Catenary — an elementary and complete approach

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Dedicated to Professor Svetozar Kurepa on the occasion of his eightieth birthday

Abstract

The equilibrium of a standard catenary is solved without previous knowledge of the Variational Calculus. An elementary proof of the strict global minimum is provided.

The equilibrium of the catenary is serving as a beautiful example of constrained minimisation. We have to minimise the functional

$$\Psi(y) = \rho g \int_0^{x_1} y\sqrt{1+y'^2}dx, \tag{1}$$

where ρ (mass density) and g (acceleration of gravity) are positive constants, The minimisation is made among all continuously differentiable functions y satisfying the constraint

$$\int_0^{x_1} \sqrt{1+y'^2}dx = d. \tag{2}$$

We pose the boundary conditions as either

$$y(0) = 0, \quad y(x_1) = y_1 \tag{3}$$

(both ends fixed) or

$$y(0) = 0, \quad y(x_1) = -\alpha x_1, \quad \alpha > 0 \tag{4}$$

(the right end sliding along the line $y = -\alpha x$).

After introducing the fundamentals of the classical variational calculus the extremal hyperbolic cosine is readily found, at least for the boundary conditions (3); the boundary conditions (4) need more theory because the definition interval varies with the function y. Again more theory is needed to prove that the obtained function is the unique minimiser. A way out of these inconveniencies is to reformulate the problem in representing the sought curve not as a single function $y = y(x)$ but in the parameter form (cf. e.g. Troutman [1], Ch. 3)

$$x = x(s), \quad y = y(s), \quad s \text{ the arc length,}$$

with x, y continuously differentiable and

$$x'(s)^2 + y'(s)^2 - 1 = 0, \quad s \in (0, d). \tag{5}$$

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This leads to the problem of minimising the functional

$$\Psi(x,y) = \rho g \int_0^d y ds \tag{6}$$

under the constraint (5) and the boundary conditions

$$x(0) = 0, \quad y(0) = 0, \quad x(d) = x_1, \quad y(d) = y_1, \tag{7}$$

$$x(0) = 0, \quad y(0) = 0, \quad y(d) = -\alpha x(d), \quad \alpha > 0, \tag{8}$$

respectively. Already in [1] the obvious convexity simplifies establishing the global minimiser (at least for (7)). However, the fact that the penalty (6) is a linear and the constraint a quadratic function allows a further simplification which we present in the following.

Any pair x, y of continuously differentiable functions satisfying (5) and (7)/(8) will be shortly called *a configuration*. For any two configurations x, y and $\tilde{x}, \tilde{y}$ and any function $\lambda = \lambda(s)$ we have

$$\Psi(\tilde{x},\tilde{y}) = \int_0^d \left(\rho g \tilde{y} + (\tilde{x}'^2 + \tilde{y}'^2 - 1)\lambda\right) ds$$

$$= \int_0^d \left((y + \tilde{y} - y)\rho g + (y' + \tilde{y}' - y')^2\lambda + (x' + \tilde{x}' - x')^2\lambda - \lambda\right) ds$$

$$= \Psi(x,y) + \int_0^d \left((\tilde{y} - y)\rho g + 2y'\lambda(\tilde{y}' - y') + 2x'\lambda(\tilde{x}' - x') + (\tilde{y}' - y')^2\lambda + (\tilde{x}' - x')^2\lambda\right) ds.$$

By partial integration, assuming that x, y are twice continuously differentiable,

$$\Psi(\tilde{x},\tilde{y}) = \Psi(x,y) + \tag{9}$$

$$+ \int_0^d \left(\rho g(\tilde{y} - y) - 2(\lambda y')'(\tilde{y} - y) - 2(\lambda x')'(\tilde{x} - x)\right) ds \tag{10}$$

$$+ 2\lambda(d)y'(d)(\tilde{y}(d) - y(d)) + 2\lambda(d)x'(d)(\tilde{x}(d) - x(d)) \tag{11}$$

$$+ \int_0^d \lambda\left((\tilde{x}' - x')^2 + (\tilde{y}' - y')^2\right) ds. \tag{12}$$

It is immediately seen: if, we find x, y such that both (10) and (11) vanish for any configuration $\tilde{x}, \tilde{y}$ and some positive function λ then x, y is the unique minimising configuration. That is, we have to solve the following system of equations for x, y, λ

$$-(2\lambda x')' = 0, \quad \rho g - (2\lambda y')' = 0 \tag{13}$$

with the boundary conditions

$$x(0) = 0, \quad y(0) = 0, \quad y(d) = y_1, \quad x(d) = x_1, \tag{14}$$

$$x(0) = 0, \quad y(0) = 0, \quad y(d) = -\alpha x(d), \quad y'(d) = x(d)/\alpha, \tag{15}$$

respectively, together with the equation (2). (The last boundary conditions at $s = d$ say that at the right end the catenary stays orthogonal to the sliding line.)

The rest is more or less standard. The equations are immediately integrated:

$$x(s) = c_1\left(\operatorname{arsinh}\frac{s - c_2}{c_1} + \operatorname{arsinh}\frac{c_2}{c_1}\right) \tag{16}$$

$$y(s) = \sqrt{c_1^2 + (s - c_2)^2} - \sqrt{c_1^2 + c_2^2}, \tag{17}$$

$$\lambda(s) = \rho g \sqrt{c_1^2 + (s - c_2)^2}. \tag{18}$$

Here the constants c_1, c_2 are to be obtained from the boundary conditions on the right end. This is best done by turning to the representation $y = y(x)$. By setting $b = -\mathrm{arsinh}(c_2/c_1)$ we obtain

$$y = c_1 \left(\cosh\left(\frac{x}{c_1} + b\right) - \cosh b \right).$$

The necessary constants are determined from

$$\begin{aligned} d = \int_0^{x_1} \sqrt{1 + y'^2} dx &= c_1 \sinh\left(\frac{x_1}{b} + b\right) - c_1 \sinh b && (19) \\ y(x_1) &= y_1, \text{ or} && (20) \\ y(x_1) = -\alpha x_1, \quad y'(x_1) &= \sinh\left(\frac{x_1}{b} + b\right) = 1/\alpha. && (21) \end{aligned}$$

The case (20) is well-known; for completeness we give the formulae. If we divide (20) by (19) we obtain

$$\tanh \mu = \frac{y_1}{d}, \quad \mu = \frac{x_1}{2c_1} + b \tag{22}$$

which has a unique solution $\mu > 0$ whereas (20), (19) also give

$$\frac{\sqrt{d^2 - y_1^2}}{x_1} = \frac{\sinh \nu}{\nu}, \quad \nu = \frac{x_1}{2c_1} \tag{23}$$

which has a unique solution ν. This completely determines c_1, b.

In the case (21) there is one more unknown x_1:

$$\begin{aligned} -\alpha x_1 &= 2c_1 \sinh\left(\frac{x_1}{2c_1} + b\right) \sinh \frac{x_1}{2c_1} && (24) \\ 1/\alpha &= \sinh\left(\frac{x_1}{c_1} + b\right) && (25) \\ d &= 2c_1 \cosh\left(\frac{x_1}{2c_1} + b\right) \sinh \frac{x_1}{2c_1}. && (26) \end{aligned}$$

Again by dividing above the first equation with the third one,

$$\begin{aligned} -\frac{\alpha x_1}{d} &= \tanh\left(\frac{x_1}{2c_1} + b\right) && (27) \\ b &= \mathrm{arsinh}\frac{1}{\alpha} - \frac{x_1}{c_1} && (28) \\ d &= 2c_1 \cosh\left(\frac{x_1}{2c_1} + b\right) \sinh \frac{x_1}{2c_1} && (29) \end{aligned}$$

and by eliminating b,

$$\begin{aligned} -\frac{\alpha x_1}{d} &= \tanh\left(\mathrm{arsinh}\frac{1}{\alpha} - \frac{x_1}{2c_1}\right) && (30) \\ c_1 &= \frac{d\sqrt{1 - \frac{\alpha^2 x_1^2}{d^2}}}{2 \sinh \frac{x_1}{2c_1}}. && (31) \end{aligned}$$

We set $x_1 = 2c_1 z$, then

$$-\frac{\alpha x_1}{d} = \tanh\left(\mathrm{arsinh}\frac{1}{\alpha} - z\right),$$

$$\frac{x_1}{z} = \frac{\sqrt{d^2 - \alpha^2 x_1^2}}{\sinh z}, \quad \text{or} \quad x_1 = \frac{d}{\sqrt{\left(\frac{\sinh z}{z}\right)^2 + \alpha^2}}$$

and eliminating x_1 we finally obtain

$$\frac{\alpha}{\sqrt{\left(\frac{\sinh z}{z}\right)^2 + \alpha^2}} = \tanh\left(z - \operatorname{arsinh}\frac{1}{\alpha}\right)$$

with an obviously unique solution z which is positive. This determines the unknowns x_1, c_1, b as well. Their numerical computation is straightforward.

Conclusion. The present derivation (i) gives a complete answer to the stated problem: existence, computation and the uniqueness of the minimiser, (ii) avoids the complication with the variable end point in a natural way and (iii) uses the fact that the constraint is quadratic which allows the 'completing to squares' and an algebraic proof of the strict global minimum.

The catenary is an example of a mechanical system under gravity with rigid constraints; such systems can oft be described by a quadratic Lagrangian and then an elementary proof of strict global minimum is possible, see for instance [2], where a finite catenary was considered.

One mostly takes this example *after* presenting fundamentals of variational calculus. In the present approach it could also be used *as a prelude* because of its elementarity — we handle quadratic functionals and solve linear differential equations (the only nonlinearities appear with constants of integration). On the other hand, our argument in (9)–(12) already contains some essential steps of what will be the method of Lagrange.

References

[1] J. L. Troutman, Variational Calculus and Optimal Control, Springer 1983.

[2] K. Veselić, Finite catenary and the method of Lagrange, SIAM R., 37 (1995) 224-229.