Bounds for exponentially stable semigroups

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Abstract

We give bounds for the decay as well as perturbation bounds for an exponentially stable semigroup e^{At} in a Hilbert space. The bounds are given in terms of only three quantities: the solution X of the corresponding Lyapunov equation $A^*X + XA = -I$ and the upper and lower bound of the spectrum of $A + A^*$. Our estimate is a consequence of a stronger, local estimate on $e^{At}\psi$ which nicely depends on the quantity $(X\psi, \psi)$ showing that the spectral geometry of the Lyapunov solution X — which is always Hermitian and positive definite — replaces, at least partly the possibly poor spectral geometry of the generator A. The local estimate remains meaningful also in some cases in which the semigroup is non-exponentially stable and the operator X is unbounded. The Lyapunov solution also gives new perturbation bounds which do not contain any exponentially growing factor.

A set of examples illustrates the power of our decay estimate for finite matrices; our bound appears to be never drastically worse and is not seldom drastically better than the existing estimates. We also illustrate our bound on an infinite dimensional example, that of the transport equation.

1 Introduction

Let e^{At} be a strongly continuous semigroup in a Hilbert space $\mathcal{X}$. Then, as it is well known, the generator A is closed and defined on a dense subspace $\mathcal{D}(A) \subset \mathcal{X}$ and has the property

$$\frac{d}{dt}e^{At}\psi = Ae^{At}\psi = e^{At}A\psi \quad \text{for any } \psi \in \mathcal{D}(A) \tag{1}$$

(see e.g. [9]). The semigroup (or the generator A itself) is called *exponentially stable*,[1] if

$$\|e^{At}\| \leq ce^{-\beta t}, \quad t \geq 0 \tag{2}$$

for some $c, \beta > 0$. If the space is finite dimensional e^{At} is just called the matrix exponential.

The aim of this paper is to provide new estimates for the numbers β, c above as well as for perturbations of such semigroups. The natural requirement to make the estimates independent of, possibly sensitive, intermediate quantities, like the eigenvector condition or the space dimension (for existing bounds see [6],[4] [12] and [8]) led us to formulate and prove our results in the infinite dimensional setting from the very beginning and to use tools from the Lyapunov theory. The proofs of our results are only little affected by this circumstance and the reader accustomed to finite matrices, should have no special difficulties to follow them.

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[1] Some authors call this property the *uniform* exponential stability.

The exponential stability of a semigroup e^{At} is equivalent to any of the following three properties:

$$\int_0^\infty \|e^{At}\psi\|^2 dt < \infty, \quad \text{all } \psi \tag{3}$$

$$2\Re(X\psi, A\psi) = -(\psi, \psi), \quad \psi \in \mathcal{D}(A) \tag{4}$$

for some bounded selfadjoint positive operator X (see Datko [2]) and

$$\|(\lambda I - A)^{-k}\| \le c(\beta - \lambda)^{-k}, \quad \lambda < -\beta, \quad k = 1, 2, \dots \tag{5}$$

(see Kato [9], Ch. IX). From this one immediately sees that the exponential stability of A is equivalent to that of its adjoint A^*.

In this case X is given by the strong integral

$$X = \int_0^\infty e^{A^* t} e^{At} dt. \tag{6}$$

If the space is finite dimensional there are further two characterisations of the exponential stability (then commonly called *asymptotic stability*)

$$e^{At} \to 0, \quad t \to 0 \tag{7}$$

and

$$\alpha(A) = \Re\sigma(A) < 0, \tag{8}$$

where $\sigma(A)$ denotes the spectrum of A. In this case the equation (4) may be written in the familiar form

$$A^* X + XA = -I \tag{9}$$

which is called *the (standard) Lyapunov equation.* Even in the general case we will call X from (6) *the formal solution* of the equation (9).

A typical finite-dimensional model of an exponentially stable semigroup is given by the vector differential equation

$$M\ddot{x} + C\dot{x} + Kx = 0 \tag{10}$$

where the matrices M, C, K are real symmetric, M, K positive definite and C positive semidefinite. An equivalent phase space formulation reads

$$\dot{y} = Ay, \quad A = \begin{bmatrix} 0 & L_1^T L_2^{-T} \\ -L_2^{-1} L_1 & -L_2^{-1} C L_2^{-T} \end{bmatrix}, \tag{11}$$

$$y = \begin{bmatrix} L_1^T x \\ L_2^T \dot{x} \end{bmatrix}, \quad K = L_1 L_1^T, \quad M = L_2 L_2^T \tag{12}$$

with the solution

$$y = e^{At} y_0.$$

The following theorem is easily proved.[2]

Theorem 1 *The system above is exponentially stable, if and only if the form $x^T Cx$ is non-degenerate on every eigenspace of the matrix $M^{-1}K$.*

[2]No infinite dimensional analogue of this theorem seems to have been proved yet.

An elegant estimate for the decay was obtained in [5]:

$$\|e^{At}\|^2 \le \|X\|\|X^{-1}\|e^{-\frac{t}{\|X\|}}. \tag{13}$$

The main shortcoming of this estimate is its dependence on the inverse of X (in the infinite dimensional case X^{-1} may well be unbounded). Indeed, in [16, 17] estimates without X^{-1} were obtained. For the above dissipative system the estimate from [17] reads

$$\|e^{At}\|^2 \le \frac{1}{h}\|X\|e^{-\beta_h t}\left(1+\frac{h}{\|X\|}\right)^2, \quad \beta_h = \frac{1}{h}\log\left(1+\frac{h}{\|X\|}\right) \tag{14}$$

for any $h > 0$ and $t \ge h$. This reaches the exponential factor $1/\|X\|$ only asymptotically, in this case the multiplicative factor becomes infinite. Our new estimate has the factor $1/\|X\|$ and completely reduces to

$$\|e^{At}\|^2 = e^{-t/\|X\|} \tag{15}$$

on any normal semigroup (this also shows that the factor $1/\|X\|$ cannot be improved in general).

A convenient control of the constant c in (2) is provided by the quantities

$$\delta(A) = 2 \sup_{\|x\|=1} (Ax, x)$$

and

$$\gamma(A) = 2 \inf_{\|x\|=1} (Ax, x).$$

For the above vibrational system $\delta(A)$ is equal to zero; in this case our semigroup norm estimate simply reads

$$\|e^{At}\|^2 \le \begin{cases} e^{1+\frac{1}{\|X\|\gamma(A)}}e^{-\frac{t}{\|X\|}}, & t \ge \|X\| + 1/\gamma(A) \\ 1, & t \le \|X\| + 1/\gamma(A) \end{cases}$$

This is better than (13) whenever

$$\|X\|\|X^{-1}\| > e^{1+1/(\gamma(A)\|X\|)}$$

which may be seen as an important improvement because $e \approx 2.71$ is relatively small, as a condition number (note that $\gamma(A)$ is always negative). Our norm estimate above is a consequence of the stronger, local estimate

$$\|e^{At}\psi\|^2 \le \begin{cases} \frac{(X\psi,\psi)}{\|X\|}e^{-\frac{t-\|X\|-1/\gamma(A)}{\|X\|}}, & t \ge \|X\| + 1/\gamma(A) \\ \min\left\{\|\psi\|^2, \frac{(X\psi,\psi)}{t-1/\gamma(A)}\right\}, & t \le \|X\| + 1/\gamma(A) \end{cases}$$

which is a key result of the present paper. It shows that the Lyapunov solution X controls the bound not only by its norm, but also by the quantity $(X\psi, \psi)$, that is by the spectral geometry of X. This geometry is always non-trivial because X is Hermitian and positive definite. This is in contrast to the spectral geometry of A which may be poor, especially in the infinite dimensional case. (For $\delta(A) \ne 0$ our estimates are a bit more complicated, but they are still explicit and directly computable.)

The local bound above keeps sense, if we let $\|X\| \to \infty$:

$$\|e^{At}\psi\|^2 \le \min\left\{\|\psi\|^2, \frac{(X\psi,\psi)}{t-1/\gamma(A)}\right\}.$$

thus covering some classes of non-exponentially stable semigroups as well; the operator X may be unbounded, it suffices that the quantity $(X\psi, \psi)$ be finite. The decay is not exponential anymore.

After having submitted this paper for publication we learned about closely related recent/parallel results of the Russian school, in particular those announced in [13] and proved in [14] by a method quite different from ours. This allowed us to improve our results to give the very same constants as in the above works. The improvement consisted in taking into account the quantity $\gamma(A)$ which was ignored in our original approach.[3] At present the main difference of our bounds from those from [13], [14] is twofold: (i) [13] gives bound for $\|e^{At}\|$ whereas we estimate the more refined, local quantity $\|e^{At}\psi\|^2$ from which the former estimate follows and (ii) [13] treats only finite matrices, whereas our result holds for general semigroups in a Hilbert space and extends to the non-exponentially stable case. The fact that the same norm bounds have been obtained by different methods suggests that these bounds may have a certain degree of optimality in a sense to be discovered yet.

As an illustration, we give some examples comparing our estimates with several existing ones including those from [5], [12],[4]. Our experiments show that our norm bound is competitive with the existing bounds, its advantage growing with the size of the matrix. We also include an infinite dimensional example, the transport equation. This example is both simple and illuminating, since it includes non-exponential decay, covered by our theory. To treat broader classes of partial differential operators of Mathematical Physics would be very desirable, but for that one should have a-priori estimates for $\|X\|$ which is still in want — even for the continuous version of the vibrational system above.

Finally, we show that the Lyapunov solution can also be used to obtain new perturbation bounds for such semigroups. The novelty lies in the fact that the bounds contain no exponentially growing factor as was the case with the known estimates for general semigroups. A particularly sharp bound is obtained for the second order system (11), if the perturbation is restricted to the damping matrix C alone.

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2 Exponential decay

We start by the obvious identity

$$\frac{d}{dt}(Xe^{At}\psi, e^{At}\psi) = -\|e^{At}\psi\|^2, \tag{16}$$

which follows from (4). This implies

$$\frac{d}{dt}(Xe^{At}\psi, e^{At}\psi) \leq -\frac{1}{\|X\|}(Xe^{At}\psi, e^{At}\psi)$$

By integrating the logarithm we obtain

$$(Xe^{At}\psi, e^{At}\psi) \leq e^{-\frac{t}{\|X\|}}(X\psi, \psi), \tag{17}$$

[3] The author is very obliged to the referee for calling his attention to these works as well as to Yu. M. Nechepurenko for sending his papers.

which, together with (16), gives

$$\int_t^\infty \|e^{A\tau}\psi\|^2 d\tau \le e^{-\frac{t}{\|X\|}}(X\psi,\psi), \quad \text{for all } \psi. \tag{18}$$

Thus, on the basis of the operator X alone a neat and natural bound for the 'averaged decay' of the semigroup is obtained.

Note that a complementary estimate is obtained in an analogous way, if X is invertible:

$$\int_t^\infty \|e^{A\tau}\psi\|^2 d\tau \ge e^{-t\|X^{-1}\|}(X\psi,\psi), \quad \text{for all } \psi. \tag{19}$$

To obtain a pointwise bound from (18), some more information on the semigroup is needed. Set

$$v_h = \sup_{0\le t\le h} \|e^{At}\|^2, \quad h>0. \tag{20}$$

Then for $0 \le x' - x \le h$ we have

$$\|e^{Ax'}\psi\|^2 \le v_h \|e^{Ax}\psi\|^2. \tag{21}$$

Obviously v_h is non-decreasing in h and $v_h \ge 1$.

Lemma 1 *Let ρ be a non-negative integrable function on $[0,\infty)$ such that*

$$\rho(x') \le v_h \rho(x) \quad \text{for } 0 \le x' - x \le h, \tag{22}$$

where $v_h \ge 1$ is non-decreasing in h and

$$\int_t^\infty \rho(x)dx \le e^{-\beta t} \quad \text{for some } \beta > 0. \tag{23}$$

Then

$$\rho(t) \le \frac{v_h}{h} e^{-\beta(t-h)} \quad \text{for } t \ge h. \tag{24}$$

Proof. Set

$$\widetilde{\rho} = \frac{1}{h}\int_{t-h}^t \rho(x)dx.$$

Then for $t-h \le x \le t$

$$\rho(t) \le v_h \rho(x),$$

$$\widetilde{\rho} \ge \frac{\rho(t)}{hv_h}\int_{t-h}^t dx = \frac{\rho(t)}{v_h}$$

and

$$\rho(t) \le \frac{v_h}{h}\int_{t-h}^t \rho(x)dx \le \frac{v_h}{h}e^{-\beta(t-h)}.$$

Q.E.D.

This can be immediately applied to semigroups.

Theorem 2 *Let A be exponentially stable and let X be the corresponding Lyapunov solution from (4). Then*

$$\|e^{At}\| \le \left(\sup_{0\le\tau\le\|X\|} \|e^{A\tau}\|\right) e^{-\frac{t-\|X\|}{2\|X\|}}. \tag{25}$$

Proof. By (21) and (18) the function $\rho(t) = \|e^{At}\psi\|^2/(X\psi,\psi)$ satisfies Lemma 1 with $\beta = 1/\|X\|$; it follows

$$\|e^{At}\psi\|^2 \leq \frac{v_h(X\psi,\psi)}{h} e^{-\frac{t-h}{\|X\|}}, \quad t \geq h$$

and then

$$\|e^{At}\| \leq \sqrt{\frac{v_h\|X\|}{h}} e^{-\frac{t-h}{2\|X\|}}, \quad t \geq h$$

with v_h from (20). Take $h = \|X\|$ and observe that now the inequality above holds for all t. Q.E.D.

For estimates of importance in applications we have to replace the rather abstract quantity v_h by something more suitable. Set

$$\delta(A) = 2 \sup_{\psi \in \mathcal{D}(A), \|\psi\|=1} \Re(A\psi,\psi) \tag{26}$$

From now on we will restrict ourselves to operators A for which $\delta(A)$ is finite. This is always fullfilled, if the space is finite dimensional; then we have

$$\delta(A) = \max \sigma(A + A^*) \tag{27}$$

and $\delta(A)$ is twice the logarithmic norm of A [15].[4] Now,

$$\frac{d}{dt}(e^{At}\psi, e^{At}\psi) = 2\Re(Ae^{At}\psi, e^{At}\psi) \leq \delta(A)\|e^{At}\psi\|^2, \quad \psi \in \mathcal{D}(A)$$

and by integrating the logarithm,

$$(e^{At}\psi, e^{At}\psi) \leq e^{\delta(A)t}(\psi,\psi), \tag{28}$$

$$\|e^{At}\|^2 \leq e^{\delta(A)t}. \tag{29}$$

For $\delta(A) < 0$, using the fact that (18) is an equality for $t = 0$, (28) can be integrated to give

$$(X\psi,\psi) \leq -\frac{1}{\delta(A)}\|\psi\|^2, \quad \text{for all } \psi$$

or, equivalently,

$$\frac{1}{\|X\|} + \delta(A) \geq 0. \tag{30}$$

Another consequence of (28) is

$$\|e^{Ax'}\psi\|^2 \leq e^{\delta(A)(x'-x)}\|e^{Ax}\psi\|^2, \quad x' \geq x. \tag{31}$$

Similarly we set

$$\gamma(A) = 2 \inf_{\psi \in \mathcal{D}(A), \|\psi\|=1} \Re(A\psi,\psi). \tag{32}$$

In the case of the system (11) we have

$$\gamma(A) = -2\|L_2^{-1}CL_2^{-T}\| = -2\sup_{\psi} \frac{(C\psi,\psi)}{(M\psi,\psi)}.$$

[4] In the infinite dimensional space exponentially stable semigroups can be found which have $\delta(A) = \infty$ (see our example with the transport equation below).

We will assume

$$\gamma(A) < \delta(A), \tag{33}$$

(othervise the semigroup is trivial: $e^{At} = e^{\delta(A)t/2}e^{(A-A^*)t/2}$). As above we obtain

$$\frac{d}{dt}(e^{At}\psi, e^{At}\psi) \geq \gamma(A)\|e^{At}\psi\|^2,$$

$$\|e^{At}\psi\|^2 \geq e^{\gamma(A)t}\|\psi\|^2. \tag{34}$$

Thus, $\gamma(A)$ is always negative and

$$(X\psi,\psi) \geq -\frac{(\psi,\psi)}{\gamma(A)}, \quad \|X\| \geq \|X^{-1}\|^{-1} \geq -\frac{1}{\gamma(A)}. \tag{35}$$

Finally, from (34) it follows

$$\|e^{Ax'}\psi\|^2 \geq e^{\gamma(A)(x'-x)}\|e^{Ax}\psi\|^2, \quad x' \geq x. \tag{36}$$

We allow the possibility $\gamma(A) = -\infty$ (this is always the case, if X has no bounded inverse). Then the inequalities (34), (35) and (36) become trivial.

Lemma 2 *Let ρ be as in Lemma 1, but replace the condition (22) by*

$$e^{\gamma(x'-x)}\rho(x) \leq \rho(x') \leq e^{\delta(x'-x)}\rho(x) \quad \text{for } x' \geq x \tag{37}$$

with

$$\beta + \delta > 0, \quad \delta - \gamma > 0, \quad \beta + \gamma < 0. \tag{38}$$

Then

$$\rho(t) \leq g(\beta,\delta,\gamma,t) = \begin{cases} \beta\left(\frac{1+\delta/\beta}{1-\delta/\gamma}\right)^{1+\beta/\delta} e^{-\beta t}, & t \geq \frac{1}{\delta}\ln\frac{1+\delta/\beta}{1-\delta/\gamma} \\ \frac{\delta}{1-e^{-\delta t}-\delta/\gamma}, & t \leq \frac{1}{\delta}\ln\frac{1+\delta/\beta}{1-\delta/\gamma} \end{cases} \tag{39}$$

for $\delta \neq 0$ and

$$\rho(t) \leq g(\beta,0,\gamma,t) = \begin{cases} \beta e^{1+\beta/\gamma-\beta t}, & t \geq \frac{1}{\beta}+\frac{1}{\gamma} \\ \frac{1}{t-1/\gamma} & t \leq \frac{1}{\beta}+\frac{1}{\gamma} \end{cases} \tag{40}$$

for $\delta = 0$ (note that $g(\beta,\delta,\gamma,t) \to g(\beta,0,\gamma,t)$ as $\delta \to 0$, whereas for $\gamma = -\infty$ we set $1/\gamma = 0$).

Proof. Using (23) and (37) and taking first $\delta \neq 0$ we obtain

$$\widetilde{\rho} = \frac{1}{h}\int_{t-h}^{t}\rho(x)dx \geq \frac{\rho(t)}{h}\int_{t-h}^{t}e^{-\delta(t-x)}dx = \frac{\rho(t)}{h\delta}(1-e^{-\delta h}).$$

So, for any $t > 0$ and $0 < h < t$,

$$\rho(t) \leq \frac{\delta}{1-e^{-\delta h}}\left(\int_{t-h}^{\infty}\rho(x)dx - \int_{t}^{\infty}\rho(x)dx\right) =$$

$$\frac{\delta}{1-e^{-\delta h}}\left(e^{-\beta(t-h)} + \rho(t)/\gamma\right).$$

Hence

$$\rho(t) \leq f(\beta,\delta,h,\gamma)e^{-\beta t} = \frac{\delta e^{-\beta(t-h)}}{1-e^{-\delta h}-\delta/\gamma}.$$

Since $h \in (0,t]$ is arbitrary the optimal bound is

$$\min_{h\in(0,t]} f(\beta,\delta,h,\gamma).$$

We have

$$\frac{\partial f}{\partial h} = \frac{\delta e^{\beta h}}{(1-e^{-\delta h}-\delta/\gamma)^2}\left(\beta(1-e^{-\delta h}-\delta/\gamma)-\delta e^{-\delta h}\right).$$

Now, $f(h)\to\infty$ for $h\to\infty$ and the unique zero of $\frac{\partial f}{\partial h}$ is

$$h = h_0 = \frac{1}{\delta}\ln\frac{1+\frac{\delta}{\beta}}{1-\frac{\delta}{\gamma}}. \tag{41}$$

Obviously, the quantities $1-\delta/\gamma$ and h_0 are always positive. For $h_0 \le t$ the minimum is taken at $h = h_0$ and it equals

$$f(\beta,\delta,h_0,\gamma) = \beta\left(\frac{1+\delta/\beta}{1-\delta/\gamma}\right)^{1+\beta/\delta}.$$

For $h_0 \ge t$ the minimum is taken at $t = h$ and reads

$$f(\beta,\delta,t,\gamma) = \frac{\delta e^{\beta t}}{1-e^{-\delta t}-\delta/\gamma}.$$

and (39) follows. The case $\delta = 0$ is even simpler. For $\gamma = -\infty$ we have just to set $1/\gamma = 0$. (Note the interesting fact that both g and $\partial g/\partial t$ are continuous as functions of $t\in(0,\infty)$.) Q.E.D.

Remark 1 Lemma 2. allows the substitution $\beta = 0$. Indeed, if we replace (23) by the weaker assumption

$$\int_0^\infty \rho(x)dx \le 1,$$

we still obtain (as a revisiting of the proof easily shows)

$$\rho(t) \le \begin{cases} \frac{\delta}{1-e^{-\delta t}-\delta/\gamma}, & \delta \ne 0 \\[2ex] \frac{1}{t-1/\gamma}, & \delta = 0. \end{cases} \tag{42}$$

The following theorem is a key result of this paper.

Theorem 3 *Let e^{At} be an exponentially stable semigroup with a finite $\delta(A)$. Then for $\delta(A)\ne 0$*

$$\|e^{At}\psi\|^2 \le \begin{cases} \frac{(X\psi,\psi)}{\|X\|}\left(\frac{1+\delta(A)\|X\|}{1-\delta(A)/\gamma(A)}\right)^{1+\frac{1}{\delta\|X\|}-\frac{t}{\|X\|}}, & t \ge h_0(A) \\[2ex] \min\left\{e^{\delta(A)t}\|\psi\|^2, \frac{(X\psi,\psi)\delta(A)}{1-e^{-\delta(A)t}-\delta(A)/\gamma(A)}\right\}, & t \le h_0(A). \end{cases} \tag{43}$$

where

$$h_0(A) = \frac{1}{\delta(A)}\ln\frac{1+\|X\|\delta(A)}{1-\delta(A)/\gamma(A)}.$$

For $\delta(A) = 0$

$$\|e^{At}\psi\|^2 \le \begin{cases} \frac{(X\psi,\psi)}{\|X\|} e^{-\frac{t-\|X\|-1/\gamma(A)}{\|X\|}}, & t \ge \|X\| + 1/\gamma(A) \\ \min\left\{\|\psi\|^2, \frac{(X\psi,\psi)}{t-1/\gamma(A)}\right\}, & t \le \|X\| + 1/\gamma(A). \end{cases} \tag{44}$$

Proof. Just apply the previous lemma to $\rho(t) = \|e^{At}\psi\|^2/(X\psi,\psi)$ with $\beta = 1/\|X\|$, $\delta = \delta(A)$ and $\gamma = \gamma(A)$. Q.E.D.

From this we obtain the norm bound.

Theorem 4 *Let* e^{At} *be an exponentially stable semigroup with a finite* $\delta(A)$*. Then*

$$\|e^{At}\| \le \begin{cases} \left(\frac{1+\delta(A)\|X\|}{1-\delta(A)/\gamma(A)}\right)^{\frac{1}{2}+\frac{1}{2\delta(A)\|X\|}} e^{-\frac{t}{2\|X\|}}, & t \ge h_0(A) \\ e^{\frac{\delta(A)t}{2}}, & t \le h_0(A) \end{cases} \tag{45}$$

for $\delta(A) \ne 0$ *and*

$$\|e^{At}\| \le \begin{cases} e^{\frac{1}{2}+\frac{1}{2\|X\|\gamma(A)}} e^{-\frac{t}{2\|X\|}}, & t \ge \|X\| - 1/\gamma(A) \\ 1, & t \le \|X\| - 1/\gamma(A) \end{cases} \tag{46}$$

for $\delta(A) = 0$*.*

Proof. Use the previous theorem in taking into account the inequality $(X\psi,\psi) \le \|X\|\|\psi\|^2$ as well as the fact that the three functions

$$\left(\frac{1+\delta(A)\|X\|}{1-\delta(A)/\gamma(A)}\right)^{\frac{1}{2}+\frac{1}{2\delta(A)\|X\|}} e^{-\frac{t}{2\|X\|}},$$

$$e^{\frac{\delta(A)t}{2}},$$

$$\sqrt{\frac{\delta(A)\|X\|}{1-e^{-\delta(A)t}-\delta(A)/\gamma(A)}}$$

have the common value

$$\sqrt{\frac{1+\delta(A)\|X\|}{1-\delta(A)/\gamma(A)}}$$

for $t = h_0$ and that on the interval

$$(0, h_0]$$

the second function is smaller than the third one (and similarly for $\delta(A) = 0$). Q.E.D.

The estimates above immediately imply the maximum bound

$$\|e^{At}\| \le \sqrt{\frac{1+\delta(A)\|X\|}{1-\delta(A)/\gamma(A)}}. \tag{47}$$

Remark 2 Note that for $1/\|X\| + \delta(A) = 0$, which holds for any normal matrix A, we have $h_0(A) = \infty$ and the bound (45) reduces to

$$\|e^{At}\|^2 \le e^{-t/\|X\|}.$$

This goes over into equality, whenever A is normal.[5] For a normal A our local estimate (43) reduces to[6]

$$\|e^{At}\psi\|^2 \le e^{-t/\|X\|} \min\left\{\|\psi\|^2, \quad (X\psi,\psi)/\|X\|\right\}.$$

Remark 3 The upper line of (45) (the case $t \ge h_0(A)$) is identical with the bound obtained in [14] for finite matrices. The improvement for $t \le h_0(A)$ may look trivial, but it is only with it that the whole bound turns to identity for normal matrices. The distinction between these two intervals is even more important with the local bound as will be seen below.

Local bounds. Although the operator norm bound (45) is simpler than the local bound (43) the latter has its own merit because it will give better estimates on vectors ψ with $(X\psi,\psi) \ll \|X\|\|\psi\|^2$. This shows that the Lyapunov solution X controls the semigroup evolution not merely by its norm, but also by its spectral geometry which may be poor and (in the infinite dimensional case) even void. Our local bound remains non-trivial even, if A is normal, we then have

$$\|e^{At}\psi\|^2 \le e^{-t/\|X\|} \min\left\{\|\psi\|^2, \quad (X\psi,\psi)/\|X\|\right\}.$$

Which is more important, the local estimate immediately extends to the non-exponentially stable semigroups on which the norm estimate does not exist at all. The only condition on the trajectory $e^{At}\psi$ is that the integral

$$\int_0^\infty \|e^{At}\psi\|^2 dt$$

be finite. The set $\mathcal{D}(\xi)$ of all such ψ is the domain of definition of the corresponding sesquilinear form

$$\xi(\psi,\phi) = \int_0^\infty (e^{At}\psi, e^{At}\phi)dt.$$

We may now immediately apply the estimate (42) to $\|e^{At}\psi\|^2$ thus obtaining

$$\|e^{At}\psi\|^2 \le \begin{cases} \min\left\{e^{\delta(A)t}\|\psi\|^2, \quad \frac{\xi(\psi,\psi)\delta(A)}{1-e^{-\delta(A)t}-\delta(A)/\gamma(A)}\right\}, & \delta(A) \ne 0 \\ \\ \min\left\{\|\psi\|^2, \quad \frac{\xi(\psi,\psi)}{t-1/\gamma(A)}\right\}, & \delta(A) = 0. \end{cases} \tag{48}$$

This estimate is meaningful for $\delta(A) \ge 0$ and $\gamma(A) < 0$. The case $\delta(A) = 0$ is particularly interesting: for a contractive semigroup we have a slow, non-exponential decay on any initial data ψ on which "$(X\psi,\psi)$ is finite".[7] For $\delta(A) > 0$ the obtained bound does not tend to zero, but to a finite quantity

$$\xi(\psi,\psi)/(1/\delta(A) - 1/\gamma(A)).$$

The obtained bounds can be used to obtain new bounds as follows

[5] This is in contrast to [16] and [17] where the 'normal' behaviour was attained only asymptotically and even this only in the exponential factor.

[6] Formula (44) has no application in this case.

[7] In fact, one could show that the form ξ always generates a positive selfadjoint operator X in the Hilbert space $\mathcal{X}_\xi$ which is the closure of $\mathcal{D}(\xi)$ in $\mathcal{X}$ such that $\xi(\psi,\phi) = (X^{1/2}\psi, X^{1/2}\phi)$. This operator need not be bounded or even densely defined.

- Apply the same theory to the adjoint A^*. This gives the same estimates as above, but instead of the operator X there appears the operator Y, which solves the 'dual equation'
$$AY + YA^* = -I. \tag{49}$$
(On second order systems (11) this carries nothing new, since there X and Y have the same norm.)

- Apply our theory to any operator A_1 similar to A i.e. $A_1 = B^{-1}AB$ with any bi-continuous operator B. This is paid by multiplying the right hand side of (45) by the respective condition number $\kappa(B) = \|B\|\|B^{-1}\|$ which is ≥ 1 but the new Lyapunov solution and the new delta may outweigh this drawback. A systematic search for convenient B's does not seem to be an easy task. Taking $B = X^{1/2}$ leads to $\delta(A_1) = -1/\|X\|$ ([7]).

- Shifting: For a given A and the corresponding Lyapunov solution X take a shift λ between $\alpha(A)$ and zero and use
$$\|e^{At}\| \leq e^{\lambda t}\|e^{(A-\lambda I)t}\|$$
while bounding $\|e^{(A-\lambda I)t}\|$ by Theorem 4 above. This gives improvements for t large enough.

3 Examples

We will first compare our norm bound with some known norm bounds for finite matrix exponentials. For any diagonalizable A we have the eigenvector bound
$$\|e^{At}\| \leq \kappa(S)e^{\alpha(A)t} \tag{50}$$
where κ means the spectral condition and
$$S^{-1}AS$$
is diagonal. Of course, in the estimate above we should better have set the infimum of $\kappa(S)$ over all S that diagonalize A; since this is a difficult task, we simply take the concrete S returned by the Matlab function *eig* (this S has the columns normalized to unity).

Next estimates are valid for an arbitrary matrix A with its Schur decomposition
$$U^{-1}AU = D + M \tag{51}$$
with U unitary, D diagonal and M strictly upper triangular. The van Loan bounds ([12]) read
$$\|e^{At}\| \leq vL_1(t) \equiv \sum_{k=0}^{n-1} \frac{t^k\|M^k\|}{k!}e^{\alpha(A)t} \leq \tag{52}$$
$$vL_2(t) \equiv \sum_{k=0}^{n-1} \frac{(t\|M\|)^k}{k!}e^{\alpha(A)t}. \tag{53}$$
The Kågstrom bound ([8]) reads
$$\|e^{At}\| \leq e^{\left(\alpha(A)+\frac{\|M\|_F}{\sqrt{2}}\right)t}. \tag{54}$$

All these bounds require initial reduction of the matrix A to the upper triangular form, this is appropriate for our bound as well, since the most known general algorithm [1] to compute the Lyapunov solution X also needs that reduction.[8] In using shifts our method needs to compute the Schur form only once which reduces the computational effort. The complexity for the bound vL_1 is $\mathcal{O}(n^4)$ which is prohibitive even for medium size matrices. The van Loan bounds as well as the eigenvalue bound have the best asymptotic decay while the Kågstrom bound may show no decay at all.

Finally, there is the Kreiss type bound ([4]), coming from the world of pseudospectra:[9]

$$\sup_{t\ge 0}\|e^{At}\| \le e\cdot nK(A), \tag{55}$$

where n is the size of the matrix A and

$$K(A) = \sup_{\Re z>0}(\Re z)\|(zI-A)^{-1}\|.$$

Note that $K(A)$ is never smaller than 1. This is turned into a decay estimate by taking any shift λ between $\alpha(A)$ and zero, so (55) yields

$$\|e^{At}\| \le e^{\lambda t}e\cdot nK(A-\lambda I). \tag{56}$$

The computational complexity of the Kreiss bound is considerable, because of the presence of the supremum of the resolvent norm in $K(A)$. So, this bound is more of a theoretical interest. A lower bound for $K(A)$ is easier to obtain by computing the resolvent norm on few points.

There is a simple case, in which all bounds can be easily computed, namely that of a normal matrix A. In this case all bounds, including ours, turn into equalities, except the Godunov bound and the Kreiss bound. The former has the extra factor $\kappa(X)^{1/2}$ while the latter has the extra factor $e\cdot n$ which is growing with the dimension n. Indeed, taking $\lambda=\alpha(A)$ it is immediately seen that

$$K(A-\alpha(A)I) = \sup_{\Re z>0}(\Re z)\|((z+\alpha(A))I-A)^{-1}\| = 1.$$

A tight comparison of all described bounds is not an easy numerical task especially, if one notes that a best bound would be obtained as an infimum over all possible shifted bounds. In our examples in all three cases we take the minimum over two typical shifts. It has to be noted that with our bound the improvement of the shift was mostly minor to moderate.

The Van Loan and the Kågstrom bound as well as the eigenvector bound are obviously shift-invariant.

The bounds are shown graphically: the abscissa is the time; in order to reproduce well the behaviour for finite times, the ordinate shows $\log(1+y)/\log(2)$ where y is the respective bound. This insures pretty true representation for bounds in the interval, say, $[0,2]$ and compresses large bounds, just showing their order of magnitude. Wherever an ordinate is larger than 8 (this means the bound is larger than 255) it is not shown anymore.

Graphical description:

- Thick solid lines: the two van Loan bounds
- Thin solid line: the Kågstrom bound
- Dashed line: the eigenvector bound

[8] As it is well-known, the norm of the inverse of the Lyapunov map in (9) is equal to $\|X\|$, so a bad condition of the Lyapunov equation will be automatically read-off from the computed X.

[9] The author is indebted to N. Trefethen for calling his attention to this bound.

- Small balls: the Godunov bound
- Stars: our new bound from Th. 4
- Pluses: (a lower bound of) the Kreiss bound
- Big balls: the true norm, computed by the Matlab function *expm*

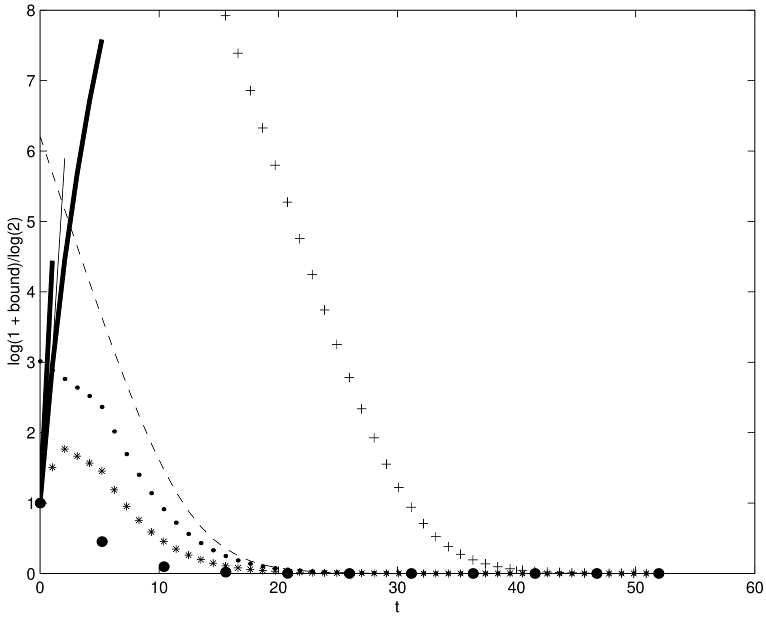

Figure 1: Random matrix of order 50

The first example is a matrix of order 50, whose elements are random numbers between $-1/2$ and $1/2$ which is made just exponentiably stable by a reasonable shift. The bounds are given on Fig. 1.

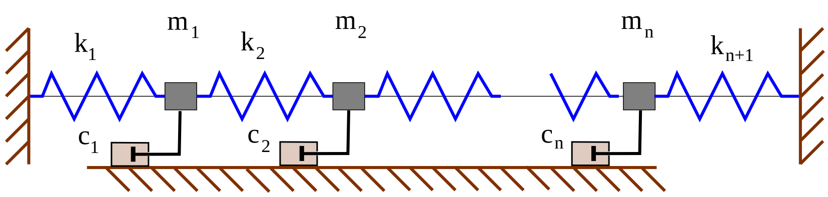

Figure 2: The n-mass oscillator with dampers

The second and the third example will be a slightly perturbed Jordan form. The bounds for

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 3 \\ 10^{-5} & 0 & 0 & -1 \end{bmatrix}$$

are on Fig. 3.

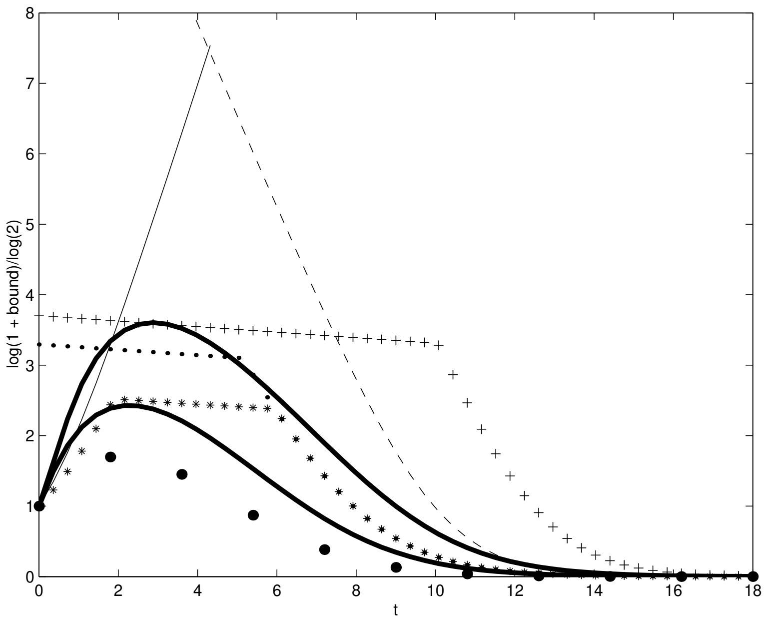

Figure 3: Perturbed Jordan form I

A similar matrix, obtained by the Matlab line

```
A=-eye(25)+diag(ones(1,24),1);A(25,1) = 1e-11;
```

is illustrated on Fig. 4.
On both pictures there are clear edges stemming from taking minimum of two exponential curves belonging to different shifts. These edges would be smoothed down by taking more shifts and minimising over all of them. The so obtained improvement would obviously be a minor one. The first of these examples also indicates that there is no much advantage in using special decay estimates which use the Jordan canonical form of A. In fact, the condition of the eigenvector matrix is about $8 \cdot 10^3$, so the matrix A goes as diagonalisable both in double and in the single precision arithmetic. In the same time this condition number already pushes the corresponding exponential bound out of competition. On the second (Fig. 4) the van Loan bounds are non-competitive as well.

Next we present two examples describing the 20-mass system from Fig. 2.

1. Homogeneous parameters, $m_i = c_i = k_i = 1$, Fig. 5.

2. As above, except $c_{11} = \cdots = c_{20} = 0$, Fig. 6.

Fig. 7 concerns the matrix of order 50

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

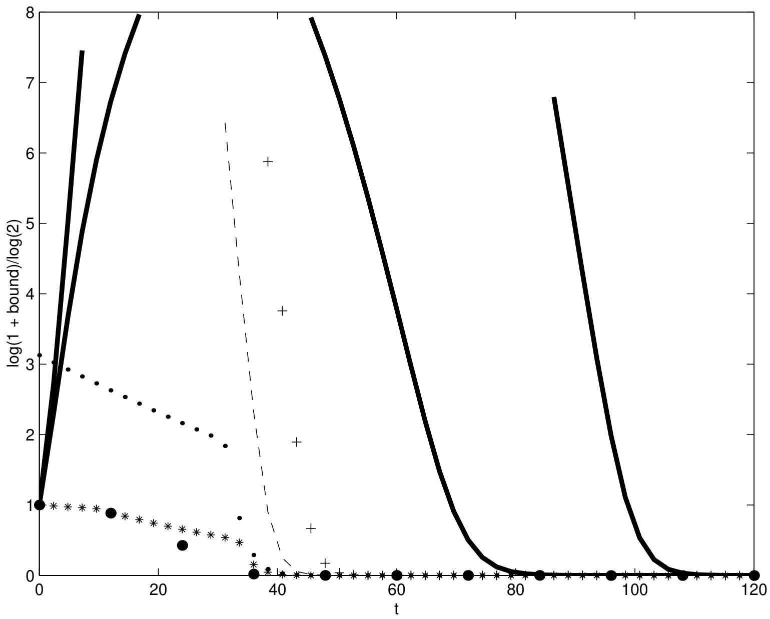

Figure 4: Perturbed Jordan form II

(this is a finite-element approximation of the continuous transport system below).

The general impression conveyed by our experiments is that our bound was never seriously worse than its competitors, the converse being often true. The advantage of our estimate grows with the dimension. It should be noted that the van Loan bounds, which grow quite drastically with the dimension, could be made more useful by trying to block-diagonalize A using a well-conditioned similarity (cf. e.g. [3]). If the obtained diagonal blocks are small the subsequent application of the van Loan estimates to the respective diagonal blocks would give better bounds. In fact, in such cases the best of all competitors to our method would be the lower of the two van Loan bounds, since its high complexity would not harm on small blocks. If the dimension of the blocks are large then our bound again wins.

Of course, as we have already said, when estimating the decay of a single trajectory $\|\exp(At)\psi\|^2$ our local estimate (48) will carry additional advantage whenever $(X\psi,\psi) \ll \|X\|\|\psi\|^2$.

A continuous example. Consider the transport equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}, \quad u = u(x,t), \tag{57}$$
$$0 < x < a, \quad t > 0,$$

under the conditions

$$u(x,0) = u_0(x), \quad u(a,t) = t, \tag{58}$$

in the Hilbert space

$$\mathcal{X} = \left\{ \psi \text{ measurable}; \quad \int_0^a g(x)|\psi(x)|^2 dx < \infty \right\}.$$

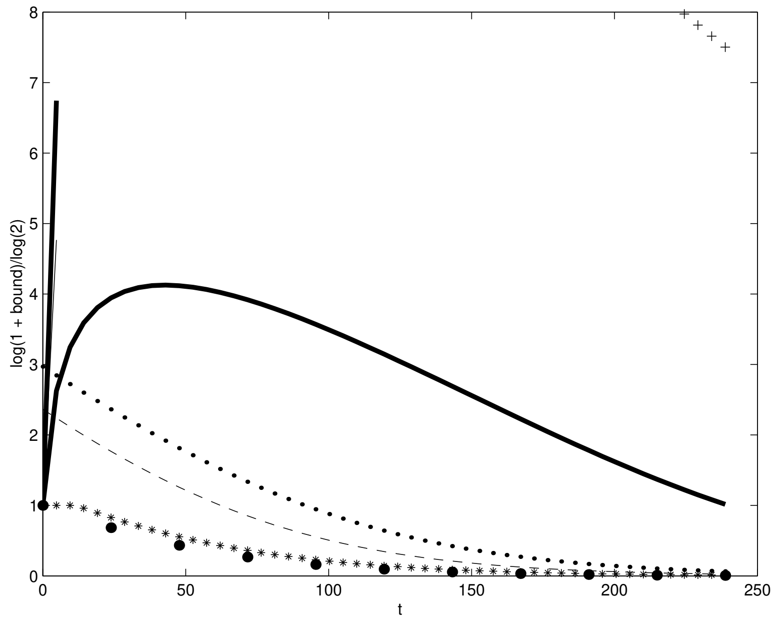

Figure 5: Proportional damping

The density function g is supposed to be positive and, say, differentiable on the open interval $(0, a)$. Under the conditions (58) the formal solution of (57)

$$u(x,t) = \begin{cases} u_0(x+t), & x+t < a \\ 0, & \text{else} \end{cases} \tag{59}$$

defines a semigroup in $\mathcal{X}$ given by

$$\left(e^{At} u_0\right)(x) = u(x,t)$$

with the property

$$e^{At} = 0, \quad t > a. \tag{60}$$

As a matter of fact, for $t < a$ we have

$$\| \left(e^{At} u_0\right)(x) \|^2 = \int_0^a g(x) |u(x,t)|^2 dx = \int_t^a g(x-t) |u_0(x)|^2 dx. \tag{61}$$

Thus, for $u_0 \in \mathcal{X}$ the function $u(\cdot, t)$ stays in $\mathcal{X}$ for any $t > 0$ whereas the semigroup property is obvious. The exponential stability follows from (60). Moreover, if g is non-decreasing then the semigroup is obviously contractive. Its generator

$$A = \frac{d}{dx}$$

is defined on

$$\mathcal{D}(A) = \{\psi, \psi' \in \mathcal{X}, \psi(a) = 0\}.$$

Hence

$$(A\psi, \psi) = -\int_0^a g' |\psi|^2 dx - g(0) |\psi(0)|^2$$

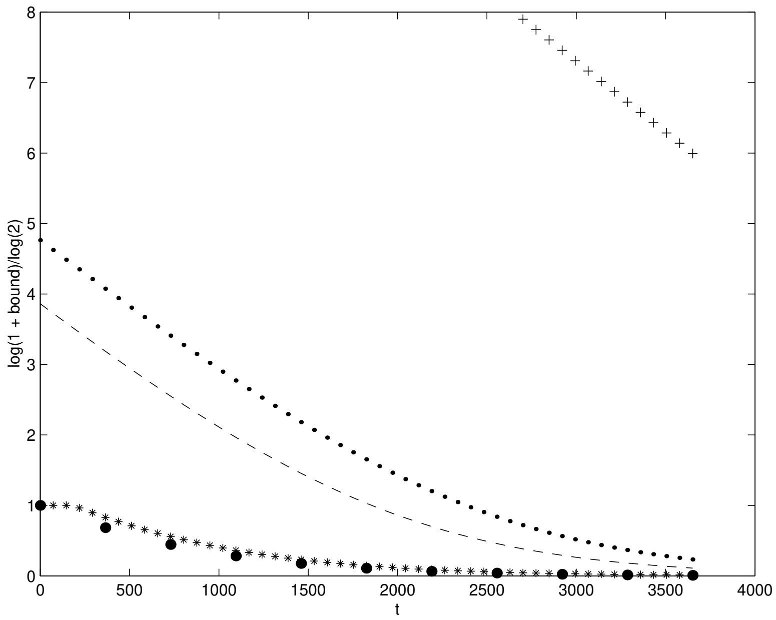

Figure 6: Non-proportional damping

and

$$\gamma(A) = -\infty, \quad \delta(A) = -\inf g'$$

thus confirming the dissipativity of A for $g' \geq 0$. Moreover, by choosing, for instance,

$$g(x) = 2\sqrt{a} - \sqrt{x}$$

then by $g'(x) = -\frac{1}{2\sqrt{x}}$ we obtain an exponentially stable semigroup e^{At} with $\delta(A) = \infty$. In any case the spectrum of A is an empty set.

By (60) and (61) we have

$$(X\psi, \psi) = \int_0^\infty \|e^{At}\|^2 dt = \int_0^a dt \int_t^a g(x-t)|\psi(x)|^2 dx =$$

$$= \int_0^a |\psi(x)|^2 dx \int_0^x g(x-t)dt = \int_0^a F(x)|\psi(x)|^2 dx$$

with

$$F(x) = \int_0^x g(\xi)d\xi.$$

Thus,

$$X\psi(x) = F(x)\psi(x).$$

We will now compute our bounds, including the shifts, for the case

$$g(x) = 1.$$

Obviously then

$$\|e^{At}\| = \begin{cases} 1, & t < a \\ 0, & t > a \end{cases}$$

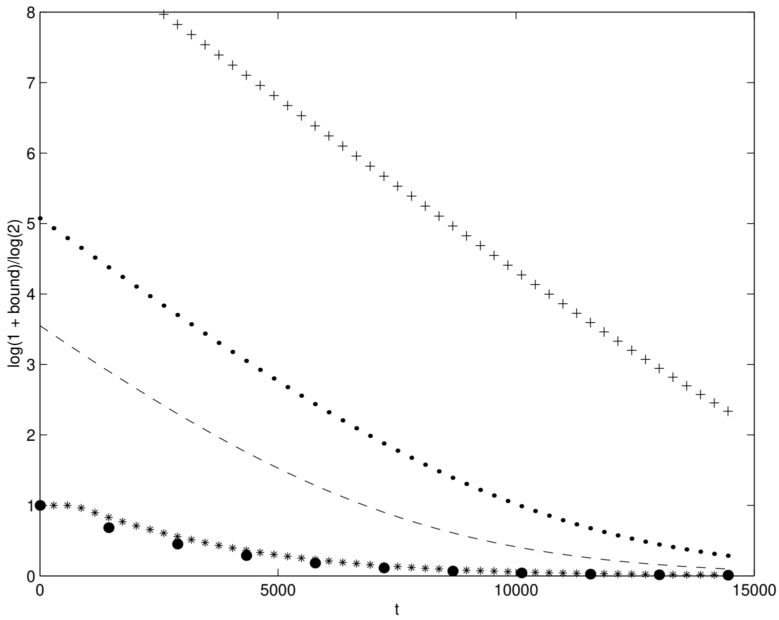

Figure 7: Transport discretised

Our estimate (46) yields $\|X\| = a$ and

$$\|e^{At}\| \leq \begin{cases} e^{-\frac{t-a}{2a}} & t > a \\ 1, & t < a \end{cases}$$

which is sharp for $t < a$ but not for $t > a$. For any $t > a$, however, we may use the shifted estimate by taking any $\lambda < 0$; and applying (45), to $A - \lambda I$. To this end we first provide the necessary quantities $\delta(A - \lambda I) = -2\lambda$, $\gamma(A - \lambda I) = -\infty$; the corresponding Lyapunov solution is given by

$$X_\lambda \psi(x) = \frac{e^{-2\lambda x} - 1}{-2\lambda} \psi(x)$$

with

$$\|X_\lambda\| = \frac{e^{-2\lambda a} - 1}{-2\lambda}, \quad h_0 = \frac{1}{-2\lambda} \ln(1 - 2\lambda \|X_\lambda\|) = a.$$

After a straightforward computation (45) yields

$$\|e^{At}\| = e^{\lambda t} \|e^{(A - \lambda I)t}\| \leq$$

$$\leq \begin{cases} \exp \dfrac{\lambda e^{-2\lambda a}(t-a)}{e^{-2\lambda a} - 1}, & t > a \\ 1, & t < a \end{cases}$$

Since

$$\frac{\lambda e^{-2\lambda a}}{e^{-2\lambda a} - 1} \to -\infty, \quad \lambda \to -\infty$$

taking infimum over all $\lambda < 0$ our bounds exactly reproduce the norm of this semigroup. Such a fine behaviour is certainly not to be expected in general, however, our method is obviously able to capture a sudden jump of the norm to zero.

Finally, the local estimate (44) reads

$$\|e^{At}\psi\|^2 \le \begin{cases} \frac{(X\psi,\psi)}{a} e^{-\frac{t-a}{a}}, & t \ge a \\ \min\left\{\|\psi\|^2, \frac{(X\psi,\psi)}{t}\right\}, & t \le a \end{cases}$$

with

$$(X\psi,\psi) = \int_0^a x|\psi(x)|^2 dx$$

so the decay is faster on functions ψ whose support is close to zero. Here we see the clear advantage of the rich spectral geometry of X over the non-existing one of A (which has no spectrum at all).

In the case $a = \infty$ the semigroup is not exponentially stable anymore, but the local estimate above keeps sense under the condition that $(X\psi,\psi)$ be finite.

4 Perturbation bounds

Known perturbation bounds in the literature contain exponentially growing factors on the right hand side ([6], [12] and [8]). With decaying semigroups such exponential factors should be removed and this is what we are going to do. Again the main ingredient will be Lyapunov solutions. We begin with a simple estimate for $e^{Bt} - e^{At}$. We decompose the perturbation as follows:

$$B - A = W = GUG^*. \tag{62}$$

Theorem 5 *If both A and B generate exponentially stable semigroups then*

$$\|e^{Bt} - e^{At}\| \le \|U\| \times$$
$$\min\{\sqrt{\|X_{B^*}(GG^*)\|\|X_A(GG^*)\|}, \sqrt{\|X_{A^*}(GG^*)\|\|X_B(GG^*)\|}\}, \tag{63}$$

where

$$X_C(D) = \int_0^\infty e^{C^*t} D e^{Ct} dt$$

is the formal solution of the Lyapunov equation

$$C^*X + XC = -D.$$

In particular, choosing $G = G^* = I$, $U = W$ we obtain

$$\|e^{Bt} - e^{At}\| \le \|W\| \min\{\sqrt{\|X_{B^*}\|\|X_A\|}, \sqrt{\|X_{A^*}\|\|X_B\|}\} \tag{64}$$

with $X_A = X_A(I)$ etc.

Proof. We start by the Duhamel equality

$$e^{Bt} - e^{At} = \int_0^t e^{A(t-\tau)} W e^{B\tau} d\tau. \tag{65}$$

For arbitrary vectors x, y we have

$$|(e^{Bt} - e^{At})x, y)|^2 \le \left(\int_0^t |(e^{A(t-\tau)} W e^{\beta\tau} x, y)| d\tau\right)^2 \le$$
$$\le \|U\|(X_{A^*}(GG^*, t)x, x)(X_B(GG^*, t)y, y) \tag{66}$$

where

$$X_B(GG^*,t) = \int_0^t e^{B^*t} GG^* e^{Bt} dt \tag{67}$$

is the formal solution of

$$B^* X_B(GG^*,t) + X_B(GG^*,t)B = -GG^* + e^{B^*t} GG^* e^{Bt} \tag{68}$$

and analogously for $X_{A^*}(GG^*,t)$. Interchanging the roles of A and B and taking norms we obtain

$$\|e^{Bt} - e^{At}\| \leq \|U\| \times$$
$$\min\{\sqrt{\|X_{B^*}(GG^*,t)\| \|X_A(GG^*,t)\|}, \sqrt{\|X_{A^*}(GG^*,t)\| \|X_B(GG^*,t)\|}\}, \tag{69}$$

By stretching the integrals from zero to infinity we obtain the statement of the theorem.
Q. E. D.

A certain shortcoming of our estimates is that they need both semigroups to be exponentially stable. With bounded perturbations W there is a known perturbation result in [9] which implies that

$$\|W\| \leq \frac{1}{2\|X\|}$$

insures the exponential stability of $e^{(A+W)t}$. The same result may be obtained with our "Lyapunov"-techniques.

On the other hand, with some classes of semigroups, like those generated by damped systems of the second order (11) we know in advance that the system is exponentially stable, or, at least dissipative. Our technique provides very sharp estimates, if in the damping matrix

$$C = \sum_i c_i w_i w_i^T, \qquad c_i \geq 0 \tag{70}$$

the viscosity parameters are perturbed as

$$c_i \mapsto c_i + \delta c_i, \qquad |\delta c_i| \leq \varepsilon c_i$$

for some $\varepsilon < 1$. We factorise the perturbation matrix as

$$G = \begin{bmatrix} 0 & 0 \\ 0 & L_2^{-1} Z \end{bmatrix}$$

with

$$Z = \begin{bmatrix} \sqrt{|\delta c_1|} w_1 & \sqrt{|\delta c_2|} w_2 & \cdots \end{bmatrix},$$
$$U = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{diag}(\mathrm{sign}(\delta c_i)) \end{bmatrix}.$$

Then $\|U\| = 1$ and

$$GG^* \leq -\varepsilon(A + A^*)/2,$$
$$GG^* \leq -\frac{\varepsilon}{1-\varepsilon}(B + B^*)/2,$$

Now,

$$X_A(GG^*) \leq -\varepsilon \int_0^\infty e^{A^*t}(A + A^*) e^{At} dt/2 = \varepsilon I/2$$

and similarly for other solutions. Now (63) gives

$$\|e^{Bt} - e^{At}\| \le \frac{\varepsilon}{2\sqrt{1-\varepsilon}}. \tag{71}$$

This estimate contains no trace of the standard Lyapunov solution X and indeed the latter need not exist at all as a revisiting of our proofs easily shows. In other words, we do not need the exponential stability to establish (71). The last is a "relative" error estimate and it would be desirable to obtain estimates of similar quality under the perturbation of the matrices M and K as well.

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