Bounds for contractive semigroups and second order systems

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Abstract

We derive a uniform bound for the difference of two contractive semigroups, if the difference of their generators is form-bounded by the Hermitian parts of the generators themselves. We construct a semigroup dynamics for second order systems with fairly general operator coefficients and apply our bound to the perturbation of the damping term. The result is illustrated on a dissipative wave equation. As a consequence the exponential decay of some second order systems is proved.

1 Introduction

The aim of this paper is to derive a new perturbation bound for strongly continuous contractive semigroups in a Hilbert space and to apply it to damped systems of second order. Let e^{At}, e^{Bt} be strongly continuous contractive semigroups in a Hilbert space $\mathcal{X}$. Their generators are maximal dissipative in the sense that $\Re(A\psi, \psi) \le 0$ and that A is maximal with this property and similarly with B. (That is, $-A$, $-B$ are maximal accretive as defined in [3]. In this paper we will follow the notations and the terminology of the Kato's monograph.)

We consider a rather restricted kind of perturbation, it reads formally

$$|(x, (B - A)y)|^2 \le \varepsilon^2 \Re(-Bx, x)\Re(-Ay, y), \quad \varepsilon > 0. \tag{1}$$

As a result we obtain a uniform estimate for the semigroups:

$$\|e^{Bt} - e^{At}\| \le \frac{\varepsilon}{2}.$$

Note that here we have not the classical situation: 'unperturbed object plus a small perturbation' in which the perturbed object often has first to be constructed and then the distance between the two is measured (see e.g. [3] Ch. XI, Th. 2.1). We impose no condition whatsoever on the size of the positive constant ε but we know that both A and B are dissipative, and both operators appear in a symmetric way. Moreover, no requirements are made about the size of the subspace $\mathcal{D}(A) \cap \mathcal{D}(B)$, it could even be trivial. To this end, (1) is rewritten in a 'weak form' as

$$|(B^*x, y) - (x, Ay)|^2 \le \varepsilon^2 \Re(-B^*x, x)\Re(-Ay, y).$$

This kind of perturbation will appear to be the proper setting for treating semigroups, generated by second order systems

$$M\ddot{x} + C\dot{x} + Kx = 0. \tag{2}$$

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Here M, C, K can be finite symmetric matrices, with *the mass matrix* M positive semidefinite, *the stiffness matrix* K positive definite and *the damping matrix* C accretive[1] (our result seems to be new even in the matrix case). Or, M, C, K may be differential operators with similar properties. As a next result, a contractive semigroup, naturally attached to (2), will be constructed, where M, C, K are understood as sesquilinear forms satisfying some mild natural regularity conditions. This construction seems to cover damped systems, more general than those treated in previous literature (cf. e.g. [1], [2], [4]), for instance, M is allowed to have a nontrivial null-space and C need not be symmetric. We then use our abstract semigroup bound to derive a bound for second order systems in which the damping term C is subject to a perturbation of the same type as (1).[2] As a consequence, the exponential decay of some damped systems will be proved. In particular, under the additional assumption that C be sectorial, a second order system is exponentially stable, if and only if the system with the 'pure symmetric damping' $\widehat{C} = (C^* + C)/2$ is such. In these applications an important property of the condition (1) will be used: it is invariant under the inversion of both operators.

The article is organised as follows. In Sect. 2 we prove the main result in a 'local' and a 'global' version. We also include an analogous bound for discrete semigroups, although we have no application for it as yet.

In Sect. 3 we apply this theory to abstract damped systems of the form (2), including the construction of the semigroup itself. In Sect. 4 we apply our theory to the damped wave equation in one dimension.

2 An abstract perturbation bound

Let A be the generator of a strongly continuous semigroup in a Hilbert space $\mathcal{X}$. By $\mathcal{T}(A)$ we denote the set of all differentiable semigroup trajectories

$$S = \left\{x = e^{At}x_0, t \geq 0\right\}, \text{ for some } x_0 \in \mathcal{D}(A).$$

Theorem 1 *Let A, B be the generators of strongly continuous semigroups in a Hilbert space $\mathcal{X}$ (then A^*, B^* are also such). Suppose that there exist trajectories $S \in \mathcal{T}(A)$, $T \in \mathcal{T}(B^*)$ and an $\varepsilon > 0$ such that for any $y \in S$, $x \in T$*

$$|(B^*x, y) - (x, Ay)|^2 \leq \varepsilon^2 \Re(-B^*x, x)\Re(-Ay, y). \tag{3}$$

Then for all such x, y

$$\left|\left(x, (e^{Bt} - e^{At})y\right)\right| \leq \frac{\varepsilon}{2}\|x\|\|y\|. \tag{4}$$

(Note that in (3) it is tacitly assumed that the factors on the right hand side are non-negative.)

Proof. For $y \in S$, $x \in T$ we have

$$\frac{d}{ds}\left(e^{B^*s}x, e^{A(t-s)}y\right) =$$

$$\left(e^{B^*s}B^*x, e^{A(t-s)}y\right) - \left(e^{B^*s}x, e^{A(t-s)}Ay\right),$$

which is continuous in s, so by integrating from 0 to t we obtain the weak Duhamel formula

$$(e^{B^*t}x, y) - (x, e^{At}y) = \int_0^t \left[(B^*e^{B^*s}x, e^{A(t-s)}y) - (e^{B^*s}x, Ae^{A(t-s)}y)\right] ds.$$

[1] For simplicity we use the term 'damping matrix' for C although it is not necessarily symmetric and thus may include a gyroscopic component.

[2] A related perturbation result for finite matrices was proved in [5].

By using (3) and the Cauchy-Schwarz inequality it follows

$$| \left(x, (e^{Bt} - e^{At} y)\right) |^2 \leq$$

$$\left(\int_0^t |(B^* e^{B^* s} x, e^{A(t-s)} y) - (e^{B^* s} x, A e^{A(t-s)} y)| ds \right)^2 \leq$$

$$\varepsilon^2 \left(\int_0^t \sqrt{\Re(-B^* e^{B^* s} x, e^{B^* s} x) \Re(-A e^{A(t-s)} y, e^{A(t-s)} y)} ds \right)^2 \leq$$

$$\varepsilon^2 \int_0^t \Re(-B^* e^{B^* s} x, e^{B^* s} x) ds \int_0^t \Re(-A e^{As} y, e^{As} y) ds.$$

By partial integration we compute

$$\mathcal{I}(A, y, t) = \int_0^t \Re(-A e^{As} y, e^{As} y) ds =$$

$$-\|e^{As} y\|^2 \Big|_0^t - \mathcal{I}(A, y, t),$$

$$\mathcal{I}(A, y, t) = \frac{1}{2} \left((y, y) - \|e^{At} y\|^2\right). \tag{5}$$

Obviously

$$0 \leq \mathcal{I}(A, y, t) \leq \frac{1}{2}(y, y)$$

and $\mathcal{I}(A, y, t)$ increases with t. Thus, there exist limits

$$\mathcal{I}(A, y, t) \nearrow \mathcal{I}(A, y, \infty) = \frac{1}{2} \left(y, y) - P(A, y)\right), \quad t \to \infty$$

$$\|e^{At} y\|^2 \searrow P(A, y), \quad t \to \infty$$

with

$$0 \leq \mathcal{I}(A, y, \infty) \leq \frac{1}{2}(y, y).$$

(and similarly for B^*). Altogether

$$| \left(x, (e^{Bt} - e^{At} y)\right) |^2 \leq \frac{\varepsilon^2}{4} \left((x, x) - P(B^*, x)\right) \left((y, y) - P(A, y)\right) \tag{6}$$

$$\leq \frac{\varepsilon^2}{4} (x, x)(y, y).$$

Q.E.D.

Remark 1 As a matter of fact, in the proof above neither of the operators need be densely defined. In this case the assertion of the theorem is valid only in the weak form

$$|(e^{B^* t} x, y) - (x, e^{At} y)| \leq \frac{\varepsilon}{2} \|x\| \|y\|.$$

Corollary 1 *Suppose that (3) holds for all y from some $S \in \mathcal{T}(A)$ and all $x \in \mathcal{D}$, where $\mathcal{D}$ is a dense subspace, invariant under $e^{B^* t}$, $t \geq 0$. Then*

$$\| \left(e^{Bt} - e^{At} y\right) \| \leq \frac{\varepsilon}{2} \|y\|. \tag{7}$$

By setting $\varepsilon = 0$ in (7) we obtain the known uniqueness of the solution of a first order differential equation:

$$e^{Bt}y = e^{At}y, \quad t \geq 0.$$

If there is a dense subspace $\mathcal{D} \in \mathcal{D}(A)$, which is invariant under e^{At}, and on which A is dissipative we set

$$P(A, y) = (P(A)y, y), \quad P(A) = \text{s-}\lim_{t\to\infty} e^{A^*t}e^{At}. \tag{8}$$

The strong limit $P(A)$ above exists by the dissipativity and obviously $0 \leq P(A) \leq I$ in the sense of forms (and similarly for B^*).

Corollary 2 *If (3) holds for all $x \in \mathcal{D}$, $y \in \mathcal{E}$, where $\mathcal{D}$, $\mathcal{E}$ are dense subspaces, invariant under e^{B^*t}, e^{At}, respectively, then*

$$|\left(x, (e^{Bt} - e^{At})y\right)|^2 \leq \frac{\varepsilon^2}{4}\left((x,x) - (P(B^*)x, x)\right)\left((y,y) - (P(A)y, y)\right). \tag{9}$$

In particular,

$$\|e^{Bt} - e^{At}\| \leq \frac{\varepsilon}{2}. \tag{10}$$

Remark 2 The corollary above certainly holds, if (3) is fulfilled for all $x \in \mathcal{D}(B^*)$ and all $y \in \mathcal{D}(\mathcal{A})$ (it is enough to require the validity of (3) on respective cores) and this will be the situation in our applications. In any of these cases both B^* and A (and then also B and A^*) are, in fact, maximal dissipative.

The condition (3) has a remarkable property of being *inversion invariant* i.e. B^* and A may be replaced by their inverses.

Proposition 1 *Suppose that both B^* and A (and then also B and A^*) are (not necessarily boundedly) invertible. Then (3), valid for all $x \in \mathcal{D}(B^*)$ and all $y \in \mathcal{D}(\mathcal{A})$ is equivalent to*

$$|(B^{-*}\xi, \eta) - (\xi, A^{-1}\eta)|^2 \leq \varepsilon^2 \Re(-B^{-*}\xi, \xi)\Re(-A^{-1}\eta, \eta), \quad \xi \in \mathcal{D}(B^{-*}), \quad \eta \in \mathcal{D}(A^{-1}). \tag{11}$$

(here B^{-*} is an abbreviation for $(B^{-1})^* = (B^*)^{-1}$).

Proof. Just set $B^*x = \xi,\ Ay = \eta$. Q.E.D.

Note that in all our results above no further restriction to the constant ε was imposed. This is partly due to the fact that the perturbation is measured by both the "perturbed" and the "unperturbed" operator in a completely symmetric way. This kind of perturbation bound will prove particularly appropriate for our applications below. If ε is further restricted important new conclusions can be drawn.

A semigroup is called *exponentially stable*[3] or *exponentially decaying*, if

$$\|e^{At}\| \leq ce^{-\beta t}, \quad t \geq 0 \tag{12}$$

for some $c, \beta > 0$.

Corollary 3 *If in Corollary 2 we have $\varepsilon < 2$ then the exponential decay of one of the semigroups implies the same for the other.*

Proof. Just recall that the exponential stability follows, if $\|e^{At}\| < 1$ for some $t > 0$. Q.E.D.

[3]Some authors call this property the *uniform* exponential stability.

Remark 3 In all that was said thus far there is an obvious symmetry: in (3) we may replace $A,\ B^*$ by $B,\ A^*$, thus obtaining the dual estimate

$$|(Bx,y)-(x,A^*y)|^2 \le \varepsilon^2 \Re(-Bx,x)\Re(-A^*y,y). \tag{13}$$

with completely analogous results. Obviously, (3) and (13) are equivalent, if $\mathcal{D}(A)=\mathcal{D}(A^*)$ and $\mathcal{D}(B)=\mathcal{D}(B^*)$.

Discrete semigroups. Every step of the perturbation theory, developed above can be correspondingly extended to discrete semigroups. An operator T is called a contraction, if $\|T\|\le 1$. For any such operator T the strong limit

$$Q(T)=\text{s-}\lim_{n\to\infty} T^{*n}T^n$$

obviously exists and satisfies

$$0\le Q(T)\le 1.$$

The following theorem sums up the most important facts.

Theorem 2 *Let A, B be contractions and*

$$|((B-A)x,y)|^2 \le \varepsilon^2((1-B^*B)x,x)((1-AA^*)y,y) \tag{14}$$

for all x,y and some $\varepsilon\ge 0$ (note that in (14) the right hand side is always non-negative). Then

$$|((B^n-A^n)x,y)|^2 \le \varepsilon^2((1-Q(B))x,x)((1-Q(A^*))y,y) \tag{15}$$

and, in particular,

$$\|B^n-A^n\| \le \varepsilon\sqrt{\|1-Q(A^*)\|\|1-Q(B)\|} \le \varepsilon. \tag{16}$$

Proof. For any x,y we have

$$|((B^n-A^n)x,y)|^2 = |(\sum_{k=0}^{n-1} A^k(B-A)B^{n-k-1}x,y)|^2 \le$$

$$\left(\sum_{k=0}^{n-1} |((B-A)B^{n-k-1}x,A^{*k}y)|\right)^2$$

$$\le \varepsilon^2\left(\sum_{k=0}^{n-1}\sqrt{(A^k(1-AA^*)A^{*k}y,y)(B^{*n-k-1}(1-B^*B)B^{n-k-1}x,x)}\right)^2$$

$$\le \varepsilon^2\sum_{k=0}^{n-1}(A^k(1-AA^*)A^{*k}y,y)\sum_{k=0}^{n-1}(B^{*k}(1-B^*B)B^k y,y)$$

$$=\varepsilon^2((1-A^nA^{*n})y,y)((1-B^{*n}B^n)x,x)$$

and (15) follows. Here we have used the identity

$$\sum_{n=0}^{n-1} A^k(1-AB)B^k = 1-A^nB^n. \tag{17}$$

Q.E.D.

It may be interesting to note that (17) appears to be a discrete analog of

$$\int_0^t e^{A\tau}(A+B)e^{B\tau}d\tau = -\left(1 - e^{At}e^{Bt}\right) \tag{18}$$

on which (5) was based.

Any contraction A is exponentially stable, if and only if $\|A^n\| < 1$ for some n. This leads to a result, analogous to Cor. 3.

Corollary 4 *Let A and B be contractions satisfying (15) with $\varepsilon < 1$. Then the exponential stability of one of them implies the same for the other.*

One might wonder that the bound (10) is uniform in t although the involved semigroups need not be exponentially decaying. As a simple example consider dissipative operators A, B in a finite dimensional space. Then each of these operators is known to be an orthogonal sum of a skew-Hermitian part and an exponentially stable part. By (3) (which is now equivalent to (1)) the skew-Hermitian parts of A and B coincide and the difference $e^{Bt} - e^{At}$ decays exponentially. The situation with discrete semigroups is similar.

In the infinite dimensional case the uniformity of the bound (10) is a more serious fact as will be illustrated on applications from Mathematical Physics below.

3 Application to damped systems

An abstract damped linear system is governed by a formal second order differential equation in a vector space $\mathcal{Y}_0$

$$\mu(\ddot{y}, v) + \theta(\dot{y}, v) + \kappa(y, v) = 0, \tag{19}$$

where μ, θ, κ are sesquilinear forms with the following properties:

- κ symmetric, strictly positive,
- μ symmetric, positive, κ-closable,
- θ κ-bounded, accretive.

A possible way to turn (19) into an operator equation is to take $(u, v) = \kappa(u, v)$ as the scalar product and to complete accordingly $\mathcal{Y}_0$ to a Hilbert space $\mathcal{Y}$. By the known representation theorems ([3]) we have

$$\mu(u, v) = (Mu, v), \quad \theta(u, v) = (Cu, v), \tag{20}$$

where M is (possibly unbounded) selfadjoint and positive and C is bounded accretive. We now replace (19) by

$$M\ddot{y} + C\dot{y} + y = 0, \tag{21}$$

where the time derivatives $\dot{y}$, $\ddot{y}$ are taken in $\mathcal{Y}$.[4]

To the equation (21) one naturally associates the phase space system, obtained by the formal substitution

$$x_1 = y, \quad x_2 = M^{1/2}\dot{y} \tag{22}$$

which leads to the first order equation

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathcal{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

[4]Our choice of the underlying scalar product in $\mathcal{Y}_0$ is fairly natural but not the only relevant one. One could show that very different, even topologically non-equivalent, choices of the scalar product still lead to the essentially same semigroup dynamics, see [4].

with

$$\mathcal{A} = \begin{pmatrix} 0 & M^{-1/2} \\ -M^{-1/2} & -M^{-1/2}CM^{-1/2} \end{pmatrix}. \tag{23}$$

which then should generate a contractive semigroup which realises the dynamics. Our conditions are far too general for this $\mathcal{A}$ to make sense as it stays (note that M may have a nontrivial null-space). However, the formal inverse

$$\mathcal{A}^+ = \begin{pmatrix} -C & -M^{1/2} \\ M^{1/2} & 0 \end{pmatrix} \tag{24}$$

is more regular, although not necessarily bounded. Considered in the 'total energy' Hilbert space $\widehat{\mathcal{X}} = \mathcal{Y} \oplus \mathcal{Y}$, $\mathcal{A}^+$ has the following properties

$$\mathcal{A}^+ \text{ is maximal dissipative,} \tag{25}$$

$$\mathcal{D}(\mathcal{A}^+) = \mathcal{D}((\mathcal{A}^+)^*) = \mathcal{D}(M^{1/2}) \oplus \mathcal{D}(M^{1/2}), \tag{26}$$

$$\mathcal{N}(\mathcal{A}^+) = \mathcal{N}((\mathcal{A}^+)^*). \tag{27}$$

All this follows from the fact that $\mathcal{A}^+$ is a sum of the skew-selfadjoint operator

$$\begin{pmatrix} 0 & -M^{1/2} \\ M^{1/2} & 0 \end{pmatrix}$$

and a bounded dissipative operator

$$-\begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, $\mathcal{N}(\mathcal{A}^+)$ reduces both $\mathcal{A}^+$ and its adjoint, the same is the case with the space $\mathcal{X}$, defined as

$$\mathcal{X} = \mathcal{N}(\mathcal{A}^+)^\perp. \tag{28}$$

More precisely, $\mathcal{A}^+$ is a direct sum of the null operator and a maximal dissipative invertible operator $\mathcal{A}^{-1}$ in the Hilbert space $\mathcal{X}$, defined on

$$\mathcal{X} \cap \mathcal{D}(\mathcal{A}^+)$$

which is dense in $\mathcal{X}$. Obviously, the operator $\mathcal{A}$ is again maximal dissipative and this is by definition the generator of our semigroup. The space $\mathcal{X}$ may be called *the physical phase space* for the system (21).[5]

Denoting by Q the orthogonal projection onto the space $\mathcal{X}$ in $\widehat{\mathcal{X}}$ we have, in fact,

$$(\lambda - \mathcal{A})^{-1}Q = \mathcal{A}^+(1 - \lambda\mathcal{A}^+)^{-1} = \frac{1}{\lambda} - \frac{1}{\lambda^2}\left(\frac{1}{\lambda} - \mathcal{A}^+\right)^{-1}, \quad \Re\lambda > 0. \tag{29}$$

Another useful identity is valid for the case of M bounded

$$(\lambda - \mathcal{A})^{-1}Q = \begin{pmatrix} \frac{1}{\lambda} - \frac{K(\lambda)^{-1}}{\lambda} & K(\lambda)^{-1}M^{-1/2} \\ -M^{-1/2}K(\lambda)^{-1} & \lambda M^{-1/2}K(\lambda)^{-1}M^{-1/2} \end{pmatrix}, \tag{30}$$

whenever $K(\lambda) = \mu^2 M + \mu C + 1$ is positive definite. Both are immediately verified.

[5]A different but related construction was used in [4] where both M and C are symmetric, but possibly unbounded.

Proposition 2 *The null-space* $\mathcal{N}(\mathcal{A}^+)$ *satisfies the inclusion*

$$\mathcal{N}(\mathcal{A}^+) \supseteq (\mathcal{N}(C) \cap \mathcal{N}(M)) \oplus \mathcal{N}(M). \tag{31}$$

If, in addition, C *is sectorial then we have the equality*

$$\mathcal{N}(\mathcal{A}^+) = (\mathcal{N}(C) \cap \mathcal{N}(M)) \oplus \mathcal{N}(M). \tag{32}$$

Proof.[6] Now, $\mathcal{N}(\mathcal{A}^+)$ is given by the equations

$$-Cx_1 - M^{1/2}x_2 = 0, \quad M^{1/2}x_1 = 0, \quad x_{1,2} \in \mathcal{D}(M^{1/2}).$$

From this the inclusion (31) follows. Let now C be sectorial. The above equations imply $(Cx_1, x_1) = -(M^{1/2}x_2, x_1) = 0$. By the assumed sectoriality it follows $Cx_1 = 0$, so (32) follows. Q.E.D.

The fact that the semigroup dynamics exists only on a closed subspace $\mathcal{X}$ of $\widehat{\mathcal{X}}$ is quite natural, even in the finite dimensional space: one cannot prescribe velocity initial data on the parts of the space where the mass is vanishing. If M is injective — no matter how singular M^{-1} may be — our dynamics exists on the whole space $\widehat{\mathcal{X}}$.

It can be shown ([4]) that this semigroup provides an appropriate solution to the second order system (21) via the formulae (22), at least in the special case of M, C bounded symmetric. In our, more general situation we can show that $\mathcal{A}$ yields the "true" dynamics by way of approximation. We approximate the operator M by a sequence M_n of bounded, positive operators such that

$$M_n^{1/2}x \to M^{1/2}x, \quad x \in \mathcal{D}(M^{1/2}). \tag{33}$$

If, in addition, all M_n are positive definite the operator (23)

$$\mathcal{A}_n = \begin{pmatrix} 0 & M_n^{-1/2} \\ -M_n^{-1/2} & -M_n^{-1/2}CM_n^{-1/2} \end{pmatrix} \tag{34}$$

is bounded dissipative in $\widehat{\mathcal{X}}$ and its semigroup trivially reproduces the solution of the so modified second order system

$$M_n\ddot{y} + C\dot{y} + y = 0, \tag{35}$$

An example of such sequence is

$$M_n = f_n(M), \quad f_n(\lambda) = \begin{cases} \frac{1}{n}, & 0 \le \lambda \le \frac{1}{n} \\ \lambda, & \frac{1}{n} \le \lambda \le n \\ n, & n \le \lambda \end{cases}$$

Note that here, in addition, the operators M_n are both bounded and boundedly invertible, being positive definite.

Proposition 3 *For any* $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{X}$ *and any approximation sequence (33) we have*

$$e^{\mathcal{A}_n}x \to e^{\mathcal{A}}x, \quad n \to \infty. \tag{36}$$

uniformly on any compact interval in t. *Choose, in addition,* M_n *as positive definite and set*

$$\begin{pmatrix} y_n(t) \\ u_n(t) \end{pmatrix} = e^{\mathcal{A}_n t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} y(t) \\ u(t) \end{pmatrix} = e^{\mathcal{A}t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

[6] In the case of C symmetric and M bounded this formula was proved in [4].

Then $y_n(t)$ solves (35) with $u_n(t) = M_n^{1/2}\dot{y}_n(t)$ and

$$y_n(t) \to y(t), \quad M_n^{1/2}\dot{y}_n(t) \to M^{1/2}\dot{y}(t), \quad n \to \infty.$$

Proof. By (33) we have $\mathcal{A}_n^{-1} \to \mathcal{A}^+$ in the strong resolvent sense (see [3], Ch. VIII, Th, 1.5) i.e.

$$(\lambda - \mathcal{A}_n^{-1})^{-1} \to (\lambda - \mathcal{A}^+)^{-1}, \quad \Im\lambda \neq 0.$$

Hence by (29),

$$(\lambda - \mathcal{A}_n)^{-1} = \frac{1}{\lambda} - \frac{1}{\lambda^2}\left(\frac{1}{\lambda} - \mathcal{A}_n^{-1}\right)^{-1} \to$$

$$\frac{1}{\lambda} - \frac{1}{\lambda^2}\left(\frac{1}{\lambda} - \mathcal{A}^+\right)^{-1} = (\lambda - \mathcal{A})^{-1}Q.$$

all in the strong sense. Now the Trotter-Kato convergence theory ([3]) can be applied to give

$$e^{\mathcal{A}_\eta t}x \to e^{\mathcal{A}t}x, \quad \eta \to 0 \tag{37}$$

for all $x \in \mathcal{X}$. (The original Trotter-Kato theorem requires the injectivity of the strong limit in (34), but the same proof is easily seen to accomodate our slightly more general setting.) The remaining assertions are now straightforward. Q.E.D.

We now apply our abstract theory from Sect. 2 to a second order system with variable damping.

Theorem 3 *Let*

$$\mathcal{A}^+ = \begin{pmatrix} -C & -M^{1/2} \\ M^{1/2} & 0 \end{pmatrix}, \quad \widehat{\mathcal{A}}^+ = \begin{pmatrix} -\widehat{C} & -M^{1/2} \\ M^{1/2} & 0 \end{pmatrix}$$

where M is bounded, positive selfadjoint and C, $\widehat{C}$ are bounded accretive operators satisfying

$$\left|\left((\widehat{C} - C)x, y\right)\right|^2 \leq \varepsilon^2 \Re(Cy, y)\Re(\widehat{C}x, x) \tag{38}$$

for all $x, y \in \mathcal{Y}$ and some $\varepsilon > 0$. Then $\mathcal{A}^+$ and $\widehat{\mathcal{A}}^+$ have the same null-space and the respective contractive semigroup generators $\mathcal{A}$ and $\widehat{\mathcal{A}}$ in $\mathcal{X}$ from (28) satisfy the assumptions of Cor. 2, in particular,

$$\|e^{\widehat{\mathcal{A}}t} - e^{\mathcal{A}t}\| \leq \frac{\varepsilon}{2}. \tag{39}$$

Proof. Obviously (38) is equivalent to

$$\left|\left((\widehat{\mathcal{A}}^+ - \mathcal{A}^+)x, y\right)\right|^2 \leq \varepsilon^2 \Re(-\mathcal{A}^+y, y)\Re(-\widehat{\mathcal{A}}^+x, x) \tag{40}$$

for all $x, y \in \mathcal{D}(\mathcal{A}^+) = \mathcal{D}(\widehat{\mathcal{A}}^+)$. From this it follows that $\mathcal{A}^+$ and $\widehat{\mathcal{A}}^+$ have the same null-space. Furthermore, by (26) the domains of the four operators $\mathcal{A}^+, \widehat{\mathcal{A}}^+, (\mathcal{A}^+)^*, (\widehat{\mathcal{A}}^+)^*$ coincide and (40) is equivalent to both (3) and (13) for for $A = \mathcal{A}^+$ and $B = \widehat{\mathcal{A}}^+$ and then also for $A = \mathcal{A}^{-1}$ and $B = \widehat{\mathcal{A}}^{-1}$ (note that in our situation we have $\Re(-\mathcal{A}^+y, y) = \Re(-(\mathcal{A}^+)^*y, y)$ and $\Re(-\widehat{\mathcal{A}}^+x, x) = \Re(-(\widehat{\mathcal{A}}^+)^*x, x)$). Now apply Prop. 1 and Cor. 2. Q.E.D.

Note the important role of the 'inverse-invariance property' in Prop. 1 in the proof above because we have no explicit formulae for the generators $\mathcal{A}$ and $\widehat{\mathcal{A}}$ and there is no control on their domains of definition.

We now prove some stability results for second order systems.

Theorem 4 *Let the system (21) be exponentially stable*[7] *with a symmetric* $C = C^{(1)}$ *and let*

$$0 \le C^{(1)} \le D \le \alpha C^{(1)}, \quad \alpha \ge 1.$$

Then the exponential stability holds with $C = D$ *and vice versa.*

Proof. Set

$$C_k = C^{(1)} + \frac{k}{n}(D - C^{(1)}), \quad k = 0, \ldots, n.$$

Then $C_0 = C^{(1)}$, $C_n = D$ and

$$0 \le C_{k+1} - C_k \le \frac{\alpha - 1}{n} C^{(1)}$$

and

$$|((C_{k+1} - C_k)x, y)|^2 \le ((C_{k+1} - C_k)x, x)((C_{k+1} - C_k)y, y)$$

$$\le \left(\frac{\alpha - 1}{n}\right)^2 (C^{(1)}x, x)(C^{(1)}y, y) \le \left(\frac{\alpha - 1}{n}\right)^2 (C_{k+1}x, x)(C_k y, y).$$

Now choose $n > (\alpha - 1)/2$ and use Theorem 3 and Corollary 3. Use induction: the exponential stability carries over from C_k to C_{k+1} and vice versa. Q.E.D.

In particular, the exponential stability with C implies the same with αC for any positive α. A similar technique can be applied to gyroscopic systems:

Theorem 5 *Suppose that in (21) the operator* C *is sectorial. Then the exponential stability of this system is equivalent to the exponential stability of the 'purely damped' system*

$$M\ddot{y} + \widehat{C}\dot{y} + y = 0, \text{ with } \widehat{C} = \frac{C^* + C}{2}. \tag{41}$$

Proof. By sectoriality there exists $N > 0$ such that

$$|\Im(Cy, y)| \le N \Re(Cy, y) \text{ for all } x, y. \tag{42}$$

We have

$$\Re(Cx, x) = (\widehat{C}x, x), \quad -i\Im(Cx, x) = -i((\widehat{C} - C)x, x).$$

The operators $\widehat{C}$ and $-i(\widehat{C} - C)$ are symmetric, so the inequality (42) may be polarised to read

$$((\widehat{C} - C)x, y)|^2 \le N^2 \Re(Cy, y)\Re(Cx, x) = N^2 \Re(Cy, y)\Re(\widehat{C}x, x).$$

Assume first $N < 2$. Then apply Theorem 3 and Corollary 3 to obtain the exponential stability with C. Now drop the condition $N < 2$ and proceed by induction. Introduce the sequence

$$C_k = \widehat{C} + \frac{k}{n}(C - \widehat{C}), \quad k = 0, \ldots, n.$$

Then obviously

$$|((C_{k+1} - C_k)x, y)|^2 = \frac{1}{n^2}|((\widehat{C} - C)x, y)|^2 \le$$

$$\frac{N^2}{n^2}\Re(Cy, y)\Re(\widehat{C}x, x) = \frac{N^2}{n^2}\Re(C_k y, y)\Re(C_{k+1}x, x).$$

[7] By the exponential stablity of a second order system we mean the exponential stablity of the generated semigroup.

Now choose $n < 2/N$ and apply the above consideration to the consecutive pairs C_k, C_{k+1}. We may begin at the bottom with $C_0 = \widehat{C}$ or at the top with $C_n = C$. Q.E.D.

The foregoing theorem can be nicely combined with the spectral shift techniques from [6] to obtain further results on exponential stability.

Suppose first that in (21) both M and C are bounded and symmetric (the boundedness of both operators is imediately seen to be a necessary condition for the exponential stability). The key relation in the following will be

$$\mathcal{L}(\mu)\widehat{\mathcal{A}}_\mu^+\mathcal{L}(\mu)^{-1} = (\mathcal{A} - \mu)^{-1} Q. \tag{43}$$

with $\mu < 0$ and

$$\mathcal{L}(\mu) = \begin{pmatrix} K(\mu)^{-1/2} & 0 \\ \mu M^{1/2} K(\mu)^{-1/2} & 1 \end{pmatrix}, \quad K(\mu) = \mu^2 M + \mu C + 1, \tag{44}$$

$$\widehat{\mathcal{A}}_\mu^+ = \begin{pmatrix} K(\mu)^{-1/2}(C + 2\mu M)K(\mu)^{-1/2} & -K(\mu)^{-1/2}M^{1/2} \\ M^{1/2}K(\mu)^{-1/2} & 0 \end{pmatrix}, \tag{45}$$

where μ is chosen in such a way that $K(\mu)$ remains positive definite.[8] In this way both $\mathcal{L}(\mu)$ and $\mathcal{L}(\mu)^{-1}$ are everywhere defined and bounded. The relation (43) is immediately verified (use (30)) and it is an immediate generalisation of the one obtained in [6], (16).

Under the additional assumption that $C + 2\mu M$ be positive the operator $\widehat{\mathcal{A}}_\mu^+$ is bounded and dissipative, so it is reduced by the subspaces $\mathcal{N}(\widehat{\mathcal{A}}_\mu^+)$ and $\mathcal{N}(\widehat{\mathcal{A}}_\mu^+)^\perp$ and we may write

$$\widehat{\mathcal{A}}_\mu^+ = \widehat{\mathcal{A}}_\mu^{-1} P, \tag{46}$$

where P is the orthogonal projection onto $\mathcal{N}(\widehat{\mathcal{A}}_\mu^+)^\perp$ and $\widehat{\mathcal{A}}_\mu$ is maximal dissipative in the Hilbert space $\mathcal{N}(\widehat{\mathcal{A}}_\mu^+)^\perp$. Thus, (43) implies

$$\mathcal{S}^{-1}\mathcal{A}\mathcal{S} = \widehat{\mathcal{A}}_\mu + \mu. \tag{47}$$

Here the linear operator

$$\mathcal{S} = \mathcal{L}(\mu)\Big|_{\mathcal{N}(\widehat{\mathcal{A}}_\mu^+)^\perp} : \mathcal{N}(\widehat{\mathcal{A}}_\mu^+)^\perp \to \mathcal{N}(\mathcal{A}^+)^\perp$$

is bijective and bicontinuous. We summarise:

Theorem 6 *Let the operators M and C from (21) have additional properties that M is bounded, C sectorial and $C - \alpha M$ accretive for some $\alpha > 0$. Then the system (21) is exponentially stable. If, in addition, C is symmetric then*

$$\gamma = \sup_{x \in \mathcal{Y}, (Mx,x) > 0} \Re \frac{-(Cx, x) + \sqrt{(Cx, x)^2 - 4(Mx, x)(x, x)}}{2(Mx, x)} < 0 \tag{48}$$

and

$$\|e^{\mathcal{A}t}\| \le \|\mathcal{L}(\mu)\|\|\mathcal{L}(\mu)^{-1}\|e^{\mu t}. \tag{49}$$

for any $\mu \in (\gamma, 0)$.

[8] By the assumed boundedness of both M and C there are negative μ's with $K(\mu)$ positive definite.

Proof. Take first C as symmetric. The value γ is the infimum of all μ such that $C + 2\mu M$ is positive and $K(\mu)$ is positive definite. The proof of this fact is the same as that of [6], Proposition 1, so we omit it here. So, for any $\mu \in (\gamma, 0)$ (47) implies (49).

For non-symmetric C the above considerations will obviously be valid for its symmetric part $\widehat{C}$ from (41). By using Theorem 5 the exponential stability follows. Q.E.D.

Remark 4 (i) All conditions on M, C, $\widehat{C}$ in the foregoing theorems can be readily expressed in the language of the original forms in (19). For instance, (38) is equivalent to

$$\left|(\widehat{\gamma} - \gamma)(x, y)\right|^2 \leq \varepsilon^2 \Re\gamma(y, y) \Re\widehat{\gamma}(x, x). \tag{50}$$

and so on.

(ii) Explicit bounds for the condition number, appearing in (49) may be taken over from [6], Lemma 1.

4 The damped wave equation

Here we apply our general theory to the wave equation in one dimension

$$\rho(x)u_{tt} + \gamma(x)u_t - (d(x)u_{tx})_x - (k(x)u_x)_x = 0 \tag{51}$$

for the unknown function $u = u(x, t)$, $a < x < b$ and $0 < t < \infty$. The functions $\rho(x),\ \gamma(x),\ d(x),\ k(x)$ are assumed to be non-negative and measurable; in addition, $\rho(x),\ \gamma(x)$ are bounded and

$$\text{ess inf}_{a<x<b} k(x) > 0, \quad \text{ess sup}_{a<x<b} \frac{d(x)}{k(x)} < \infty. \tag{52}$$

The boundary conditions are

$$u(a, t) = 0, \quad u_x(b, t) + \zeta u_t(b, t) = 0, \quad \zeta \geq 0. \tag{53}$$

This is a formally dissipative equation which we shall understand in its weak form

$$\mu(u_{tt}, v) + \theta(u_t, v) + \kappa(u, v) = 0 \tag{54}$$

with $u(a) = v(a) = 0$ and

$$\mu(u, v) = \int_a^b \rho(x) u\bar{v} dx, \tag{55}$$

$$\theta(u, v) = \int_a^b (\gamma(x)u\bar{v} + d(x)u'\bar{v}')\, dx + \zeta u(a)\bar{v}(b), \tag{56}$$

$$\kappa(u, v) = \int_a^b k(x)u'\bar{v}' dx. \tag{57}$$

The forms $\mu,\ \theta$ are symmetric and positive. θ is obviously κ-bounded while μ is κ-closable. As the underlying Hilbert space $\mathcal{Y}$ we take the functions with the scalar product

$$(u, v) = \kappa(u, v) = \int_a^b k(x)u'\bar{v}' dx, \quad u(a) = v(a) = 0. \tag{58}$$

Then under our conditions,

$$\mu(u, v) = (Mu, v), \quad \theta(u, v) = (Cu, v) \tag{59}$$

where $M,\ C$ are positive selfadjoint operators, with bounded C and M. Thus, we end up with the second order system (21) and (54) gives rise to a contractive semigroup on the space $\mathcal{X}$ which is determined from the null-spaces of $M,\ C$.

Note that in order for M to have a non-trivial null-space it is not sufficient that the function ρ vanishes just on a set of positive measure, rather ρ must vanish on an interval (and similarly for C). If ρ vanishes on an interval and γ does not, then (51) is of mixed type (hyperbolic - parabolic). All such cases are covered by our theory.

Now for the perturbation. We perturb the damping parameters $\gamma(x),\ d(x),\ \zeta$ into $\widehat{\gamma}(x),\ \widehat{d}(x),\ \widehat{\zeta}$, which satisfy the same conditions as $\gamma(x),\ d(x),\ \zeta$ above and are such that

$$|\widehat{\gamma}(x) - \gamma(x)| \le \varepsilon\sqrt{\widehat{\gamma}(x)\gamma(x)} \tag{60}$$

$$|\widehat{d}(x) - d(x)| \le \varepsilon\sqrt{\widehat{d}(x)d(x)} \tag{61}$$

$$|\widehat{\zeta} - \zeta| \le \varepsilon\sqrt{\widehat{\zeta}\zeta} \tag{62}$$

This is a 'relatively small' change of the damping parameters, commonly encountered in practice. The corresponding operators C and $\widehat{C}$ are immediately seen to satisfy (38). Hence Theorem 3 applies and the corresponding semigroups satisfy (39).

One might be interested to obtain perturbation results under the more common assumptions involving only the 'unperturbed' data and the perturbation:

$$|\widehat{\gamma}(x) - \gamma(x)| \le \eta\gamma(x) \tag{63}$$

$$|\widehat{d}(x) - d(x)| \le \eta d(x) \tag{64}$$

$$|\widehat{\zeta} - \zeta| \le \eta\zeta \tag{65}$$

with $\eta < 1$. This implies (60) – (62) with

$$\varepsilon = \frac{\eta}{\sqrt{1-\eta}}.$$

But the real use of (63) – (65) consists merely in insuring the non-negativity of the perturbed damping parameters and the conditions (52); all this is usually known in advance, so there is no need to abandon the much less restrictive conditions (60) – (62).

In view of Corollary 1 we conclude that *if the equation (51) decays exponentially with the damping parameters* $\gamma(x),\ d(x),\ \zeta$, *then the same will be the case with* $\widehat{\gamma}(x),\ \widehat{d}(x),\ \widehat{\zeta}$, *if the constant* ε *is less than* 2. Theorem 4 also applies accordingly.

The situation in higher dimensions is similar and the results are completely analogous and straightforward. The interval $[a,b] \subseteq R$ will be replaced by a bounded domain Ω, so the only additional issue is to insure that the boundary $\partial\Omega \subseteq R^n$ be smooth enough to accomodate any of the boundary conditions from (52).

As a second example consider the equation (51) on the infinite interval $0 < x < \infty$ with the boundary condition

$$u(0,t) = 0. \tag{66}$$

For simplicity we take

$$k(x) = \rho(x) \equiv 1, \tag{67}$$

whereas $\gamma(x) \ge 0$ is supposed to satisfy

$$D = \sup_{u\in\mathcal{Y}} \frac{\int_0^\infty \gamma(x)|u(x)|^2 dx}{\int_0^\infty |u'(x)|^2 dx} < \infty, \tag{68}$$

where $\mathcal{Y}$ is the set of all u which are absolutely continuous, vanish at zero and have a square integrable u'; this is obviously a Hilbert space with the scalar product

$$(u, v) = \int_0^\infty u'(x)\bar{v}'(x)dx.$$

The class of functions γ satisfying (68) is not void since it includes

$$\gamma(x) = \frac{1}{x^2}, \text{ with } D = 4$$

([3] Ch. VI 4.1). The form

$$\mu(u, v) = \int_0^\infty u(x)\bar{v}(x)dx$$

defined on $\mathcal{D}(\mu) = L^2(0, \infty) \cap \mathcal{Y}$ is closed in $\mathcal{Y}$, so (59) yields a positive unbounded operator M with a trivial null-space and a bounded C. Hence our semigroup construction applies and under the perturbation (60) the bounds (10), (9) hold.

Such semigroups are in general not exponentially decaying (they are usually extended to uniformly bounded groups) and will give rise to a non-trivial scattering theory on an 'absorbing obstacle' represented by the short range damping function γ. Further considerations along these lines go beyond the scope of this article.

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