, $\mathbf{M}$ as selfadjoint realizations of the differential operators

$$-\frac{\partial}{\partial x}\alpha(x)\frac{\partial}{\partial x}, \quad -\frac{\partial}{\partial x}\beta(x)\frac{\partial}{\partial x},$$

respectively, in the Hilbert space $\mathcal{H} = L^2(I)$, I a (finite or infinite) interval with, say, Dirichlet boundary conditions and non-negative bounded measurable functions $\alpha(x)$, $\beta(x)$ which satisfy

$$|\beta(x) - \alpha(x)| \le \eta\sqrt{\beta(x)\alpha(x)}.$$

Now, the form

$$\delta(u, v) = h(u, v) - m(u, v)$$

is not—in general—representable by a bounded operator. This rules out an application of the subspace perturbation theorems from [1, 2]. On the other hand both of our Theorems 3.2 and 3.3 apply and yield the corresponding estimates, e.g.

$$\|E_\alpha(D_1) - E_\beta(D_1)\| \le \frac{\sqrt{D_2 D_1}}{D_2 - D_1}\ \eta\ ,$$

when we know that $[D_1, D_2]$ is contained in the resolvent sets of both $\mathbf{H}$ and $\mathbf{M}$.

Theorem 2.7 can also be applied to yield a Hilbert–Schmidt version of Theorems 3.2 and 3.3.

Theorem 3.5. *Let the positive definite forms m and h be given such that (1.1) holds. Assume P and Q are projections which commute with the operators $\mathbf{H}$ and $\mathbf{M}$ respectively and let $\mathbf{A}$, M, $\mathbf{W}$, H as in (3.1). If both $\| Q_\perp SP \|_{\mathrm{HS}} < \infty$, $\| P_\perp S^* Q \|_{\mathrm{HS}} < \infty$ and both*

$$\mathsf{gap}(\sigma(\mathbf{A}), \sigma(M)), \qquad \mathsf{gap}(\sigma(\mathbf{W}), \sigma(H)),$$

are positive. Then $Q_\perp P$, $P_\perp Q$ and $P - Q$ are Hilbert–Schmidt operators and

$$\| Q_\perp P \|_{\mathrm{HS}}^2 \le \frac{\| Q_\perp SP \|_{\mathrm{HS}}^2}{\mathsf{gap}(\sigma(\mathbf{A}), \sigma(M))^2} \tag{3.7}$$

$$\| P_\perp Q \|_{\mathrm{HS}}^2 \le \frac{\| QSP_\perp \|_{\mathrm{HS}}^2}{\mathsf{gap}(\sigma(\mathbf{W}), \sigma(H))^2} \tag{3.8}$$

$$\| P - Q \|_{\mathrm{HS}}^2 \le \frac{\| Q_\perp SP \|_{\mathrm{HS}}^2}{\mathsf{gap}(\sigma(\mathbf{A}), \sigma(M))^2} + \frac{\| QSP_\perp \|_{\mathrm{HS}}^2}{\mathsf{gap}(\sigma(\mathbf{W}), \sigma(H))^2}\ . \tag{3.9}$$

Proof. by construction (3.1) the operator $T = T_1 = Q_\perp$ satisfies the Sylvester equation (3.3), which in this setting has the form, cf. (2.16),

$$(\mathbf{A}^{1/2}v, TM^{-1/2}u) - (\mathbf{A}^{-1/2}v, TM^{1/2}u) = (v, Q_\perp SPu), \quad v \in \mathcal{D}(\mathbf{A}^{1/2}), \quad u \in \mathcal{D}(M^{1/2}). \tag{3.10}$$

On the other hand, the operator $T = T_2 = P_\perp Q$ satisfies the "dual" equation, cf. Proposition 1.11,

$$(\mathbf{W}^{1/2}v, TH^{-1/2}u) - (\mathbf{W}^{-1/2}v, TH^{1/2}u) = (v, P_\perp S^* Qu), \quad v \in \mathcal{D}(\mathbf{W}^{1/2}), \quad u \in \mathcal{D}(H^{1/2}). \tag{3.11}$$

Now,

$$(P - Q)^2 = Q_\perp P + P_\perp Q,$$

where by Theorem 2.7 both $Q_\perp P$ and $P_\perp Q$ are Hilbert–Schmidt[2] and

$$\begin{aligned} \| P - Q \|_{HS}^2 &= \mathsf{Tr}(Q_\perp P + P_\perp Q) = \mathsf{Tr}(PQ_\perp P) + \mathsf{Tr}(QP_\perp Q) \\ &= \| Q_\perp P \|_{HS}^2 + \| P_\perp Q \|_{HS}^2 \end{aligned}$$

Using (2.17), we see that estimates (3.7)–(3.9) hold. □

Corollary 3.6. *Let the positive definite forms m and h be given such that (1.1) holds. If $\| S \|_{\mathrm{HS}} < \infty$ and the other conditions of Theorem 3.5 hold. Then*

$$\| P - Q \|_{\mathrm{HS}}^2 \leq \frac{\| S \|_{\mathrm{HS}}^2}{\min\left\{\mathsf{gap}(\sigma(\mathbf{A}), \sigma(M))^2, \mathsf{gap}(\sigma(\mathbf{W}), \sigma(H))^2\right\}} . \tag{3.12}$$

Proof. Just note that

$$\| Q_\perp SP \|_{HS}^2 + \| QSP_\perp \|_{HS}^2 \leq \| S \|_{HS}^2 .$$

□

4. Further properties of the operator S — an application in the numerical analysis

We will now present an application of Theorem 2.2 in numerical analysis. This will also demonstrate a role played by the new Hilbert–Schmidt norm estimates.

Assume now that we are given a positive definite operator $\mathbf{H}$ such that (1.2) holds. Let P be an orthogonal projection such that $\mathsf{R}(P) \subset \mathcal{Q}(h)$ and $\dim \mathsf{R}(P) = n$. We aim to obtain estimates of

$$\| E_{\mathbf{H}}(D) - P \| \tag{4.1}$$

for $\| \cdot \| = \| \cdot \|$ and $\| \cdot \| = \| \cdot \|_{HS}$.

We estimate (4.1) by an application of Theorem 2.2 (equivalently Theorem 3.5). Theorem 2.2 will allow us to improve [7, Theorem 3.2] inasmuch as that we establish estimates for the Hilbert–Schmidt norm and not just the spectral norm.

The properties of the main perturbation construction from [7], cf. [5, 6], will be summarized for reader's convenience.

We start by defining the positive definite form

$$h_P(u, v) = h(Pu, Pv) + h(P_\perp u, P_\perp v) \tag{4.2}$$

---

[2]To prove this equality one can use the singular value analysis from [2]. Alternatively, one could use the property $(\mathcal{P})$ from Theorem 2.8.

and a selfadjoint operator $\mathbf{H}_P$ which represents the form h_P in the sense of Kato. It was shown, see [5, 6, 7] that

(1) the form h_P is positive definite , hence there exists the positive definite operator $\mathbf{H}_P$ which represents h_P in the sense of Kato.
(2) $\mathcal{Q}(h) = \mathcal{Q}(h_P)$
(3) $\mathsf{R}(P)$ reduces $\mathbf{H}_P$.
(4) $\mathbf{H}^{-1} - \mathbf{H}_P^{-1}$ is a degenerate selfadjoint operator.
(5) Let $\delta h^P := h - h_P$ and let δH_s^P be the bounded selfadjoint operator such that

$$(u, \delta H_s^P v) = \delta h^P(\mathbf{H}_P^{-1/2} u, \mathbf{H}_P^{-1/2} v) \tag{4.3}$$

then δH_s^P is a degenerate operator and $\mathsf{dim}\mathsf{R}(\delta H_s^P) = 2n$.
(6) The values

$$\eta_i = \max_{\mathcal{S}\subset \mathsf{R}(P),\ \mathsf{dim}\mathcal{S}=n-i} \min \left\{ \frac{(\psi, \mathbf{H}^{-1}\psi) - (\psi, \mathbf{H}_P^{-1}\psi)}{(\psi, \mathbf{H}^{-1}\psi)} : \psi \in \mathcal{S} \right\}^{1/2} \tag{4.4}$$

together with their negatives are all non-zero eigenvalues of δH_s^P. Furthermore, η_i are all the singular values of the operator $\delta H_s^P P$.
(7)

$$|\delta h^P(\phi, \psi)| \leq \eta_n \sqrt{h_P[\psi] h_P[\phi]} \tag{4.5}$$

The estimates from [7, Theorem 3.2] only use information which is contained in η_n. New theory allows us to take advantage of other η_i.

Proposition 4.1. *Let P and h_P be as in (4.2) and let*

$$S = \mathbf{H}^{1/2}\mathbf{H}_P^{-1/2} - \overline{\mathbf{H}^{-1/2}\mathbf{H}_P^{1/2}}$$

then

$$\| SQ \| \leq \frac{\| \delta H_s^P Q \|}{\sqrt{1-\eta_n}}$$
$$\| S \| \leq \frac{\| \delta H_s^P \|}{\sqrt{1-\eta_n}}.$$

Here δH_s^P is the degenerate operator from (4.3), Q is any projection, η_n is given by (4.4) and $\| \cdot \|$ is any unitary invariant norm.

Proof.

$$\begin{aligned}(\psi, SQ\phi) &= \delta h^P(\mathbf{H}^{-1/2}\psi, \mathbf{H}_P^{-1/2}Q\phi) = \delta h^P(\mathbf{H}_P^{-1/2}(\mathbf{H}_P^{1/2}\mathbf{H}^{-1/2})\psi, \mathbf{H}_P^{-1/2}Q\phi) \\ &= (\psi, (\mathbf{H}_P^{1/2}\mathbf{H}^{-1/2})^* \delta H_s^P Q\phi), \qquad \phi, \psi \in \mathcal{H}\end{aligned}$$

(4.5) and (1.20) imply $\|\mathbf{H}_P^{1/2}\mathbf{H}^{-1/2}\| \leq 1/\sqrt{1-\eta_n}$. Property **(i)** of the symmetric norm $\| \cdot \|$ and the fact that $\delta H_s^P Q$ is a degenerate operator allow us to complete the proof. □

This proposition leads to an improved version of [7, Theorem 3.2]. Observe that $\| \delta H_s^P \|$ depends only on η_i from (4.4).

Theorem 4.2. *Let h be as in (1.2) and let P and h_P be as in (4.2) and η_i as in (4.4). Set*

$$D_P := \max_{\psi \in \mathsf{R}(P)} \frac{h[\psi]}{\|\psi\|^2}$$

and assume $\eta_n(1-\eta_n)^{-1} < (D - D_P)(D + D_P)$ then

$$\| E_{\mathbf{H}}(D) P_\perp \| \leq \frac{\sqrt{D D_P}}{D - D_P} \frac{\| \delta H_s^P P \|}{\sqrt{1-\eta_n}} . \tag{4.6}$$

Here $\| \cdot \|$ is any unitary invariant norm which has Property $(\mathcal{P})$.

Proof. Set $T = (E_{\mathbf{H}}(D))_\perp P$ and apply Theorem 2.8 to estimate the norm $\| (E_{\mathbf{H}}(D))_\perp P \|$. Proposition 4.1 now implies (4.6), cf. Corollary 3.6, [2, Corollary 3.1] and [2, Proposition 6.1]. □

Assume $\| \cdot \| = \| \cdot \|_{HS}$, then Theorem 4.2 yields the estimate

$$\| (E_{\mathbf{H}}(D))_\perp P \|_{HS} \leq \frac{\sqrt{D D_P}}{D - D_P} \frac{\sqrt{\eta_1^2 + \cdots + \eta_n^2}}{\sqrt{1-\eta_n}} . \tag{4.7}$$

Remark 4.3. If $\| \cdot \| = \| \cdot \|$ then under the conditions of Theorem 4.2 the identity $\|(E_{\mathbf{H}}(D))_\perp P\| = \|E_{\mathbf{H}}(D) - P\|$ holds, cf. Theorem 3.1. A similar relation holds for a general unitary invariant norm since according to [2, Corollary 3.1] and [2, Section 2] we have $\| (E_{\mathbf{H}}(D))_\perp P \| = \| P_\perp E_{\mathbf{H}}(D) \|$ and

$$\| E_{\mathbf{H}}(D) - P \| = \| (E_{\mathbf{H}}(D))_\perp P + P_\perp E_{\mathbf{H}}(D) \| . \tag{4.8}$$

Theorem 4.2 is therefore our version (generalization) of the $\sin\Theta$ theorem from [2, Appendiy 6.]. Same as in [2, Proposition 6.1], an estimate of (4.8) is obtained by a combination of Proposition 4.1 and available (depending on an application) information on the separation of the involved spectra, cf. Corollary 3.6. We have not specified a general estimate on $\| E_{\mathbf{H}}(D) - P \|$ since we consider such an estimate to be highly application dependent and we would not like to prejudice its form.

N	5	6	7	8	9	10
$\| (E_{\mathbf{H}}(D))_\perp P \|_{HS}$	4.4e-3	2.0e-3	1.1e-3	6.0e-4	3.7e-4	2.4e-4
$\frac{\sqrt{\lambda_3 D_{P_N}}}{\lambda_3 - D_{P_N}} \frac{\sqrt{\eta_1^2+\eta_2^2}}{\sqrt{1-\eta_2}}$	2.2e-2	1.0e-2	5.3e-3	3.3e-3	2.2e-3	1.5e-3
$\frac{\sqrt{s_1(R_2^N) + s_2(R_2^N)}}{\lambda_3 - D_{P_N}}$	2.0e-2	1.4e-2	9.6e-3	7.2e-3	5.5e-3	4.4e-3

TABLE 1. Error estimate from Theorem 4.2 and the true error

We will now evaluate (4.7) on the example from [7, Section 4]. There we have considered the positive definite operator $\mathbf{H}$ which is defined by the symmetric form

$$h(u,v) = \int_0^{2\pi} \left(u'\overline{v'} - \alpha u\overline{v}\right)\,dt$$
$$u,v \in \{f : f, f' \in L^2[0,2\pi], \mathrm{e}^{\mathrm{i}\theta} f(0) = f(2\pi)\} = \mathcal{D}(h).$$

The eigenvalues and eigenvectors of the operator $\mathbf{H}$ are

$$\begin{aligned}\omega_{\pm k} &= \left(\pm k + \frac{\theta}{2\pi}\right)^2 - \alpha, \quad z_{\pm k}(t) = \mathrm{e}^{-\mathrm{i}\left(\pm k + \frac{\theta}{2\pi}\right)t}, \quad k \in \mathbb{N}\\ \omega_0 &= \left(\frac{\theta}{2\pi}\right)^2 - \alpha, \qquad z_0(t) = \mathrm{e}^{-\mathrm{i}\frac{\theta}{2\pi}t}.\end{aligned}$$

In standard notation we have

$$\lambda_1(\mathbf{H}) = \omega_0,\ \ \lambda_2(\mathbf{H}) = \omega_{-1},\ \ \lambda_3(\mathbf{H}) = \omega_1,$$
$$u_1 = z_0,\ \ u_2 = z_{-1},\ \ u_3 = z_1.$$

For numerical experiments we chose $\theta = \pi - 10^{-4}$ and $\alpha = 0.2499$ so that the eigenvalues λ_1 and λ_2 are "small" and tightly clustered. As a test space we chose $\mathcal{Y}_N^3 = \mathsf{span}\{w_1^N, w_2^N\}$, where w_1^N and w_2^N are generated by the *smooth* N point equidistant *cubic* interpolation of the known eigenfunctions u_1 and u_2. Take P_N such that $\mathsf{R}(P_N) = \mathcal{Y}_N^3$. Since $\mathcal{Y}_N^3 \subset \mathcal{D}(\mathbf{H})$ both Theorem 4.2 and the bounds from [2] apply. Set $r_\phi = \mathbf{H}\phi + (\phi, \mathbf{H}\phi)\phi$. Since $w_1^N, w_2^N \in \mathcal{D}(\mathbf{H})$ we conclude that $r_{w_1^N}$ and $r_{w_2^N}$ are bona fide vectors. Set

$$R_2^N = \begin{bmatrix}(r_{w_1^N}, r_{w_1^N}) & (r_{w_1^N}, r_{w_2^N})\\ (r_{w_2^N}, r_{w_1^N}) & (r_{w_2^N}, r_{w_2^N})\end{bmatrix}.$$

The competing bound from [2] is

$$\| (E_{\mathbf{H}}(D))_\perp P_N \|_{HS} \leq \frac{\sqrt{s_1(R_2) + s_2(R_2)}}{\lambda_3 - D_{P_N}}. \tag{4.9}$$

We see that with the improvement of the approximation the advantage of the bound from Theorem 4.2 over (4.9) grows, see Table 1. On Table 1 we have displayed the actual measured error in the first line, in the second line we display the bound from (4.7) and in the third line Davis–Kahan bound (4.9). Further examples, where a numerical advantage of (4.6) over (4.9) is more stunning, are given in [7]. We repeat the results of the numerical experiments from [7] on Table 2. There we try to estimate the approximation error in the vector w_1^N in the $\|\cdot\|$-norm by an application of Theorem 3.2. Otherwise the makeup of Table 2 is the same as the makeup of Table 1.

We now present a variation on this example where (4.9) does not apply whereas (4.6) still gives useful information. We chose $\mathcal{Y}_N^1 = \mathsf{span}\{l_1^N, l_2^N\}$, where l_1^N and l_2^N are generated by the N point equidistant *continuous linear* interpolation of u_1 and u_2 then $r_{l_1^N}$ and $r_{l_2^N}$ are no longer bona fide vectors. Subsequently, (4.9) does not apply any more but Theorem 4.2 is still applicable. Take now Q_N such that $\mathsf{R}(Q_N) = \mathcal{Y}_N^1$. The results are presented on Table 3.

The performance of the bound (4.6) is influenced by the quotient

$$\frac{|D_{P_N} - \lambda_2|}{D_{P_N}}.$$

N	6	7	8	9	10
$\|E_{\mathbf{H}}(D) - P_N\|$	2.0e-3	1.1e-3	6.0e-4	3.7e-4	2.4e-4
$\frac{\sqrt{\lambda_2 d_{P_N}}}{\lambda_2 - d_{P_N}} \frac{\eta_2}{\sqrt{1-\eta_2}}$	1.5e0	6.2e-1	3.5e-1	2.2e-1	1.5e-1
$\frac{\sqrt{s_1(R_2^N)}}{\lambda_2 - d_{P_N}}$	3.6e+2	2.1e+2	1.5e+2	1.1e+2	8.9e+1

TABLE 2. Approximations for u_1 (here we use $d_{P_N} := \min_{\psi \in \mathsf{R}(P_N)} \frac{h[\psi]}{\|\psi\|^2}$)

N	100	120	140
$\interleave (E_{\mathbf{H}}(D))_{\perp} Q_N \interleave_{HS}$	5.2024e-005	3.6126e-005	2.6541e-5
$\frac{\sqrt{\lambda_3 D_{Q_N}}}{\lambda_3 - D_{Q_N}} \frac{\interleave \delta H_s^{Q_N} \interleave_{HS}}{\sqrt{1-\eta_n}}$	8.7374e-003	6.9293e-003	5.7302e-003

TABLE 3

D_{P_N} is an approximation[3] of λ_2 and in this example we have measured

$$\frac{|D_{P_N} - \lambda_2|}{D_{P_N}} > 0.17, \qquad N = 100, 120, 140.$$

The (under)performance of the bound (4.6) correctly detects this approximation feature of $\mathsf{R}(Q_N)$, cf. Table 3.

5. ESTIMATES FOR PERTURBATIONS OF THE SQUARE ROOT OF A NONNEGATIVE OPERATOR

In this section we will show that there are interesting applications of the equation (2.1) even when all of the coefficients $\mathbf{A}, M$ and F are unbounded. To demonstrate this we will generalize the known scalar inequality [4]

$$\frac{|\sqrt{m} - \sqrt{h}|}{\sqrt[4]{mh}} \leq \frac{|m-h|}{2\sqrt{mh}}, \quad h, m > 0. \tag{5.1}$$

[3] To be more precise D_{P_N} is Rayleigh–Ritz approximation to $\lambda_2(\mathbf{H})$ from the subspace $\mathsf{R}(P_N)$. For more on the Rayleigh–Ritz eigenvalue approximations see [7].

[4] "The relative error in the square root is bounded by the half relative error in the radicand".

to positive definite Hermitian matrices or, more generally, to positive, possibly unbounded, operators in an arbitrary Hilbert space. One of the obtained bounds is

$$\|M^{-1/4}(M^{-1/2} - H^{-1/2})H^{-1/4}\| \le \frac{1}{2}\|M^{-1/2}(M - H)H^{-1/2}\|. \tag{5.2}$$

In [13] a related bound for finite matrices was obtained. It reads

$$\|H^{-1/4}(M^{-1/2} - H^{-1/2})H^{-1/4}\| \le \frac{\eta}{2} + O(\eta^2), \tag{5.3}$$

$$\eta = \|H^{-1/2}(M - H)H^{-1/2}\|.$$

This is a more common type of estimate — the error is measured by the "unperturbed operator" only — while in our estimate the error is measured by H and M in a symmetric way. The latter type of estimate is convenient, if both operators H and M are known equally well and we are interested in a possibly sharp bound. Our bound is obviously as sharp as its scalar pendant. It is also rigorous, in contrast to (5.3) which is only asymptotic. Moreover, (5.2) will retain its validity for fairly general positive selfadjoint operators in a Hilbert space. The bound (5.2) is a "relative bound" which may be convenient in computing or measuring environments (cf. related bounds obtained for the eigenvalues and eigenvectors of the Hermitian matrices in [14] and the literature cited there). Also, this bound is readily expressed in terms of quadratic forms, which will be convenient for application with elliptic differential operators as will be shown below.

The idea of the proof is very simple, especially in the finite dimensional case which we present first, also in order to accommodate readers not interested in infinite dimension technicalities.

The basis of our proof is the obvious Sylvester equation (cf. [16])

$$M^{1/2}(M^{1/2} - H^{1/2}) + (M^{1/2} - H^{1/2})H^{1/2} = M - H, \tag{5.4}$$

valid for any Hermitian, positive definite matrices H and M. We rewrite this equation in the equivalent form

$$M^{1/4}TH^{-1/4} + M^{-1/4}TH^{1/4} = F \tag{5.5}$$

with

$$F = M^{-1/2}(M - H)H^{-1/2}, \quad T = M^{-1/4}(M^{-1/2} - H^{-1/2})H^{-1/4}, \tag{5.6}$$

which is immediately verified. This equation has a unique solution

$$T = \int_0^\infty e^{-M^{-1/2}t}M^{-1/4}FH^{-1/4}e^{-H^{-1/2}t}dt. \tag{5.7}$$

(just premultiply (5.5) by $e^{-M^{-1/2}t}M^{-1/4}$, postmultiply by $e^{-H^{-1/2}t}H^{-1/4}$, integrate from 0 to ∞ and perform partial integration on its left hand side). Hence for arbitrary vectors ϕ, ψ we have

$$|(T\psi,\phi)|^2 \le \|F\|^2 \left(\int_0^\infty \|e^{-M^{-1/2}t}M^{-1/4}\phi\|\|e^{-H^{-1/2}t}H^{-1/4}\psi\|dt\right)^2$$

$$\le \|F\|^2 \int_0^\infty \|e^{-M^{-1/2}t}M^{-1/4}\phi\|^2dt \int_0^\infty \|e^{-H^{-1/2}t}H^{-1/4}\psi\|^2dt$$

$$= \frac{\|F\|^2}{4}\|\|\psi\|^2\phi\|^2, \tag{5.8}$$

where we have used the obvious identity

$$\int_0^\infty e^{-2Ct} C dt = \frac{1}{2} I \tag{5.9}$$

for $C = H^{-1/2},\ M^{-1/2}$. Thus, (5.2) holds true.

We now turn to the Hilbert space $\mathcal{H}$ of arbitrary dimension. We assume that $\mathbf{H}$ and $\mathbf{M}$ are positive selfadjoint operators. This implies that all fractional powers of $\mathbf{H}$ and $\mathbf{M}$ are also positive. Neither of these operators need be bounded (or have bounded inverse).

Theorem 5.1. *Let* $\mathbf{H}$ *and* $\mathbf{M}$ *be positive selfadjoint operators in a Hilbert space* $\mathcal{X}$ *having the following property (A):* $\mathcal{D}(\mathbf{H}^{1/2}) = \mathcal{D}(\mathbf{M}^{1/2})$ *and the norms* $\|\mathbf{H}^{1/2} \cdot \|$ *and* $\|\mathbf{M}^{1/2} \cdot \|$ *are topologically equivalent. Then the same property is shared by* $\mathbf{H}^{1/2}$ *and* $\mathbf{M}^{1/2}$*. The operators*

$$\overline{\mathbf{M}^{-1/2}\mathbf{H}^{1/2}},\quad \overline{\mathbf{M}^{1/2}\mathbf{H}^{-1/2}},\quad \overline{\mathbf{M}^{-1/4}\mathbf{H}^{1/4}},\quad \overline{\mathbf{M}^{1/4}\mathbf{H}^{-1/4}},$$

$$\overline{\mathbf{H}^{-1/2}\mathbf{M}^{1/2}},\quad \overline{\mathbf{H}^{1/2}\mathbf{M}^{-1/2}},\quad \overline{\mathbf{H}^{-1/4}\mathbf{M}^{1/4}},\quad \overline{\mathbf{H}^{1/4}\mathbf{M}^{-1/4}} \tag{5.10}$$

are well defined and bounded. Let

$$F = \overline{\mathbf{M}^{1/2}\mathbf{H}^{-1/2}} - \overline{\mathbf{M}^{-1/2}\mathbf{H}^{1/2}} \tag{5.11}$$

and

$$T = \overline{\mathbf{M}^{1/4}\mathbf{H}^{-1/4}} - \overline{\mathbf{M}^{-1/4}\mathbf{H}^{1/4}} \tag{5.12}$$

then

$$\|T\| \le \frac{1}{2}\|F\|. \tag{5.13}$$

Proof. The fact that the square roots inherit the property (A) is a consequence of Löwner type theorems (see e.g. [8], Ch.V, Th. 4.12). The corresponding pairs of operators in (5.10) are mutually adjoint e.g. $\overline{\mathbf{M}^{-1/2}\mathbf{H}^{1/2}}^* = \overline{\mathbf{H}^{1/2}\mathbf{M}^{-1/2}}$ etc. Obviously, (5.11) and (5.12) reduce to F, T from (5.11), if the space is finite dimensional. The equation (5.5) becomes here

$$(T\mathbf{H}^{-1/4}u, \mathbf{M}^{1/4}v) + (T\mathbf{H}^{1/4}u, \mathbf{M}^{-1/4}v) = (Fu, v) \tag{5.14}$$

for $u \in \mathcal{D}_A = \mathcal{D}(\mathbf{H}^{1/4}) \cap \mathcal{D}(\mathbf{H}^{-1/4})$ and similarly for v and $\mathbf{M}$. We will now prove this.

The left hand side of (5.14) equals

$$(\overline{\mathbf{M}^{1/4}\mathbf{H}^{-1/4}}\mathbf{H}^{-1/4}u, \mathbf{M}^{1/4}v) - (\mathbf{H}^{-1/4}u, \overline{\mathbf{H}^{1/4}\mathbf{M}^{-1/4}}\mathbf{M}^{1/4}v)$$

$$+(\mathbf{H}^{1/4}u, \overline{\mathbf{H}^{-1/4}\mathbf{M}^{1/4}}\mathbf{M}^{-1/4}v) - (\overline{\mathbf{M}^{-1/4}\mathbf{H}^{1/4}}\mathbf{H}^{1/4}u, \mathbf{M}^{-1/4}v) =$$

$$(\mathbf{H}^{-1/2}u, \mathbf{M}^{1/2}v) - (u, v) + (u, v) - (\mathbf{H}^{1/2}u, \mathbf{M}^{-1/2}v) =$$

$$(\mathbf{M}^{1/2}\mathbf{H}^{-1/2}u, v) - (\mathbf{M}^{-1/2}\mathbf{H}^{1/2}u, v) = (Fu, v).$$

Now, substitute in (5.14)

$$v = e^{-\mathbf{M}^{-1/2}t}\mathbf{M}^{-1/4}\phi, \quad u = e^{-\mathbf{H}^{-1/2}t}\mathbf{H}^{-1/4}\psi \tag{5.15}$$

for any $\phi \in \mathcal{D}(\mathbf{M}^{-1/2})$, $\psi \in \mathcal{D}(\mathbf{H}^{-1/2})$. Note that subspaces $\mathbf{M}^{-1/4}\mathcal{D}(\mathbf{M}^{-1/2})$ and $\mathbf{H}^{-1/4}\mathcal{D}(\mathbf{H}^{-1/2})$ are invariant under $e^{-\mathbf{M}^{-1/2}t}$, $e^{-\mathbf{H}^{-1/2}t}$, respectively so, in (5.15) we have $u \in \mathcal{D}_A$ and $v \in \mathcal{D}_{\mathbf{M}}$. Then integrate (5.15) and use partial integration:

$$\int_0^s (Te^{-\mathbf{H}^{-1/2}t}\mathbf{H}^{-1/2}\psi, e^{-\mathbf{M}^{-1/2}t}\phi)dt + \int_0^s (Te^{-\mathbf{H}^{-1/2}t}\psi, e^{-\mathbf{M}^{-1/2}t}\mathbf{M}^{-1/2}\phi)dt =$$

$$-\int_0^s (T\frac{d}{dt}e^{-\mathbf{H}^{-1/2}t}\psi, e^{-\mathbf{M}^{-1/2}t}\phi)dt + \int_0^s (Te^{-\mathbf{H}^{-1/2}t}\psi, e^{-\mathbf{M}^{-1/2}t}\mathbf{M}^{-1/2}\phi)dt =$$

$$(T\psi,\phi) - (Te^{-\mathbf{H}^{-1/2}s}\psi, e^{-\mathbf{M}^{-1/2}s}\phi) + \int_0^s (Te^{-\mathbf{H}^{-1/2}t}\psi, (-e^{-\mathbf{M}^{-1/2}t}\mathbf{M}^{-1/2})\phi)dt$$

$$+\int_0^s (Te^{-\mathbf{H}^{-1/2}t}\psi, e^{-\mathbf{M}^{-1/2}t}\mathbf{M}^{-1/2}\phi)dt =$$

$$(T\psi,\phi) - (Te^{-\mathbf{H}^{-1/2}s}\psi, e^{-\mathbf{M}^{-1/2}s}\phi) =$$

$$\int_0^s (Fe^{-\mathbf{H}^{-1/2}t}\mathbf{H}^{-1/4}\psi, e^{-\mathbf{M}^{-1/2}t}\mathbf{M}^{-1/4}\phi)dt.$$

In the limit $s \to \infty$ by using the functional calculus for $\mathbf{H}$, $\mathbf{M}$, respectively and monotone convergence for spectral integrals we obtain

$$e^{-\mathbf{H}^{-1/2}s}\psi \to 0, \quad e^{-\mathbf{M}^{-1/2}s}\phi \to 0$$

in the norm. Hence

$$(T\psi,\phi) = \int_0^\infty (Fe^{-\mathbf{H}^{-1/2}t}\mathbf{H}^{-1/4}\psi, e^{-\mathbf{M}^{-1/2}t}\mathbf{M}^{-1/4}\phi)dt \tag{5.16}$$

where the integral on the right hand side is, in fact, Lebesgue as shows the chain of inequalities in (5.8) which are valid in this general case as well. Here the identity (5.9) is used in the weak sense:

$$\int_0^\infty (e^{-2Ct}C\phi,\phi)dt = (\phi,\phi)/2, \quad \phi \in \mathcal{D}(C)$$

for any positive selfadjoint C. Thus,

$$|(T\psi,\phi)|^2 \le \|F\|^2(\psi,\psi)(\phi,\phi)/4.$$

□

Remark 5.2. The main assertion (5.13) of Theorem 5.1 is obviously equivalent to the following statement: the inequality

$$|m(\phi,\psi) - h(\phi,\psi)| \le \varepsilon\sqrt{h(\phi,\phi)m(\phi,\psi)}$$

implies

$$|m_2(\phi,\psi) - h_2(\phi,\psi)| \le \frac{\varepsilon}{2}\sqrt{h_2(\phi,\phi)m_2(\psi,\psi)}$$

where the sesquilinear forms h, m, h_2, m_2 belong to the operators $\mathbf{H}, \mathbf{M}, \mathbf{H}^{1/2}, \mathbf{M}^{1/2}$, respectively. Thus, our theorem will be directly applicable to differential operators given in weak form.

Example 5.3. Let again $\mathbf{H}$ and $\mathbf{M}$ be as in Example 3.4. That is to say take $\mathbf{H}$, $\mathbf{M}$ as selfadjoint realizations of the differential operators

$$-\frac{\partial}{\partial x}\alpha(x)\frac{\partial}{\partial x}, \quad -\frac{\partial}{\partial x}\beta(x)\frac{\partial}{\partial x},$$

in the Hilbert space $\mathcal{H} = L^2(I)$ (again I can be a finite or infinite interval) with the Dirichlet boundary conditions and non-negative bounded measurable functions $\alpha(x)$, $\beta(x)$ which satisfy

$$|\beta(x) - \alpha(x)| \leq \varepsilon\sqrt{\beta(x)\alpha(x)}$$

Now

$$|(\mathbf{M}^{1/2}\phi, \mathbf{M}^{1/2}\psi) - (\mathbf{H}^{1/2}\phi, \mathbf{H}^{1/2}\psi)|^2 \leq \left(\int_I |\beta(x) - \alpha(x)||\psi'(x)\phi'(x)|dx\right)^2 \leq$$

$$\varepsilon^2 \int_I \alpha(x)|\psi'(x)|^2 dt \int_I \beta(x)|\phi'(x)|^2 dt = \varepsilon^2 \|\mathbf{H}^{1/2}\phi\|^2 \|\mathbf{M}^{1/2}\psi\|^2$$

hence $\|F\| \leq \varepsilon$ and Theorem 5.1 applies yielding

$$|(\mathbf{M}^{1/4}\phi, \mathbf{M}^{1/4}\psi) - (\mathbf{H}^{1/4}\phi, \mathbf{H}^{1/4}\psi)| \leq \frac{\varepsilon}{2}\|\mathbf{H}^{1/4}\phi\|\|\mathbf{M}^{1/4}\psi\|$$

or, equivalently, in the terms as in Remark 5.2

$$|m_2(\phi, \psi) - h_2(\phi, \psi)| \leq \frac{\varepsilon}{2}\sqrt{h_2(\phi, \phi)m_2(\psi, \psi)}.$$

6. Conclusion

With this work we complete our study of the weak Sylvester equation which started in [7]. A notion of a weak Sylvester equation was introduced in [7] as a tool on a way to obtain invariant subspace estimates for unbounded positive definite operators. With this paper we show that there are applications of the concept of a weak Sylvester equation outside the theory of Rayleigh–Ritz spectral approximations. We have extended out theory to infinite dimensional invariant subspaces and have obtained estimates of the difference between the corresponding spectral projections in all unitary invariant norms. With this results we have developed a counterpart of the $\sin\Theta$ theorems from [1] for perturbations of operators which are only defined as quadratic forms.

Due to the very singular nature of integral representations (which can not be avoided by reformulation of the integrals) of the solution to the equation (2.4), cf. formula (2.5), we were not able to extend the technique from [1] to prove that in the setting of Theorem 2.7 assumption $\|\| F \|\| < \infty$ also implies that there exists a bounded solution T such that $\|\| T \|\| < \infty$. We believe that this statement is true, but the proof will have to remain a task for the future and will most likely require another technique. The technique behind [12, Theorem 1.] could be a way to overcome this difficulty since the inequality (1.5) holds for bounded and invertible operators H and M. In order to complete this agenda a new way to regularize the weak Sylvester equation has to be found. We believe that the results of this article well illustrate the advantages and limitations of our form theoretic approach to weak Sylvester equation.

An application of the concept to a perturbation of the square root of a positive definite operator shows that there are other application areas for weakly formulated operator equations and that the developed techniques are (and hopefully will be) easily adaptable to new situations. The applications which we have reported in this paper are presented as an illustration only. Further applications will be the subject of the future work, cf. [6].

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