Lyapunov optimization of a damped system

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Abstract

Our aim is to optimize the damping of a linear vibrating system. The penalty function is the average total energy, which is equal to the trace of the corresponding Lyapunov solution. We prove the existence and the uniqueness of the global minimum, if the damping varies over the set of all possible positive definite matrices. The minimum is shown to be taken on the so-called modal critical damping, thus confirming a long existing conjecture. We also give some preliminary results concerning dampings which depend linearly on the viscosity parameters whereas the damper positions are kept fixed. We produce physical examples on which the minimum is taken on a negative viscosity or which have several local minima. Both phenomena seem to be a consequence of a bad choice of the damper positions.

Key words: damped vibrational systems, optimal damping, parametrized damping

1 Introduction and mathematical model

We consider a vibrational system described by the differential equation

$$M\ddot{x} + C\dot{x} + Kx = 0, \tag{1a}$$

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$$x(0) = x_0, \ \dot{x}(0) = \dot{x}_0, \tag{1b}$$

where M, C and K (called mass, damping and stiffness matrix, respectively) are real, symmetric matrices of order n with M, K positive definite, and C positive semi–definite. Optimization of such systems was recently investigated in e.g. [9], [10], [8] and [1]. Infinite–dimensional linear vibrational systems were discussed in, for example, [4], [5], [6], [3], [2] and [7]. To (1) there corresponds the eigenvalue equation

$$(\lambda^2 M + \lambda C + K)x = 0. \tag{2}$$

By introducing

$$y_1 = L_1^T x, \quad y_2 = L_2^T \dot{x},$$

where

$$K = L_1 L_1^T, \quad M = L_2 L_2^T \tag{3}$$

(1) is obviously equivalent to

$$\dot{y} = Ay,$$
$$y(0) = y_0,$$

where $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$, $y_0 = \begin{pmatrix} L_1^T x_0 \\ L_2^T \dot{x}_0 \end{pmatrix}$,

$$A = \begin{pmatrix} 0 & L_1^T L_2^{-T} \\ -L_2^{-1} L_1 & -L_2^{-1} C L_2^{-T} \end{pmatrix}, \tag{4}$$

with the solution $y(t) = e^{At} y_0$. The eigenvalue problem $Ay = \lambda y$ is obviously equivalent to (2). Moreover,

$$y(t)^T y(t) = \|y(t)\|^2 = x(t)^T K x(t) + \dot{x}(t)^T M \dot{x}(t) = 2E(t, y_0),$$

where $E(t, y_0)$ is energy of the system (1) in time t. Thus, the Euclidian norm of this phase-space representation equals twice the total energy of the system.

In this note we consider the stabilizing the system (1) by requiring

$$\int_{\|y_0\|=1} \int_0^\infty E(t, y_0) \, \mathrm{d}t \, \mathrm{d}S \to \min, \tag{5}$$

where $\mathrm{d}S$ is the surface measure on the unit sphere S^{2n} and C varies over some prescribed set of admissible dampings. [2] That is, we minimize the average total energy of the system. The relation (5) can be written as

$$\int_{\|y_0\|=1} y_0^T X y_0 \, \mathrm{d}S \to \min, \tag{6}$$

[2] The case in which $\mathrm{d}S$ is an arbitrary positive measure on S^{2n} is more difficult and shall be treated in a subsequent work.

where

$$X = \int_0^\infty e^{A^T t} e^{At} \mathrm{d}t$$

solves the Lyapunov equation

$$A^T X + XA = -I. \tag{7}$$

It is well known that the solution X of the equation (7), with A given in (4), is positive definite if and only if A is stable, i.e. if the real part of the spectrum $\sigma(A)$ of A is negative. We note the useful inequality

$$\|X\| \geq -\frac{1}{2 \max \Re\sigma(A)}. \tag{8}$$

Lemma 1 *If* $\|C\|^2 \geq 4\|K\|\|M\|$ *then*

$$\max \Re\sigma(A) \geq -\frac{2\|K\|}{\|C\|}.$$

PROOF. Let x be a unit vector with $Cx = \|C\|x$. Then

$$x^T(\lambda_+^2 M + \lambda_+ C + K)x = 0$$

with

$$\lambda_+ = -\frac{2x^T K x}{\|C\| + \sqrt{\|C\|^2 - 4x^T x x^T M x}} \geq -\frac{2\|K\|}{\|C\|}.$$

Since $\lambda^2 M + \lambda C + K$ is positive definite for λ negative and close to zero, there must be an eigenvalue of A in the interval $[\lambda_+, 0)$ i.e. $\max \Re\sigma(A) \geq \lambda_+$. This completes the proof.

It can easily be seen that (6) is equivalent to

$$\mathrm{Tr}(X) \to \min.$$

This penalty function has many advantages over the time-honored 'spectral abscissa' $-\max \Re\sigma(A)$, for a detailed discussion see [8].

If all positive semi–definite dampings are allowed, K. Brabender [1] (see also [8]) showed the existence of a generic strong local minimum at the 'modal critical damping'

$$C = 2L_2\sqrt{L_2^{-1} K L_2^{-T}} L_2^T,$$

(note that here the right hand side does not depend on the choice of L_2 in (3)). This minimum was long conjectured to be global and unique and the proof of this fact is a main result of this article. Although this result has not an immediate bearing in practice it strongly speaks for the choice of TrX as the global measure of the stability.

The set of admissible dampings is typically parametrized as

$$C = C(a) = \sum_{j=1}^{s} a_j C_j \tag{9}$$

where $a_j \geq 0$ are viscosity parameters and C_j are fixed positive semi–definite matrices which usually describe damper positions. The problem was positively solved in the case of rank–one damping i.e. for $s = 1$ in (9) ([10]). With many–parameter damping the situation is more complicated.

Consider the simple system on Fig. 1.

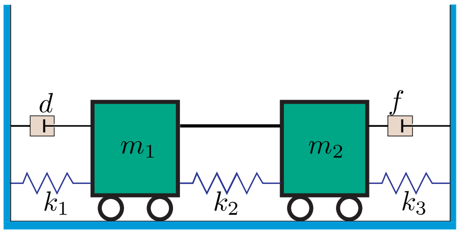

Fig. 1. Simple system

If we suppose unit masses and stiffnesses, we arrive at the matrices

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} d & 0 \\ 0 & f \end{pmatrix}.$$

By solving the equation (7) symbolically after d and f, we find

$$\tau(d,f) \equiv \mathrm{Tr}(X(d,f)) = \frac{(2fd+13)(f+d)^2+24}{3(f+d)(2fd+1)}.$$

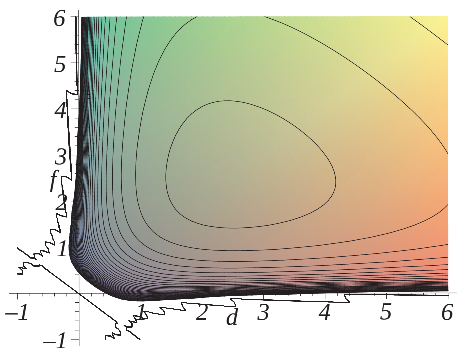

Fig. 2. Contours of τ

The contours of the function τ are plotted on Fig. 2. One can eye–ball the unique minimizer. Proceeding more carefully we note that equating $\frac{\partial\tau}{\partial d}$ and $\frac{\partial\tau}{\partial f}$ brings

$$(d-f)((d+f)^2+2)=0,$$

and hence $d=f$ at every critical point. By

$$\tau(d,d)=\frac{2(d^2+6)}{3d},$$

τ attains its unique minimum at $d=f=\sqrt{6}$.

In Section 2 we solve the problem of optimal damping in the case when the set of admissible damping matrices consists of all positive semidefinite symmetric matrices C for which the corresponding system is stable. We prove that $\mathrm{Tr}X(C)$ takes its minimum at the modal critical damping C and nowhere else. In Section 3 we prove a general theorem on the existence of the minimum of the function $C \mapsto \mathrm{Tr}X(C)$, in the case of structural damping. More precisely, if the matrix C is parametrized as in (9) then a function $a \mapsto X(C(a))$ attains its minimum on a generic set of admissible viscosities.

Finally, in Section 4 we give some examples consisting of tractable, yet representative, low dimensional systems. We produce cases in which an admissible minimum is attained at a C from (9), where some of the viscosities, a_j, are negative, i.e., unphysical. This can naturally be interpreted as a consequence of a bad distribution of damping positions C_j. In some such cases also the existence of several local minima was observed. This suggests other related problems: (i) give the conditions on C_i which insure the uniqueness and physicality of the minimum and, more generally, (ii) determine the optimal position of a damper — an equally important, but more difficult task for future research.

2 Optimal damping

Obviously, in (4) we may immediately assume $L_2=M=I$ and $K=\Omega^2$ with $\Omega = diag(\omega_1,\ldots,\omega_n)$, $\omega_i>0$. Thus,

$$A=A(C)=\begin{pmatrix} 0 & \Omega \\ -\Omega & -C \end{pmatrix}. \tag{10}$$

The Lyapunov equation (7) can be written as

$$(A_0-BCB^T)^TX+X(A_0-BCB^T)=-I, \tag{11}$$

where

$$A_0 = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The so–called dual Lyapunov equation is given by

$$(A_0 - BCB^T)Y + Y(A_0 - BCB^T)^T = -I. \tag{12}$$

Let

$$\mathcal{C}_s = \{C \in \mathbb{R}^{n\times n} : C = C^T, \Re\{\sigma(A_0 - BCB^T)\} < 0\},$$
$$D_s^+ = \{C \in \mathbb{R}^{n\times n} : C = C^T,\ C \geq 0,\ \Re\{\sigma(A_0 - BCB^T)\} < 0\},$$

and let D_s be the connected component of $\mathcal{C}_s$ which contains D_s^+. To emphasize the dependence of X and Y on C we write $X(C)$ and $Y(C)$. We are interested in the following optimization problems:

$$(OD) \qquad \text{minimize } \operatorname{tr}(X(C)) \text{ subject to } C \in D_s \text{ and (11)},$$

and

$$(OD^+) \qquad \text{minimize } \operatorname{tr}(X(C)) \text{ subject to } C \in D_s^+ \text{ and (11)} \qquad .$$

Let us define the function $f : D_s \to \mathbb{R}$ by

$$f(C) = \operatorname{tr}(X(C)), \text{ where } X(C) \text{ solves (11)}. \tag{13}$$

To prove these assertions we first need the following Lemma, which is implicitly contained in [8].

Lemma 2 *$C_0 \in \mathcal{C}_s$ is a stationary point of f, if and only if*

$$B^T X(C_0) Y(C_0) B = 0. \tag{14}$$

PROOF. Let us first calculate the differential $X'(C)$ of $X(C)$ as a function of the matrix C. Let us define the function

$$\psi(X, C) = (A_0 - BCB^T)^T X + X(A_0 - BCB^T) + I.$$

One can easily see that the function ψ satisfies the assumptions of the implicit function theorem and that

$$\frac{\partial \psi}{\partial X}\mathrm{d}X = (A_0 - BCB^T)^T \mathrm{d}X + \mathrm{d}X(A_0 - BCB^T),$$
$$\frac{\partial \psi}{\partial C}\mathrm{d}C = -(B\mathrm{d}CB^T)^T X - X(B\mathrm{d}CB^T),$$

hence the implicit function theorem yields the following representation of $X'(C)$:

$$(A_0 - BCB^T)^T X'(C)\mathrm{d}C + X'(C)\mathrm{d}C(A_0 - BCB^T) = \\ - X(C)B\mathrm{d}CB^T - B\mathrm{d}C^T B^T X(C), \quad (15)$$

so the differential $X'(C)$ on the matrix $\mathrm{d}C$ can be calculated via Lyapunov equation. Observe also that the equation (15) has the same left hand side as the equation (11). The differential $f'(C)$ can be written as

$$f'(C)\mathrm{d}C = \mathrm{tr}(X'(C)\mathrm{d}C).$$

Now multiply (12) from the left by $X'(C)\mathrm{d}C$ and multiply (15) from the right by $Y(C)$ and then take the trace of the left and right hand side of both equations. This yields

$$\mathrm{tr}(X'(C)\mathrm{d}C) = \mathrm{tr}(2B^T X(C)Y(C)B\mathrm{d}C)$$

thus completing the proof.

Theorem 3 *The problems (OD) and (OD^+) have the same solution* $2\mathrm{tr}(\Omega^{-1})$*, and in both cases the unique minimizer is* $C = 2\Omega$.

PROOF. First observe that the solution of the equation (12) can be easily calculated if we know the solution of the equation (11). Indeed, if we multiply from the left and the right hand side (11) by J , where $J = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$, we obtain

$$AJXJ + JXJA^T = -I,$$

since A is J–symmetric. This implies that the solution of (12) is $Y = JXJ$, where X is the solution of (11). Further, by (8) and Lemma 1 TrX becomes infinite on the boundary of D_s. Hence there exists a minimizer and it is a stationary point. Let now C be any such stationary point. By (14) and $Y = JXJ$ we obtain

$$X_{22}^2 = X_{12}^T X_{12}, \quad (16)$$

where $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{smallmatrix}\right]$. Let us decompose (11) into components. We get

$$-\Omega X_{12}^T - X_{12}\Omega = -I, \quad (17)$$

$$\Omega X_{11} - C X_{12}^T - X_{22}\Omega = 0, \quad (18)$$

$$\Omega X_{12} + X_{12}^T\Omega - X_{22}C - CX_{22} = -I. \quad (19)$$

From (17) it follows

$$X_{12} = \frac{1}{2}(I - S)\Omega^{-1}, \quad (20)$$

where S is skew–symmetric matrix. From the equation (18) and the previous formula it follows

$$X_{11} = \frac{1}{2}\Omega^{-1}C\Omega^{-1} + \frac{1}{2}\Omega^{-1}C\Omega^{-1}S + \Omega^{-1}X_{22}\Omega.$$

Since $\mathrm{tr}(\Omega^{-1}C\Omega^{-1}S) = 0$ we have

$$\mathrm{tr}(X) = \mathrm{tr}(X_{11}) + \mathrm{tr}(X_{22}) = \frac{1}{2}\mathrm{tr}(\Omega^{-1}C\Omega^{-1}) + 2\mathrm{tr}(X_{22}). \tag{21}$$

From (16) it follows

$$X_{12} = UX_{22} \tag{22}$$

where U is an orthogonal matrix. Note that (22) implies that X_{22} is nonsingular. Now (19) can be written as

$$-X_{22}C - CX_{22} = -I - \Omega U X_{22} - X_{22}U^T\Omega.$$

Hence C is the solution of a Lyapunov equation, and we have

$$C = \int_0^\infty e^{-X_{22}t}(I + \Omega U X_{22} + X_{22}U^T\Omega)e^{-X_{22}t}\mathrm{d}t.$$

This implies

$$C = \frac{1}{2}X_{22}^{-1} + \int_0^\infty e^{-X_{22}t}\Omega U e^{-X_{22}t}\mathrm{d}t\, X_{22} + X_{22}\int_0^\infty e^{-X_{22}t}U^T\Omega e^{-X_{22}t}\mathrm{d}t.$$

From (17) and (22) follows

$$\Omega U = \frac{1}{2}\left(X_{22}^{-1} - X_{22}^{-1}U^TSU\right), \tag{23}$$

hence

$$C = X_{22}^{-1} - \frac{1}{2}X_{22}^{-1}\int_0^\infty e^{-X_{22}t}U^TSUe^{-X_{22}t}\mathrm{d}t\, X_{22} + \frac{1}{2}X_{22}\int_0^\infty e^{-X_{22}t}U^TSUe^{-X_{22}t}\mathrm{d}t\, X_{22}^{-1}.$$

So, we have obtained

$$C = X_{22}^{-1} - \frac{1}{2}X_{22}^{-1}YX_{22} + \frac{1}{2}X_{22}YX_{22}^{-1}, \tag{24}$$

where Y is the solution of the Lyapunov equation

$$-X_{22}Y - YX_{22} = -U^TSU. \tag{25}$$

Since U^TSU is skew–symmetric, so is Y.

From (25) it follows

$$X_{22}YX_{22}^{-1} = -Y + U^TSUX_{22}^{-1},$$

which together with (24) implies

$$C = X_{22}^{-1} + \frac{1}{2}U^TSUX_{22}^{-1} - \frac{1}{2}X_{22}^{-1}U^TSU.$$

Now taking into account relation (23) we obtain

$$C = \Omega U + U^T\Omega,$$

hence

$$\Omega^{-1}C\Omega^{-1} = U\Omega^{-1} + \Omega^{-1}U^T. \tag{26}$$

From (20) and (22) we obtain

$$S = I - 2UX_{22}\Omega, \tag{27}$$

and from this and the fact that S is skew–symmetric it follows that

$$\Omega X_{22}U^T + UX_{22}\Omega = I. \tag{28}$$

This implies

$$\Omega^{-1}U^T = \Omega^{-1}X_{22}^{-1}\Omega^{-1} - \Omega^{-1}X_{22}^{-1}\Omega^{-1}UX_{22}\Omega.$$

From the previous relation it follows that

$$\mathrm{tr}(\Omega^{-1}U) = \mathrm{tr}(\Omega^{-1}X_{22}^{-1}\Omega^{-1}) - \mathrm{tr}(U\Omega^{-1}).$$

Now (21), (26) and the previous relation imply

$$\mathrm{tr}(X(C)) = g(X_{22}(C))$$

with

$$g(Z) = \frac{1}{2}\mathrm{tr}(\Omega^{-1}Z^{-1}\Omega^{-1}) + 2\mathrm{tr}(Z).$$

for any C satisfying (14). We compute g', g'' as functionals:

$$g'(Z)(Y) = \mathrm{tr}(-\frac{1}{2}\Omega^{-1}Z^{-1}YZ^{-1}\Omega^{-1} + 2Y),$$
$$g''(Z)(Y,Y) = \mathrm{tr}(\Omega^{-1}Z^{-1}YZ^{-1}YZ^{-1}\Omega^{-1}) > 0$$

for any symmetric positive definite Z and any symmetric $Y \neq 0$. The equation $g'(Z) = 0$ has the unique solution $Z_0 = \frac{1}{2}\Omega^{-1}$ and this is the unique minimizer with the minimum value $2\mathrm{Tr}(\Omega^{-1})$ (here, too, one easily sees that $g(X)$ tends to infinity as X approaches the boundary of the set of positive definite symmetric matrices). Now, $\frac{1}{2}\Omega^{-1} = X_{22}(2\Omega)$ and $C_0 = 2\Omega$ is readily seen to satisfy (14). Hence the value

$$2\mathrm{Tr}(\Omega^{-1}) = g(X_{22}(C_0)) = \mathrm{Tr}(X(C_0))$$

is strictly less than any other stationary value $\mathrm{Tr}(X(C))$. Thisr completes the proof.

3 Parametrized damping

For $a = (a_1, \ldots, a_s) \in \mathbb{R}^s$, $s \le n^2$, let

$$C = C(a) = \sum_{j=1}^{s} a_j C_j,$$

where C_j, $1 \le j \le s$ are linearly independent positive semi–definite matrices. Let

$$A(a) = \begin{pmatrix} 0 & B^T \\ -B & -C(a) \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

Lemma 4 *The $2n \times 2n$ block matrix*

$$A = \begin{pmatrix} 0 & B^T \\ -B & -C \end{pmatrix},$$

with B invertible, and C positive semi–definite matrix, is stable if and only if $Cy \ne 0$ holds for each eigenvector y of BB^T.

PROOF. Let us suppose that $BB^T y = \mu^2 y$, $y \ne 0$ and $Cy = 0$. Set $z = \left(\begin{smallmatrix} \frac{1}{i\mu} B^T y \\ y \end{smallmatrix}\right)$. Then $Az = i\mu z$.

On the other hand, one can easily see that if $Az = \lambda z$, $z = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, $z^T z = 1$ holds, then

$$\Re\lambda = -y^T C y. \tag{29}$$

Hence, if $Az = i\mu z$, $z = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \ne 0$ holds, then

$$B^T y = i\mu x, \tag{30}$$

$$-Bx - Cy = i\mu y \tag{31}$$

holds. Relation (29) implies $Cy = 0$, hence (30) and (31) imply $BB^T y = \mu^2 y$.

We set $\mathbb{R}^s_+ = \{a \in \mathbb{R}^s : a_i > 0, 1 \le i \le s\}$.

Theorem 5 *We have*

(i) If $A(a)$ is stable for some $a \in \mathbb{R}^s_+$, then $A(a)$ is stable for all $a \in \mathbb{R}^s_+$.

(ii) If $A(a)$ is stable for some $a \in \mathbb{R}^s_+$, and if $\mathcal{C}_+$ denotes the open component containing $\mathbb{R}^s_+$, of all $a \in \mathbb{R}^s$ for which the equation $A(a)^T X + XA(a) = -I$ is solvable, then $a \mapsto \mathrm{Tr}X(a)$ takes its minimum there.

PROOF. Obviously, $a \mapsto X(a)$ is a rational function, therefore so is its trace.

(i) Let $A(a^0)$ be stable for some $a^0 \in \mathbb{R}^s_+$. Then, for each eigenvector x of BB^T, from Lemma 4 follows

$$0 < x^T C(a^0)x = \sum_{j=1}^{s} (a^0)_j x^T C_j x,$$

hence $x^T C_k x > 0$, for at least one $k \in \{1, \dots, s\}$. This implies $x^T C(a)x > 0$ for all $a \in \mathbb{R}^s_+$, which, again from Lemma 4, implies that $A(a)$ is stable for all $a \in \mathbb{R}^s_+$.

(ii) First we show that $X(a)$ is positive definite (i.e. $A(a)$ is stable) on the whole of $\mathcal{C}_+$. Obviously, $X(a)$ is positive definite on $\mathbb{R}^s_+$. Suppose that there exists $b \in \mathcal{C}_+$ such that $X(b)$ is not positive definite. Then there exists an eigenvalue $\lambda(b)$ such that $\lambda(b) \leq 0$. Since $\lambda(b) = 0$ contradicts the solvability of

$$A(b)^T X(b) + X(b)A(b) = -I,$$

we must have $\lambda(b) < 0$. But, due to connectedness of $\mathcal{C}_+$ and continuity of eigenvalues of $X(a)$, there exists $a' \in \mathcal{C}_+$ such that $X(a')$ is singular, which again contradicts the solvability of Lyapunov equation. Hence, $A(a)$ is stable for all $a \in \mathcal{C}_+$. Moreover, the boundary of $\mathcal{C}_+$ consists of matrices A with $\max \Re\sigma(A) = 0$. Again by (8) and Lemma 1 we conclude that $\mathrm{Tr}(X(a)) \to \infty$ as a tends to the boundary of $\mathcal{C}_+$. Thus, $\mathrm{Tr}(X(a))$ takes its minimum on $\mathcal{C}_+$. This proves the theorem.

4 Examples

Example 6 *Let*

$$M = I, K = \begin{pmatrix} 1 & -1 \\ 1 & 201 \end{pmatrix}, C = \begin{pmatrix} d+f & -f \\ -f & f \end{pmatrix}.$$

The Figure 3 displays the function $(d, f) \mapsto \mathrm{Tr}X(d, f)$.

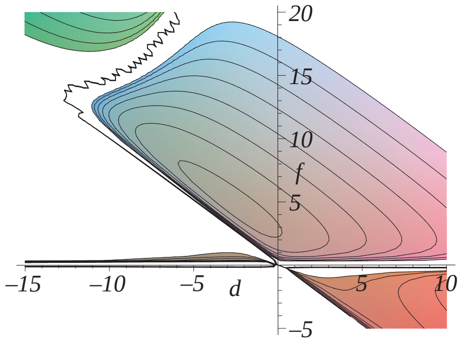

Fig. 3. $\mathrm{Tr}X(d,f)$

In the minimum we have $d<0$ *and* $f>0$.

Example 7 *Let*

$$M=I, K=\begin{pmatrix} \frac{3}{5} & -\frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{31}{50} & -\frac{1}{50} \\ 0 & -\frac{1}{50} & \frac{51}{50} \end{pmatrix}, C=\begin{pmatrix} d+f & -f & 0 \\ -f & f & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Figure 4 displays the function $(d,f)\mapsto \mathrm{Tr}X(d,f)$.

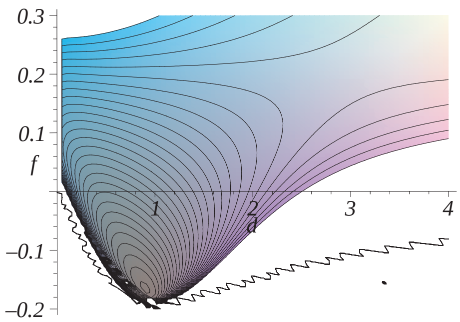

Fig. 4. $\mathrm{Tr}X(d,f)$

Hence, the minimum is outside the positive quadrant.

Example 8 *Let*

$$M=I, K=\begin{pmatrix} 16 & -15 & 0 \\ -15 & \frac{61}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{5}{4} \end{pmatrix}, C=\begin{pmatrix} d & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The Figure 5 displays the function $(d,f)\mapsto \mathrm{Tr}X(d,f)$. *One can see two local minima, both outside the positive quadrant.*

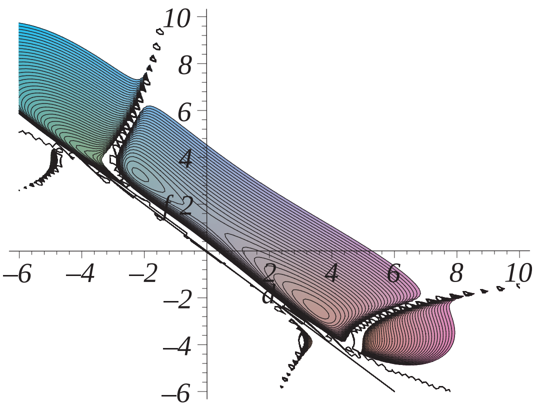

Fig. 5. Tr$X(d, f)$

Negative viscosity is not physical — it would add to the energy of the system. In cases with unphysical optimal viscosities a physical minimum will lie on a coordinate axis, i.e. we may throw away one damper without changing the value of the penalty function. Another possibility is to choose better damper positions. Optimizing them is an important problem which is beyond the scope of the present note.

As a final example, consider the truss given in the Figure 6.

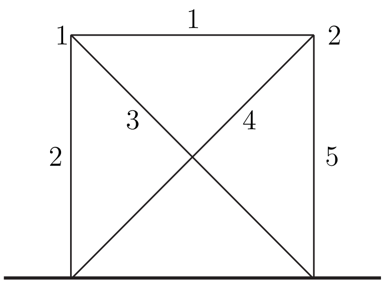

Fig. 6. Truss

With unit masses and stiffness and with dampers on bars 3 and 4, we find

$$M = I, K = \frac{1}{2}\begin{pmatrix} 3 & 1 & -2 & 0 \\ 1 & 3 & 0 & 0 \\ -2 & 0 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}, C = \frac{1}{2}\begin{pmatrix} d & d & 0 & 0 \\ d & d & 0 & 0 \\ 0 & 0 & f & -f \\ 0 & 0 & -f & f \end{pmatrix}$$

and a consequent average total energy

$$\tau(d, f) \equiv \mathrm{Tr}X(d, f) = \frac{896 + 9194df + 24882d^2f^2 + 16894d^3f^3 + 2290d^4f^4}{14(d + f)(5d^3f^3 + 43d^2f^2 + 13df + 1)} + \frac{(d^2 + f^2)(25d^3f^3 + 3855d^2f^2 + 9865df + 1461)}{14(d + f)(5d^3f^3 + 43d^2f^2 + 13df + 1)}.$$

This becomes singular at $d = -f$ and when fd is a root of $5z^3 + 43z^2 + 13z + 1$. Solving $\nabla\tau = 0$ symbolically gives 48 roots, two of which are real:

$$(d, f) = \pm\sqrt{\frac{224}{117}}(1, 1).$$

The positive one gives the unique global solution to the problem.

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