Note on interlacing for hyperbolic quadratic pencils

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Dedicated to the memory of Peter Jonas, dear colleague and friend.

Abstract. We prove interlacing inequalities for the eigenvalues of the submatrices of (weakly) hyperbolic and gyroscopic quadratic pencils.

In [1], Theorem 4.3 an interlacing result for quadratic pencils was derived. It is the aim of this note to offer an elementary proof of this fact. Our main tool will be the Sylvester theorem of inertia. A Hermitian quadratic pencil

$$K(\lambda) = \lambda^2 M + \lambda C + K \tag{0.1}$$

of order n is called *hyperbolic*, if M is positive definite and $K(\mu)$ is negative definite for some real μ. It is well known, see [2], that the eigenvalues of $K(\cdot)$ can be written as

$$\lambda_n^- \le \cdots \le \lambda_1^- < \mu < \lambda_1^+ \le \cdots \le \lambda_n^+. \tag{0.2}$$

Let

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix} \tag{0.3}$$

be any given partition, where K_{11} is of order m (and similarly for $M, C, K(\lambda)$). Then $K_{11}(\mu)$ is again negative definite and $K_{11}(\cdot)$ itself hyperbolic with the eigenvalues

$$\alpha_m^- \le \cdots \le \alpha_1^- < \mu < \alpha_1^+ \le \cdots \le \alpha_m^+. \tag{0.4}$$

The interlacing inequalities are

$$\alpha_k^+ \ge \lambda_k^+, \quad \alpha_{k+m-n}^+ \le \lambda_k^+ \tag{0.5}$$

$$\alpha_k^- \le \lambda_k^-, \quad \alpha_{k+m-n}^- \ge \lambda_k^- \tag{0.6}$$

for all possible k. To prove these inequalities we will use the formula

$$\Pi_+(K(\lambda)) = k, \quad \text{whenever } \lambda_k^+ < \lambda < \lambda_{k+1}^+ \tag{0.7}$$

where

$$\Pi(H) = \{\Pi_+(H), \Pi_0(H), \Pi_-(H)\}$$

is the inertia of a Hermitian matrix H.

To prove this formula note that without loss of generality we may assume that C, K are positive definite. Otherwise make the spectral shift $\lambda = \lambda_0 + \nu$, $\lambda_0 > 0$ such that both $2\lambda_0 M + C$ and $K(\lambda_0)$ become positive definite and the new pencil

$$\hat{K}(\nu) = K(\lambda) = \nu^2 M + \nu(2\lambda_0 M + C) + K(\lambda_0)$$

has the eigenvalues shifted by $-\lambda_0$ so that all of them become negative. Set

$$A = \begin{bmatrix} 0 & K^{1/2}M^{-1/2} \\ M^{-1/2}K^{1/2} & M^{-1/2}CM^{-1/2} \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{0.8}$$

Then

$$JA - \lambda J = Z \begin{bmatrix} -\lambda & 0 \\ 0 & \frac{K(\lambda)}{\lambda} \end{bmatrix} Z^*, \quad Z = \begin{bmatrix} 1 & 0 \\ -M^{-1/2}K^{1/2}/\lambda & M^{-1/2} \end{bmatrix}.$$

Thus, $\Pi_+(K(\lambda)) = \Pi_-(JA - \lambda J)$ and the latter is equal to k; this is best seen in diagonalising the J-Hermitian matrix A by a J-unitary similarity.

We proceed to prove the first inequality from (0.5). Supposing the contrary there would exist a λ, not an eigenvalue of $K_{11}(\cdot)$, with $\alpha_k^+ < \lambda < \lambda_k^+$. Now

$$K(\lambda) = W \begin{bmatrix} K_{11}(\lambda) & 0 \\ 0 & K_{22}(\lambda) - K_{12}(\lambda)K_{11}(\lambda)^{-1}K_{11}(\lambda)^* \end{bmatrix} W^* \tag{0.9}$$

with

$$W = \begin{bmatrix} 1 & 0 \\ K_{11}(\lambda)^{-1}K_{12}(\lambda) & 1 \end{bmatrix} \tag{0.10}$$

By the Sylvester inertia theorem

$$\Pi_+(K_{11}(\lambda)) \le \Pi_+(K(\lambda)) \le \Pi_+(K_{11}(\lambda)) + n - m. \tag{0.11}$$

Now, $\alpha_k^+ < \lambda < \lambda_k^+$ would imply

$$\Pi_+(K_{11}(\lambda)) \ge k, \quad \Pi_+(K(\lambda)) \le k - 1$$

which contradicts the first inequality in (0.11). In a similar way would $\lambda_k^+ < \lambda < \alpha_{k+m-n}^+$ imply

$$\Pi_+(K(\lambda)) \ge k, \quad \Pi_+(K_{11}(\lambda)) \le k + m - n - 1$$

which contradicts the second inequality in (0.11). This proves (0.5). The proof of (0.6) is completely analogous.

Weakly hyperbolic pencils. The pencil $K(\cdot)$ is *weakly hyperbolic*, if M is positive definite and $K(\mu)$ is negative semidefinite for some real μ. In this case the two strict inequalities appearing in (0.2) and (0.4) are correspondingly weakened. The proof of (0.5) and (0.6) will immediately be carried over, if the validity of (0.7) is extended to the weakly hyperbolic case. This can be done by the continuity argument. The pencil $K(\lambda, \epsilon) = \lambda^2 \epsilon M + \lambda C + K$ will be hyperbolic for every $\epsilon < 1$. In the limit $\epsilon \to 1$ the inequality (0.7) is established by the known continuity of the eigenvalues. Thus, the interlacings (0.5), (0.6) hold in the the weakly hyperbolic case as well.

Gyroscopic pencils. The pencil $K(\cdot)$ is *gyroscopic*, if C is skew-Hermitian and M, K Hermitian positive definite. By substituting $\lambda = i\mu$ we obtain

$$\hat{K}(\mu) = K(i\mu) = -\mu^2 M + i\mu C + K$$

and $-\hat{K}(\cdot)$ is hyperbolic, so (0.5), (0.6) immediately apply.

References

[1] Azizov, T., Dijksma, A., Förster, K-H., Jonas, P., Quadratic (weakly) hyperbolic matrix polynomials: direct and inverse spectral problems, preprint 2008, see also this volume.

[2] Gohberg, I., Lancaster, P., Rodman, L., Matrices and indefinite scalar products, Birkhäuser Basel 1983.

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