OPTIMAL DAMPING OF VIBRATIONAL SYSTEMS

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Ivica Nakić
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1. Berichterstatter und Mentor: Prof. Dr. K. Veselić, Hagen

2. Berichterstatter: Prof. Em. Dr. A. Suhadolc, Ljubljana

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To Marcela

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Chapter 1

Introduction

1.1 Motivation

Scientific treatment of vibration and damping goes back at least to the celebrated German physicist H. Helmholtz who wrote a book entitled ”On the Sensations of Tone”, first published in 1885, constituting one of the first scientific treatises of the subject of vibration. The study of vibration and damping is concerned with oscillatory motions of bodies and particles and the attenuation of those vibrations. Both engineered machines and natural physical systems have the property of being subject to a vibration all or part of the time.

Dangerous vibrations are frequent practical problem. In the design of a bridge, for example, one must pay attention to the resonances of the bridge with the wind–induced oscillatory forces. For the majority of the structural vibrations, engineering applications dictate that resonance and sustained oscillations are not desirable because they may result in structural damage.

One way to reduce resonance is through damping. Actually, the damping is present in nearly all vibrational systems. Its presence may be attributed to medium impurities, small friction and viscous effects, or artificially imposed damping materials or devices.

Recent scientific and technological advances have generated strong interest in the field of vibrational systems governed by the partial differential equations.

There is a vast literature in this field of research. Particulary the engineering literature is very rich. For a brief insight we give some older references: [Bes76], [Mül79], [BB80],[IA80], [Mül88] and [NOR89], and some more recent: [FL98], [DFP99], [PF00b] and [PF00a]. Among mathematical references we emphasize two books by G. Chen and J. Zhen [CZ93a] and [CZ93b], where a thorough presentation of techniques and results in this area is given. Among references concerning abstract second order differential equations which will be our abstract setting, we mention classical books by H. O. Fattorini [Fat83] and [Fat85]. The book [Shk97] possesses more then 350 references on the abstract second order differential equations. Other authors which significantly improved the knowledge in this field are S. J. Cox, J. Lagnese, D. L. Russell, R. Triggiani and K. Veselić, among others. Some more recent references, besides those cited in the thesis, are [CZ94], [CZ95], [LG97], [Fre99], [FL99], [Gor97], and [BDMS98].

For conservative systems without damping, the mathematical analysis has been developed to a substantial degree of completeness. Such systems usually correspond to evolution equations with an skew–adjoint generator with an associated complete set of orthonormal eigenfunctions, yielding a Fourier representation for the solution. In applications such eigenfunctions are refereed to as the normal modes of vibration.

When the vibrational system described by a partial differential equation contains damping terms, whether appearing in the equation per se (called distributed damping) or in the boundary conditions (called boundary damping), the generating operator in the evolution system will no longer be skew–adjoint. Thus the powerful Fourier series expansion method is not available any more. In the extreme case, the operator can have empty spectrum, hence the spectral methods are inadequate.

The initial motivation for the research described in this thesis was to develop a functional–analytic theory for a particular type of the vibrational systems, those which can be described by an abstract second order differential equation with symmetric non–negative sesquilinear forms as coefficients. Examples of such vibrational systems are mechanical systems.[1]

The simplest system of this type is the mass–spring–damper system described by

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = 0, \qquad (1.1.1)$$
$$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0,$$

where $m, d, k > 0$ are the mass, damping and stiffness coefficient, respectively, and $x(t)$ is the displacement from the equilibrium position.

We will concentrate on the following subjects:

- stability of the system,
- optimal damping of the system.

Among many types of stability, we will use the most desirable one: uniform

[1] The important class, the class of electromagnetic damping systems unfortunately does not satisfy our abstract setting since in these systems the damping form usually contains an imaginary part.

exponential stability. This means that there is a uniform exponential decay of the solutions of the system.

As an optimization criterion we will use the criterion governed by the minimization of the average total energy of the system, but also some other optimization criteria will be mentioned.

The two subjects mentioned above are closely connected. This will become evident by our choice of the Hilbert space setting. We will use the most natural Hilbert space – the one having the energy of the system as the scalar product. More precisely, in our setting the system will be stable if and only if the energy of the system has an uniform exponential decay. Hence, in the treatment of both subjects, the energy of the system plays a prominent role.

To emphasize the importance of choosing the appropriate scalar product, we give a simple example.

Example 1.1.1. It is well–known (see, for example page 141 of this thesis) that the finite–dimensional system has an energy decay, if and only if the corresponding generating operator A satisfies

$$\mathrm{Re}(Ax, x) \leq 0. \tag{1.1.2}$$

If we set $y(t) = \dot{x}(t)$ and $z(t) = \left(\begin{smallmatrix} x(t) \\ y(t) \end{smallmatrix}\right)$, the equation (1.1.1) can be written as

$$\dot{z}(t) = Az(t),$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix}.$$

If we choose the usual scalar product on $\mathbb{R}^2$, (1.1.2) is not satisfied. However, we could introduce the new scalar product given by $\langle \left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right) \rangle = kx_1x_2 + y_1y_2$.

In this case $\frac{1}{2}\left\|\binom{x(t)}{y(t)}\right\|^2 = \frac{1}{2}kx(t)^2 + \frac{1}{2}\dot{x}(t)^2$, i.e. the norm of the solution equals one half of the energy of the system. With this scalar product (1.1.2) holds.

For the equation (1.1.1) the most popular optimization criteria lead to the same optimal damping parameter given by $d = 2\sqrt{mk}$.

Although our primary aim is the investigation of infinite–dimensional systems, we also present some new results in the finite–dimensional case.

The central results of this thesis are:

- Theorem 2.2.1, which treats the problem of finding the optimal damping in the matrix case,
- its infinite–dimensional analogue, Theorem 4.2.2, and
- the characterization of the uniform exponential stability of the abstract vibrational systems given in Subsections 3.3.3 and 3.3.4.

The obtained results will be applied to a number of concrete problems.

Most results in this thesis can be interpreted as a generalization of the results from [Ves88], [Ves90] and [VBD01] to the infinite–dimensional case. An approach similar to ours is also taken in [FI97].

Throughout this thesis results will be applied to a vibrating string which serves as our basic model.

1.2 Organization of the thesis

In this section we give an overview of the organization of this thesis, so that the interested reader can set his or her favorite route through the results.

Chapter 2: The matrix case

This chapter deals with the vibrational systems described by a second order matrix differential equation. These are so–called lumped parameter systems. This chapter consists of two parts. The first part (Sections 2.1 and 2.3) is primarily used as a motivation for our study of the infinite–dimensional systems, and our motivation indeed was to try to generalize known result in finite–dimensional case to the infinite–dimensional case. The main importance of the matrix case lies in the approximation of the infinite–dimensional problems.

The second part (Section 2.2) is devoted to the finding of a global minimum of a penalty function corresponding to the optimization of the damping. Especially, this result gives an affirmative answer to a conjecture of Veselić–Brabender–Delinić from [VBD01]. This result also has importance in the study of infinite–dimensional systems.

The results of this chapter, with exception of Section 2.2 are mostly well–known and are collected for the sake of the completeness.

Chapter 3: Abstract vibrational system

This is a central chapter of this thesis. Here a general framework for the stability of an abstract second order differential equation involving sesquilinear forms is given. We show how to translate this equation into a linear Cauchy problem, and we solve this problem in terms of a semigroup. The scalar product is chosen in such a way that the norm of the semigroup equals one half of the energy of the solution of the original differential equation, and that the original differential equation has a solution if and only if there exists a solution for the Cauchy problem. We also give necessary and sufficient conditions for an uniform exponential stability of this semigroup. It is shown that in the generic situation we can investigate the stability of the system in terms of the eigenvalues and eigenvectors of the corresponding undamped system and the damping operator.

The main ideas and nearly all proofs of the first two sections of this chapter are due to A. Suhadolc and K. Veselić [SV02]. We express our gratitude for their permission to include their results into this thesis.

Chapter 4: Optimal damping

In this chapter the ideas from Chapter 2 are generalized to the infinite–dimensional case. For the system which is uniformly exponentially stable it is shown that, analogously to the matrix case, an optimal damping problem can be algebraically described. It is also shown how our previous knowledge about dangerous vibrations can be implemented into the mathematical model.

In the case of systems which possess an internal damping, we find the optimal damping operators.

Also, an optimal damping in the case of the so–called modal damping is found, hence generalizing the result from the matrix case.

The author thanks prof. Veselić for the help in the process of writing Section 4.4.

Chapter 5: Applications

In this chapter we illustrate the theory developed in the previous chapters by applying it to various concrete problems. Those include the problems described by one–dimensional as well as multidimensional models.

Appendix: Semigroup theory

In the appendix we introduce the basic concepts and results of the semigroup theory which we use in this thesis.

Notation

Here we give a list of notation which is used but not defined in the thesis.

$\partial\Omega$	the boundary of the set Ω
$D(A)$	the domain of the operator A
$\mathcal{R}(A)$	the range of the operator A
$\mathcal{N}(A)$	the null–space of the operator A
$\sigma(A)$	the spectrum of the operator A
$\rho(A)$	the resolvent set of the operator A
$\sigma_p(A)$	the point spectrum of the operator A
$\sigma_r(A)$	the residual spectrum of the operator A
$\sigma_c(A)$	the continuous spectrum of the operator A
$\sigma_{ap}(A)$	the approximate spectrum of the operator A
$\sigma_{ess}(A)$	the essential spectrum of the operator A
$*$	the adjoint (on $\mathbb{C}$) or the transpose (on $\mathbb{R}$)
$\mathrm{L}^2(H)$	the Lebesgue space of H–valued functions
∇	the gradient
div	the divergence operator
tr	the trace
$\searrow$	the right limit
$\mathbb{R}_+$	the set $\{x \in \mathbb{R} : x \geq 0\}$

Chapter 2

The matrix case

In this chapter we give an introduction to the optimal damping problem in the finite–dimensional case, as well as present a brief survey of the main ideas and results. We also find a solution to the optimal damping problem, i.e. we find the set of matrices for which the system produces minimal damping.

In the Section 2.3 we give a brief survey of numerical methods.

2.1 Introduction and a review of known results

We consider a damped linear vibrational system described by the differential equation

$$M\ddot{x} + D\dot{x} + Kx = 0, \tag{2.1.1a}$$

$$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0, \tag{2.1.1b}$$

where M, D and K (called mass, damping and stiffness matrix, respectively) are real, symmetric matrices of order n with M, K positive definite, and D positive semi–definite matrices. In some important applications (e.g. with so–called lumped masses in vibrating structures) M, too, is only semi–definite.

This can be easily reduced to the case with a non–singular M at least if the null-space of M is contained in the one of D.

To (2.1.1) there corresponds the eigenvalue equation

$$(\lambda^2 M + \lambda D + K)x = 0. \tag{2.1.2}$$

An example is the so–called n–mass oscillator or oscillator ladder given on the Figure 2.1, where

$$M = \operatorname{diag}(m_1, \dots, m_n), \tag{2.1.3a}$$

$$K = \begin{bmatrix} k_0 + k_1 & -k_1 & & & \\ -k_1 & k_1 + k_2 & -k_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \\ & & & -k_{n-1} & k_{n-1} + k_n \end{bmatrix}, \tag{2.1.3b}$$

$$D = de_1 e_1^*, \tag{2.1.3c}$$

where e_1 is the first canonical basis vector.

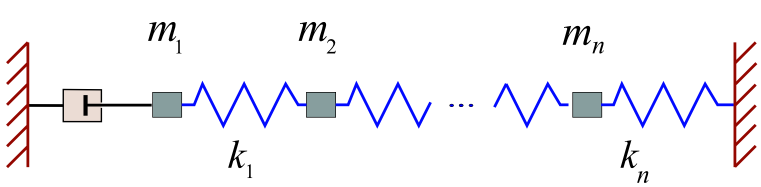


Figure 2.1: The n–mass oscillator with one damper

Here $m_i > 0$ are the masses, $k_i > 0$ the spring constants or stiffnesses and $d > 0$ is the damping constant of the damper applied to the mass m_1 (Fig. 2.1). Note that the rank of D is one. Such damping is called one–dimensional.

Obviously, all eigenvalues of (2.1.2) lie in the left complex half–plane. Equation (2.1.2) can be written as a $2n$–dimensional linear eigenvalue problem. This can be done by introducing

$$y_1 = L_1^* x, \quad y_2 = L_2^* \dot{x},$$

where

$$K = L_1 L_1^*, \quad M = L_2 L_2^*.$$

It can be easily seen that (2.1.1) is equivalent to

$$\dot{y} = Ay, \tag{2.1.4a}$$

$$y(0) = y_0, \tag{2.1.4b}$$

where $y = \binom{y_1}{y_2}$, $y_0 = \begin{pmatrix} L_1^* x_0 \\ L_2^* \dot{x}_0 \end{pmatrix}$, and

$$A = \begin{pmatrix} 0 & L_1^* L_2^{-*} \\ -L_2^{-1} L_1 & -L_2^{-1} D L_2^{-*} \end{pmatrix}, \tag{2.1.5}$$

with the solution $y(t) = e^{At} y_0$. The eigenvalue problem $Ay = \lambda y$ is obviously equivalent to (2.1.2).

The basic concept in the vibration theory is its stability. We say that the matrix A is *stable* if $\mathrm{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$.

In the literature the term "asymptotically stable" is also used. The famous result of Lyapunov states that the solution $y(t)$ of the Cauchy problem (2.1.4) satisfies $y(t) \to 0$ for $t \to \infty$ if and only if A is stable. The following result is easily proved ([MS85], [Bra98]).

Proposition 2.1.1. *The matrix* (2.1.5) *is stable if and only if the form $x^* D x$ is non–degenerate on every eigenspace of the matrix $M^{-1} K$.*

This result will be generalized to the infinite–dimensional case in Subsections 3.3.3 and 3.3.4.

Our aim is to optimize the vibrational system described by (2.1.1) in the sense of finding an optimal damping matrix D so as to insure an optimal evanescence.

There exist a number of optimality criteria. The most popular is the spectral abscissa criterion, which requires that the (penalty) function

$$D \mapsto s(A) := \max_k \mathrm{Re}\lambda_k$$

is minimized, where λ_k are eigenvalues of A (so they are the phase space complex eigenfrequencies of the system). This criterion concerns the asymptotic behavior of the system and it is not a priori clear that it will favorably influence the behavior of the system in finite times, too. More precisely, the asymptotic formula $\|y(t)\| \leq Me^{-s(A)t}$ holds for all $t \geq 0$, where $M \geq 1$. There are examples in which for the minimizing sequence D_n the corresponding coefficients M_n tend to infinity.

The shortcoming of this criterion is that in the infinite–dimensional case the quantity $s(A)$ does not describe the asymptotic behavior of the system accurately (see Remark A.1).

A similar criterion is introduced in [MM91], and is given by the requirement that the function

$$D \mapsto \max_k \frac{\mathrm{Re}\lambda_k}{|\lambda_k|}$$

is minimized, where λ_k are as above. This criterion is designed to minimize the number of oscillations before the system comes to rest.

Both of these criteria are independent of the initial conditions of the system.

Another criterion is given by the requirement for the minimization of the total energy of the system. The energy of the system (as a sum of kinetic and potential energy) is given by the formula

$$E(t;x_0,\dot{x}_0)\ (= E(t;y_0)) = \frac{1}{2}\dot{x}(t)^* M\dot{x}(t) + \frac{1}{2}x(t)^* Kx(t).$$

Note that

$$y(t)^* y(t) = \|y(t)\|^2 = 2E(t;y_0).$$

In other words, the Euclidian norm of this phase-space representation equals twice the total energy of the system. The total energy of the system is given by

$$\int_0^\infty E(t;y_0)\mathrm{d}t. \tag{2.1.6}$$

Note that this criterion, in the contrast to the criteria mentioned above, does depend on the initial conditions. The two most popular ways to correct this defect are:

(i) maximization (2.1.6) over all initial states of unit energy, i.e.

$$\max_{\|y_0\|=1} \int_0^\infty E(t;y_0)\mathrm{d}t, \tag{2.1.7}$$

(ii) taking the average of (2.1.6) over all initial states of unit energy, i.e.

$$\int_{\|y_0\|=1} \int_0^\infty E(t;y_0)\mathrm{d}t\mathrm{d}\sigma, \tag{2.1.8}$$

where σ is some probability measure on the unit sphere in $\mathbb{R}^{2n}$.

In some simple cases all these criteria lead to the same optimal matrix D, but in general, they lead to different optimal matrices.

The criterion with the penalty function (2.1.8), introduced in [Ves90], will be used throughout this thesis. The advantage of this criterion is that we can, by the choice of the appropriate measure σ, implement our knowledge about the most dangerous input frequencies.

To make this criteria more applicable we proceed as follows.

$$\int_0^\infty E(t;y_0)\mathrm{d}t = \frac{1}{2}\int_0^\infty y(t;y_0)^* y(t;y_0)\mathrm{d}t = \frac{1}{2}\int_0^\infty y_0^* e^{A^*t}e^{At}y_0\mathrm{d}t$$
$$= \frac{1}{2}y_0^* X y_0,$$

where

$$X = \int_0^\infty e^{A^*t}e^{At}\mathrm{d}t. \tag{2.1.9}$$

The matrix X is obviously positive definite. By the well–known result (see, for example [LT85]) the matrix X is the solution of the Lyapunov equation

$$A^*X + XA = -I. \tag{2.1.10}$$

There exists another integral representation of X [MS85] given by

$$X = \frac{1}{2\pi}\int_{-\infty}^\infty (-i\eta - A^*)^{-1}(i\eta - A)^{-1}\mathrm{d}\eta. \tag{2.1.11}$$

This formula will be generalized to the infinite–dimensional case in Section 4.5. The equation (2.1.10) has the unique solution if and only if A is stable. The expression (2.1.7) now reads

$$\frac{1}{2}\max_{\|y_0\|=1} y_0^* X y_0,$$

which is simply $\frac{1}{2}\|X\|$, so in this case we minimize the greatest eigenvalue of X. The expression (2.1.8) now can be written as

$$\frac{1}{2}\int_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma.$$

Since with the map

$$X \mapsto \int\limits_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma$$

is given a linear functional on the space of the symmetric matrices, by Riesz theorem there exists a symmetric matrix Z such that

$$\int\limits_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma = \mathrm{tr}(XZ), \text{ for all symmetric } X.$$

Let $y \in \mathbb{R}^{2n}$ be arbitrary. Set $X = yy^*$. Then

$$0 \leq \int\limits_{\|y_0\|=1} y_0^* X y_0 \mathrm{d}\sigma = \mathrm{tr}(XZ) = \mathrm{tr}(yy^* Z) = \mathrm{tr}(y^* Z y),$$

hence Z is always positive semi–definite.

For the measure σ generated by the Lebesgue measure on $\mathbb{R}^{2n}$, we obtain $Z = \frac{1}{2n} I$.

Hence the criterion given by the penalty function (2.1.8) can be written as

$$\mathrm{tr}(XZ) \to \min, \tag{2.1.12}$$

where X solves (2.1.10), and the matrix Z depends on the measure σ.

Since A is J–symmetric, where $J = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$, it follows that

$$\mathrm{tr}(XZ) = \mathrm{tr}(Y), \tag{2.1.13}$$

where Y is the solution of another, so–called "dual Lyapunov equation"

$$AY + YA^* = -Z. \tag{2.1.14}$$

In some cases, instead of (2.1.9) one can use the expression

$$\int\limits_0^\infty e^{A^* t} Q e^{At} \mathrm{d}t,$$

where Q is symmetric (usually positive semi–definite) matrix. We interpret the matrix Q as a weight function in the total energy integral. In this case the corresponding Lyapunov equation reads

$$A^*X + XA = -Q.$$

A possible choice of the matrix Q is given in Section 4.5.

To fix the ideas let us consider the simple problem described on Figure 2.2.

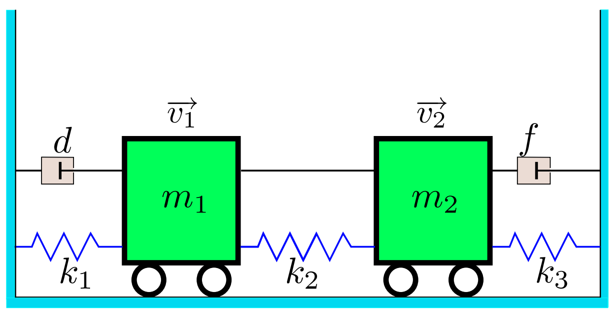


Figure 2.2: Simple system

If we suppose unit masses and stiffnesses, we arrive at the mass, stiffness and damping matrices

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, D = \begin{pmatrix} d & 0 \\ 0 & f \end{pmatrix}.$$

We plotted the contours of the function $(d, f) \mapsto \mathrm{tr}(X(D))$ on Figure 2.3. This strictly convex function attains its minimum at $d = f = \sqrt{6}$.

The solution X of (2.1.10) also gives an estimate [Ves02a] of the exponential decay of the solution $y(t)$:

$$\|y(t)\| \leq e^{\frac{1}{2} + \frac{1}{2\|X\|\gamma(A)}} e^{-\frac{t}{2\|X\|}},$$

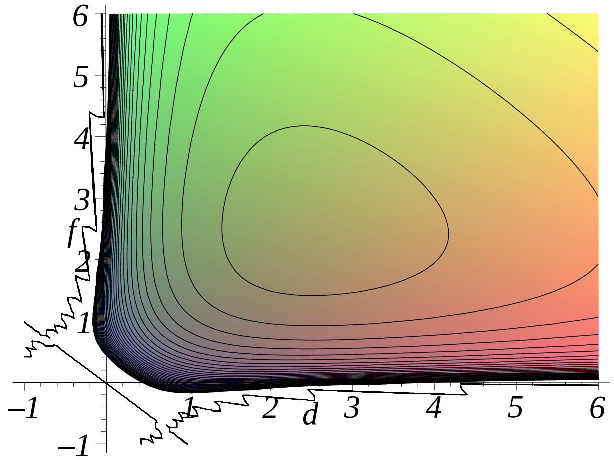


Figure 2.3: Contours of the function $(d, f) \mapsto \mathrm{tr}(X(D))$

where $\gamma(A) = 2\inf_{\|x\|=1} x^*Ax$.

Let $L_2^{-1}L_1 = U_2\Omega U_1^*$ be SVD–decomposition of the matrix $L_2^{-1}L_1$, with $\Omega = \mathrm{diag}(\omega_1, \ldots, \omega_n) > 0$. We can assume $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$. Set $U = \left(\begin{smallmatrix} U_1 & 0 \\ 0 & U_2 \end{smallmatrix}\right)$. Then

$$\widehat{A} = U^*AU = \begin{bmatrix} 0 & \Omega \\ -\Omega & -C \end{bmatrix}, \tag{2.1.15}$$

where $C = U_2^*L_2^{-1}DL_2^{-*}U_2$ is positive semi–definite. If we denote $F = L_2^{-*}U_2$, then $F^*MF = I$, $F^*KF = \Omega$. Thus we have obtained a particularly convenient, the so–called *modal representation* of the problem (2.1.2). In the following we assume that the matrix A has the form given in (2.1.15).

Modally damped system are characterized by the generalized commutativity property

$$DK^{-1}M = MK^{-1}D.$$

In the modal representation (2.1.15) this means that C and Ω commute. It has been shown in [Cox98b] that

$$X = \begin{bmatrix} \frac{1}{2}C\Omega^{-2} + C^{-1} & \frac{1}{2}\Omega^{-1} \\ \frac{1}{2}\Omega^{-1} & C^{-1} \end{bmatrix}. \tag{2.1.16}$$

Hence, the optimal matrix C, for the criterion with the penalty function (2.1.7) with $Z = I$, as well as for the criterion given by (2.1.12), is $C = 2\Omega$. This can be easily seen in the case when $\omega_i \neq \omega_j$, $i \neq j$, since then the matrix C must be diagonal.

This result is generalized to the infinite–dimensional case in Section 4.4. In the matrix case, this result can be generalized to the case when the matrix Z has the form $Z = \left(\begin{smallmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{Z} \end{smallmatrix}\right)$, where $\widetilde{Z}$ is diagonal with zeros and ones on the diagonal.

The case of the friction damping i.e. when $D = 2aM$, $a > 0$ was considered in [Cox98b], where it was shown that the optimal parameter for the criterion with the penalty function (2.1.7) is $a = \sqrt{\omega_1}\sqrt{\frac{\sqrt{5}-1}{2}}$.

The set of damping matrices over which we optimize the system is determined by the physical properties of the system. The maximal admissible set is the set of all symmetric matrices C for which the corresponding matrix A is stable. Usually, the admissible matrices must be positive semi–definite. The important case is when the admissible set consists of all positive semi–definite matrices C for which the corresponding matrix A is stable. For this case Brabender [Bra98] (see also [VBD01]) had shown that the following theorem holds.

Theorem 2.1.2. *Let the matrix Z be of the form $Z = \left(\begin{smallmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{Z} \end{smallmatrix}\right)$, where $\widetilde{Z} = \left(\begin{smallmatrix} I_s & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $1 \leq s \leq n$. Denote by $\mathcal{M}$ the set of all matrices of the form*

$$2\begin{bmatrix} \Omega_s & 0 \\ 0 & H \end{bmatrix}, \Omega_s = \mathrm{diag}(\omega_1, \ldots, \omega_s),$$

where H varies over the set of all symmetric positive semi–definite matrices of order $n-s$ such that the corresponding matrix A is stable. On the set $\mathcal{M}$ the function $X \mapsto \operatorname{tr}(XZ)$, where X solves (2.1.10), achieves a strict local minimum. In particular, this function is constant on $\mathcal{M}$.

For $s=n$ the set $\mathcal{M}$ reduces to a single matrix 2Ω, hence in this case, the function $X \mapsto \operatorname{tr}(XZ)$ attains in $C = 2\Omega$ local minimum.

In [VBD01] it was conjectured that the minimum from Theorem 2.1.2 is global. We will give an affirmative answer to this conjecture in Section 2.2.

In the applications the set of admissible matrices is often much smaller. In the particular case when the set of admissible matrices is parameterized by

$$C = C(a) = \sum_{j=1}^{k} a_j C_j, \tag{2.1.17}$$

where $a = (a_1, \ldots, a_s) \in \mathbb{R}^k$, $k \le n^2$ and C_j, $1 \le j \le k$ are linearly independent positive semi–definite matrices, the following theorem has been proved in [CNRV02].

Theorem 2.1.3. *Set $\mathbb{R}^k_+ = \{a \in \mathbb{R}^k : a_i > 0, 1 \le i \le k\}$. We have*

(i) If $A(a)$ is stable for some $a \in \mathbb{R}^k_+$, then $A(a)$ is stable for all $a \in \mathbb{R}^k_+$.

(ii) If $A(a)$ is stable for some $a \in \mathbb{R}^k_+$, and if $\mathcal{C}_+$ denotes the open component containing $\mathbb{R}^k_+$ on which the equation $A(a)^*X + XA(a) = -I$ is solvable, then $a \mapsto \operatorname{tr} X(a)$ takes its minimum there.

In [Rit] examples are given where the optimal damping is achieved for the damping matrix which is not positive semi–definite. This phenomenon occurs when the matrices C_j are badly chosen.

In the case of one–dimensional damping, i.e when $C = cc^*$ Veselić had shown in [Ves90] that the corresponding penalty function is convex and a (unique) minimum has been found. Also in the one–dimensional case, in [Ves88] it has been shown that for any sequence of $2n$ eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ (symmetric with respect to the real axis), situated in the left half–plane, there exists a corresponding Ω and a vector c such that the eigenvalues of the corresponding matrix A are $\lambda_1, \ldots, \lambda_{2n}$. An analogous inverse spectral problem was solved in [Ves90] for the case of n–mass oscillator.

2.2 Global minimum

The Lyapunov equation (2.1.10) can be written as

$$(A_0 - BCB^*)^* X + X(A_0 - BCB^*) = -I, \tag{2.2.1}$$

where

$$A_0 = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The dual Lyapunov equation (2.1.14) is given by

$$(A_0 - BCB^*)Y + Y(A_0 - BCB^*)^* = -Z \tag{2.2.2}$$

Let

$$\mathcal{C}_s = \{C \in \mathbb{R}^{n\times n} : C = C^*, \mathrm{Re}\{\sigma(A_0 + BCB^*)\} < 0\},$$

$$D_s^+ = \{C \in \mathbb{R}^{n\times n} : C \geq 0,\ \mathrm{Re}\{\sigma(A_0 + BCB^*)\} < 0\},$$

and let D_s be the connected component of $\mathcal{C}_s$ which contains D_s^+.

To emphasize the dependence of X and Y of the parameter C we write $X(C)$ and $Y(C)$. We are interested in the following optimization problems:

$$(OD) \qquad \text{minimize } \operatorname{tr}(X(C)) \text{ subject to } C \in D_s \text{ and (2.2.1)},$$

and

$$(OD^+) \qquad \text{minimize } \operatorname{tr}(X(C)) \text{ subject to } C \in D_s^+ \text{ and (2.2.1)} \qquad .$$

Let us define the function $f : D_s \to \mathbb{R}$ by

$$f(C) = \operatorname{tr}(X(C)Z), \text{ where } X(C) \text{ solves (2.2.1).} \tag{2.2.3}$$

Let $\widetilde{Z} = \operatorname{diag}(\alpha_1, \ldots, \alpha_s, 0, \ldots, 0)$, where $1 \le s \le n$ and $\alpha_i > 0$, $i = 1, \ldots, s$. Set $Z = \left(\begin{smallmatrix} \widetilde{Z} & 0 \\ 0 & \widetilde{Z} \end{smallmatrix}\right)$. We also define $\Omega_s = \operatorname{diag}(\omega_1, \ldots, \omega_s, 0, \ldots, 0)$.

Theorem 2.2.1. *For the matrix Z given above, the problem (OD^+) has a solution, and the set on which the minimum is attained is*

$$\mathcal{C}_{\min} = \left\{ C = \begin{bmatrix} 2\Omega_s & 0 \\ 0 & H \end{bmatrix} : H \ge 0,\ C \in \mathcal{C}_s \right\}.$$

Note that the solution of (OD^+) depends only on number s.

Proof. Let $C \in D_s^+$ be arbitrary. Since Z commutes with $J = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$ and $A_0 - BCB^*$ is J-symmetric,

$$f(C) = \operatorname{tr}(X(C)) \tag{2.2.4}$$

holds, where $X(C)$ solves the Lyapunov equation

$$(A_0 - BCB^*)^* X + X(A_0 - BCB^*) = -Z. \tag{2.2.5}$$

Let $\widetilde{Z}_i$ be a diagonal matrix with all entries zero except the i-th which is α_i. Set $Z_i = \left(\begin{smallmatrix} \widetilde{Z}_i & 0 \\ 0 & \widetilde{Z}_i \end{smallmatrix}\right)$. Let X_i be the solution of the Lyapunov equation

$$(A_0 - BCB^*)^* X + X(A_0 - BCB^*) = -Z_i. \tag{2.2.6}$$

Then it is easy to see that the solution of the Lyapunov equation (2.2.5) is

$$X = \sum_{i=1}^{s} X_i. \tag{2.2.7}$$

Our aim is to show

$$\min\{\mathrm{tr}(X(C)) : X(C) \text{ solves } (2.2.6),\ C \in D_S^+\} \geq \frac{2\alpha_i}{\omega_i}, \quad i = 1, \ldots, s \tag{2.2.8}$$

First observe that by simple permutation argument we can assume $i = 1$. Secondly, we can assume $\alpha_i = 1$ (just multiply (2.2.6) by $1/\alpha_i$). Let us decompose the matrix $X \in \mathbb{R}^{2n\times 2n}$ in the following way:

$$X = \begin{bmatrix} x_{11} & X_{12} & x_{13} & X_{14} \\ X_{12}^* & X_{22} & X_{23} & X_{24} \\ x_{13} & X_{23}^* & x_{33} & X_{34} \\ X_{14}^* & X_{24}^* & X_{34}^* & X_{44} \end{bmatrix}, \tag{2.2.9}$$

where $x_{11}, x_{33}, x_{13} \in \mathbb{R}$, $X_{12}, X_{14}, X_{34} \in \mathbb{R}^{1\times(n-1)}$, $X_{22}, X_{24}, X_{44} \in \mathbb{R}^{(n-1)\times(n-1)}$, and $X_{23} \in \mathbb{R}^{(n-1)\times 1}$. Next we partition the Lyapunov equation

$$(A_0 - BCB^*)^* X + X(A_0 - BCB^*) = -Z_1$$

in the same way as we did with X. We obtain

$$-x_{13}\omega_1 - \omega_1 x_{13}^* + 1 = 0 \tag{1,1}$$

$$-\omega_1 X_{23}^* - X_{14}\Omega_{n-1} = 0 \tag{1,2}$$

$$-\omega_1 x_{33} + x_{11}\omega_1 - x_{13}c_{11} - X_{14}C_{12}^* = 0 \tag{1,3}$$

$$-\omega_1 X_{34} + X_{12}\Omega_{n-1} - x_{13}C_{12} - X_{14}C_{22} = 0 \tag{1,4}$$

$$-\Omega_{n-1}X_{24}^* - X_{24}\Omega_{n-1} = 0 \tag{2,2}$$

$$-\Omega_{n-1}X_{34}^* + X_{12}^*\Omega_1 - X_{23}c_{11} - X_{24}C_{12}^* = 0 \tag{2,3}$$

$$-\Omega_{n-1}X_{44} + X_{22}\Omega_{n-1} - X_{23}C_{12} - X_{24}C_{22} = 0 \tag{2,4}$$

$$\omega_1 x_{13} - c_{11}x_{33} - C_{12}XC_{34}^* + x_{13}^*\omega_1 - x_{33}c_{11} - X_{34}C_{12}^* + 1 = 0 \tag{3,3}$$

$$\omega_1 X_{14} - c_{11}X_{34} - C_{12}X_{44} + X_{23}^*\Omega_{n-1} - x_{33}C_{12} - X_{34}C_{22} = 0 \tag{3,4}$$

$$\Omega_{n-1}X_{24} - C_{12}^*X_{34} - C_{22}X_{44} + X_{24}^*\Omega_{n-1} - X_{34}^*C_{12} - X_{44}C_{22} = 0, \tag{4,4}$$

where $\omega_1, c_{11} \in \mathbb{R}$, $C_{12} \in \mathbb{R}^{1\times(n-1)}$, and $C_{22}, \Omega_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$.

From (1,1) we obtain $x_{13} = \frac{1}{2\omega_1}$. One can easily see that $c_{11} = 0$ implies $C_{12} = 0$, hence (3,3) reads $2 = 0$, a contradiction. Hence, $c_{11} > 0$. From (3,3) we now get

$$x_{33} = \frac{1 - X_{34}C_{12}^*}{c_{11}}. \tag{2.2.10}$$

The relation (4,4), together with the facts $X_{44} \geq 0$, $C_{22} \geq 0$, implies

$$\mathrm{tr}(C_{12}^*X_{34} + X_{34}^*C_{12}) \leq \mathrm{tr}(\Omega_{n-1}X_{24} + X_{24}^*\Omega_{n-1}),$$

and the relation (2,2) implies $\mathrm{tr}(X_{24}\Omega_{n-1}) = 0$, hence we obtain

$$\mathrm{tr}(X_{34}^*C_{12}) = \mathrm{tr}(X_{34}C_{12}^*) \leq 0. \tag{2.2.11}$$

From the relation (1,3) we obtain

$$x_{11} = x_{33} + x_{13}c_{11}\omega_1^{-1} + \omega_1^{-1}X_{14}C_{12}^*.$$

From relation (2,4) we obtain

$$X_{22} = \Omega_{n-1} X_{44} \Omega_{n-1}^{-1} + X_{23} C_{12} \Omega_{n-1}^{-1} + X_{24} C_{22} \Omega_{n-1}^{-1},$$

hence

$$\mathrm{tr} X_{22} = \mathrm{tr} X_{44} + \mathrm{tr}(X_{23} C_{12} \Omega_{n-1}^{-1}) + \mathrm{tr}(X_{24} C_{22} \Omega_{n-1}^{-1}).$$

From the relation (2,2) we obtain

$$X_{24} = \frac{1}{2} S \Omega_{n-1}^{-1},$$

where $S \in \mathbb{R}^{(n-1)\times(n-1)}$ is a skew–symmetric matrix.

Hence

$$\begin{aligned}
\mathrm{tr} X &= x_{11} + \mathrm{tr} X_{22} + x_{33} + \mathrm{tr} X_{44} \\
&= 2x_{33} + 2\mathrm{tr} X_{44} + \frac{c_{11}}{2\omega_1^2} + \frac{1}{\omega_1} X_{14} C_{12}^* + \mathrm{tr}(X_{23} C_{12} \Omega_{n-1}^{-1}) + \frac{1}{2}\mathrm{tr}(S \Omega_{n-1}^{-1} C_{22} \Omega_{n-1}^{-1}) \\
&= 2x_{33} + 2\mathrm{tr} X_{44} + \frac{c_{11}}{2\omega_1^2} + \frac{1}{\omega_1} X_{14} C_{12}^* + \mathrm{tr}(X_{23} C_{12} \Omega_{n-1}^{-1}).
\end{aligned}$$

From the relation (1,2) follows $X_{23} = -\frac{1}{\omega_1} \Omega_{n-1} X_{14}^*$, hence

$$\mathrm{tr} X = 2x_{33} + 2\mathrm{tr} X_{44} + \frac{c_{11}}{2\omega_1^2}.$$

Now (2.2.10) and (2.2.11) imply

$$\begin{aligned}
\mathrm{tr} X = 2\frac{1 - X_{34} C_{12}^*}{c_{11}} + \frac{c_{11}}{2\omega_1^2} + 2\mathrm{tr} X_{44} \geq \\
\geq \frac{2}{c_{11}} + \frac{c_{11}}{2\omega_1^2} \geq \frac{2}{\omega_1}.
\end{aligned} \tag{2.2.12}$$

The last inequality follows from the following observation. Let us define the function $g(x) = \frac{2}{x} + \frac{x}{2\omega_1^2}$. Then the function g attains its unique minimum $\frac{2}{\omega_1}$ in $x = 2\omega_1$.

Hence, we have shown (2.2.8). Now (2.2.7) and (2.2.4) imply

$$\mathrm{tr}(X(C)Z) \geq 2 \sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}.$$

Since $\mathrm{tr}(X(2\Omega)Z) = 2\sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}$, this is indeed the global minimum.

Assume that $C \in D_s^+$ is such that $\mathrm{tr}(X(C)Z) = 2\sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}$. Then (2.2.12) and (2.2.7) imply $\mathrm{tr}X_i = \frac{2}{\omega_i}$. Observe the matrix X_1 which we decompose as in (2.2.9). Then (2.2.12) implies $X_{44} = 0$. Since $X_1 \geq 0$, it follows $X_{14} = X_{24} = X_{34} = 0$. From the relation (1,2) follows immediately $X_{23} = 0$. The relation (1,3) implies $x_{11} = \frac{3}{2}\frac{1}{\omega_1}$, which implies $\mathrm{tr}X_{22} = 0$. Hence $X_{22} = 0$. This implies $X_{12} = 0$. Finally, from (1,4) now follows $C_{12} = 0$.

By repeating this procedure for $i = 2, \ldots, s$ we obtain $C \in \mathcal{C}_{\min}$.

On the other hand, it is easy to see that $\mathrm{tr}(X(C)Z) = 2\sum_{i=1}^{s} \frac{\alpha_i}{\omega_i}$, for all $C \in \mathcal{C}_{\min}$. $\square$

2.3 Numerical aspects

A general algorithm for the optimization of damping does not exist. Available algorithms optimize only the viscosities of dampers, not their positions, i.e. they deal with the cases when D is parameterized by some function $\mathbb{R}^k \ni a \mapsto C(a) \in \mathbb{R}^{n\times n}$, where k is usually much smaller then n.

Two types of numerical optimization algorithms are currently in use. One type are the algorithms which rely on the use of Newton–type algorithms for higher–dimensional (constrained or unconstrained) problems on the function f defined by (2.2.3) with the use of some Lyapunov solver. The second type algorithms are ones which bypass the solving of the Lyapunov equation by explicitly calculating the function f.

The algorithms of the second type, when available, are faster, but harder to construct. One algorithm of this type is constructed for the case when rank of

D is one. In [Ves90] Veselić has given an efficient algorithm which calculates an optimal ε, where $D = \varepsilon dd^*$. Moreover, the optimal viscosity ε is given by a closed formula.

Recently, a sort of generalization of this algorithm is given in [TV02]. This algorithm can be used to calculate the optimal viscosity ε if $D = \varepsilon D_0$.

Note that both of these algorithms only deal with the case when the set of admissible matrices is parameterized by a real–valued function.

If we want to treat more complex cases, only the algorithms of the first type are available so far. A major computational effort is spent on the calculation of the solution of the Lyapunov equation. The standard algorithm is given in [BS72]. A variant of this general algorithm designed for the Lyapunov equation is given in [Ham82]. Another algorithm is given in [GNVL79]. There also exist algorithms based on the iterative methods, see, for example [Saa90], [Wac88] and [HTP96].

A Newton–type algorithm is developed in [Bra98]. The gradient and the Hessian of the penalty function can be explicitly calculated.

Chapter 3

Abstract vibrational system

In this chapter we will introduce the notion of an abstract vibrational system in terms of an abstract second–order differential equation involving quadratic forms. We will show how we can translate this equation into a linear Cauchy problem, and we will solve this problem in terms of a semigroup. It turns out that the norm of the semigroup equals one half of the energy of the solution of the original differential equation, and that the original differential equation has a solution if and only if there exists a solution for the Cauchy problem. We will give necessary and sufficient conditions for uniform exponential stability of this semigroup in terms of the original coefficients. In generic situation, uniform exponential stability can be easily checked, if we know the eigenvalues and eigenvectors of the corresponding undamped system.

3.1 Setting the problem

We start with three symmetric, non–negative sesquilinear forms μ, γ and κ in a vector space $\mathcal{X}_0$ over the complex field. The forms μ, γ and κ correspond to the mass, damping and stiffness, respectively. We assume that the sesquilinear form κ is positive, i.e. $\kappa(x,x) > 0$ for all $x \neq 0$. We also assume that $\mathcal{X}_1 = D(\mu) \cap D(\gamma) \cap D(\kappa)$ is a non–trivial subspace of $\mathcal{X}_0$. We equip the vector space $\mathcal{X}_1$ with the scalar product

$$(x, y) = \kappa(x, y), \ x, y \in \mathcal{X}_1.$$

This is a natural choice, since it enables us to work in a Hilbert space setting with the scalar product which corresponds to the energy of the system (see Example 1.1.1).

Let $\mathcal{Z}$ denote the completion of the space $(\mathcal{X}_1, (\cdot, \cdot))$. The norm generated by this scalar product will be denoted by $\| \cdot \|$. Obviously, $\mathcal{X}_1$ is dense in $\mathcal{Z}$.

For our purposes it will suffice to suppose that the Hilbert space $\mathcal{Z}$ is separable.

We also assume that μ and γ are closable in $\mathcal{Z}$, and we denote these closures also by μ and γ. Set $\mathcal{X} = D(\mu) \cap D(\gamma)$. The subspace $\mathcal{X}$ is obviously dense in $\mathcal{Z}$. The following definition slightly generalizes the notion of the closability of forms, introduced in [Kat95] for positive forms.

Definition 3.1.1. Let χ and ξ be symmetric non–negative sesquilinear forms in a vector space $\mathcal{X}$. We say that χ is ξ–closable if for an arbitrary sequence (x_n), $x_n \in D(\xi) \cap D(\chi)$ with

$$\xi(x_n, x_n) \to 0, \quad \chi(x_n - x_m, x_n - x_m) \to 0, \text{ as } n, m \to \infty,$$

there exist $x \in \mathcal{X}$ such that

$$\xi(x, x) = 0, \quad \chi(x - x_n, x - x_n) \to 0.$$

The following proposition is obvious.

Proposition 3.1.1. *The sesquilinear forms μ and γ are closable in $\mathcal{Z}$ if and only if μ and γ are κ–closable.*

The following formal Cauchy problem

$$\begin{gathered}\mu(\ddot{x}(t), z) + \gamma(\dot{x}(t), z) + (x(t), z) = 0, \text{ for all } z \in \mathcal{Z},\ t \geq 0, \\ x(0) = x_0,\ \ \dot{x}(0) = \dot{x}_0,\end{gathered} \tag{3.1.1}$$

where $x : [0, \infty) \to \mathcal{Z}$, $x_0, \dot{x}_0 \in \mathcal{Z}$, and μ, γ and κ satisfy the assumptions given above, is called an *abstract vibrational system*.

Note that the stiffness form κ is indirectly present in (3.1.1); it defines the geometry of the underlying space.

We introduce the energy function of the system described by (3.1.1) by

$$E(t; x_0, \dot{x}_0) = \frac{1}{2}\mu(\dot{x}(t), \dot{x}(t)) + \frac{1}{2}\kappa(x(t), x(t)), \tag{3.1.2}$$

where $x(t)$ is the solution of (3.1.1).

The Cauchy problem (3.1.1) and the energy function (3.1.2) will be central objects of the investigation in this thesis.

The corresponding undamped system is described by

$$\begin{gathered}\mu(\ddot{x}(t), z) + (x(t), z) = 0, \text{ for all } z \in \mathcal{Z},\ t \geq 0, \\ x(0) = x_0,\ \ \dot{x}(0) = \dot{x}_0.\end{gathered}$$

Example 3.1.1 *(Vibrating string).* As an illustrative example, throughout this chapter we will study the following differential equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - \frac{\partial^2}{\partial x^2}u(x,t) = 0,\ x \in [0, \pi],\ t \geq 0, \tag{3.1.3a}$$

$$u(0,t) = 0, \tag{3.1.3b}$$

$$\frac{\partial}{\partial x}u(\pi, t) + \varepsilon\frac{\partial}{\partial t}u(\pi, t) = 0, \tag{3.1.3c}$$

$$u(x, 0) = u_0(x),\ \ \frac{\partial}{\partial t}u(x, 0) = u_1(x), \tag{3.1.3d}$$

where $\varepsilon > 0$, and u_0, u_1 initial data. The equation (3.1.3) describes a homogeneous vibrating string with linear boundary damping on its right end. We multiply equation (3.1.3a) by any continuously differentiable function v with $v(0) = 0$, and then partially integrate. Using (3.1.3c) we obtain

$$\int_0^\pi \frac{\partial^2}{\partial t^2} u(x,t)v(x)\mathrm{d}x + \varepsilon \frac{\partial}{\partial t} u(\pi,t) + \int_0^\pi \frac{\partial}{\partial x} u(x,t)v'(x)\mathrm{d}x = 0.$$

Thus, we set

$$\mu(u,v) = \int_0^\pi u(x)\overline{v(x)}\mathrm{d}x,$$

$$\gamma(u,v) = \varepsilon u(\pi)\overline{v(\pi)},$$

$$\kappa(u,v) = \int_0^\pi u'(x)\overline{v'(x)}\mathrm{d}x,$$

with $\mathcal{X}_0 := D(\mu) = D(\gamma) = D(\kappa) = \{u \in C^2([0,\pi]) : u(0) = 0\}$. Hence $\mathcal{X}_1 = \mathcal{X}_0$. Poincar inequality implies $\mathcal{Z} = \{u \in L^2([0,\pi]) : u' \in L^2([0,\pi]), u(0) = 0\}$, i.e. $\mathcal{Z}$ is the space of all functions from the Sobolev space $\mathcal{H}_1([0,\pi])$ which vanish in π. Also, the scalar product generated by κ and standard scalar product in $\mathcal{H}_1([0,\pi])$ are equivalent.

Next we will show that μ and γ are κ–closable. Indeed, $\kappa(u_n, u_n) \to 0$ implies $u_n' \to 0$ in L^2–norm, and $\mu(u_n - u_m, u_n - u_m) \to 0$ for $n, m \to \infty$ implies the existence of $u \in L^2([0,\pi])$ such that $u_n \to u$ in L^2–norm. From Poincar inequality now follows $u = 0$, hence μ is κ–closable. From $\gamma(u_n - u_m, u_n - u_m) \to 0$ and Poincar inequality follows $|u_n(\pi)|^2 \leq \|u_n'\|_{L^2}^2 \to 0$, so γ is also κ–closable. Obviously, the closures of μ and γ (which we again denote by μ and γ) are defined on all $\mathcal{Z}$. Hence, the system (3.1.3) can be written as

$$\begin{aligned} &\mu(\ddot{u}, v) + \gamma(\dot{u}, v) + (u, v) = 0, \text{ for all } v \in \mathcal{Z}, \\ &u(0) = u_0, \quad \dot{u}(0) = u_1, \quad u_0, u_1 \in \mathcal{Z}. \end{aligned} \tag{3.1.4}$$

For (3.1.1), the formal substitution $x(t) = e^{\lambda t} y$ leads to

$$\lambda^2 \mu(y, z) + \lambda\gamma(y, z) + (y, z) = 0.$$

The form $\mu_\lambda = \lambda^2\mu + \lambda\gamma + I$ is densely defined (with the domain $\mathcal{X}$) closed (as a sum of closed symmetric forms) positive form for all $\lambda > 0$.

For $\lambda \geq 0$, the second representation theorem [Kat95, pp. 331] implies the existence of selfadjoint non–negative operators M and C such that

$$D(M^{1/2}) = D(\mu), \quad \mu(x, y) = (M^{1/2}x, M^{1/2}y), \; x, y \in D(\mu),$$
$$D(C^{1/2}) = D(\gamma), \quad \gamma(x, y) = (C^{1/2}x, C^{1/2}y), \; x, y \in D(\gamma),$$

and a selfadjoint positive operator M_λ such that

$$D({M_\lambda}^{1/2}) = D(\mu_\lambda), \quad \mu_\lambda(x, y) = ({M_\lambda}^{1/2}x, {M_\lambda}^{1/2}y), \; x, y \in D(\mu_\lambda).$$

Obviously,

$$D({M_\lambda}^{1/2}) \subset D(M^{1/2}) \text{ and } D({M_\lambda}^{1/2}) \subset D(C^{1/2}) \text{ for all } \lambda > 0,$$

and

$$\mu_\lambda(x, x) \geq \|x\|^2 \text{ for all } \lambda \geq 0. \tag{3.1.5}$$

We write

$$M_\lambda = \lambda^2 M + \lambda C + I,$$

in the form–sum sense. From (3.1.5) follows that $M_\lambda^{-1/2}$ exists and is everywhere defined bounded operator, and $\|M_\lambda^{-1/2}\| \leq 1$ holds for all $\lambda \geq 0$. This obviously implies $\|M_\lambda^{-1}\| \leq 1$ for all $\lambda \geq 0$. Note also that $M_0 = I$.

Now (3.1.1) can be written as

$$(M^{1/2}\ddot{x}(t), M^{1/2}z) + (M^{1/2}\dot{x}(t), M^{1/2}z) + (x(t), z) = 0, \text{ for all } z \in \mathcal{Z},\ t \geq 0,$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \tag{3.1.6}$$

and the energy function (3.1.2) can be written as

$$E(t; x_0, \dot{x}_0) = \frac{1}{2}\|M^{1/2}\dot{x}(t)\|^2 + \frac{1}{2}\|x(t)\|^2 \tag{3.1.7}$$

Example 3.1.2 *(Continuation of Example* 3.1.1*).* A straightforward computation gives

$$(Mu)(x) = \int_0^\pi G(x,\xi)u(\xi)\mathrm{d}\xi,$$
$$(Cu)(x) = \varepsilon u(\pi)x,$$

where G is the Green function of the corresponding undamped system given by

$$G(x,\xi) = \begin{cases} x, & x \leq \xi, \\ \xi, & x \geq \xi \end{cases}.$$

Both operators are everywhere defined, and M is compact. Obviously, $\mathcal{N}(M) = \{0\}$. The operator C has one–dimensional range spanned by the identity function. Observe that $M^{1/2}$ and $C^{1/2}$ cannot be written in a closed form.

The energy function of the system (3.1.3) is given by

$$E(t; u_0, u_1) = \frac{1}{2}\int_0^\pi \left(|x(t)|^2 + |\dot{x}(t)|^2\right)\mathrm{d}x. \tag{3.1.8}$$

3.2 The pseudoresolvent

From $D({M_\lambda}^{1/2}) \subset D(M^{1/2})$ follows that $M^{1/2}M_\lambda^{-1}$ is a closed operator defined on all $\mathcal{Z}$, hence by

$$B_\lambda = M^{1/2}M_\lambda^{-1/2}$$

is given a bounded operator in $\mathcal{Z}$, for all $\lambda > 0$.

The adjoint B_λ^* of B_λ is then also bounded, and it is the closure of the densely defined operator $M_\lambda^{-1/2}M^{1/2}$, i.e.

$$B_\lambda^* = \overline{M_\lambda^{-1/2}M^{1/2}}.$$

To prove this, first note that $D(M_\lambda^{-1/2}M^{1/2}) = D(M^{1/2})$. From [Wei76, Satz 4.19] we have

$$(M_\lambda^{-1/2}M^{1/2})^* = M^{1/2}M_\lambda^{-1/2} = B_\lambda.$$

Hence from [Wei76, Satz 4.13] follows

$$\overline{M_\lambda^{-1/2}M^{1/2}} = (M_\lambda^{-1/2}M^{1/2})^{**} = B_\lambda^*.$$

For $\lambda > 0$ we have

$$\mu \leq \mu + \frac{1}{\lambda}\gamma + \frac{1}{\lambda^2}I = \frac{\mu_\lambda}{\lambda^2},$$

hence

$$\begin{aligned}\|B_\lambda x\|^2 = (M^{1/2}M_\lambda^{-1/2}x, M^{1/2}M_\lambda^{-1/2}x) = \mu(M_\lambda^{-1/2}x) \leq \\ \leq \frac{\mu_\lambda(M_\lambda^{-1/2}x)}{\lambda^2} = \frac{\|x\|^2}{\lambda^2}.\end{aligned}$$

This implies

$$\|B_\lambda\| \leq \frac{1}{\lambda}, \;\; \|B_\lambda^*\| \leq \frac{1}{\lambda}, \text{ for all } \lambda > 0.$$

In the finite–dimensional case the corresponding equation (3.1.1) reads

$$M\ddot{x} + C\dot{x} + x = 0. \tag{3.2.1}$$

Substituting $y_1 = x$, $y_2 = M^{1/2}\dot{x}$, it is easy to see that (3.2.1) is equivalent with the matrix differential equation

$$\dot{y} = A^f y,$$

where $y = \binom{y_1}{y_2}$, and

$$A^f = \begin{bmatrix} 0 & M^{-1/2} \\ -M^{-1/2} & -M^{-1/2}CM^{-1/2} \end{bmatrix}.$$

By a straightforward computation one can obtain

$$(A^f - \lambda)^{-1} = \begin{bmatrix} \frac{1}{\lambda}(M_\lambda^{-1} - I) & -M_\lambda^{-1}M^{1/2} \\ M^{1/2}M_\lambda^{-1} & -\lambda M^{1/2}M_\lambda^{-1}M^{1/2}, \end{bmatrix}, \tag{3.2.2}$$

which is a well-known formula.

In the general case the operator given by (3.2.2) may not exist. Hence, to bypass this, we define a family of bounded operators in $\mathcal{Z} \oplus \mathcal{Z}$, which, in some sense, generalizes (3.2.2), by

$$R_0(\lambda) = \begin{bmatrix} \frac{1}{\lambda}M_\lambda{}^{-1} - \frac{1}{\lambda} & -M_\lambda{}^{-1/2}B_\lambda^* \\ B_\lambda M_\lambda{}^{-1/2} & -\lambda B_\lambda B_\lambda^* \end{bmatrix}, \ \lambda > 0. \tag{3.2.3}$$

In the matrix case, it is easy to see that (3.2.3) is equivalent with (3.2.2).

Next we will show that $R_0(\lambda)$ is a pseudoresolvent.

Lemma 3.2.1. *For all $\lambda, \nu > 0$, and all $x, y \in \mathcal{Z}$ we have*

$$(\lambda M_\lambda{}^{-1}x - \nu M_\nu^{-1}x, y) = (\lambda - \nu)\left[-\lambda\nu(B_\nu M_\nu^{-1/2}x, B_\lambda M_\lambda{}^{-1/2}y) + (M_\nu^{-1}x, M_\lambda{}^{-1}y)\right]$$

Proof. We have

$$\mu_\lambda(M_\nu^{-1}x, M_\lambda^{-1}y) = ({M_\lambda}^{1/2}M_\nu^{-1}x, {M_\lambda}^{-1/2}y) = (M_\nu^{-1}x, y),$$

$$\mu_\nu(M_\nu^{-1}x, M_\lambda^{-1}y) = (M_\lambda^{-1}x, y).$$

Hence,

$$\begin{aligned}(\lambda M_\lambda^{-1}x - \nu M_\nu^{-1}x, y) &= \lambda(M_\lambda^{-1}x, y) - \nu(M_\nu^{-1}x, y) \\ &= \lambda\mu_\nu(M_\nu^{-1}x, M_\lambda^{-1}y) - \nu\mu_\lambda(M_\nu^{-1}x, M_\lambda^{-1}y) \\ &= (\lambda - \ \mu)\left[-\lambda\nu\mu(M_\nu^{-1}x, M_\lambda^{-1}y) + (M_\nu^{-1}x, M_\lambda^{-1}y)\right].\end{aligned}$$

Now, from

$$\mu(M_\nu^{-1}x, M_\lambda^{-1}y) = (M^{1/2}M_\nu^{-1}x, M^{1/2}M_\lambda^{-1}y) = (B_\nu {M_\nu}^{-1/2}x, B_\lambda {M_\lambda}^{-1/2}y),$$

our claim follows. □

Proposition 3.2.2. *The operator family* $\{R_0(\lambda) : \lambda > 0\}$ *satisfies the resolvent equation*

$$R_0(\lambda) - R_0(\nu) = (\lambda - \nu)R_0(\lambda)R_0(\nu), \qquad (3.2.4)$$

i.e. $R_0(\lambda)$ *is a pseudoresolvent.*

Proof. Let us denote the operator matrix on the left hand side and the right hand side of (3.2.4) by

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}, \end{bmatrix},$$

respectively. From the straightforward computation we get

$$
\begin{aligned}
a_{11} &= \frac{\lambda-\nu}{\lambda\nu}(M_\lambda^{-1}M_\nu^{-1} - M_\lambda^{-1} - M_\nu^{-1} + I) - (\lambda-\nu)(M_\lambda{}^{-1/2}B_\lambda^* B_\nu M_\nu^{-1/2}),\\
a_{12} &= \frac{\lambda-\nu}{\nu}(B_\lambda M_\lambda{}^{-1/2}M_\nu^{-1} - B_\lambda M_\lambda{}^{-1/2}) - \lambda(\lambda-\nu)B_\lambda B_\lambda^* B_\nu M_\nu^{-1/2},\\
a_{21} &= -\frac{\lambda-\nu}{\lambda}M_\lambda^{-1}M_\nu^{-1/2}B_\nu^* + \frac{\lambda-\nu}{\lambda}M_\nu^{-1/2}B_\nu^* + \nu(\lambda-\nu)M_\lambda{}^{-1/2}B_\lambda^* B_\nu B_\nu^*,\\
a_{22} &= (\lambda-\nu)(-B_\lambda M_\lambda{}^{-1/2}M_\nu^{-1/2}B_\nu^* + \lambda\nu B_\lambda B_\lambda^* B_\nu B_\nu^*).
\end{aligned}
$$

Let $x, y \in D(M^{1/2})$ be arbitrary, and let us denote $u = M^{1/2}x$, $v = M^{1/2}y$. Then $B_\nu^* x = M_\nu^{-1/2}u$ and $B_\lambda^* y = M_\lambda{}^{-1/2}v$.

First we show that $a_{22} = b_{22}$. Let us now calculate $(a_{22}x, y)$. Using Lemma 3.2.1 we obtain

$$
\begin{aligned}
(a_{22}x, y) &= (\lambda-\nu)\left[-(B_\lambda M_\lambda{}^{-1/2}M_\nu^{-1/2}B_\nu^* x, y) + \lambda\nu(B_\lambda B_\lambda^* B_\nu B_\nu^* x, y)\right]\\
&= (\lambda-\nu)\left[-(M_\lambda{}^{-1/2}M_\nu^{-1}u, M_\lambda{}^{-1/2}v) + \lambda\nu(B_\lambda^* B_\nu M_\nu^{-1/2}u, M_\lambda{}^{-1/2}v)\right]\\
&= (\lambda-\nu)\left[\lambda\nu(B_\nu M_\nu^{-1/2}u, B_\lambda M_\lambda{}^{-1/2}v) - (M_\nu^{-1}u, M_\lambda^{-1}v)\right]\\
&= -\lambda(M_\lambda^{-1}u, v) + \nu(M_\nu^{-1}u, v)\\
&= -\lambda(M^{1/2}M_\lambda^{-1}M^{1/2}x, y) + \nu(M^{1/2}M_\nu^{-1}M^{1/2}x, y)\\
&= -\lambda(B_\lambda B_\lambda^* x, y) + \nu(B_\nu B_\nu^* x, y) = (b_{22}x, y).
\end{aligned}
$$

Since $D(M^{1/2})$ is dense in $\mathcal{Z}$, and operators a_{22} and b_{22} are bounded we have $(a_{22}x, y) = (b_{22}x, y)$ for all $x, y \in \mathcal{Z}$, hence $a_{22} = b_{22}$.

Next we show $a_{12} = b_{12}$. Again, using Lemma 3.2.1 we obtain

$$\begin{aligned}
(a_{12}x, y) &= -\frac{\lambda-\nu}{\lambda}\left[(M_\lambda^{-1}M_\nu^{-1}u, y) - (M_\nu^{-1}u, y) - \lambda\nu(M_\lambda{}^{-1/2}B_\lambda^* B_\nu M_\nu^{-1/2}u, y)\right] \\
&= -\frac{\lambda-\nu}{\lambda}\left[(M_\nu^{-1}u, M_\lambda^{-1}y) - (M_\nu^{-1}u, y) - \lambda\nu(B_\nu M_\nu^{-1/2}u, B_\lambda M_\lambda{}^{-1/2}y)\right] \\
&= -\frac{1}{\lambda}(\lambda M_\lambda^{-1}u - \nu M_\nu^{-1}u, y) + \frac{\lambda-\nu}{\lambda}(M_\nu^{-1}u, y) \\
&= -\frac{1}{\lambda}\left[\lambda(M_\lambda^{-1}u, y) - \nu(M_\nu^{-1}u, y) - \lambda(M-\nu^{-1}u, y) + \nu(M-\nu^{-1}u, y)\right] \\
&= (M_\nu^{-1}u, y) - (M_\lambda^{-1}u, y) = (M_\nu^{-1}uB_\nu^* x, y) - (M_\lambda^{-1}uB_\lambda^* x, y) = (b_{12}x, y),
\end{aligned}$$

hence again $a_{12} = b_{12}$.

Similarly, one can also prove that $a_{21} = b_{21}$.

The term $(a_{11}x, y)$ can be written as

$$\begin{aligned}
&(a_{11}x, y) = -\frac{1}{\lambda} + \frac{1}{\nu} + \\
&+ \frac{\lambda-\nu}{\lambda\nu}\left[(M_\nu^{-1}x, M_\lambda^{-1}y) - (M_\lambda^{-1}x, y) - (M_\nu^{-1}x, y) - \lambda\nu(B_\nu M_\nu^{-1/2}x, B_\lambda M_\lambda^{-1/2}y)\right],
\end{aligned}$$

and since

$$(b_{11}x, y) = -\frac{1}{\lambda} + \frac{1}{\nu} + \frac{1}{\lambda}(M_\lambda^{-1}x, y) - \frac{1}{\nu}(M_\nu^{-1}x, y),$$

to see that $(a_{11}x, y)$ equals $(b_{11}x, y)$, we have to show that

$$\begin{aligned}
&\frac{\lambda-\nu}{\lambda\nu}\left[(M_\nu^{-1}x, M_\lambda^{-1}y) - (M_\lambda^{-1}x, y) - (M_\nu^{-1}x, y) - \lambda\nu(B_\nu M_\nu^{-1/2}x, B_\lambda M_\lambda^{-1/2}y)\right] \\
&= \frac{1}{\lambda}(M_\lambda^{-1}x, y) - \frac{1}{\nu}(M_\nu^{-1}x, y).
\end{aligned}$$

But this easily follows from Lemma 3.2.1. □

Let us define

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \text{ in } \mathcal{Z} \oplus \mathcal{Z}.$$

Proposition 3.2.3. *The operators $R_0(\lambda)$ and $R_0(\lambda)^*$ are dissipative and J–selfadjoint, for all $\lambda > 0$.*

Proof. Let $u = \binom{x}{y} \in \mathcal{Z} \oplus \mathcal{Z}$. Then

$$(R_0(\lambda)\binom{x}{y}, \binom{x}{y}) = -\frac{1}{\lambda}\|x\|^2 + \frac{1}{\lambda}(M_\lambda^{-1}x, x) + 2i\mathrm{Im}(B_\lambda M_\lambda^{-1/2}x, y) - \|B_\lambda^* y\|^2.$$

Now,

$$\begin{aligned}\mathrm{Re}\,(R_0(\lambda)\binom{x}{y}, \binom{x}{y}) &= -\frac{1}{\lambda}\|x\|^2 + \frac{1}{\lambda}(M_\lambda^{-1}x, x) - \|B_\lambda^* y\|^2 \leq \\ &\leq -\frac{1}{\lambda}\|x\|^2 + \frac{1}{\lambda}\|x\|^2 - \|B_\lambda^* y\|^2 \leq 0,\end{aligned}$$

hence $R_0(\lambda)$ is dissipative.

On the other hand

$$\mathrm{Re}(R_0(\lambda)^* x, x) = \mathrm{Re}(x, R_0(\lambda)x) = \mathrm{Re}\overline{(R_0(\lambda)x, x)} = \mathrm{Re}(R_0(\lambda)x, x) \leq 0.$$

The fact that $R_0(\lambda)$ is J–selfadjoint is obvious. □

Proposition 3.2.4. *The null–space of the operator $R_0(\lambda)$ is given by*

$$\mathcal{N}(R_0(\lambda)) = \left\{u \in \mathcal{Z} \oplus \mathcal{Z} : u = \binom{x}{y}, x \in \mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}), y \in \mathcal{N}(B_\lambda^*)\right\},$$

where

$$\mathcal{N}(B_\lambda^*) = \left(M^{1/2}(D(M^{1/2}) \cap D(C^{1/2}))\right)^\perp. \tag{3.2.5}$$

Remark 3.2.1.

1. The set $\mathcal{N}(R_0(\lambda))$ is independent of λ, which is a property shared by all pseudoresolvents, see for example [Paz83].

2. If $D(M^{1/2}) \cap D(C^{1/2})$ is a core of the operator $M^{1/2}$ then $\mathcal{N}(B_\lambda^*) = \mathcal{N}(M^{1/2})$, but in general we have only $\mathcal{N}(B_\lambda^*) \supset \mathcal{N}(M^{1/2})$. This is a mild regularity condition which will be fulfilled in most applications.

Proof of Proposition 3.2.4. From the equation $R_0(\lambda)u = 0$, where $u = \binom{x}{y}$, it follows

$$\begin{aligned}\frac{1}{\lambda}M_\lambda^{-1}x - \frac{1}{\lambda}x - M_\lambda^{-1/2}B_\lambda^* y = 0,\\ B_\lambda M_\lambda^{-1/2}x - \lambda B_\lambda B_\lambda^* y = 0.\end{aligned}$$

Multiplying the first equation by λx, and second by λy and then conjugate, we get

$$(M_\lambda^{-1}x, x) - (x, x) - \lambda(M_\lambda^{-1/2}B_\lambda^* y, x) = 0, \tag{3.2.6}$$

$$\lambda(M_\lambda^{-1/2}B_\lambda^* y, x) - \lambda^2(B_\lambda B_\lambda^* y, y) = 0. \tag{3.2.7}$$

Adding (3.2.6) and (3.2.7) we obtain

$$(M_\lambda^{-1}x, x) - (x, x) - \lambda^2(B_\lambda B_\lambda^* y, y) = 0,$$

which implies

$$(M_\lambda^{-1}x, x) - (x, x) = \lambda^2(B_\lambda B_\lambda^* y, y) \geq 0. \tag{3.2.8}$$

Since $\|M_\lambda^{-1}\| \leq 1$, it follows $(x, x) \geq (M_\lambda^{-1}x, x) \geq (x, x)$, hence

$$(M_\lambda^{-1}x, x) = (x, x), \tag{3.2.9}$$

and since $\mathcal{N}(R_0(\lambda))$ is independent of the choice of λ, this equation holds for all $\lambda > 0$. Also, from (3.2.8) follows $B_\lambda^* y = 0$.

Let us denote $z = M_\lambda^{-1/2}x$. Then (3.2.9) reads $(z, z) = ({M_\lambda}^{1/2}z, {M_\lambda}^{1/2}z)$. Hence,

$$(z, z) = \lambda^2(M^{1/2}z, M^{1/2}z) + \lambda(C^{1/2}z, C^{1/2}z) + (z, z).$$

This implies

$$\lambda^2(M^{1/2}z, M^{1/2}z) + \lambda(C^{1/2}z, C^{1/2}z) = 0, \text{ for all } \lambda > 0,$$

so we obtain $M^{1/2}z = C^{1/2}z = 0$. Hence, $(\lambda^2 M^{1/2} + \lambda C^{1/2})z = 0$, for all $\lambda > 0$, which implies ${M_\lambda}^{1/2}z = z$. This implies $M_\lambda^{-1/2}x = x$, hence $\lambda^2 M^{1/2}x + \lambda C^{1/2}x = 0$ for all $\lambda > 0$, so we finally obtain $M^{1/2}x = C^{1/2}x = 0$. Hence we have shown

$$\mathcal{N}(R_0(\lambda)) \subset \left\{u \in \mathcal{Z} \oplus \mathcal{Z} : u = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), x \in \mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}), y \in \mathcal{N}(B_\lambda^*)\right\}.$$

The other inclusion is straightforward.

And finally, from $\mathcal{N}(B_\lambda^*) = \mathcal{R}(B_\lambda)^\perp$ and

$$\begin{aligned}\mathcal{R}(B_\lambda) &= M^{1/2}M_\lambda^{-1/2}\mathcal{Z} = M^{1/2}D(M_\lambda^{1/2}) = M^{1/2}D(\mu_\lambda) = M^{1/2}(D(\mu)\cap D(\gamma))\\ &= M^{1/2}(D(M^{1/2})\cap D(C^{1/2})),\end{aligned}$$

(3.2.5) follows. □

Remark 3.2.2. If $D(M^{1/2}) \cap D(C^{1/2})$ is a core of $M^{1/2}$ and $\mathcal{N}(M^{1/2}) = \{0\}$, then $\mathcal{N}(R_0(\lambda)) = \{0\}$, hence $R_0(\lambda)$ is a resolvent.

Let us denote $\mathbf{Y} = (\mathcal{N}(R_0(\lambda)))^\perp$. This is the so–called *phase space*. Obviously, the subspace $\mathbf{Y}$ reduces the operator $R_0(\lambda)$. Let us denote by $P_\mathbf{Y} : \mathcal{Z}\oplus\mathcal{Z} \to \mathbf{Y}$ the corresponding orthogonal projector to the subspace $\mathbf{Y}$.

Let $R(\lambda) = P_\mathbf{Y} R_0(\lambda)|_\mathbf{Y}$ denote the corresponding restriction of the operator $R_0(\lambda)$ to the phase space. It is easy to see that $R(\lambda)$ also satisfies the resolvent equation and has trivial null space. Then from the theory of pseudoresolvents [Paz83] it follows that there exists an unique closed operator $A : \mathbf{Y} \to \mathbf{Y}$ such that $R(\lambda) = (\lambda - A)^{-1}$ for all $\lambda \geq 0$. Since $R(\lambda)$ is dissipative and bounded, Proposition A.2 implies that A is maximal dissipative.

The subspace $\mathbf{Y}$ can be decomposed by $\mathcal{Z}_1 \oplus \mathcal{Z}_2$, where $\mathcal{Z}_1 = (\mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}))^\perp$, $\mathcal{Z}_2 = (\mathcal{N}(B_\lambda^*))^\perp$. Since $\mathcal{N}(M^{1/2}) = \mathcal{N}(M)$ and $\mathcal{N}(C^{1/2}) = \mathcal{N}(C)$, we can also write $\mathcal{Z}_1 = (\mathcal{N}(M) \cap \mathcal{N}(C))^\perp$.

3.3 Uniform exponential stability

3.3.1 The operator A^{-1}

Since A is maximal dissipative, the Lumer–Phillips theorem A.3 implies that A generates a contractive strongly continuous semigroup.

Our next aim is to explore properties of the semigroup generated by the operator A and to explain the connection of this semigroup with our quadratic problem. We also find necessary and sufficient conditions which ensure that our semigroup is uniformly exponentially stable.

To achieve this goal, we have to find some suitable representation of A in the space $\mathcal{Z}_1 \oplus \mathcal{Z}_2$. Unfortunately, in general, the operator A does not have block–matrix representation in the product space $\mathcal{Z}_1 \oplus \mathcal{Z}_2$, but its inverse has.

First we will show that a necessary condition for the uniform exponential stability of the semigroup generated by A is that M and C are bounded.

We will need the following lemma.

Lemma 3.3.1. *For an arbitrary $x \in \mathcal{Z}$ and $\lambda > 0$ we have*

$$\lim_{\lambda \to 0} M_\lambda^{-1/2} x - x \to 0, \tag{3.3.1}$$

$$\lim_{\lambda \to 0} M_\lambda^{-1} x - x \to 0. \tag{3.3.2}$$

Proof. Let $x \in D({M_\lambda}^{1/2})$ be arbitrary. We have

$$\|M_\lambda^{-1/2} x - x\| = \|M_\lambda^{-1} {M_\lambda}^{1/2} x - {M_\lambda}^{1/2} x + {M_\lambda}^{1/2} x - x\| \leq$$
$$\leq (\|M_\lambda^{-1}\| + 1)\|{M_\lambda}^{1/2} x - x\| \leq 2\|{M_\lambda}^{1/2} x - x\|.$$

Now first observe that $\|x\| \leq \|M_\lambda x\|$ implies (using Heinz inequality) $(x, x) \leq$

$({M_\lambda}^{1/2}x, x)$. Next we have

$$\begin{aligned}\|{M_\lambda}^{1/2}x - x\|^2 &= ({M_\lambda}^{1/2}x - x, {M_\lambda}^{1/2}x - x)\\ &= ({M_\lambda}^{1/2}x, {M_\lambda}^{1/2}x) - 2({M_\lambda}^{1/2}x, x) + (x, x)\\ &= \lambda^2(M^{1/2}x, M^{1/2}x) + \lambda(C^{1/2}x, C^{1/2}x) + 2((x, x) - ({M_\lambda}^{1/2}x, x)) \leq\\ &\leq \lambda^2(M^{1/2}x, M^{1/2}x) + \lambda(C^{1/2}x, C^{1/2}x) \to 0, \text{ for } \lambda \to 0.\end{aligned}$$

This implies (3.3.1) for all $x \in D({M_\lambda}^{1/2})$. Since $D({M_\lambda}^{1/2})$ is dense in $\mathcal{Z}$ and $\|M_\lambda^{-1/2}\|$ is bounded as a function in λ, (3.3.1) follows for all $x \in \mathcal{Z}$.

From the sequentially strong continuity of operator product [Kat95] (3.3.2) immediately follows. □

Theorem 3.3.2. *Assume that the operator A generates an uniformly exponentially stable semigroup. Then the operators M and C are bounded.*

Proof. A necessary condition for the uniform exponential stability is $0 \in \rho(A)$, so let us assume $0 \in \rho(A)$.

Since $R(\lambda)$ is a continuous function in λ, we have

$$R(\lambda) \to R(0) = A^{-1}, \text{ as } \lambda \searrow 0, \tag{3.3.3}$$

hence $R(\lambda)x$ has a norm limit for $\lambda \searrow 0$. This implies that $B_\lambda M_\lambda^{-1/2}x$ also has a norm limit for $\lambda \searrow 0$, for all $x \in \mathcal{Z}_1$.

Note that $B_\lambda M_\lambda^{-1/2} = M^{1/2}M_\lambda^{-1}$. Next we introduce the operator F in $\mathcal{Z}$ defined by $F = \lim_{\lambda\searrow 0} M^{1/2}M_\lambda^{-1}$, i.e. by

$$D(F) = \{x \in \mathcal{Z} : \exists \lim_{\lambda\searrow 0} M^{1/2}M_\lambda^{-1}x\},\ Fx = \lim_{\lambda\searrow 0} M^{1/2}M_\lambda^{-1}x.$$

Let $x \in D(F)$ be arbitrary. From (3.3.2) and from the fact that $M^{1/2}$ is closed, it follows that $M^{1/2}{M_\lambda}^{-1}x$ has a norm limit for $\lambda \searrow 0$, so $D(F) \cap \mathcal{Z}_1 = \mathcal{Z}_1$.

This implies $\mathcal{Z}_1 \subset D(F) \subset D(M^{1/2})$, i.e. $(\mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}))^{\perp} \subset D(M^{1/2})$. Since $\mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}) \subset D(M^{1/2})$, we obtain $D(M^{1/2}) = \mathcal{Z}$, i.e. M is a bounded operator.

Now we turn our attention to the entry $(1,1)$ in (3.2.3). For $x \in D(M_\lambda)$ $(= D(C)$ now) we have

$$\frac{1}{\lambda}(M_\lambda^{-1} - I)x = -M_\lambda^{-1}(\lambda M + C)x,$$

hence

$$\left\| \frac{1}{\lambda}(M_\lambda^{-1} - I)x + Cx \right\| \leq \lambda \|M_\lambda^{-1}\| \|Mx\| + \|Cx - M_\lambda^{-1}Cx\| \to 0,$$

as $\lambda \searrow 0$.

Let us define $C_0 = -\lim_{\lambda \searrow 0} \frac{1}{\lambda}(M_\lambda^{-1} - I)$. Obviously, C_0 is a symmetric operator (as a limit of symmetric operators) and $D(C) \subset D(C_0)$. Since C is a selfadjoint operator, we have $C_0 = C$, i.e.

$$C = -\lim_{\lambda \searrow 0} \frac{1}{\lambda}(M_\lambda^{-1} - I).$$

Using similar considerations as given above, we obtain that operator C is also bounded. □

Remark 3.3.1.

1. If the operator M is bounded, then $\mathcal{N}(B_\lambda^*) = \mathcal{N}(M^{1/2})$, hence $\mathcal{Z}_2 = (\mathcal{N}(M^{1/2}))^{\perp}$.

2. If M is bounded, then M_λ can be regarded as a usual operator–sum, and $D(M_\lambda) = D(C)$ holds.

Remark 3.3.2. In terms of the forms μ, γ and κ, Theorem 3.3.2 says that if A generates an uniformly exponentially stable semigroup, then μ and γ are necessarily κ–bounded, i.e. there exists $\Delta > 0$ such that

$$\mu(x,x) \leq \Delta\kappa(x,x) \text{ and } \gamma(x,x) \leq \Delta\kappa(x,x), \text{ for all } x \in \mathcal{Z}.$$

From now on, in the rest of thesis, we assume the following:

The operators M and C are bounded.

Note that now $M_\lambda^{-1/2} B_\lambda^* = M_\lambda^{-1} M^{1/2}$.

Using previous considerations and the fact that $\mathcal{R}(C) \subset \mathcal{Z}_1$ and $\mathcal{R}(M^{1/2}) \subset \mathcal{Z}_1$, we obtain

$$A^{-1} = \begin{bmatrix} -C|_{\mathcal{Z}_1} & -M^{1/2}|_{\mathcal{Z}_2} \\ M^{1/2}|_{\mathcal{Z}_1} & 0 \end{bmatrix}.$$

Note that the phase space now becomes $\mathbf{Y} = \mathcal{Z}_1 \oplus \mathcal{Z}_2 = (\mathcal{N}(M^{1/2}) \cap \mathcal{N}(C^{1/2}))^\perp \oplus (\mathcal{N}(M^{1/2}))^\perp$. The operators $C|_{\mathcal{Z}_1}$ and $M^{1/2}|_{\mathcal{Z}_1}$ are obviously bounded symmetric operators in the space $\mathcal{Z}_1$, hence they are selfadjoint.

To avoid technicalities and simplify the proofs, we assume $\mathcal{N}(M^{1/2}) \subset \mathcal{N}(C^{1/2})$, i.e. there is no damping on the positions where mass vanishes. All the results of this thesis *remain valid* also in the case when $\mathcal{N}(M^{1/2}) \not\subseteq \mathcal{N}(C^{1/2})$.

This assumption implies $\mathcal{Y} := \mathcal{Z}_1 = \mathcal{Z}_2$, so now $\mathbf{Y} = \mathcal{Y} \oplus \mathcal{Y}$. Also, from now on let C denote the operator $C : \mathcal{Y} \to \mathcal{Y}$, and let M denote the operator $M : \mathcal{Y} \to \mathcal{Y}$. Letters M and C will denote operators in the spaces $\mathcal{Z}$ or $\mathcal{Y}$ depending on the context.

Hence we can write

$$A^{-1} = \begin{bmatrix} -C & -M^{1/2} \\ M^{1/2} & 0 \end{bmatrix}.$$

From

$$A^{-1} \begin{bmatrix} x \\ M^{1/2} y \end{bmatrix} = \begin{bmatrix} -Cx - M^{1/2} y \\ M^{1/2} x \end{bmatrix}$$

follows that $D(A) = \mathcal{R}(A^{-1}) = (\mathcal{R}(C) + \mathcal{R}(M^{1/2})) \oplus \mathcal{R}(M^{1/2})$, hence the operator A is not bounded in general.

Remark 3.3.3. The operator A cannot, in general, be written in the block-matrix form in $\mathcal{Y} \oplus \mathcal{Y}$. Indeed, let us suppose that the operator A can be written in the block–matrix form. Then we obtain $\mathcal{R}(C) \subset \mathcal{R}(M^{1/2})$, and this does not hold in general, as we can see from the following example.

Example 3.3.1 *(Continuation of Example* 3.1.2*).* As we have seen, M and C are bounded, hence satisfy our assumption. Since $\mathcal{N}(M) = \{0\}$ we have $\mathcal{Y} = \mathcal{Z}$.

Next we show that A cannot be written in block–matrix form. To see this, it is sufficient to show that the identity function is not in $\mathcal{R}(M^{1/2})$. First observe that

$$(M^{-1/2}u, M^{-1/2}v) = \int_0^\pi u''(x)\overline{v''(x)}\mathrm{d}x.$$

Let us assume that $u(x) = x$ is in $\mathcal{R}(M^{1/2})$. Then there exists $v \in \mathcal{Y}$ such that $u = M^{1/2}v$. Hence

$$(v, v) = (M^{-1/2}u, M^{-1/2}u) = \int_0^\pi |u''(x)|^2\mathrm{d}x = 0,$$

a contradiction.

Observe also that $\mathcal{R}(C) + \mathcal{R}(M^{1/2}) \neq \mathcal{Y}$, hence A is not bounded.

Next we want to compute the resolvent of the operator A. Since A is maximal dissipative, the spectrum of A is contained in the left complex half–plane. Our candidate for the resolvent of A is $R(\lambda)$. Obviously, $R(\lambda)$ is resolvent of A for all $\lambda \geq 0$. Next we show that $R(\lambda)$ can be written as a block–matrix in $\mathcal{Y}$ $\mathcal{Y}$ by

$$(A-\lambda)^{-1}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = R(\lambda)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \begin{bmatrix} \frac{1}{\lambda}(M_\lambda^{-1} - I) & -M_\lambda^{-1}M^{1/2} \\ M^{1/2}M_\lambda^{-1} & -\lambda M^{1/2}M_\lambda^{-1}M^{1/2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{for all } x, y \in \mathcal{Y}, \tag{3.3.4}$$

i.e.

$$(M_\lambda^{-1} - I)x \in \mathcal{Y}, \quad M_\lambda^{-1} M^{1/2} x \in \mathcal{Y},$$
$$M^{1/2} M_\lambda^{-1} x \in \mathcal{Y}, \quad M^{1/2} M_\lambda^{-1} M^{1/2} x \in \mathcal{Y},$$

holds for all $x \in \mathcal{Y}$ and all $\lambda \geq 0$. The last two assertions are trivial. To prove the first one, decompose $(M_\lambda^{-1} - I)x$ as $(M_\lambda^{-1} - I)x = y_1 + y_2$, where $y_1 \in \mathcal{N}(M)$, $y_2 \in \mathcal{Y}$. Then $-\lambda^2 Mx - \lambda Cx = y_1 + M_\lambda y_2$, hence $y_1 \in \mathcal{Y}$, so $y_1 = 0$. The second assertion can be proved similarly. It is evident that $R(\lambda)$ is well defined and bounded operator for all λ such that $0 \in \rho(M_\lambda)$.

Let $\lambda \in \rho(A)$ be such that $0 \in \rho(M_\lambda)$. Then it follows $(I - \lambda A^{-1})R(\lambda) = A^{-1}$, hence $R(\lambda)x \in D(A)$, for all $x \in \mathcal{Y}$, which implies $(A - \lambda)R(\lambda)x = x$ for all $x \in \mathcal{Y}$.

Similarly one can also prove $R(\lambda)A^{-1} = A^{-1}R(\lambda)$, hence $R(\lambda)(I - \lambda A^{-1}) = A^{-1}$. This implies $R(\lambda)(A - \lambda)x = x$, for all $x \in D(A)$.

3.3.2 The spectrum of the operator A

Our aim in this section is to establish the correspondence between various types of spectra of the operator A and the operator function $\lambda \mapsto M_\lambda$.

Let us denote by $\rho, \sigma, \sigma_p, \sigma_{ap}$ and σ_r the resolvent set, spectrum, point spectrum, approximate point spectrum and residual spectrum of the operator function $\lambda \mapsto M_\lambda = \lambda^2 M + \lambda C + I$ in the space $\mathcal{Z}$, respectively. The point λ is in the point spectrum of the operator function L if zero is in the point spectrum of the operator $L(\lambda)$, and analogous definitions hold for the other parts of the spectrum. Note that now M_λ can be viewed as an operator sum.

Theorem 3.3.3. *The following holds:*

$$\rho = \rho(A), \sigma = \sigma(A), \sigma_p = \sigma_p(A), \sigma_{ap} = \sigma_{ap}(A) \text{ and } \sigma_r = \sigma_r(A).$$

Proof. Since $0 \in \rho(A)$ and $M_0 = I$, we can assume $\lambda \neq 0$.

First we consider the point spectrum. Let $\lambda \in \sigma_p$, $\lambda \neq 0$ be arbitrary. Then there exists $x \neq 0$ such that $\lambda^2 Mx + \lambda Cx + x = 0$, hence $x = -\lambda^2 Mx - \lambda Cx \in \mathcal{Y}$. Let us denote $y = \lambda M^{1/2} x \in \mathcal{Y}$. Then $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbf{Y}$. From

$$A^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -Cx - \lambda Mx \\ M^{1/2} x \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} x \\ y \end{bmatrix}$$

follows $\frac{1}{\lambda} \in \sigma_p(A^{-1})$, hence $\lambda \in \sigma_p(A)$.

On the other hand, let $\lambda \in \sigma_p(A)$, $\lambda \neq 0$ be arbitrary. Then there exists $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \neq 0$ such that

$$\begin{bmatrix} -C & -M^{1/2} \\ M^{1/2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} x \\ y \end{bmatrix},$$

hence

$$-Cx - M^{1/2} y = \frac{1}{\lambda} x,$$
$$M^{1/2} x = \frac{1}{\lambda} y.$$

From this one can easily obtain $\lambda^2 Mx + \lambda Cx + x = 0$. We can assume $x \neq 0$, since $x = 0$ implies $y = 0$, so $\lambda \in \sigma_p$ holds.

Let now $\lambda \in \sigma_{ap}$, $\lambda \neq 0$ be arbitrary. Then there exists a sequence (x_n), $\|x_n\| = 1$ such that $M_\lambda x_n \to 0$ as $n \to \infty$. We decompose x_n as $x_n = x_n^1 + x_n^2$, with $x_n^1 \in \mathcal{Y}$, $x_n^2 \in \mathcal{N}(M) = \mathcal{N}(M) \cap \mathcal{N}(C)$. Then

$$M_\lambda x_n = \lambda^2 M x_n^1 + \lambda C x_n^1 + x_n^1 + x_n^2 \to 0,$$

which implies $\lambda^2 M x_n^1 + \lambda C x_n^1 + x_n^1 \to 0$ and $x_n^2 \to 0$, hence we can assume $x_n \in \mathcal{Y}$. Let us define $y_n = \lambda M^{1/2} x_n$. We have

$$(A^{-1} - \frac{1}{\lambda}) \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} -Cx_n - \frac{1}{\lambda}x_n - M^{1/2}y_n \\ M^{1/2}x_n - \frac{1}{\lambda}y_n \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} -\lambda^2 M x_n - \lambda C x_n - x_n \\ 0 \end{bmatrix} \to 0,$$

as $n \to \infty$. Since $\|\binom{x_n}{y_n}\| \geq 1$, we have $\lambda \in \sigma_{ap}(A)$.

On the other hand, let be $\lambda \in \sigma_{ap}(A)$, $\lambda \neq 0$ be arbitrary. Then there exists a sequence $\binom{x_n}{y_n} \in \mathbf{Y}$, $\|x_n\|^2 + \|y_n\|^2 = 1$ such that $(A^{-1} - \frac{1}{\lambda})\binom{x_n}{y_n} \to 0$ as $n \to \infty$. This implies

$$Cx_n - \frac{1}{\lambda}x_n - M^{1/2}y_n \to 0, \tag{3.3.5}$$

$$M^{1/2}x_n - \frac{1}{\lambda}y_n \to 0. \tag{3.3.6}$$

Multiplying (3.3.6) by $\lambda M^{1/2}$ we obtain

$$\lambda M x_n - M^{1/2} y_n \to 0. \tag{3.3.7}$$

Subtracting (3.3.7) from (3.3.6), we obtain

$$\lambda^2 M x_n + \lambda C x_n + x_n \to 0,$$

so to prove $\lambda \in \sigma_{app}$, we have to exclude the possibility of x_n tending to zero. But $x_n \to 0$ and (3.3.6) imply $y_n \to 0$, which leads to a contradiction.

Let us denote by $B^{[*]}$ the adjoint of the operator B in the scalar product generated by the operator $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} : \mathbf{Y} \to \mathbf{Y}$. Obviously, A^{-1} (and hence also A) is a J–selfadjoint operator. Hence the following sequence of equivalences hold: if $\lambda \neq 0$ then

$$\lambda \in \sigma_r \iff 0 \in \sigma_r(M_\lambda) \iff 0 \in \sigma_p({M_\lambda}^*) \iff 0 \in \sigma_p(M_{\overline{\lambda}}),$$

and

$$0 \in \sigma_p(M_{\overline{\lambda}}) \Longleftrightarrow \frac{1}{\overline{\lambda}} \in \sigma_p(A^{-1}) \Longleftrightarrow \frac{1}{\lambda} \in \sigma_r(A^{-[*]}) \Longleftrightarrow \frac{1}{\lambda} \in \sigma(A^{-1}) \Longleftrightarrow \lambda \in \sigma_r(A).$$

□

Theorem 3.3.3 implies that (3.3.4) for all $\lambda \in \rho(A)$.

Note that the spectrum of the corresponding undamped system (i.e. when we set $C = 0$) is given by

$$\left\{ \pm i \frac{1}{\sqrt{\lambda}} : \lambda \in \sigma(M) \setminus \{0\} \right\}.$$

3.3.3 Characterization of the uniform exponential stability

The main result of this subsection will be the following theorem.

Theorem 3.3.4. *The operator A generates an uniformly exponentially stable semigroup if and only if*

(i) for each $\beta \in \sigma(M)$, and for each singular (the other term in use is "approximate eigenvector") sequence (x_n) (i.e. a sequence such that $\|x_n\| = 1$, $(M - \beta)x_n \to 0$, as $n \to \infty$) we have $\inf \|Cx_n\| > 0$, and

(ii) $\sup_{\lambda \in i\mathbb{R}} \|M^{1/2} M_\lambda^{-1}\| < \infty$ holds.

Remark 3.3.4. 1. The condition (i) is equivalent to the assumption that $i\mathbb{R} \subset \rho(A)$, or, equivalently, $i\mathbb{R} \subset \rho$.

2. If the operator M has discrete spectrum, the condition (i) is equivalent to the assumption that

$$Cx \neq 0 \text{ for all eigenvectors } x \text{ of } M, \tag{3.3.8}$$

hence the operator C does not vanish on the eigenvectors of the corresponding undamped system.

3. In the finite–dimensional case the assumption (3.3.8) is necessary and sufficient condition for the uniform exponential stability of the corresponding semigroup (see Proposition 2.1.1).

4. The condition (ii) is the consequence of the well–known fact that in the infinite–dimensional case the location of the spectrum of the generator does not characterize the behavior of the semigroup.

5. Since $(M^{1/2}M_\lambda^{-1})^* = M_{-\lambda}^{-1}M^{1/2}$, the condition (ii) is equivalent to

   (ii)' $\sup_{\lambda \in i\mathbb{R}} \|M_\lambda^{-1}M^{1/2}\| < \infty$.

6. The conditions (i) and (ii) are equivalent with the condition

$$\sup_{\mathrm{Re}\lambda>0} \|M^{1/2}{M_\lambda}^{-1}\| < \infty.$$

Proof of the Theorem 3.3.4. For A to generate an uniformly exponentially stable semigroup it is necessary that $i\mathbb{R} \subset \rho(A)$.

First we will show that the condition (i) is necessary and sufficient for the absence of the spectrum of the operator A on the imaginary axis.

Since $M_0 = I$ and $0 \in \rho(A)$, we exclude zero from our observations.

If there exists $\beta \in \sigma(M)$ and a sequence (x_n), $\|x_n\| = 1$ such that $(M - \beta)x_n \to 0$ and $Cx_n \to 0$, then obviously $M_\lambda x_n \to 0$ for $\lambda = \frac{i}{\sqrt{\beta}}$, hence $\lambda \in \sigma(A)$.

On the other hand, let us assume that $i\lambda \in \sigma_{ap}(A)$ for some $\lambda \neq 0$. Then $i\gamma := \frac{1}{i\lambda} \in \sigma_{ap}$, hence there exists a sequence (x_n), $\|x_n\| = 1$, such that

$$-\gamma^2 Mx_n - i\gamma Cx_n + x_n \to 0$$

as $n \to \infty$. Multiplying by x_n we obtain

$$-\gamma^2(Mx_n, x_n) - i\gamma(Cx_n, x_n) + (x_n, x_n) \to 0,$$

which implies $(Cx_n, x_n) \to 0$, hence $Cx_n \to 0$. This implies that $(I - \gamma^2 M)x_n \to 0$, hence $(M - \frac{1}{\gamma^2})x_n \to 0$.

If $i\lambda \in \sigma_r(A)$, then $i\frac{1}{\lambda} \in \sigma_p$, so this case reduces to the already proved case.

From Corollary A.5 we know that a sufficient and necessary condition for the uniform exponential stability is that $\sup_{\lambda \in i\mathbb{R}} \|R(\lambda)\| < \infty$ holds.

From the representation of the resolvent (3.3.4) it is obvious that condition (ii) is necessary. To see that condition (ii) is also sufficient, first note that it is sufficient to show that $(1, 1)$ and $(2, 2)$ entries in (3.3.4) are bounded on the imaginary axis.

Let us assume that the condition (ii) is satisfied.

Let $\lambda \in i\mathbb{R}$, $x \in \mathcal{Y}$ be arbitrary. Set

$$y = M_\lambda^{-1} M^{1/2} x. \tag{3.3.9}$$

Then

$$\begin{aligned}(M^{1/2}x, y) = (M_\lambda y, y) &= \lambda^2 \|M^{1/2}y\|^2 + \lambda(Cy, y) + \|y\|^2 \\ &= -\|\lambda M^{1/2} y\|^2 + \lambda(Cy, y) + \|y\|^2.\end{aligned}$$

Hence

$$\|\lambda M^{1/2} y\|^2 = \|y\|^2 - \mathrm{Re}(M^{1/2}x, y). \tag{3.3.10}$$

From (3.3.10) follows

$$\|\lambda M^{1/2} y\|^2 \le \|M^{1/2}x\| \|y\| + \|y\|^2. \tag{3.3.11}$$

From (3.3.11) and (3.3.9) we get

$$\|\lambda M^{1/2} M_\lambda^{-1} M^{1/2} x\|^2 \le \|M_\lambda^{-1} M^{1/2} x\|^2 + \|M^{1/2}x\| \|M_\lambda^{-1} M^{1/2} x\|. \tag{3.3.12}$$

Hence, the condition (ii) implies that entry $(2,2)$ is bounded.

To show that the entry $(1,1)$ is bounded, we proceed similarly. Let $\lambda \in i\mathbb{R}$, $\lambda \neq 0$, $x \in \mathcal{Y}$ be arbitrary. Set

$$y = M_\lambda^{-1} x. \tag{3.3.13}$$

Then

$$(x, y) = \lambda^2 \|M^{1/2} y\|^2 + \lambda(Cy, y) + (y, y). \tag{3.3.14}$$

From (3.3.14) follows

$$(y-x, y-x) = (y,y) + (x,x) - 2\mathrm{Re}(x,y) = -\mathrm{Re}(x,y) - \lambda^2 \|M^{1/2} y\|^2 - \lambda(Cy, y),$$

hence

$$\left\| \frac{1}{\lambda}(y - x) \right\| = \|M^{1/2} y\|^2 - \frac{1}{|\lambda|^2} \mathrm{Re}(x, y). \tag{3.3.15}$$

If $\mathrm{Re}(x, y) \geq 0$ holds then obviously

$$\left\| \frac{1}{\lambda}(y - x) \right\| \leq \|M^{1/2} y\|^2,$$

and hence (3.3.13) implies

$$\left\| \frac{1}{\lambda}(M_\lambda{}^{-1} - I)x \right\| \leq \|M^{1/2} M_\lambda^{-1/2} x\|.$$

The case $\mathrm{Re}(x, y) < 0$ we treat in the following way. From (3.3.14) and $\mathrm{Re}(x, y) < 0$ it follows

$$\|y\| < |\lambda| \|M^{1/2} y\| \tag{3.3.16}$$

Suppose $|\lambda| \geq 1$. Then from (3.3.15) and (3.3.16) follows that

$$\left\| \frac{1}{\lambda}(y - x) \right\|^2 \leq \|M^{1/2} y\|^2 + \|M^{1/2} y\|^2,$$

hence

$$\left\| \frac{1}{\lambda}(M_\lambda{}^{-1} - I)x \right\|^2 \leq \|M^{1/2} M_\lambda^{-1/2} x\|^2 + \|M^{1/2} M_\lambda^{-1/2} x\|.$$

Suppose now $|\lambda| < 1$. Then from (3.3.16) follows $\|y\| \le \|M^{1/2}y\|$, and hence

$$\|M_\lambda^{-1/2}x\| \le \|M^{1/2}M_\lambda^{-1/2}x\|.$$

This implies

$$\left\|\frac{1}{\lambda}({M_\lambda}^{-1} - I)x\right\| \le \|M^{1/2}M_\lambda^{-1/2}x\|(\|M\| + \|C\|).$$

When we put together the estimates given above, we get

$$\left\|\frac{1}{\lambda}({M_\lambda}^{-1} - I)x\right\| \le \max\left\{\sqrt{\|M^{1/2}M_\lambda^{-1/2}x\|^2 + \|M^{1/2}M_\lambda^{-1/2}x\|}, \|M^{1/2}M_\lambda^{-1/2}x\|(\|M\| + \|C\|)\right\}. \tag{3.3.17}$$

Hence we have proved that the entry $(1,1)$ is also bounded. □

Corollary 3.3.5. *Assume that* $0 \in \rho(M)$ *and that the condition (i) from Theorem* 3.3.4 *is satisfied. Then the condition (ii) from Theorem* 3.3.4 *is also satisfied.*

Proof. Obviously, it is sufficient to show that

$$\sup_{\lambda \in i\mathbb{R}} \|M_\lambda^{-1}\| < \infty.$$

Let us assume that the last relation is not satisfied. From the uniform boundedness theorem it follows that there exists x, $\|x\| = 1$ such that $\sup_{\lambda \in i\mathbb{R}} \|M_\lambda^{-1}x\| = \infty$, hence there exists a sequence (β_n), $\beta_n \in \mathbb{R}$, such that

$$\|M_{i\beta_n}^{-1}x\| \to \infty. \tag{3.3.18}$$

Obviously, $|\beta_n| \to \infty$. By choosing a subsequence, if necessary (and denoting this subsequence again by β_n) relation (3.3.18) implies $\|M_{i\beta_n}^{-1}x\| \ge n =$

$n\|x\|$, for all $n \in \mathbb{N}$. Let us define $x_n = \frac{M_{i\beta_n}^{-1}x}{\|M_{i\beta_n}^{-1}x\|}$. Then $M_{i\beta_n}x_n \to 0$, $\|x_n\| = 1$ follows, and we have

$$-\beta_n^2 Mx_n + x_n + i\beta_n Cx_n \to 0.$$

Multiplying the previous relation by x_n, and using the fact that C is bounded, we obtain

$$\beta_n Cx_n \to 0,$$

hence

$$-\beta_n^2 Mx_n + x_n \to 0. \tag{3.3.19}$$

Let us define $y_n = Mx_n$. Then (3.3.19) reads

$$(M^{-1} - \beta_n^2)y_n \to 0. \tag{3.3.20}$$

Since $0 \in \rho(M)$, the sequence y_n does not tend to zero, hence we can assume that (3.3.20) holds with $\|y_n\| = 1$. But then

$$\|M^{-1}y_n - \beta_n^2 y_n\| \geq \beta_n^2 - \|M^{-1}y_n\| \geq \beta_n^2 - \|M^{-1}\| \to \infty,$$

a contradiction with (3.3.20). □

Assume that the assumptions of Theorem 3.3.4 are satisfied, and set

$$\Delta = \sup_{\lambda \in i\mathbb{R}} \|M_\lambda^{-1} M^{1/2}\|. \tag{3.3.21}$$

Then the following proposition holds.

Proposition 3.3.6. *We have*

$$\omega(A) \leq -\frac{1}{\max\{\sqrt{\Delta^2 + \Delta}, \Delta(\|M\| + \|C\|)\} + 2\Delta + \sqrt{\Delta^2 + \|M^{1/2}\|\Delta}}.$$

Proof. From Lemma A.7 follows that it is enough to show

$$\|R(\lambda)\| \leq \max\{\sqrt{\Delta^2+\Delta}, \Delta(\|M\|+\|C\|)\} + 2\Delta + \sqrt{\Delta^2 + \|M^{1/2}\|\Delta}. \quad (3.3.22)$$

Since

$$\begin{aligned}
\|R(\lambda)\| = \sup_{\|x\|^2+\|y\|^2=1} & \left\| \begin{bmatrix} \frac{1}{\lambda}(M_\lambda^{-1}-I)x - M_\lambda^{-1}M^{1/2}y \\ M^{1/2}M_\lambda^{-1}x - \lambda M^{1/2}M_\lambda^{-1}M^{1/2}y \end{bmatrix} \right\| \leq \\
& \leq \sqrt{\left\|\frac{1}{\lambda}(M_\lambda^{-1}-I)\right\|^2 + 2\|M_\lambda^{-1}M^{1/2}\|^2 + \|\lambda M^{1/2}M_\lambda^{-1}M^{1/2}\|^2} \leq \\
& \leq \left\|\frac{1}{\lambda}(M_\lambda^{-1}-I)\right\| + 2\|M_\lambda^{-1}M^{1/2}\| + \|\lambda M^{1/2}M_\lambda^{-1}M^{1/2}\|,
\end{aligned}$$

estimates (3.3.12) and (3.3.17) imply (3.3.22). □

Proposition 3.3.6 roughly says:

smaller $\Delta \Longleftrightarrow$ faster exponential decay of the semigroup.

Our next goal is to show how the condition (ii) from Theorem 3.3.4 can be written out in terms of the operators M and C.

Proposition 3.3.7. *If the condition (i) from Theorem* 3.3.4 *holds, the condition (ii) from Theorem* 3.3.4 *is equivalent to the following:*

(ii)" *Let (x_n) be a sequence such that $\|M^{1/2}x_n\| = 1$, $\|x_n\|^2 - \frac{1}{\|Mx_n\|^2} \to 0$ and $\|x_n\|\|Cx_n\| \to 0$. Then $\|x_n\|$ is a bounded sequence.*

Proof. First observe that the condition (ii) from Theorem 3.3.4 is not satisfied if and only if there exists a sequence (β_n) such that $\beta_n \in i\mathbb{R}$, $|\beta_n| \to \infty$ and there exists $x \in \mathcal{Y}$ such that $\|M^{1/2}M_{\beta_n}^{-1}x\| \to \infty$. Here we used the uniform boundedness theorem, and the fact $\|M^{1/2}M_\lambda^{-1}\|$ is bounded for λ from any bounded subset of $i\mathbb{R}$. By a simple substitution, this condition (as in Corollary

3.3.5) can be rewritten as:

there exists a sequence (β_n) such that $\beta_n \in i\mathbb{R}$, $|\beta_n| \to \infty$ and a sequence (x_n) such that $\|M^{1/2}x_n\| = 1$ and $M_{\beta_n}x_n \to 0$.

Let us assume that (ii)" is not satisfied. Then there exists a sequence (x_n) such that

$$\|M^{1/2}x_n\| = 1, \tag{3.3.23}$$

$$\|x_n\|^2 - \frac{1}{\|Mx_n\|^2} \to 0, \tag{3.3.24}$$

$$\|x_n\|\|Cx_n\| \to 0 \tag{3.3.25}$$

and

$$\|x_n\| \to \infty. \tag{3.3.26}$$

Let us define $\beta_n = \frac{1}{\|Mx_n\|}$. From (3.3.24) and (3.3.26) it follows $\beta_n \to \infty$. Now

$$\|M_{i\beta_n}x_n\| \leq \|-\beta_n^2 Mx_n + x_n\| + \beta_n\|Cx_n\|.$$

Relations (3.3.23) and (3.3.24) imply

$$\begin{aligned}
\|-\beta_n^2 Mx_n + x_n\|^2 &= (-\beta_n^2 Mx_n + x_n, -\beta_n^2 Mx_n + x_n) \\
&= \beta_n^4\|Mx_n\|^2 - 2\beta_n^2\|M^{1/2}x_n\|^2 + \|x_n\|^2 \\
&= \|x_n\|^2 - \frac{1}{\|Mx_n\|^2} \to 0.
\end{aligned}$$

Also from (3.3.24) follows

$$\|Mx_n\|\|x_n\| \to 1,$$

which implies

$$\frac{\beta_n}{\|x_n\|} \to 1. \tag{3.3.27}$$

Now (3.3.25) and (3.3.27) imply

$$\beta_n\|Cx_n\| = \frac{\beta_n}{\|x_n\|}\|x_n\|\|Cx_n\| \to 0,$$

hence $M_{i\beta_n}x_n \to 0$.

On the other hand, let us assume that there exist sequences (β_n) and (x_n) such that $|\beta_n| \to \infty$, $\|M^{1/2}x_n\| = 1$ and

$$M_{i\beta_n}x_n \to 0. \tag{3.3.28}$$

Multiplying relation (3.3.28) by $\frac{x_n}{\|x_n\|}$ we get

$$-\beta_n^2 \frac{1}{\|x_n\|}\|M^{1/2}x_n\|^2 + i\frac{\beta_n}{\|x_n\|}(Cx_n, x_n) + \|x_n\| \to 0,$$

which implies

$$-\frac{\beta_n^2}{\|x_n\|} + \|x_n\| \to 0. \tag{3.3.29}$$

Hence, for an arbitrary $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$-\varepsilon < -\frac{\beta_n^2}{\|x_n\|} + \|x_n\| < \varepsilon.$$

Multiplying the last equation by $\frac{1}{\|x_n\|}$ we obtain

$$1 - \frac{\varepsilon}{\|x_n\|} < \left(\frac{\beta_n}{\|x_n\|}\right)^2 < 1 + \frac{\varepsilon}{\|x_n\|}.$$

Since $\|x_n\| \to \infty$ (from (3.3.29)), it follows

$$\frac{|\beta_n|}{\|x_n\|} \to 1.$$

Multiplying (3.3.28) by $\frac{\beta_n}{\|x_n\|}$ we get

$$-\beta_n^3 M\frac{x_n}{\|x_n\|} + i\beta_n^2 C\frac{x_n}{\|x_n\|} + \beta_n\frac{x_n}{\|x_n\|} \to 0. \tag{3.3.30}$$

Set $\widehat{x}_n = \frac{x_n}{\|x_n\|}$, and multiply relation (3.3.30) by $\widehat{x}_n$. We get

$$-\beta_n^3(M\widehat{x}_n, \widehat{x}_n) + i\beta_n^2(C\widehat{x}_n, \widehat{x}_n) + \beta_n(\widehat{x}_n, \widehat{x}_n) \to 0,$$

which implies

$$\beta_n^2 C\widehat{x}_n \to 0, \tag{3.3.31}$$

$$-\beta_n^3 M\widehat{x}_n + \beta_n\widehat{x}_n \to 0. \tag{3.3.32}$$

Multiplying the relations (3.3.31) and (3.3.32) by $\frac{\|x_n\|}{\beta_n}$ we obtain

$$\beta_n C x_n \to 0, \tag{3.3.33}$$

$$-\beta_n^2 M x_n + x_n \to 0. \tag{3.3.34}$$

Now, multiplying (3.3.33) by $\frac{\|x_n\|}{\beta_n}$ we get

$$\|x_n\| C x_n \to 0.$$

From (3.3.34) follows

$$\|x_n\|^2 - 2\beta_n^2 + \beta_n^4 \|Mx_n\|^2 \to 0,$$

which, due to the fact that $1 = \|M^{1/2}x_n\|^2 = (Mx_n, x_n) \leq \|Mx_n\|\|x_n\|$, can be written as

$$\left(\beta_n^2 - \frac{1}{\|Mx_n\|^2} + i\frac{\sqrt{\|Mx_n\|^2\|x_n\|^2 - 1}}{\|Mx_n\|^2}\right) \cdot \\ \cdot \left(\beta_n^2 - \frac{1}{\|Mx_n\|^2} - i\frac{\sqrt{\|Mx_n\|^2\|x_n\|^2 - 1}}{\|Mx_n\|^2}\right) \to 0,$$

hence

$$\beta_n^2 - \frac{1}{\|Mx_n\|^2} \to 0, \tag{3.3.35}$$

$$\frac{\|x_n\|^2}{\|Mx_n\|^2} - \frac{1}{\|Mx_n\|^4} \to 0. \tag{3.3.36}$$

From (3.3.35) follows $Mx_n \to 0$. And finally, we multiply (3.3.36) by $\|Mx_n\|$ to obtain

$$\|x_n\|^2 - \frac{1}{\|Mx_n\|^2} \to 0.$$

□

Corollary 3.3.8. *Assume that* $0 \in \rho(C)$. *Then both conditions from Theorem* 3.3.4 *are satisfied.*

Proof. It is clear that the condition (i) is satisfied. The fact that the condition (ii) is satisfied immediately follows from the inequality $\|x_n\|\|Cx_n\| \geq \|C^{-1}\|\|x_n\|^2$. □

Remark 3.3.5. From the proof of Proposition 3.3.7 one can obtain that the condition (ii) from Theorem 3.3.4 is equivalent to the following condition:

(ii)^a Let sequences (β_n), $\beta_n \in \mathbb{R}$ and (x_n), $x_n \in \mathcal{Y}$, be such that $|\beta_n| \to \infty$, $\|M^{1/2}x_n\| = 1$, $\frac{|\beta_n|}{\|x_n\|} \to 1$, $-\beta_n^2 M x_n + x_n \to 0$ and $\|x_n\|Cx_n \to 0$. Then x_n is a bounded sequence.

3.3.4 Characterization of the uniform exponential stability via eigenvectors of M

Corollary 3.3.5 implies that the condition (ii) from Theorem 3.3.4 is trivial when $0 \in \rho(M)$, where M acts in $\mathcal{Z}$ or $\mathcal{Y}$.

Hence we assume that $0 \in \sigma_c(M)$, for M acting in the space $\mathcal{Y}$. This implies that zero is an accumulation point of the spectrum of M.

The following theorem can be seen as a quadratic problem analogue of Theorem 5.4.1. from [CZ93a].

Theorem 3.3.9. *Let us assume that there exists an open interval around zero such that there are no essential spectrum of* M *in this interval, i.e. there exists* $\delta > 0$ *such that* $(0, \delta) \cap \sigma(M)$ *consists only of eigenvalues with finite multiplicities with no accumulation points on* $(0, \delta)$.

Denote the eigenvalues of M on $(0, \delta)$ by $\lambda_1 \geq \lambda_2 \geq \dots$, where we have taken multiple eigenvalues into account. Denote the corresponding normalized eigenvectors by ϕ_n, i.e. $M\phi_n = \lambda_n \phi_n$, $\|\phi_n\| = 1$.

Set

$$\Sigma = \{\psi = \sum_{n \in I_m} a_n \phi_n : \sum_{n \in I_m} |a_n|^2 = 1, m \in \mathbb{N}, a_n \in \mathbb{C}\}, \tag{3.3.37}$$

where

$$I_m = \{n \in \mathbb{N} : \lambda_m = \lambda_n\}, \; m \in \mathbb{N}. \tag{3.3.38}$$

Then the operator A generates an uniformly exponentially stable semigroup if and only if

$$\inf_{\psi \in \Sigma} \frac{\|C\psi\|}{\|M\psi\|} > 0. \tag{3.3.39}$$

Remark 3.3.6. Theorem 3.3.9 implies that if the operator C is such that the corresponding operator A generates an uniformly exponentially stable semigroup, then the operator αC has the same property, for all $\alpha > 0$.

Remark 3.3.7. Using Theorem 3.3.9 and a spectral shift technique which was introduced in [Ves02b] one can prove $\omega(A) \leq -\Delta$, where $\Delta = \inf\{\beta \in \mathbb{R} : 2\beta M + C \geq 0\}$. This result is proved in [Ves02b] in the case of an abstract second order system

$$M\ddot{x} + C\dot{x} + Kx = 0,$$

where M, C and K are selfadjoint positive operators, and M and C are bounded.

Note that this result is void in the case that operator C has a non–trivial null-space.

Theorem 3.3.9 is a considerable improvement of the similar results from [CZ93a], since Theorem 3.3.9 can be applied to the systems with boundary damping, and the results from [CZ93a] cannot. For example, the results from [CZ93a] cannot be used to characterize uniform exponential stability of the system from Example 3.1.1.

Also, in our case the corresponding undamped system can posses continuous spectrum.

Although the results from [CZ93a] formally can be applied also to the systems with non–selfadjoint damping operator, it appears that the assumption (H5) [CZ93a, pp. 277]:

$$\lim_{n\to\infty} \mathrm{Re}(Cy_n, y_n) = 0 \Longrightarrow \lim_{n\to\infty} Cy_n = 0$$

is very restrictive. In fact, we do not know any concrete application with non–selfadjoint damping operator which satisfies (H5).

An improvement of the results from [CZ93a] is obtained recently in [LLR01], where it was shown that the assumption (H5) can be dropped, if the damping operator has a sufficiently small norm.

When M is compact, Theorem 3.3.9 obviously reduces to the following.

Corollary 3.3.10. *Let M be compact. Denote by λ_n the eigenvalues of M and by ϕ_n the corresponding normalized eigenvectors. Then the operator A generates an uniformly exponentially stable semigroup if and only if*

$$\inf \frac{1}{\lambda_n}\|C\phi_n\| > 0. \tag{3.3.40}$$

Proof of Theorem 3.3.9. First note that (3.3.39) obviously implies the condition (i) from Theorem 3.3.4.

Let us assume that the condition (ii)$^{\mathrm{a}}$ from Remark 3.3.5 is not satisfied. Then there exist sequences (β_n), $\beta_n \in \mathbb{R}$ and (x_n), $x_n \in \mathcal{Y}$, with $|\beta_n| \to \infty$, $\|x_n\| \to \infty$, $\|M^{1/2}x_n\| = 1$, $\frac{|\beta_n|}{\|x_n\|} \to 1$, $-\beta_n^2 Mx_n + x_n \to 0$ and $\|x_n\|Cx_n \to 0$.

Set $\widetilde{x}_n = \beta_n x_n$. Then

$$-\beta_n M\widetilde{x}_n + \frac{1}{\beta}\widetilde{x}_n \to 0, \tag{3.3.41}$$

$$C\widetilde{x}_n \to 0. \tag{3.3.42}$$

The relation (3.3.41) can be written as

$$\sum_{p=1}^{\infty} \left(\frac{1}{\beta_n} - \beta_n \lambda_p\right)^2 |(\widetilde{x}_n, \phi_p)|^2 + \int_{\delta}^{\|M\|} \left(\frac{1}{\beta_n} - \beta_n t\right)^2 \mathrm{d}\|E(t)\widetilde{x}_n\|^2 \to 0, \tag{3.3.43}$$

where $E(t)$ is the spectral function of M, and δ is from the statement of the theorem.

For n big enough we have

$$\left(\frac{1}{\beta_n} - \beta_n t\right)^2 = \frac{1}{\beta_n^2} - 2t + \beta_n^2 t^2 \geq \beta_n^2 \delta^2 - 2\|M\| \geq 1,$$

hence

$$\int_{\delta}^{\|M\|} \mathrm{d}\|E(t)\widetilde{x}_n\|^2 \to 0. \tag{3.3.44}$$

Choose $p(n) \in \mathbb{N}$ such that

$$\left|\frac{1}{\beta_n} - \beta_n \lambda_{p(n)}\right| = \min\left\{\left|\frac{1}{\beta_n} - \beta_n \lambda_p\right| : p \in \mathbb{N}\right\}.$$

Then there exists $\gamma > 0$ such that

$$\left|\frac{1}{\beta_n} - \beta_n \lambda_p\right| \geq \gamma, \text{ for all } p \notin I_{p(n)}. \tag{3.3.45}$$

Indeed, let us assume that (3.3.45) is not satisfied. Then there exists a subsequence (p_k) such that

$$\frac{1}{\beta_n} - \beta_n \lambda_{p_k} \to 0 \text{ as } k \to \infty.$$

This implies $\lambda_{p_k} \to \frac{1}{\beta_n^2}$, which is in the contradiction with the assumption that the eigenvalues of M do not have accumulation points in $(0, \delta)$.

Now (3.3.45) and (3.3.43) imply

$$\sum_{p \notin I_{p(n)}} |(\widetilde{x}_n, \phi_p)|^2 \to 0. \tag{3.3.46}$$

Set $z_n = \sum_{q \in I_{p(n)}} (\widetilde{x}_n, \phi_q)\phi_q$. Then

$$\begin{aligned}
\widetilde{x}_n - z_n &= \sum_{p \notin I_{p(n)}} (\widetilde{x}_n, \phi_p)\phi_p + \int_{\delta}^{\|M\|} \mathrm{d}E(t)\widetilde{x}_n + \sum_{p \in I_{p(n)}} (\widetilde{x}_n - z_n, \phi_p)\phi_p \\
&= \sum_{p \notin I_{p(n)}} (\widetilde{x}_n, \phi_p)\phi_p + \int_{\delta}^{\|M\|} \mathrm{d}E(t)\widetilde{x}_n.
\end{aligned}$$

Using (3.3.46) and (3.3.44) we obtain

$$\|\widetilde{x}_n - z_n\|^2 = \sum_{p \notin I_{p(n)}} |(\widetilde{x}_n, \phi_p)|^2 + \int_{\delta}^{\|M\|} \mathrm{d}\|E(t)\widetilde{x}_n\|^2 \to 0. \tag{3.3.47}$$

Now (3.3.42) and (3.3.47) imply

$$Cz_n \to 0,$$

which is equivalent to

$$\beta_n \sum_{q \in I_{p(n)}} (x_n, \phi_q) C\phi_q \to 0. \tag{3.3.48}$$

Let us assume that

$$\beta_n^2 \lambda_{p(n)}^2 \sum_{q \in I_{p(n)}} |(x_n, \phi_q)|^2 \to 0. \tag{3.3.49}$$

Then (3.3.49) implies

$$\sum_{q \in I_{p(n)}} \beta_n^2 |(Mx_n, \phi_q)|^2 \to 0. \tag{3.3.50}$$

On the other hand, we have

$$\|\beta_n M x_n\|^2 = \sum_{p\notin I_{p(n)}} \beta_n^2 |(Mx_n, \phi_p)|^2 + \int\limits_{\delta}^{\|M\|} \beta_n^2 t^2 \mathrm{d}\|E(t)x_n\|^2 + \sum_{q\in I_{p(n)}} \beta_n^2 |(Mx_n, \phi_q)|^2 \leq$$

$$\leq \lambda_1 \sum_{p\notin I_{p(n)}} |(\widetilde{x}_n, \phi_p)|^2 + \|M\|^2 \int\limits_{\delta}^{\|M\|} \mathrm{d}\|E(t)\widetilde{x}_n\|^2 + \sum_{q\in I_{p(n)}} \beta_n^2 |(Mx_n, \phi_q)|^2.$$

From the previous relation and relations (3.3.50), (3.3.46) and (3.3.44) follows

$$\beta_n M x_n \to 0. \tag{3.3.51}$$

Since $\frac{|\beta_n|}{\|x_n\|} \to 1$, (3.3.51) implies

$$\|x_n\| M x_n \to 0.$$

This implies

$$1 = \|M^{1/2} x_n\|^2 = (Mx_n, x_n) \leq \|Mx_n\| \|x_n\| \to 0,$$

a contradiction.

Hence, the sequence

$$\frac{1}{\beta_n \lambda_{p(n)} \sqrt{\sum_{q\in I_{p(n)}} |(x_n, \phi_q)|^2}}$$

is bounded, which together with (3.3.48) implies

$$\frac{1}{\lambda_{p(n)}} \sum_{q\in I_{p(n)}} \frac{(x_n, \phi_q)}{\sqrt{\sum_{q\in I_{p(n)}} |(x_n, \phi_q)|^2}} C\phi_q \to 0.$$

Then obviously (3.3.39) is not satisfied.

On the other hand, let us assume that (3.3.39) is not satisfied. Then there exists a sequence (ψ_n), $\psi_n \in \Sigma$ such that

$$\frac{\|C\psi_n\|}{\|M\psi_n\|} \to 0.$$

We can assume that $\|\psi\| = 1$. By β_n we denote a number for which $M\psi_n = \frac{1}{\beta_n^2}\psi_n$. Then $\beta_n \to \infty$ (otherwise there would be an accumulation point of the spectrum of M in $(0, \delta)$). Set $x_n = \beta_n \psi_n$.

We will show that the sequences (β_n) and (x_n) violate the condition (ii)$^{\mathrm{a}}$ from Remark 3.3.5.

We have $\|M^{1/2}x_n\| = \beta_n \|M^{1/2}\psi_n\| = 1$, $\|x_n\| = \beta_n$, hence $\|x_n\| \to \infty$ and $\frac{\beta_n}{\|x_n\|} = 1$. Also, $-\beta_n^2 M x_n + x_n = 0$, and

$$\|x_n\| \|Cx_n\| = \beta_n^2 \|C\psi_n\| = \frac{\|C\psi_n\|}{\|M\psi_n\|} \to 0,$$

which all together implies that the condition (ii)$^{\mathrm{a}}$ from Remark 3.3.5 is violated.

□

Remark 3.3.8. From Corollary 3.3.10 we immediately have the following.
The operator A generates a uniformly exponentially stable semigroup if there exists a sequence $\delta_n > 0$ such that

$$\delta_n \leq \|C\phi_n\| \quad \text{and} \quad \inf \frac{\delta_n}{\lambda_n} > 0,$$

where λ_n and ϕ_n are eigenvalues and normalized eigenvectors of M, respectively. As a special case, note that

$$\inf \frac{\gamma(\phi_n, \phi_n)}{\lambda_n} > 0$$

is a sufficient condition for the uniform exponential stability.

Example 3.3.2 *(Continuation of Example* 3.3.1*).* First we want to find eigenvalues and eigenfunctions of M. This can be achieved by solving the eigenproblem for the operator M^{-1}. One can easily obtain that the operator M^{-1} is given by

$$M^{-1}u(x) = -u''(x), \quad D(M^{-1}) = \{u \in \mathcal{Y} : u'' \in L^2([0, \pi]), u'(\pi) = 0\}.$$

Now, from a straightforward calculation follows that the eigenvalues of M^{-1} are $(n+\frac{1}{2})^2$, $n \in \mathbb{N}$, with the corresponding eigenfunctions $\psi_n(x) = \sin(n+\frac{1}{2})x$. Hence, the eigenvalues of M are $\lambda_n = \frac{1}{(n+\frac{1}{2})^2}$, $n \in \mathbb{N}$, with the corresponding eigenfunctions ψ_n.

Now we calculate $\|\psi_n\| = \frac{\sqrt{2}}{2}\sqrt{\pi}(n+\frac{1}{2})$, and $(C\psi_n)(x) = (-1)^n \varepsilon x$. Hence

$$\frac{1}{\lambda_n}\frac{\|C\psi_n\|}{\|\psi_n\|} = (n+\frac{1}{2})^2 \frac{\varepsilon\sqrt{\pi}}{\frac{\sqrt{2}}{2}\sqrt{\pi}(n+\frac{1}{2})} = \sqrt{2}\varepsilon(n+\frac{1}{2}),$$

so the assumption of Corollary 3.3.10 is satisfied and A generates an uniformly exponentially stable semigroup for all $\varepsilon > 0$.

3.4 The solution of the abstract vibrational system

In this section we will solve the equation (3.1.1) using the semigroup generated by the operator A.

Note that (3.1.6) (and hence (3.1.1)) can be written as

$$\begin{aligned} &M\ddot{x} + C\dot{x} + x = 0, \\ &x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0, \end{aligned} \tag{3.4.1}$$

and the energy function (3.1.2) can be written as

$$E(t; x_0, \dot{x}_0) = \frac{1}{2}(M\dot{x}(t), \dot{x}(t)) + \frac{1}{2}(x(t), x(t)). \tag{3.4.2}$$

Since the Cauchy problem (3.1.6) is equivalent with the problem (3.4.1), we will solve (3.4.1). First we define what do we exactly mean by a solution of (3.4.1). We introduce two kinds of solutions.

Definition 3.4.1. A *classical solution* of the Cauchy problem (3.4.1) is a function $x : [0, \infty) \to \mathcal{Z}$ such that $x(t)$ is twice continuously differentiable on $[0, \infty)$, with respect to $\mathcal{Z}$, and satisfies (3.4.1).

A *mild solution* of the Cauchy problem (3.4.1) is a function $x : [0, \infty) \to \mathcal{Z}$ such that $x(t)$ is continuous, $Mx(t)$ is continuously differentiable, and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mx(t)) + Cx(t) + \int_0^t x(s)\mathrm{d}s - Cx_0 - M\dot{x}_0 = 0, \text{ for all } t \geq 0. \tag{3.4.3}$$

Obviously, a classical solution is also a mild solution.

The main result of this section is the following theorem.

Theorem 3.4.1. *The Cauchy problem (*3.4.1*) has a mild solution if and only if $x_0 \in \mathcal{Y}$, and a classical solution if and only if $x_0 \in \mathcal{R}(M^{1/2}) + \mathcal{R}(C)$. If solution (mild or classical) exists, it is unique.*

Proof. First we treat the case of the classical solutions. Since the operator A generates a strongly continuous semigroup, the Cauchy problem

$$\begin{aligned} \dot{u}(t) &= Au(t), \\ u(0) &= u_0, \end{aligned} \tag{3.4.4}$$

has a unique classical solution if and only if $u_0 \in D(A)$, and a unique mild solution for all $u_0 \in \mathcal{Y}$ (Theorem A.8). We will connect Cauchy problems (3.4.1) and (3.4.4).

For the rest of the proof, let $\dot{x}_0 \in \mathcal{Z}$ be arbitrary.

Let $x_0 \in \mathcal{R}(M^{1/2}) + \mathcal{R}(C)$ be arbitrary. Set $u_0 = \left(\begin{smallmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{smallmatrix}\right)$. Obviously $u_0 \in D(A)$, hence there exists a unique classical solution $u(t) = \left(\begin{smallmatrix} u_1(t) \\ u_2(t) \end{smallmatrix}\right)$ of (3.4.4) for the initial condition $u(0) = u_0$. Hence $A^{-1}\dot{u}(t) = u(t)$ holds. This implies

$$-C\dot{u}_1(t) - M^{1/2}\dot{u}_2 = u_1(t),$$
$$M^{1/2}\dot{u}_1(t) = u_2(t).$$

Hence $u_1(t)$ is twice continuously differentiable, and $M\ddot{u}_1(t)+C\dot{u}_1(t)+u_1(t) = 0$. Since $M^{1/2}\dot{u}_1(0) = u_2(0) = M^{1/2}\dot{x}_0$ and $u_1(0) = x_0$, the function $u_1(t)$ is a classical solution of (3.4.1).

Conversely, let $x(t)$ be a classical solution of (3.4.1), and $x_0 \in \mathcal{R}(M^{1/2}) + \mathcal{R}(C)$. Set $u(t) = \binom{x(t)}{M^{1/2}\dot{x}}$ and $u_0 = \binom{x_0}{M^{1/2}\dot{x}_0}$. The function $u(t)$ is obviously continuously differentiable, and from $x(t) = -M\ddot{x} - C\dot{x}$ it follows that $u(t) \in D(A)$. One can easily prove that $u(t)$ and u_0 satisfy (3.4.4). Hence we established a bijective correspondence between the classical solutions of (3.4.1) and (3.4.4).

In case $x_0 \in \mathcal{Y}$, for $u_0 = \binom{x_0}{M^{1/2}\dot{x}_0}$ the Cauchy problem (3.4.4) in general has only a mild solution. Let us denote this solution by $u(t) = \binom{u_1(t)}{u_2(t)}$. From $u(t) = A\int_0^t u(s)\mathrm{d}s + u_0$ follows

$$A^{-1}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \int_0^t \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} + A^{-1}\begin{pmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{pmatrix}.$$

This implies

$$-Cu_1(t) - M^{1/2}u_2(t) = \int_0^t u_1(s)\mathrm{d}s - Cx_0 - M\dot{x}_0, \tag{3.4.5}$$

$$M^{1/2}u_1(t) = \int_0^t u_2(s)\mathrm{d}s + M^{1/2}x_0. \tag{3.4.6}$$

The relation (3.4.6) implies that $M^{1/2}u_1(t)$ (and hence $Mu_1(t)$) is continuously differentiable and that $u_2(t) = \frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}u_1(t))$. Then (3.4.5) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}u_1(t)) + Cu_1(t) + \int_0^t u_1(s) - M\dot{x}_0 - Cx_0 = 0,$$

hence $u_1(t)$ is a mild solution of (3.4.1).

On the other hand, let $x(t)$ be a mild solution of (3.4.1) for $x_0 \in \mathcal{Y}$. Set $u(t) = \begin{pmatrix} x(t) \\ M^{1/2}x(t) \end{pmatrix}$ and $u_0 = \left(\begin{smallmatrix} x_0 \\ \dot{x}_0 \end{smallmatrix}\right)$. Obviously $u(t) \in \mathcal{Y}$ and $u(t)$ is continuous. One can easily prove that $A^{-1}u(t) = \int_0^t u(s)\mathrm{d}s + A^{-1}u_0$ holds, hence $u(t)$ is a mild solution of (3.4.4).

Finally, let us assume that there exists a mild solution of (3.4.1) for $x_0 \in \mathcal{Z}$. We decompose x_0 as $x_0 = y_0 + w_0$, where $y_0 \in \mathcal{Y}$ and $w_0 \in \mathcal{N}(M)$. For the initial conditions $y(0) = y_0$, $\dot{y}(0) = \dot{x}_0$ there exists a unique mild solution $y(t)$ of (3.4.1). Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}x(t)) + Cx(t) + \int_0^t x(s) - M\dot{x}_0 - Cx_0 = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}y(t)) + Cy(t) + \int_0^t y(s) - M\dot{x}_0 - Cy_0 = 0.$$

By subtracting these two equations, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(M^{1/2}z(t)) + Cz(t) + \int_0^t z(s) = 0,$$

where $z(t) = x(t) - y(t)$. This implies that $z(t)$ is a mild solution of (3.4.1) for initial conditions $z(0) = 0$, $\dot{z}(0) = 0$. From the uniqueness of the solutions, it follows $z(t) \equiv 0$, hence $w_0 = 0$, i.e. $x_0 \in \mathcal{Y}$. □

Remark 3.4.1. Let us denote by $P : \mathbf{Y} \to \mathcal{Y}$ the orthogonal projector on the first component of $\mathbf{Y}$. Then from Theorem 3.4.1 it follows that if the Cauchy problem (3.4.1) has a solution $x(t)$ for initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, it is given by $x(t) = PT(t)\left(\begin{smallmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{smallmatrix}\right)$, where $T(t)$ denotes semigroup generated by operator A.

Remark 3.4.2. Theorem 3.4.1 also implies

$$E(t; x_0, \dot{x}_0) = \frac{1}{2}\left\| T(t)\begin{pmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{pmatrix}\right\|^2 \leq \|T(t)\|^2 E(0; x_0, \dot{x}_0), \tag{3.4.7}$$

i,e, the rate of the exponential decay of the energy (3.4.2) of the system (3.4.1) is given by the growth bound of the semigroup $T(t)$.

Hence, the energy of the system (3.4.1) decay uniformly exponentially if and only if the semigroup $T(t)$ is uniformly exponentially stable, and the sufficient and necessary conditions for this are given in Subsections 3.3.3 and 3.3.4.

Note that the equation (3.4.7) also implies that energy of the system (3.4.1) always decays in time.

Exponential decay of the energy of the system (3.4.1) can also be expressed as follows. Multiply (3.4.1) by $\dot{x}(t)$. Since $\frac{\mathrm{d}}{\mathrm{d}t}(M\dot{x}(t), \dot{x}(t)) = 2(M\ddot{x}(t), \dot{x}(t))$ and $\frac{\mathrm{d}}{\mathrm{d}t}(x(t), x(t)) = 2(\dot{x}(t), x(t))$, it follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((M\dot{x}(t), \dot{x}(t)) + (x(t), x(t))\right) = -(C\dot{x}(t), \dot{x}(t)) \leq 0,$$

hence for all classical solutions $x(t)$ we have

$$E(t; x_0, \dot{x}_0) = -(C\dot{x}(t), \dot{x}(t)).$$

Also, the energy decays exponentially in time if and only if there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t}\int_0^t \|C^{1/2}\dot{x}(s)\|^2 \mathrm{d}s = 0.$$

Example 3.4.1 *(Continuation of Example* 3.3.2*).* The results of this section imply that the energy (3.1.8) of the system described by (3.1.3) has uniform exponential decay for all $\varepsilon > 0$.

Moreover, it was shown in [Ves88] that in the case $\varepsilon = 1$ the system comes to rest for $t = 2\pi$ independently of the initial conditions. Also, for $\varepsilon = 1$ the corresponding operator A has empty spectrum. This is not accidental, since Proposition A.6 implies that the emptiness of the spectrum is a necessary condition for this behavior.

Chapter 4

Optimal damping

In this chapter we will give a precise mathematical formulation of our optimal damping criterion. This criterion is designed in such a way that our knowledge of the most dangerous input frequencies for the system could be naturally implemented. This procedure uses the theory developed in the matrix case in a natural way.

In the commutative case, i.e. when M and C commute, the optimal damping will be found, and this result generalizes the well-known result in the matrix case.

An alternative approach to the problem of cutting–off low–risk frequencies is also given.

4.1 Minimization of the average total energy

As in the finite–dimensional case, our aim is to minimize the total energy of the system described by (3.4.1) as the function of the damping form γ (hence of the operator C), where γ goes over some prescribed set of admissible damping forms (this set is always contained in the set of all damping forms for which the corresponding operator A generates an uniformly exponentially stable semigroup). In case of the initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, relation (3.4.7) implies that the total energy (given by $\int_0^\infty E(t; x_0, \dot{x}_0)\mathrm{d}t$) of the system (3.4.1) is given by

$$\int_0^\infty (T(t)^* T(t) u_0, u_0)\mathrm{d}t, \tag{4.1.1}$$

where $u_0 = \left(\begin{smallmatrix} x_0 \\ M^{1/2}\dot{x}_0 \end{smallmatrix} \right)$.

From the famous Datko theorem ([Paz83]) it follows that the semigroup $T(t)$ is uniformly exponentially stable if and only if total energy of the system is finite for all $u_0 \in \mathbf{Y}$.

As in the finite–dimensional case, the expression (4.1.1) can be algebraically represented. The following result can be derived from [Phó91].

Theorem 4.1.1. *The following operator equation*

$$A^* X x + X A x = -x, \quad \text{for all } x \in D(A), \tag{4.1.2}$$

has a bounded solution, and the solution X can be expressed by

$$Xx = \int_0^\infty T(t)^* T(t) x \mathrm{d}t. \tag{4.1.3}$$

Theorem 4.1.1 immediately implies that the total energy of the system is given by (Xu_0, u_0). To make our minimization process independent of the initial conditions, we would like to minimize the average total energy over the set of admissible damping coefficients γ, i.e.

$$\int_{\|u_0\|=1} (Xu_0, u_0)\mu(\mathrm{d}u_0) \to \min, \tag{4.1.4}$$

where X is regarded as a function of γ, and μ is some measure on the unit sphere in $\mathbf{Y}$.

The general theory of integration in Hilbert spaces can be found in [Sko74]. In the rest of this section we will give the strict mathematical meaning to the formula (4.1.4).

Remark 4.1.1. Instead of the average, one can take a maximum over all initial conditions u_0, $\|u_0\| = 1$ as a measure of the efficiency of the damping, i.e. instead of (4.1.4) one can take

$$\max_{\|x\|=1}(Xx, x) \to \min . \tag{4.1.5}$$

Since X is selfadjoint positive definite, (4.1.5) is equivalent with $\|X\| \to \min$. This approach is taken in, for example, [Cox98b], [Cox98a].

Since there exists no generalization of Lebesgue measure for Hilbert spaces (see, for example [Kuo75]), the natural choice for the measure is the Gaussian measure, since it is perhaps the simplest class of measures on Hilbert spaces. Let us recall definition and basic properties of Gaussian measures.

Gaussian measure μ on $\mathbb{R}$ is a Borel measure with density $f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}$, which is absolutely continuous with respect to Lebesgue measure. In other words, $\mu(B) = \int_B f(x)\mathrm{d}x$, for all Borel sets B in $\mathbb{R}$. It is uniquely determined by the real number m (mean) and positive number σ^2 (variance). The characteristic functional of a Gaussian measure μ in $\mathbb{R}$ is easily verified to have the form

$$\varphi(z) := \int_{-\infty}^{\infty} e^{izx}\mu(\mathrm{d}x) = e^{-\frac{1}{2}\sigma^2 z^2 + imz}.$$

The measure is uniquely determined by its characteristic functional.

A non–negative measure μ in $\mathbb{R}^n$ is called Gaussian if all its one–dimensional projections μ^y, $y \in \mathbb{R}^n$, where

$$\mu^y(B) = \mu\{x \in \mathbb{R}^n : (x, y) \in B\}$$

are Gaussian. One can easily compute that the characteristic functional of μ is

$$\varphi(z) = e^{-\frac{1}{2}(Kz,z)+i(m,z)}, \tag{4.1.6}$$

where $m \in \mathbb{R}^n$ is the mean and K is a positive semi–definite operator on $\mathbb{R}^n$, called the covariance operator. It can be easily verified that for $\det K \neq 0$, the function (4.1.6) is the Fourier transform of the positive function

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det K}} e^{-\frac{1}{2}(K^{-1}(x-m),(x-m))},$$

which is the density of the measure μ, with respect to Lebesgue measure. If $\det K = 0$, then there is a orthogonal projector P in $\mathbb{R}^n$ such that $K = PK = KP$. Now μ is concentrated on the subspace $P\mathbb{R}^n$, shifted along the vector m, and is non–degenerate there.

By Gaussian measure in a real Hilbert space $\mathcal{Y}$ we understand a measure μ whose characteristic functional has the form

$$\varphi(z) := \int_{\mathcal{Y}} e^{i(z,x)}\mu(\mathrm{d}x) = e^{-\frac{1}{2}(Kz,z)+i(m,z)}, \tag{4.1.7}$$

where $m \in \mathcal{Y}$ and K is trace class operator. It follows from (4.1.7) that all finite–dimensional projections μ_L of μ are also Gaussian measures in the corresponding subspace. It turns out that one can define a set of subspaces L, the projections upon which completely determine the measure μ and on which the structure of μ is simplest.

The mean m is the unique vector from $\mathcal{Y}$ such that

$$\int_{\mathcal{Y}} (x,z)\mu(\mathrm{d}x) = (m,z), \quad \text{for all } z \in \mathcal{Y},$$

and the covariance operator K is defined by

$$\int_{\mathcal{Y}} (x,z_1)(x,z_2)\mu(\mathrm{d}x) = (Kz_1,z_2), \quad \text{for all } z_1, z_2 \in \mathcal{Y}.$$

The trace of K can be calculated by

$$\mathrm{tr}K = \int_{\mathcal{Y}} \|x\|^2 \mu(\mathrm{d}x).$$

If $m = 0$ (i.e. if measure μ is centered)

$$\int_{\mathcal{Y}} (Xx, x)\mu(\mathrm{d}x) = \mathrm{tr}(XK), \tag{4.1.8}$$

for all bounded operators X on $\mathcal{Y}$. The generalization of the formula (4.1.8) to the case of surface measures will play important role in the sequel.

For the general theory of Gaussian measures in Hilbert spaces see [Kuo75].

To define the Gaussian measure in $\mathbf{Y}$, which is a complex Hilbert space, we first define Gaussian measure in the underlying real space $\mathbf{Y}_{\mathbb{R}}$ which is defined as follows (this procedure is essentially given in [VK96], where one can also find the proofs of our assertions).

Take any orthonormal basis $\{e_1, e_2, \ldots\}$ in $\mathcal{Y}$ and denote by $\mathcal{Y}_{\mathbb{R}}$ the collection of all elements of $\mathcal{Y}$ that have real Fourier coefficients in this basis. Each element $x \in \mathcal{Y}$ can be written as $x = x' + ix''$, where $x', x'' \in \mathcal{Y}_{\mathbb{R}}$. For a fixed basis this decomposition is unique, and $\mathcal{Y}_{\mathbb{R}}$ can be regarded as a real Hilbert space. Let $\hat{\mathcal{Y}}$ denote the product $\mathcal{Y}_{\mathbb{R}} \oplus \mathcal{Y}_{\mathbb{R}}$ with the usual scalar product. The relation $\mathcal{Y} \ni x = x' + ix'' \mapsto \left(\begin{smallmatrix} x' \\ x'' \end{smallmatrix}\right) =: \hat{x} \in \hat{\mathcal{Y}}$ gives the one–to–one mapping between $\mathcal{Y}$ and $\hat{\mathcal{Y}}$. Any bounded operator G in $\mathcal{Y}$ generates the corresponding bounded operator $\hat{G}$ in $\hat{\mathcal{Y}}$. The correspondence means that $\hat{y} = \hat{G}\hat{x}$ if $y = Gx$. Each bounded operator G can be uniquely represented as $G = G' + iG''$, where both operators G' and G'' leave the set $\mathcal{Y}_{\mathbb{R}} \subset \mathcal{Y}$ invariant, and we have

$$\hat{G} = \begin{bmatrix} G' & -G'' \\ G'' & G' \end{bmatrix}.$$

The operator matrix $\hat{G}$ commutes with the operator matrix $J = \left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$. This property characterizes the class of the operators in $\hat{\mathcal{Y}}$ which correspond to some operator in $\mathcal{Y}$. In the case that G is selfadjoint, it follows that G' is selfadjoint, and G'' skew–adjoint.

Let $K_{\mathbb{R}}$ be a trace class operator in $\mathcal{Y}_{\mathbb{R}}$. The operator $K_{\mathbb{R}}$ induces a Gaussian measure $\mu_{\mathbb{R}}$ in $\mathcal{Y}_{\mathbb{R}}$ with zero mean and covariance operator $K_{\mathbb{R}}$. Set

$$\hat{K} = \begin{bmatrix} K_{\mathbb{R}} & 0 \\ 0 & K_{\mathbb{R}} \end{bmatrix}.$$

Then $\hat{K}$ induces a Gaussian measure $\hat{\mu}$ in $\hat{\mathcal{Y}}$, and, due to [VK96, Theorems 1,6 and 7], the operator K in $\mathcal{Y}$, defined by

$$Kx = K_{\mathbb{R}}x + iK_{\mathbb{R}}x'', \quad \text{where } x = x' + ix'', \tag{4.1.9}$$

induces a Gaussian measure μ in $\mathcal{Y}$. In [VK96] it was shown that the measure μ does not depend on the choice of the basis in $\mathcal{Y}$.

Finally, let ν denote the Gaussian measure in $\mathbf{Y}$ induced by the operator $\left(\begin{smallmatrix} K & 0 \\ 0 & K \end{smallmatrix}\right)$, and let $\hat{\nu}$ denote the corresponding measure in $\hat{\mathbf{Y}} = \hat{\mathcal{Y}} \oplus \hat{\mathcal{Y}}$.

Observe that for $G = G' + iG''$ selfadjoint,

$$(Gx, x) = (\hat{G}\hat{x}, \hat{x}) = (G'x', x') + (G'x'', x''), \text{ where } x = x' + ix'', \ \hat{x} = \left(\begin{smallmatrix} x' \\ x'' \end{smallmatrix}\right).$$

We treat the left hand side in (4.1.4) as an integral in $\hat{\mathbf{Y}}$, i.e.

$$\int\limits_{\|\hat{u}\|=1} (\hat{X}\hat{u}, \hat{u})\nu^S(\mathrm{d}\hat{u}),$$

where ν^S is some measure on the unit sphere in $\hat{\mathbf{Y}}$. For the measure ν^S we take the measure induced by the measure $\hat{\nu}$ via Minkowski formula (see [Fed69]):

$$\int\limits_S f \mathrm{d}\nu^S = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int\limits_{d(\hat{x}, S) \le \varepsilon} f(\hat{x})\hat{\nu}(\mathrm{d}\hat{x}), \tag{4.1.10}$$

where S denotes the unit sphere in $\hat{\mathbf{Y}}$. A justification of the formula (4.1.10) is given in [Her82] (see also [Her80]).

More precisely, it was proven in [Her82] that there exists a surface measure ν^S on S, induced by $\hat{\nu}$, and that the formula (4.1.10) holds for all continuous bounded functions f defined in $\hat{\mathbf{Y}}$.

The σ–ring of ν^S is just the σ–ring of Borel sets in S (i.e. smallest σ–ring of subsets of S which contains every compact set of S), where the topology on S is induced by the topology on $\hat{\mathbf{Y}}$.

Hence, the precise meaning of (4.1.4) will be

$$\int_S (\hat{X}\hat{u}, \hat{u})\nu^S(\mathrm{d}\hat{u}) \to \min, \tag{4.1.11}$$

where $\hat{X}$ corresponds to the operator X, and ν^S is the surface measure on S induced by the Gaussian measure $\hat{\nu}$ in $\hat{\mathbf{Y}}$.

The question arises what is the most natural way to define the operator $K_{\mathbb{R}}$ (and hence the measure ν^S). In the case of compact M (which is a generic situation in most applications) as a basis in $\mathcal{Y}$ we can take the eigenvectors e_i of M (which correspond to the eigenvectors of the undamped system which correspond to resonant frequencies of the system), and define $K_{\mathbb{R}}$ by

$$K_{\mathbb{R}}\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{\infty} \lambda_i x_i e_i, \tag{4.1.12}$$

where $\sum_{i=1}^{\infty} \lambda_i < \infty$, $\lambda_i \geq 0$. The weights λ_i should be chosen in such a manner as to implement our knowledge of the most dangerous eigenfrequencies (e_i's) for the system. In this way the choice of the measure depends on the physical properties of the system. In the case of non–compact M the basis of $\mathcal{Y}$ could be chosen in a similar way, and should contain all eigenfrequencies of the corresponding undamped system. If M possesses continuous spectrum, the basis of $\mathcal{Y}$ could contain also some approximate eigenvectors corresponding to the continuous part of the spectrum.

If we take that some λ_i's are zero, i.e. if we take a degenerate $K_{\mathbb{R}}$, that means that the frequencies which correspond to the vanishing λ_i's need not be damped. Mathematically, this case reduces to the non–degenerate case on the orthogonal complement of the null-space of $K_{\mathbb{R}}$, the measure in the null-space of $K_{\mathbb{R}}$ being Dirac measure concentrated at zero.

Let us assume for the moment that M is compact, and that $K_{\mathbb{R}}$ has the form (4.1.12). Then the functional from (4.1.11) can be written as $\mathrm{tr}(XZ)$, where the trace class operator Z is given by the following:

$$Zx = \sum_{i=1}^{\infty} \alpha_i \left(\int_S (u, e_i)^2 \nu^S(\mathrm{d}u) \right) e_i, \text{ for } x = \sum_{i=1}^{\infty} \alpha_i e_i.$$

From [Her82] we know that

$$\int_S (u, e_i)^2 \nu^S(\mathrm{d}u) \leq M\lambda_i,$$

where M does not depend on $i \in \mathbb{N}$. Hence the operator Z is well–defined and trace class. Let us compute $\mathrm{tr}(XZ)$: we obtain

$$\begin{aligned}
\mathrm{tr}(XZ) &= \sum_{i=1}^{\infty}(XZe_i, e_i) = \sum_{i=1}^{\infty}(Ze_i, Xe_i) = \sum_{i=1}^{\infty} \left(\int_S (u, e_i)^2 \nu^S(\mathrm{d}u)\, e_i, Xe_i \right) \\
&= \sum_{i=1}^{\infty} \int_S (u, e_i)^2 (e_i, Xe_i) \nu^S(\mathrm{d}u) = \int_S \sum_{i=1}^{\infty} (u, e_i)^2 (e_i, Xe_i) \nu^S(\mathrm{d}u) \\
&= \int_S \left(X(\sum_{i=1}^{\infty}(u, e_i)e_i), \sum_{i=1}^{\infty}(u, e_i)e_i \right) \nu^S(\mathrm{d}u) = \int_S (Xu, u)\nu^S(\mathrm{d}u),
\end{aligned}$$

what was claimed. Here we used Levi theorem to interchange the sum and the integral.

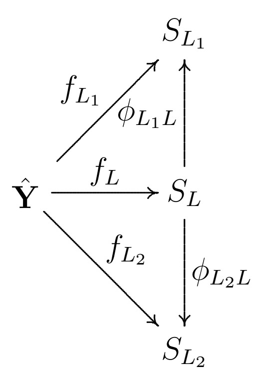
One can also easily see that $Z_N \to Z$ strongly as $N \to \infty$.

4.2 Optimization procedure

In this section we will show how (4.1.11) can be calculated by an approximation procedure, as well as solve the optimal damping problem in case of presence of internal damping.

First we show how we can approximate the surface measure ν^S by a sequence of surface measures of the unit spheres in the finite–dimensional spaces.

Let L be a finite–dimensional subspace of $\hat{\mathbf{Y}}$, and let P_L denote the corresponding orthogonal projection. Set $S_L = \{x \in L : \|x\| = 1\}$, and let us define $f_L : S \to S_L$ by $f_L(x) = \frac{P_L x}{\|P_L x\|}$. Let $\mathcal{B}_L$ denote the collection of the Borel sets in S_L. The family $(S_L, \mathcal{B}_L, f_L)$, where L goes over a set of finite–dimensional subspace of $\hat{\mathbf{Y}}$, is directed in the sense of [DF91, Section I.2.1], i.e. for each pair of subspaces L_1, L_2 there exists a subspace L and the pair of surjective connecting maps $\phi_{L_j L} : S_L \to S_{L_j}$, $j = 1, 2$ for which the following diagram commutes



If we take $L = L_1 + L_2$ and define the connecting maps by $\phi_{L_j L}(x) = \frac{P_{L_j} x}{\|P_{L_j} x\|}$, $j = 1, 2$ one can easily check that this property holds. Every map f_L determines a σ–ring $\mathcal{U}_L$ in S. We denote the union of all σ–rings $\mathcal{U}_L$ by $\mathcal{U}$. Then [Sko74] the

σ–closure of $\mathcal{U}$ coincides with the σ–ring $\mathcal{B}$ of the Borel sets in S. If we take a chain of increasing subspaces $L_n \subset L_{n+1}$ for which is $\bigcup_n L_n$ is dense in $\hat{\mathbf{Y}}$, then the σ–closure of $\bigcup_n \mathcal{U}_{L_n}$ is again $\mathcal{B}$, and the family $\{(S_{L_n}, \mathcal{B}_{L_n}, f_{L_n}) : n \in \mathbb{N}\}$ is also directed.

On S_L we define induced measure ν_L^S by

$$\nu_L^S(B) = \hat{\nu}(C(L,B)),$$

where $C(L,B) = f_L^{-1}(B) = \{x \in S : f_L(x) \in B\}$ and $B \in \mathcal{B}_L$. These measures are called a system of finite–dimensional distributions of the measure $\hat{\nu}$. One can easily check that this system satisfies the compatibility condition

$$\nu_{L'}^S(B) = \nu_L^S(\phi_{L'L}^{-1}(B)),\ B \in \mathcal{B}_{L'},\ L' \subset L.$$

Let us assume that $P_N \to I$ strongly. Then from [DF91, Example 3.3] and [Sko74, pp. 8] follows that for every bounded continuous function φ we have

$$\lim_n \int\limits_{S_{L_n}} \varphi(x)\nu_{L_n}^S(\mathrm{d}x) = \lim_n \int\limits_S \varphi(f_{L_n}(x))\nu^S(\mathrm{d}x) = \int\limits_S \varphi(x)\nu^S(\mathrm{d}x). \tag{4.2.1}$$

Proposition 4.2.1. *The surface measure ν_L^S is the measure induced by the Gaussian measure in L with zero mean and covariance operator*

$$K_L = P_L \begin{bmatrix} \hat{K} & 0 \\ 0 & \hat{K} \end{bmatrix} P_L.$$

Proof. Let us denote the Gaussian measure in L with zero mean and covariance operator K_L by $\widetilde{\nu}_L$, and let $\widetilde{\nu}_L^S$ be corresponding induced surface measure on S_L defined via Minkowski formula, i.e.

$$\widetilde{\nu}_L^S(A) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int\limits_{d(x,A) \le \varepsilon} \mathrm{d}\widetilde{\nu}_L,$$

for a Borel set A in S_L.

First note that $\widetilde{\nu}_L^S$ and ν_L^S are defined on the same σ–ring of the Borel sets in S_L. Let A be an arbitrary Borel set in S_L. Then

$$\nu_L^S(A) = \nu^S(C(L,A)) = \nu^S(\widetilde{A}),$$

where $\widetilde{A} = \{x \in S : f_L(x) \in A\}$. Hence

$$\nu_L^S(A) = \int\limits_{\widetilde{A}} \mathrm{d}\nu^S = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int\limits_{\widetilde{A}_\varepsilon} \mathrm{d}\hat{\nu} = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int\limits_{\hat{\mathbf{Y}}} \chi_{\widetilde{A}_\varepsilon} \mathrm{d}\hat{\nu},$$

where $\widetilde{A}_\varepsilon = \{x \in \hat{\mathbf{Y}} : d(x, \widetilde{A}) \le \varepsilon\}$, and $\chi_{\widetilde{A}_\varepsilon}$ is the characteristic function of the set $\widetilde{A}_\varepsilon$. From [Sko74, 1§5] follows

$$\int\limits_{\hat{\mathbf{Y}}} \chi_{\widetilde{A}_\varepsilon} \mathrm{d}\hat{\nu} = \int\limits_{L} \chi_{\widetilde{A}_\varepsilon} \mathrm{d}\hat{\nu}^L \int\limits_{\hat{\mathbf{Y}}\ominus L} \chi_{\widetilde{A}_\varepsilon} \mathrm{d}\hat{\nu}^{\hat{\mathbf{Y}}\ominus L}, \tag{4.2.2}$$

where $\hat{\nu}^L$ and $\hat{\nu}^{\hat{\mathbf{Y}}\ominus L}$ are induced measures in subspaces L and $\hat{\mathbf{Y}} \ominus L$, defined by $\hat{\nu}^L(A) = \hat{\nu}(\{x \in \hat{\mathbf{Y}} : P_L x \in A\})$ and $\hat{\nu}^{\hat{\mathbf{Y}}\ominus L}(A) = \hat{\nu}(\{x \in \hat{\mathbf{Y}} : (I - P_L)x \in A\})$, respectively. Both measures are Gaussian with zero mean and with covariance operators K_L and $(I - P_L)\left[\begin{smallmatrix} \hat{K} & 0 \\ 0 & \hat{K} \end{smallmatrix}\right](I - P_L)$, respectively.

Let us calculate the right hand side of (4.2.2). First note that

$$\widetilde{A}_\varepsilon \cap L = A_\varepsilon := \{x \in L : d(x, A) \le \varepsilon\}.$$

Indeed, let $x \in \widetilde{A}_\varepsilon \cap L$ be arbitrary. We assume $\varepsilon < 1$. Then for each $n \in \mathbb{N}$ there exists $x_n \in \widetilde{A}$ such that $\|x - x_n\| \le \varepsilon + \frac{1}{n}$. Let $x_n = a_n + b_n$, where $a_n \in L$, $b_n \in \hat{\mathbf{Y}} \ominus L$. We can take $n \in \mathbb{N}$ such that $\varepsilon + \frac{1}{n} < 1$. We have

$$\begin{aligned}(\varepsilon + \frac{1}{n})^2 &\ge \|x - x_n\|^2 = \|x - a_n\|^2 + \|b_n\|^2 = \|x\|^2 - 2(x, a_n) + \|a_n\|^2 + \|b_n\|^2 \\ &= \|x\|^2 - 2(x, a_n) + 1.\end{aligned}$$

Our choice of n and ε implies $(x, a_n) > 0$. Hence

$$\left\| x - \frac{a_n}{\|a_n\|} \right\|^2 = \|x\|^2 - 2\frac{1}{\|a_n\|}(x, a_n) + 1 \leq \|x\|^2 - 2(x, a_n) + 1 \leq (\varepsilon + \frac{1}{n})^2,$$

and since $\frac{a_n}{\|a_n\|} \in A$, we have obtained our assertion. The other inclusion is obvious.

Hence,

$$\int\limits_L \chi_{\widetilde{A}_\varepsilon} \mathrm{d}\hat{\nu}^L = \int\limits_{A_\varepsilon} \mathrm{d}\hat{\nu}^L.$$

This implies that

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int\limits_{A_\varepsilon} \mathrm{d}\hat{\nu}^L$$

exists and is equal to $\widetilde{\nu}_L^S(A)$.

Now, fix $\varepsilon > 0$ and let $x \in \hat{\mathbf{Y}} \ominus L$ be arbitrary. Take some $a \in A$. Set $\widetilde{a} := \frac{1}{2}\varepsilon a + x$. Then $f_L(a) = a \in A$, so we have $\widetilde{a} \in \widetilde{A}$. From $\|x - \widetilde{a}\| = \frac{1}{2}\varepsilon$ follows that $x \in \widetilde{A}_\varepsilon$, so we have proved $(\hat{\mathbf{Y}} \ominus L) \cap \widetilde{A}_\varepsilon = \hat{\mathbf{Y}} \ominus L$, for all $\varepsilon > 0$. This implies that the second integral on the right hand side of (4.2.2) reads

$$\int\limits_{\hat{\mathbf{Y}} \ominus L} \mathrm{d}\hat{\nu}^{\hat{\mathbf{Y}} \ominus L} = \hat{\nu}^{\hat{\mathbf{Y}} \ominus L}(\hat{\nu}^{\hat{\mathbf{Y}} \ominus L}) = 1.$$

Thus we have proved $\widetilde{\nu}_L^S(A) = \nu_L^S(A)$ for all Borel sets in S_L, which was needed.

□

From now on we assume that the measures ν^S and ν_L^S are normalized ($\nu^S(S)$ is calculated in [Her82, Theorem 1], and $\nu_L^S(S_L)$ will be calculated explicitly).

Our next aim is to approximate the operator X. Let $\mathcal{Y}_N$, $\mathcal{Y}_n \subset \mathcal{Y}_{N+1}$ be a chain of finite–dimensional subspaces of $\mathcal{Y}$. Set $\mathbf{Y}_N = \mathcal{Y}_N \oplus \mathcal{Y}_N$, which we treat as a subspace of $\mathbf{Y}$. Let P_N be the orthogonal projector from $\mathbf{Y}$ to

$\mathbf{Y}_N$. The space $\mathbf{Y}_N$ is equipped with the norm induced from $\mathbf{Y}$. Consider the sequence of operators A_N defined in $\mathbf{Y}_N$. Assume that A_N satisfy the following assumptions:

(1) there exists $\lambda \in \rho(A) \cap_n \rho(A_N)$ such that the resolvents converge

$$(\lambda - A_N)^{-1} P_N x \to (\lambda - A)^{-1} x, \text{ for all } x \in \mathbf{Y}, \tag{4.2.3}$$

(2) there exist numbers $M \geq 1$ and $\omega < 0$ such that

$$\|e^{tA_N}\| \leq Me^{\omega t} \text{ for } t \geq 0 \text{ and all } n \in \mathbb{N} \tag{4.2.4}$$

Remark 4.2.1. Generally, we can use any discretization method for the semigroups for which some Kato–Trotter–type theorem exists, and (4.2.4) is satisfied for some $\omega < 0$. In article [GKP01] is given a survey of these methods. Also a method from [LZ94] can be used. An error estimate for Kato–Trotter theorem is given in [IK98].

Under these assumptions, one can easily see that the assumptions of Theorem 2.5 from [IM98] are satisfied, which implies that the Lyapunov equation

$$A_N^* X + X A_N = -I \tag{4.2.5}$$

is solvable for all $N \in \mathbb{N}$ and for the solutions X_N we have

$$X_N P_N x \to Xx \text{ for all } x \in \mathbf{Y}.$$

From the uniform boundedness principle follows $\sup_N \|X_N\| < \infty$, hence the functions $\varphi_N(x) = (X_N P_N x, x)$ are bounded and continuous, so they are ν–measurable functions in $\mathbf{Y}$ and $\varphi_N(x) \to (Xx, x)$ holds.

This also implies that the functions $\hat{\varphi}_N(\hat{x}) = (\hat{X}_N \hat{P}_N \hat{x}, \hat{x})$ are $\hat{\nu}$–measurable and that $\hat{\varphi}_N(x) \to (\hat{X}\hat{x}, \hat{x})$ holds, where $\hat{X}_N$, $\hat{P}_N$ and $\hat{x}$ are the corresponding operators and elements in $\hat{Y}$.

We assume that we chose $\mathcal{Y}_N$ in such a way that $A''_N = 0$, i.e. such that the "imaginary part" of the operator A_N in the sense of the construction given on the page 77 is zero, so that the operator $\widehat{A}_N$ has the following matrix representation in $\mathbf{Y}_N$:

$$\hat{A}_N = \begin{bmatrix} A_N & 0 \\ 0 & A_N \end{bmatrix}. \tag{4.2.6}$$

Then

$$\hat{X}_N = \begin{bmatrix} X_N & 0 \\ 0 & X_N \end{bmatrix}.$$

Since

$$\hat{G} \mapsto \int\limits_{S_{\hat{\mathbf{Y}}_N}} (\hat{G}\hat{x}, \hat{x}) \nu^S_{\hat{\mathbf{Y}}_N}(\mathrm{d}\hat{x})$$

is a linear functional in the space of symmetric matrices $\hat{G}$ in $\hat{\mathbf{Y}}_N$, there exists a matrix $\hat{Z}_N$ such that

$$\int\limits_{S_{\hat{\mathbf{Y}}_N}} (\hat{G}\hat{x}, \hat{x}) \nu^S_{\hat{\mathbf{Y}}_N}(\mathrm{d}\hat{x}) = \mathrm{tr}(\hat{G}\hat{Z}_N), \text{ for all symmetric matrices } \hat{G}. \tag{4.2.7}$$

As is shown in the Section 2.1 $\hat{Z}_N$ is a symmetric positive semi–definite matrix. Due to the symmetry of the measure $\nu^S_{\hat{\mathbf{Y}}_N}$ we have $\hat{Z}_N = \left[\begin{smallmatrix} Z_N & 0 \\ 0 & Z_N \end{smallmatrix}\right]$, hence $\mathrm{tr}(\hat{X}_N \hat{Z}_N) = 2\mathrm{tr}(X_N Z_N)$.

In the next section it will be shown how Z_N can be calculated.

Now,

$$\mathrm{tr}(X_N Z_N) = \frac{1}{2} \int\limits_{S_{\hat{\mathbf{Y}}_N}} (\hat{X}_N \hat{x}, \hat{x}) \nu^S_{\hat{\mathbf{Y}}_N}(\mathrm{d}\hat{x}) = \frac{1}{2} \int\limits_S \hat{\varphi}_N(\hat{x}) \nu^S(\mathrm{d}\hat{x}) \to \frac{1}{2} \int\limits_S (\hat{X}\hat{x}, \hat{x}) \nu^S(\mathrm{d}\hat{x}). \tag{4.2.8}$$

The results of this section can be summarized as follows. Under the usual assumptions on convergence of the semigroups, we have proved that, instead of (4.1.11), we can use the following minimization process:

$$\lim_{N\to\infty} \operatorname{tr}(X_N Z_N) \to \min, \tag{4.2.9}$$

where X_N is the solution of the approximate Lyapunov equation (4.2.5), and Z_N depends only on $\mathcal{Y}_N$ and $K_{\mathbb{R}}$, and can be explicitly computed.

The formula (4.2.9) clearly gives rise to a numerical procedure for the optimization of the damping.

Let us assume that M is compact, and let $K_{\mathbb{R}}$ be given by (4.1.12). Let K be the corresponding operator on $\mathcal{Y}$. By μ_K we denote the corresponding (in the sense of Section 4.1) Gaussian measure on $\mathbf{Y}$. We decompose $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, where $\mathcal{Y}_1 = \mathcal{N}(K)^\perp$, $\mathcal{Y}_2 = \mathcal{N}(K)$. Then the following generalization of Theorem 2.2.1 holds.

Theorem 4.2.2. *Consider the set of operators C such that there exists $\delta > 0$ such that $C \geq \delta M$, i.e. such that*

$$(Cx, x) \geq \delta(Mx, x), \quad \text{for all } x \in \mathcal{Y}. \tag{4.2.10}$$

Then the optimal damping operators corresponding to the measure μ_K (in the sense of (4.1.11)) over this set are those operators which have the following form:

$$C_0 = \begin{bmatrix} 2M^{1/2}|_{\mathcal{Y}_1} & 0 \\ 0 & C_1 \end{bmatrix}$$

with C_1 being bounded positive definite operator on $\mathcal{Y}_2$.

Physical interpretation of the condition (4.2.10) is that the system possesses internal damping.

Proof. First observe that (4.2.10) implies that the corresponding operator $A(C)$ is uniformly exponentially stable.

Let us denote by μ_K^S the corresponding surface measure. Let us denote by $X(C_0)$ the corresponding solution of the Lyapunov equation (4.1.2). Note that $\int\limits_S (X(C_0)u,u)\mu_K^S(\mathrm{d}u)$ does not depend on the choice of C_1. Let us assume that there exists an operator C satisfying the assumption given above, and such that

$$\int\limits_S (X(C)u,u)\mu_K^S(\mathrm{d}u) < \int\limits_S (X(C_0)u,u)\mu_K^S(\mathrm{d}u).$$

Set $\mathcal{Y}_N = \operatorname{span}\{e_1,\dots,e_N\}$, where e_i are normalized eigenvectors of M. We define $\mathbf{Y}_N = \mathcal{Y}_N \oplus \mathcal{Y}_N \subset \mathbf{Y}$. Let us denote by P_N and $\widehat{P}_N$ the orthogonal projectors onto $\mathcal{Y}_N$ and $\widehat{Y}_N$, respectively. Set $A_N = \widehat{P}_N A$. Then we have

$$A_N := A_N(C) = \begin{bmatrix} 0 & \Omega_N \\ -\Omega_N & -\Omega_N C_N \Omega_N \end{bmatrix},$$

where $\Omega_N = M^{-1/2}|_{\mathcal{Y}_N}$ and $C_N = P_N C$.

First we show that the operators A_N are stable. Let us assume that A_N is not stable, for some $N \in \mathbb{N}$. Then from Proposition 2.1.1 follows that there exists $x \in \mathbf{Y}_N$ such that $\Omega_N x = \omega x$, for some $\omega \in \mathbb{R}$, and $\Omega_N C_N \Omega_N x = 0$. This implies $P_N C x = 0$, and from $0 = (P_N Cx, x) = (Cx, P_N x) = (Cx, x)$ we obtain $Cx = 0$. Since $\omega \in \sigma(M)$, this is in contradiction with Theorem 3.3.9.

One can easily prove

$$(\lambda - A_N)^{-1}\widehat{P}_N x \to (\lambda - A)^{-1}x, \text{ for all } x \in \mathbf{Y}, \mathrm{Re}\lambda \geq 0. \tag{4.2.11}$$

The relation (4.2.11) implies (4.2.3). Theorem 2.1 from [LZ94] implies that

(4.2.4) holds if and only if the following three conditions hold:

$$\sup_{N\in\mathbb{N}}\{\mathrm{Re}\lambda : \lambda \in \sigma(A_N)\} < 0, \tag{4.2.12}$$

$$\sup_{\mathrm{Re}\lambda\geq 0, N\in\mathbb{N}} \|(\lambda - A_N)^{-1}\| < \infty, \tag{4.2.13}$$

there exists $\Psi > 0$ such that

$$\|e^{tA_N}\| \leq \Psi, \text{ for all } t > 0, N \in \mathbb{N}. \tag{4.2.14}$$

The relation (4.2.14) is obviously satisfied, since e^{tA_N} are contractions, and relation (4.2.13) follows from (4.2.11) and the principle of uniform boundedness. Assume now that (4.2.12) is not satisfied. Then there exists $x_N \in \mathbf{Y}_N$, $\|x_N\| = 1$, and $\lambda_N = \alpha_N + i\beta_N$, $\alpha_N < 0$, $\beta_N \geq 0$ such that

$$A_N x_N = \lambda_N x_N \tag{4.2.15}$$

and $\alpha_N \to 0$.

Let $x_N = \binom{u_N}{v_N}$, $u_N, v_N \in \mathcal{Y}_N$. Then (4.2.15) can be written as

$$\Omega_N v_N = \lambda_N u_N, \tag{4.2.16}$$

$$\Omega_N u_N + \Omega_N C_N \Omega_N v_N + \lambda_N v_N = 0. \tag{4.2.17}$$

The relations (4.2.16) and (4.2.17) imply

$$\lambda_N^2 \Omega_N^{-2} u_N + \lambda_N C_N u_N + u_N = 0. \tag{4.2.18}$$

From (4.2.18) we obtain

$$\alpha_N = -\frac{(C_N u_N, u_N)}{2\|\Omega_N^{-1} u_N\|^2} = \frac{(C u_N, u_N)}{2(M u_N, u_N)} \to 0, \tag{4.2.19}$$

which is in contradiction with (4.2.10).

Hence for the subspace sequence $\mathbf{Y}_N$ and approximation operators A_N, $N \in \mathbb{N}$, the formula (4.2.8) holds, which implies that for N large enough there exists a subspace $\mathcal{Y}_N$ such that the corresponding projections $A_N(C)$, $A_N(C_0)$ and Z_N satisfy

$$\mathrm{tr}(X_N(C)Z_N) < \mathrm{tr}(X_N(C_0)Z_N).$$

But this is in contradiction with Theorem 2.2.1, since $P_N C_0 \in \mathcal{C}_{\min}$, $\mathcal{C}_{\min}$ being the set on which the global minimum is attained. □

4.3 Calculation of the matrix Z_N

Let us fix some $\mathcal{Y}_n$ a n–dimensional subspace of $\mathcal{Y}$. Set $N = 2n$. By $\mathbf{Y}_N$ we denote the corresponding N–dimensional real subspace of $\mathbf{Y}$ constructed analogously as the space $\mathbf{Y}_{\mathbb{R}}$ in Section 4.1. Let $\nu_{\mathbf{Y}_N}$ and $\nu^S_{\mathbf{Y}_N}$ be Gaussian measures in $\mathbf{Y}_N$ and $S_{\mathbf{Y}_N}$, respectively, their construction given in the previous section. Let K_N denote the corresponding covariance operator for the measure $\nu_{\mathbf{Y}_N}$. We decompose $\mathbf{Y}_N$ into $\mathbf{Y}_N = \mathbf{Y}^1_N \oplus \mathbf{Y}^2_N$, where $\mathbf{Y}^2_N$ is the null–space of the operator K_N, and $\mathbf{Y}^1_N$ is the orthogonal complement of $\mathbf{Y}^2_N$. Then $\nu_{\mathbf{Y}_N} = \nu_{\mathbf{Y}^1_N} \times \nu_{\mathbf{Y}^2_N}$, where $\nu_{\mathbf{Y}^1_N}$ is Gaussian measure with zero mean and covariance operator $P_{\mathbf{Y}^1_N} K_N P_{\mathbf{Y}^1_N}$, $P_{\mathbf{Y}^1_N}$ being the orthogonal projector in $\mathbf{Y}^1_N$, and $\nu_{\mathbf{Y}^2_N}$ is Dirac measure in $\mathbf{Y}^2_N$ concentrated at zero.

Let us fix a basis in $\mathbf{Y}_N$ such that K_N has a matrix representation of the form

$$K_N = \begin{bmatrix} K^1_N & 0 \\ 0 & 0 \end{bmatrix},$$

$K^1_N \in \mathbb{R}^{2t \times 2t}$ being positive definite. Then it easily follows that Z_N has the matrix representation

$$Z_N = \begin{bmatrix} Z_N^1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, our aim is to compute the matrix Z_N^1, where Z_N^1 is such that (4.2.7) holds for the measure $\nu_{\mathbf{Y}_N^1}$ in $\mathbf{Y}_N^1$.

The following formula obviously holds

$$\int_{S_{\mathbf{Y}_N^1}} \mathrm{d}\nu_{\mathbf{Y}_N^1}^S = \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=1} \left(\int_{x^*x \leq r^2} \nu_{\mathbf{Y}_N^1}(\mathrm{d}x) \right). \tag{4.3.1}$$

The density function of $\nu_{\mathbf{Y}_N^1}$ with respect to the Lebesgue measure is

$$p(x) = \frac{1}{(2\pi)^t \sqrt{\det K_N^1}} e^{-1/2x^* {K_N^1}^{-1} x},$$

hence

$$\int_{x^*x \leq r^2} \nu_{\mathbf{Y}_N^1}(\mathrm{d}x) = \frac{1}{(2\pi)^t \sqrt{\det K_N^1}} \int_{x^*x \leq r^2} e^{-1/2x^* {K_N^1}^{-1} x} \mathrm{d}x. \tag{4.3.2}$$

Let $K_N^1 = LL^*$ be Cholesky factorization of K_N^1, and let $L^*L = U^*\Lambda U$ be spectral decomposition of L^*L, where $\Lambda = \mathrm{diag}(\mu_1, \ldots, \mu_{2t})$. Note that $\mu_1, \ldots, \mu_{2t}$ are eigenvalues of K_N^1. By the use of the substitution $x = LU^*y$, from (4.3.2) we obtain

$$\int_{x^*x \leq r^2} \nu_{\mathbf{Y}_N^1}(\mathrm{d}x) = \frac{1}{(2\pi)^t} \int_{y^*\Lambda y \leq r^2} e^{-1/2y^*y} \mathrm{d}y = \Pr\{\sum_{j=1}^{2t} \mu_j X_j^2 \leq r^2\}, \tag{4.3.3}$$

where $X_i \sim N(0,1)$ are random vectors with Gaussian distribution $N(0,1)$ and Pr denotes the probability function. From the probability theory (see, for example [Fel66, pp. 48]), follows

$$\Pr\{\sum_{j=1}^{2t} \mu_j X_j^2 \leq r^2\} = \Pr\{\sum_{j=1}^{2t} \mu_j \chi_j(1) \leq r^2\} = \Pr\{\sum_{j=1}^{m} \lambda_j \chi_j(k_j) \leq r^2\}, \tag{4.3.4}$$

where $\chi(k)$ denotes the chi–squared distribution with k degrees of freedom, and $\lambda_1, \ldots, \lambda_m$ are mutually different eigenvalues of K_N^1, with their multiplicities (as eigenvalues) k_j. For our construction it is essential to note that k_j are always even.

Let us denote by f and φ the probability density function and the characteristic function of $\sum_{j=1}^m \lambda_j \chi_j(k_j)$, respectively. Then [Fel66, Chapter 15.]

$$\Pr\{\sum_{j=1}^{m} \lambda_j \chi_j(k_j) \leq r^2\} = \int_0^{r^2} f(x)\mathrm{d}x, \tag{4.3.5}$$

hence (4.3.1), (4.3.3), (4.3.4) and (4.3.5) imply

$$\int_{S_{\mathbf{Y}_N^1}} \mathrm{d}\nu^S_{\mathbf{Y}_N^1} = 2f(1). \tag{4.3.6}$$

From [Fel66, Chapter 15.] also follows

$$\varphi(t) = \prod_{j=1}^{m} \varphi_{\chi_j(k_j)}(\lambda_j t) = \prod_{j=1}^{m} (1 - 2it\lambda_j)^{-k_j/2}.$$

Set $g_j = \frac{k_j}{2}$. We want to expand $\varphi(t)$ in partial fractions, i.e. to obtain

$$\prod_{j=1}^{m} (1 - 2it\lambda_j)^{-g_j} = \sum_{j=1}^{m} \sum_{s=1}^{g_j} \alpha_{js} (1 - 2it\lambda_j)^{-s}. \tag{4.3.7}$$

To calculate the coefficients α_{js} we proceed as follows. Fix $j \in \{1, \ldots, m\}$. We can rewrite (4.3.7) as

$$(1-2it\lambda_i)^{-g_i} \prod_{j\neq i} (1-2it\lambda_j)^{-g_j} = \sum_{j=1}^{g_i} \alpha_{is}(1-2it\lambda_i)^{g_i} + \sum_{j\neq i} \sum_{w=1}^{g_j} \alpha_{jw}(1-2it\lambda_j)^{-w}.$$

Multiplying the previous relation by $(1 - 2it\lambda_i)^{g_i}$, and by substitution $y = 1 - 2it\lambda_i$ we get

$$\prod_{j\neq i} \left(\frac{\lambda_i - \lambda_j}{\lambda_i} + y\frac{\lambda_j}{\lambda_i} \right)^{-g_j} = \sum_{s=1}^{g_i} \alpha_{is} y^{g_i - s} + y^{g_i} \sum_{j\neq i} \sum_{w=1}^{g_j} \alpha_{jw} \left(\frac{\lambda_i - \lambda_j}{\lambda_i} + y\frac{\lambda_j}{\lambda_i} \right)^{-w}. \tag{4.3.8}$$

When we take $y = 0$ in (4.3.8), we obtain

$$\alpha_{ig_i} = \prod_{j \neq i} \left(\frac{\lambda_i - \lambda_j}{\lambda_i} \right)^{-g_j},$$

and when we differentiate both sides of (4.3.8) k times ($k = 1, \ldots, g_i - 1$) and take $y = 0$, we obtain

$$\alpha_{i,g_i-k} = \frac{f_i^{(k)}(0)}{k!}, \text{ where } f_i(y) = \prod_{j \neq i} \left(\frac{\lambda_i - \lambda_j}{\lambda_i} + y \frac{\lambda_j}{\lambda_i} \right)^{-g_j}.$$

Set

$$\psi_i(y) = \ln f_i(y) = -\sum_{j \neq i} g_j \ln \left| \frac{\lambda_i - \lambda_j}{\lambda_i} + y \frac{\lambda_j}{\lambda_i} \right|.$$

We calculate the derivatives in zero of the functions ψ_i and obtain

$$\psi_i^{(k)}(0) = (-1)^k (k-1)! \sum_{j \neq i} \frac{g_j}{\left| \frac{\lambda_i}{\lambda_j} - 1 \right|^k} \text{ for } k \geq 1.$$

Now we can calculate the derivatives in zero of the functions f_i by use of the following recursive procedure:

$$\begin{aligned} f_i^{(1)}(0) &= f_i(0) \psi_i^{(1)}(0), \\ f_i^{(k+1)}(0) &= \sum_{l=0}^{k} \binom{k}{l} f_i^{(k-l)}(0) \psi_i^{(l+1)}(0), \ k = 2, \ldots, g_i - 1. \end{aligned}$$

After a straightforward calculation we get the following recursive formula for the coefficients α_{ij}, $i = 1, \ldots, m$:

$$\begin{aligned} \alpha_{ig_i} &= \prod_{j \neq i} \left(1 - \frac{\lambda_j}{\lambda_i} \right)^{-g_j}, \\ \alpha_{i,g_i-1} &= -\alpha_{ig_i} \sum_{j \neq i} \frac{g_j}{\left| \frac{\lambda_i}{\lambda_j} - 1 \right|}, \\ \alpha_{i,g_i-k-1} &= \frac{1}{k+1} \sum_{l=0}^{k} (-1)^{l+1} \alpha_{i,g_i-k+l} \sum_{j \neq i} \frac{g_j}{\left| \frac{\lambda_i}{\lambda_j} - 1 \right|^{l+1}}, \ k = 1, 2, \ldots, g_i - 2. \end{aligned} \tag{4.3.9}$$

Since $f(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} e^{-itx}\varphi(t)\mathrm{d}t$, we have

$$f(x) = \sum_{j=1}^{m} \sum_{l=1}^{g_j} \alpha_{jl} f_{\lambda_j \chi(2l)}(x).$$

Now the last equation, together with (4.3.6), implies

$$\begin{aligned} \int\limits_{S_{\mathbf{Y}_N}} \mathrm{d}\nu^S_{\mathbf{Y}_N} &= 2\sum_{j=1}^{m} \sum_{l=1}^{g_j} \alpha_{jl} f_{\lambda_j \chi(2l)}(1) = 2\sum_{j=1}^{m} \sum_{l=1}^{g_j} \alpha_{jl} \frac{1}{\lambda_j} f_{\chi(2l)}\left(\frac{1}{\lambda_j}\right) \\ &= 2\sum_{j=1}^{m} \sum_{l=1}^{g_j} \alpha_{jl} \frac{1}{\lambda_j^l} \frac{1}{2^l (l-1)!} e^{-\frac{1}{2\lambda_j}} = 2\sum_{j=1}^{m} e^{-\frac{1}{2\lambda_j}} \sum_{l=1}^{g_j} \alpha_{jl} \frac{1}{\lambda_j^l} \frac{1}{2^l (l-1)!}, \end{aligned} \tag{4.3.10}$$

since the characteristic function for the chi–squared distribution with k degrees of freedom is given by

$$f_{\chi(k)}(x) = \frac{1}{2^{k/2}\Gamma(k/2)} e^{-\frac{x}{2}} x^{k/2-1}.$$

Hence we have found a recursive formula for the calculation of the surface measure of the sphere. It turns out that we can also calculate the entries of the matrix Z_N^1 by the use of the coefficients α_{ij}.

Assume for the moment that the surface measure $\nu^S_{\mathbf{Y}_N^1}$ is not normalized. Let $X = (X_{ij})$ be an arbitrary symmetric matrix in $\mathbb{R}^N$. We have

$$\begin{aligned} \mathrm{tr}(XZ_N^1) &= \sum_{i,j} X_{ij} \mathrm{tr}(Z_N^1 E_{ij}) = \sum_{i,j} X_{ij} \int\limits_{S_{\mathbf{Y}_N^1}} x^* E_{ij} x \, \nu^S_{\mathbf{Y}_N^1}(\mathrm{d}x) \\ &= \sum_{i,j} X_{ij} \int\limits_{S_{\mathbf{Y}_N^1}} x_i x_j \, \nu^S_{\mathbf{Y}_N^1}(\mathrm{d}x), \end{aligned}$$

hence

$$(Z_N^1)_{ij} = \int\limits_{S_{\mathbf{Y}_N^1}} x_i x_j \, \nu^S_{\mathbf{Y}_N^1}(\mathrm{d}x), \tag{4.3.11}$$

where E_{ij} denotes the matrix which has all entries zero except for the entry (i,j) which has value 1. Let ${K_N^1}^{-1} = V\Lambda V^*$ be a spectral decomposition of the operator ${K_N^1}^{-1}$, with V orthogonal matrix. By the use of (4.1.10) and by the substitution $x = Vy$, we obtain

$$\int\limits_{S_{\mathbf{Y}_N^1}} x_i x_j \, \nu^S_{\mathbf{Y}_N^1}(\mathrm{d}x) = \frac{1}{(2\pi)^t\sqrt{\det K_N^1}} \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon} \int\limits_{d(x,S_{\mathbf{Y}_N^1})\le\varepsilon} x_i x_j e^{-1/2x^*{K_N^1}^{-1}x}\mathrm{d}x$$
$$= \frac{1}{(2\pi)^t\sqrt{\det K_N^1}} \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon} \int\limits_{d(x,S_{\mathbf{Y}_N^1})\le\varepsilon} (Vy)_i(Vy)_j e^{-1/2y^*\Lambda y}\mathrm{d}y. \tag{4.3.12}$$

Since $(Vy)_i(Vy)_j = y^*\widetilde{E}_{ij}y$, where

$$\widetilde{E}_{ij} = V^* E_{ij} V, \tag{4.3.13}$$

to compute $(Z_N^1)_{ij}$ it is enough to calculate

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon} \int\limits_{d(x,S_{\mathbf{Y}_N^1})\le\varepsilon} y_i y_j e^{-1/2y^*\Lambda y}\mathrm{d}y. \tag{4.3.14}$$

From (4.3.2) we obtain

$$\int\limits_{S_{\mathbf{Y}_N^1}} \mathrm{d}\nu^S_{\mathbf{Y}_N^1} = \frac{1}{(2\pi)^t\sqrt{\det K_N^1}} \int\limits_{S_{\mathbf{Y}_N^1}} e^{-1/2y^*\Lambda y}\mathrm{d}y. \tag{4.3.15}$$

By the use of the polar coordinates one can easily check that (4.3.14) equals

$$\int\limits_{S_{\mathbf{Y}_N^1}} y_i y_j e^{-1/2y^*\Lambda y}\mathrm{d}y.$$

Note that this integral equals zero in the case $i \ne j$.

Let $\xi : \{1,\dots,2t\} \to \{1,\dots,m\}$ be the function such that $\xi(i) = j$ implies $\mu_i = \lambda_j$. Let us fix $i \in \{1,\dots,2t\}$. Due to the symmetry of the measure $\nu^S_{\mathbf{Y}_N^1}$, we have

$$\int\limits_{S_{\mathbf{Y}_N^1}} x_i^2 \nu^S_{\mathbf{Y}_N^1}(\mathrm{d}x) = \int\limits_{S_{\mathbf{Y}_N^1}} x_j^2 \nu^S_{\mathbf{Y}_N^1}(\mathrm{d}x) \tag{4.3.16}$$

for all $j \in \xi^{-1}(\xi(i))$.

Because of (4.3.15) we can interpret $\int\limits_{S_{\mathbf{Y}_N^1}} \mathrm{d}\nu^S_{\mathbf{Y}_N^1}$ as a function in the variables $\lambda_1, \ldots, \lambda_m$, i.e. we denote

$$F(\lambda_1, \ldots, \lambda_m) = \frac{1}{(2\pi)^t \sqrt{\det K_N^1}} \int\limits_{S_{\mathbf{Y}_N^1}} e^{-1/2 \sum_{i=1}^m \lambda_i \sum_{j \in \xi^{-1}(\xi(i))} y_j^2} \mathrm{d}y.$$

All partial derivatives of this function exist and

$$\frac{\partial}{\partial \lambda_i} F(\lambda_1, \ldots, \lambda_m) = -\frac{1}{2} \frac{1}{(2\pi)^t \sqrt{\det K_N^1}} \int\limits_{S_{\mathbf{Y}_N^1}} \sum_{j \in \xi^{-1}(\xi(i))} y_j^2 e^{-1/2 y^* \Lambda y} \mathrm{d}y.$$

The last relation, together with (4.3.16) implies

$$\int\limits_{S_{\mathbf{Y}_N^1}} y_i^2 e^{-1/2 y^* \Lambda y} \mathrm{d}y = -2 \frac{(2\pi)^t \sqrt{\det K_N^1}}{k_{\xi(i)}} \frac{\partial}{\partial \lambda_{\xi(i)}} F(\lambda_1, \ldots, \lambda_m). \tag{4.3.17}$$

Hence, the relations (4.3.11), (4.3.12), (4.3.15), (4.3.16), and (4.3.17) imply

$$(Z_N^1)_{ij} = -2 \sum_l \frac{(\widetilde{E}_{ij})_{ll}}{k_{\xi(l)}} \frac{\partial}{\partial \lambda_{\xi(l)}} F(\lambda_1, \ldots, \lambda_m), \tag{4.3.18}$$

where $\widetilde{E}_{ij}$ is given by (4.3.13).

From (4.3.10) follows

$$F(\lambda_1, \ldots, \lambda_m) = 2 \sum_{j=1}^m e^{-\frac{1}{2\lambda_j}} \sum_{l=1}^{g_j} \alpha_{jl} \frac{1}{\lambda_j^l 2^l (l-1)!},$$

where α_{jl} is interpreted as a function in variables $\lambda_1, \ldots, \lambda_m$. We calculate

$$\frac{\partial}{\partial \lambda_i} F(\lambda_1, \ldots, \lambda_m) = 2 \sum_{j=1}^m e^{-\frac{1}{2\lambda_j}} \sum_{l=1}^{g_j} \frac{\frac{\partial}{\partial \lambda_i} \alpha_{jl}}{\lambda_j^l} \frac{1}{2^l (l-1)!} + \frac{1}{\lambda_i^2} e^{-\frac{1}{2\lambda_i}} \sum_{l=1}^{g_i} \frac{\alpha_{il}}{\lambda_i^l} \frac{1}{2^l (l-1)!} -$$

$$- 2 e^{-\frac{1}{2\lambda_i}} \sum_{l=1}^{g_i} \frac{l \alpha_{il}}{\lambda_i^{l+1}} \frac{1}{2^l (l-1)!}.$$

Since $\alpha_{jl} = \frac{f_j^{(g_j-l)}(0)}{(g_j-l)!}$, we have

$$\frac{\partial}{\partial \lambda_i}\alpha_{jl} = \frac{1}{(g_j-l)!}\frac{\partial}{\partial y^{g_j-l}}\frac{\partial}{\partial \lambda_i} f_j(y,\lambda_1,\ldots,\lambda_m)\bigg|_{y=0},$$

where f_j is taken as a function in variables $y, \lambda_1, \ldots, \lambda_m$. Now

$$\begin{aligned}\frac{\partial}{\partial \lambda_i} f_j(y,\lambda_1,\ldots,\lambda_m) &= f_j(y,\lambda_1,\ldots,\lambda_m)\frac{\partial}{\partial \lambda_i}\ln f_j(y,\lambda_1,\ldots,\lambda_m) \\ &= -f_j(y,\lambda_1,\ldots,\lambda_m)\sum_{l\neq j} g_l \frac{\partial}{\partial \lambda_i}\ln\left|\frac{\lambda_j-\lambda_l}{\lambda_j}+y\frac{\lambda_l}{\lambda_j}\right|.\end{aligned}$$

In the case $i \neq j$ we obtain

$$\frac{\partial}{\partial \lambda_i} f_j(y,\lambda_1,\ldots,\lambda_m) = -f_j(y,\lambda_1,\ldots,\lambda_m)\frac{g_i(y-1)}{\lambda_j-\lambda_i+y\lambda_i},$$

and in the case $i = j$ we obtain

$$\frac{\partial}{\partial \lambda_i} f_i(y,\lambda_1,\ldots,\lambda_m) = f_i(y,\lambda_1,\ldots,\lambda_m)\frac{y-1}{\lambda_i}\sum_{l\neq i}\frac{g_l\lambda_l}{\lambda_i-\lambda_l+y\lambda_l}.$$

Let us define functions $\phi_{ji}(y) = \frac{g_i(y-1)}{\lambda_j-\lambda_i+y\lambda_i}$. From the straightforward calculation we obtain:

$$\phi_{ji}^{(k)}(0) = \frac{(-1)^{k-1}k!g_i\lambda_j\lambda_i^{k-1}}{(\lambda_j-\lambda_i)^{k+1}}, \text{ for } k>0 \text{ and } \phi_{ji}(0) = -\frac{g_i}{\lambda_j-\lambda_i}.$$

Hence in the case $i \neq j$ we have

$$\begin{aligned}\frac{\partial}{\partial \lambda_i}\alpha_{jl} &= -\frac{1}{(g_j-l)!}\sum_{k=0}^{g_j-l}\binom{g_j-l}{k} f_j^{(g_j-l-k)}(0)\phi_{ji}^{(k)}(0) \\ &= g_i\left(\frac{\alpha_{jl}}{\lambda_j-\lambda_i}+\lambda_j\sum_{k=1}^{g_j-l}(-1)^k\frac{\lambda_i^{k-1}\alpha_{j,l+k}}{(\lambda_j-\lambda_i)^{k+1}}\right) \\ &= g_i\left(\frac{\alpha_{jl}}{\lambda_j-\lambda_i}+\frac{\lambda_j}{\lambda_i(\lambda_j-\lambda_i)}\sum_{k=1}^{g_j-l}(-1)^k\left(\frac{\lambda_i}{\lambda_j-\lambda_i}\right)^k\alpha_{j,l+k}\right).\end{aligned}$$

For the case $i = j$, let us define $\phi_i(y) = \frac{y-1}{\lambda_i}\sum_{p\neq i}\frac{g_p\lambda_p}{\lambda_i-\lambda_p+y\lambda_p}$. Then

$$\phi_i^{(k)}(0) = (-1)^{k-1}k!\sum_{p\neq i}\frac{g_p\lambda_p^k}{(\lambda_i-\lambda_p)^{k+1}}, \text{ for } k>0 \text{ and } \phi_i(0) = -\frac{1}{\lambda_i}\sum_{p\neq i}\frac{g_p\lambda_p}{\lambda_i-\lambda_p}.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial\lambda_i}\alpha_{il} &= \frac{1}{(g_i-l)!}\sum_{k=0}^{g_i-l}\binom{g_i-l}{k}f_i^{(g_i-l-k)}(0)\phi_i^{(k)}(0)\\ &= -\frac{\alpha_{il}}{\lambda_i}\sum_{p\neq i}\frac{g_p\lambda_p}{\lambda_i-\lambda_p} - \sum_{k=1}^{g_i-l}(-1)^k\alpha_{i,l+k}\sum_{p\neq i}\frac{g_p\lambda_p^k}{(\lambda_i-\lambda_p)^{k+1}}.\end{aligned}$$

After a tedious but straightforward calculation we obtain

$$\frac{\partial}{\partial\lambda_i}F(\lambda_1,\ldots,\lambda_m) = \sum_{j=1}^{m}\sum_{l=1}^{g_j}\beta_{ijl}\alpha_{jl}, \tag{4.3.19}$$

where in the case $i\neq j$ we have

$$\beta_{ij1} = e^{-\frac{1}{2\lambda_j}}\frac{g_i}{\lambda_j(\lambda_j-\lambda_i)}, \tag{4.3.20}$$

and in the case $i\neq j$, $l\neq 1$

$$\beta_{ijl} = 2g_i\frac{e^{-\frac{1}{2\lambda_j}}}{\lambda_j-\lambda_i}\left(\frac{1}{2^l(l-1)!}\frac{1}{\lambda_j^l} + \frac{\lambda_j}{\lambda_i}\sum_{k=1}^{l-1}(-1)^{l-k}\left(\frac{\lambda_i}{\lambda_j-\lambda_i}\right)^{l-k}\frac{1}{\lambda_j^k}\frac{1}{2^k(k-1)!}\right). \tag{4.3.21}$$

In the case $i = j$ we have

$$\beta_{ii1} = \frac{e^{-\frac{1}{2\lambda_i}}}{\lambda_i^2}\left(\frac{1}{2\lambda_i} - 1 - \sum_{p\neq i}\frac{g_p\lambda_p}{\lambda_i-\lambda_p}\right), \tag{4.3.22}$$

and for $l\neq 1$ we have

$$\begin{aligned}\beta_{iil} = e^{-\frac{1}{2\lambda_i}}\frac{1}{\lambda_i^{l+1}}\frac{1}{2^l(l-1)!}\left(\frac{1}{\lambda_i} - 2l - 2\sum_{p\neq i}\frac{g_p\lambda_p}{\lambda_i-\lambda_p}\right) - \\ 2e^{-\frac{1}{2\lambda_i}}\sum_{k=1}^{l-1}\frac{1}{\lambda_i^k}\frac{1}{2^k(k-1)!}(-1)^{l-k}\sum_{p\neq i}\frac{g_p\lambda_p^{l-k}}{(\lambda_i-\lambda_p)^{l-k+1}}.\end{aligned} \tag{4.3.23}$$

Hence the procedure of the computation of the entries of the matrix Z_N^1 consists of four steps:

(i) compute the coefficients α_{ij} using formulae (4.3.9),

(ii) compute the coefficients β_{ijl} using (4.3.20), (4.3.21), (4.3.22) and (4.3.23),

(iii) compute $\frac{\partial}{\partial \lambda_i} F(\lambda_1, \ldots, \lambda_m)$ using (4.3.19), and

(iv) compute $(Z_N^1)_{ij}$ using (4.3.18).

This algorithm is numerically unstable in the case in which g_i's are large and λ_i is close to λ_j for some $i \neq j$. In such cases one can use a Monte Carlo method of numerical integration to compute the left hand side of (4.3.17). In our case this method is especially simple and it consists of producing a sequence of $2t$–dimensional random vectors $x^{(i)}$ with normal distribution $N(0, \Lambda)$ and calculating $\sum_i (x_j^{(i)})^2 / \|x^{(i)}\|^2$, $j = 1, \ldots, 2t$, where $x^{(i)} = (x_1^{(i)}, \ldots, x_{2t}^{(i)})$.

The serious drawback of Monte Carlo method is its slow convergence which is of the order $O(n^{-1/2})$.

There also exist so–called quasi–Monte Carlo methods of integration. They need significantly less iterations, but the computation of quasi–random vectors is much more involved.

Note that Z_N can be seen as the function of the matrix K_N. Also, the matrices Z_N and K_N have the same number of zero eigenvalues.

Example 4.3.1. If we take $\lambda_i = i$, $i = 1, \ldots, 5$ and $K = \mathrm{diag}(\lambda_1, \ldots, \lambda_5)$, then we obtain $Z = \mathrm{diag}(0.8105, 0.4258, 0.2887, 0.2183, 0.1756)$

Example 4.3.2. Let us take $K_1 = \mathrm{diag}(10, 9, \ldots, 2, 1, 1, \ldots, 1)$, where the size of K is 100. The Monte–Carlo integration with 10^6 iterations produces $Z = \mathrm{diag}(\beta_1, \ldots, \beta_{100})$, where $\beta_1 = 0.068770$, $\beta_2 = 0.062182$, $\beta_3 = 0.055647$, $\beta_4 = 0.048532$, $\beta_5 = 0.041262$, $\beta_6 = 0.034278$, $\beta_7 = 0.027652$, $\beta_8 = 0.020550$, $\beta_9 = 0.013740$, $\beta_{10} = \cdots \beta_{100} = 0.006900$.

Example 4.3.3 *(Continuation of Example* 3.3.2*).* We will approximate the system from Example 3.1.1. The eigenvalues and eigenfunctions of M are calculated in Example 3.3.2. We take $N = 50$, $\mathcal{Y}_N = \operatorname{span}\{u_1, \dots, u_n\}$, where $u_i(x) = \sin(n + \frac{1}{2})x$ are the eigenfunctions of M. We also choose covariance operator K such that its (infinite–dimensional) matrix in the basis consisting of the eigenfunctions of M has the form $\operatorname{diag}(K_1, 0, 0, \dots)$, where K_1 is from Example 4.3.2. It is easy to see that $\widehat{A}_N(\varepsilon)$ has the form (4.2.6) and is stable. We calculate

$$A_N(\varepsilon) = \begin{bmatrix} 0 & \Omega_N \\ -\Omega_N & -\varepsilon C_N \end{bmatrix},$$

where

$$\Omega_N = \operatorname{diag}\left(\frac{1}{2}, \dots, N + \frac{1}{2}\right)$$

and

$$(C_N)_{ij} = \frac{(-1)^{i+j}}{\frac{1}{2}\pi(i + \frac{1}{2})(j + \frac{1}{2})}.$$

The matrix $A_N(\varepsilon)$ is clearly of the form (4.2.6). The function $\varepsilon \mapsto \operatorname{tr}(X_N(\varepsilon)Z_N)$, where $X_N(\varepsilon)$ is the solution of the Lyapunov equation

$$A_N(\varepsilon)^* X + X A_N(\varepsilon) = -I,$$

is plotted on the Figure 4.1.

The optimal damping is attained for $\varepsilon = 0.38$, and for this viscosity we have $\operatorname{tr}(X_N(\varepsilon)Z_N) = 1.0275$.

If we choose another covariance operator K such that its (infinite–dimensional) matrix in the basis consisting of the eigenfunctions of M has the form $\operatorname{diag}(\lambda_1, \lambda_2, \dots)$, where $\lambda_i = \lambda_{40+i} = 11 - i$, $i = 1, \dots, 10$, $\lambda_i = 1$, $11 \le i \le 40$, and $\lambda_i = 0$, $i > 50$, we obtain Figure 4.2.

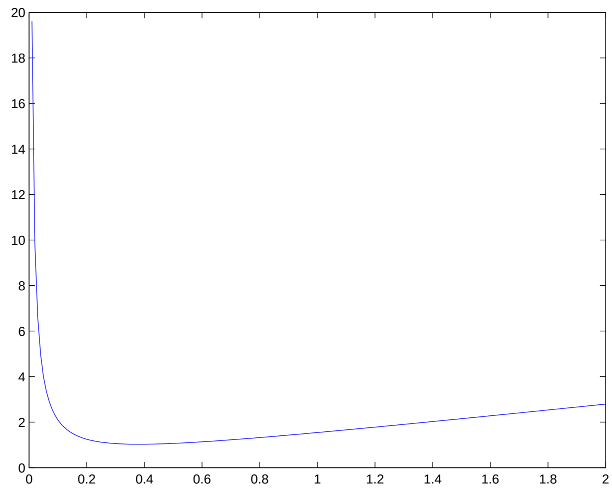


Figure 4.1: The function $\varepsilon \mapsto \mathrm{tr}(X_N(\varepsilon)Z_N)$

The optimal damping is attained for $\varepsilon = 0.53$, and for this viscosity we have $\mathrm{tr}(X_N(\varepsilon)Z_N) = 1.6618$.

4.4 The commutative case

In this section we will treat the case when operators M and C commute. In the engineering literature this is so–called modal damping case. A very systematic treatment of the abstract differential equation

$$\ddot{y}(t) + A\dot{y}(t) + By(t) = 0,$$

in terms of well-posedness of the corresponding Cauchy and boundary value problem, where A and B are normal commutative operators, is given in [Shk97].

We assume that the operators M and C are such that the corresponding operator A generates a uniformly exponentially stable semigroup $T(t)$. In this case the operator X, the solution of the Lyapunov equation (4.1.2) can be

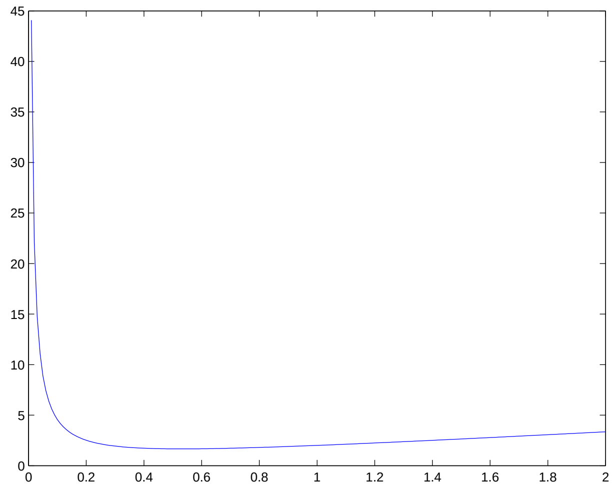


Figure 4.2: The function $\varepsilon \mapsto \operatorname{tr}(X_N(\varepsilon)Z_N)$

explicitly calculated. We start with the well-known formula [EN00, Corollary 3.5.15]

$$T(s)x = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\varepsilon-in}^{\varepsilon+in} e^{\lambda s} R(\lambda, A)x \mathrm{d}\lambda,\ x \in D(A),$$

where $\varepsilon > 0$ is arbitrary chosen, $n \in \mathbb{N}$ and $s \geq 0$. Since it is always $T(0) = I$, in the sequel we consider only $s > 0$. Recall that

$$R(\lambda, A) = \begin{bmatrix} \frac{1}{\lambda}(M_\lambda^{-1} - I) & -M_\lambda^{-1}M^{1/2} \\ M^{1/2}M_\lambda^{-1} & -\lambda M^{1/2}M_\lambda^{-1}M^{1/2} \end{bmatrix},$$

where $M_\lambda = \lambda^2 M + \lambda C + I$.

Since M and C commute, there exists a bounded selfadjoint operator G such that the operators M and C are functions of G [AG93, Theorem 76.2], hence there exists a spectral function $E(t)$ and $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ measurable functions for all Stieltjes measures ([AG93, Section 75] and [KF75, Section

36.1]) generated by $(E(t)x, x)$, $x \in \mathcal{Y}$, such that

$$M = \int_0^{\Xi} \alpha(t)\mathrm{d}E(t), \quad C = \int_0^{\Xi} \beta(t)\mathrm{d}E(t),$$

where $\Xi = \|G\|$. Since M and C are bounded, so are also the functions α and β. We have also $\alpha(t) > 0$ a.e. It follows that the resolvent $R(\lambda, A)$ can be written as

$$R(\lambda, A) = \int_0^{\Xi} \begin{bmatrix} \frac{-\lambda\alpha(t)-\beta(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1} & \frac{-\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1} \\ \frac{\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1} & \frac{-\lambda\alpha(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1} \end{bmatrix} \mathrm{d}E(t),$$

hence

$$T(s)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{\varepsilon-in}^{\varepsilon+in} e^{\lambda s} \begin{bmatrix} \int_0^{\Xi} \frac{-\lambda\alpha(t)-\beta(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)x + \int_0^{\Xi} \frac{-\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)y \\ \int_0^{\Xi} \frac{\sqrt{\alpha(t)}}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)x + \int_0^{\Xi} \frac{-\lambda\alpha(t)}{\lambda^2\alpha(t)+\lambda\beta(t)+1}\mathrm{d}E(t)y \end{bmatrix} \mathrm{d}\lambda. \tag{4.4.1}$$

We first treat the case when $x, y \in \mathcal{R}(M^{1/2})$.

We want to change the order of integration in (4.4.1) by the use of Fubini theorem. By the change of variables, we obtain

$$(T(s)\left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)) = -e^{\varepsilon s} \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} e^{i\lambda s} \Bigg[\int_0^{\Xi} p_1(\lambda, t)\mathrm{d}(E(t)x_1, x_2) -$$

$$\int_0^{\Xi} p_2(\lambda, t)\mathrm{d}(E(t)y_1, x_2) + \int_0^{\Xi} p_2(\lambda, t)\mathrm{d}(E(t)x_1, y_2) + \int_0^{\Xi} p_3(\lambda, t)\mathrm{d}\left(E(t)y_1, y_2)\right) \Bigg] \mathrm{d}\lambda,$$

where

$$p_1(\lambda, t) = \frac{-(i\lambda+\varepsilon)\alpha(t) - \beta(t)}{(i\lambda+\varepsilon)^2\alpha(t) + (i\lambda+\varepsilon)\beta(t) + 1},$$

$$p_2(\lambda, t) = \frac{\sqrt{\alpha(t)}}{(i\lambda+\varepsilon)^2\alpha(t) + (i\lambda+\varepsilon)\beta(t) + 1},$$

$$p_3(\lambda, t) = \frac{-(i\lambda+\varepsilon)\alpha(t)}{(i\lambda+\varepsilon)^2\alpha(t) + (i\lambda+\varepsilon)\beta(t) + 1}.$$

The integrals

$$\int_{-n}^{n}\int_{0}^{\Xi} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}(E(t)x,x)\mathrm{d}\lambda,\ j=1,2,3, \tag{4.4.2}$$

can be viewed as a double Lebesgue integrals in $\mathbb{R}\times\mathbb{R}_+$, with the product measure generated by the standard Lebesgue measure in $\mathbb{R}$ and by (real–valued) Stieltjes measure $(E(t)x,x)$ in $\mathbb{R}_+$. Let us now fix $n\in\mathbb{N}$. In order to use Fubini theorem on (4.4.2) we have to prove [KF75, pp. 361,362]:

(i) the functions p_j, $j=1,2,3$ are measurable in the product measure, and

(ii) the integrals

$$\int_{0}^{\Xi}\left(\int_{-n}^{n}\left|e^{i\lambda s}p_j(\lambda,t)\right|\mathrm{d}\lambda\right)\mathrm{d}(E(t)x,x),\ j=1,2,3$$

exist.

To prove (i) it suffices to show that the function $g(\lambda,t)=(i\lambda+\varepsilon)^2\alpha(t)+(i\lambda+\varepsilon)\beta(t)+1$ is measurable and vanishes only on the set of the measure zero. One can easily see that the function g does not vanish. Set $A_n=\alpha^{-1}([n,n+1])$, which is a measurable set in $\mathbb{R}_+$. Then $\mathbb{R}_+=\cup_n A_n$. To prove that g is measurable, observe that for an arbitrary $\delta>0$ the following holds

$$\{(\lambda,t):|g(\lambda,t)|<\delta\}=\bigcup_n\left(A_n\times\bigcup_{t\in A_n}\{\lambda:|g(\lambda,t)|<\delta\}\right).$$

For fixed $t\in A_n$, one can easily see that $\{\lambda:|g(\lambda,t)|<\delta\}$ is either an empty set or an open interval or an union of two open intervals, hence always an open set. It follows that $\bigcup_{t\in A_n}\{\lambda:|g(\lambda,t)|<\delta\}$ is an open set as a union of open sets. From this immediately follows that $\{(\lambda,t):|g(\lambda,t)|<\delta\}$ is measurable for all $\delta>0$, hence g is a measurable function.

Now we prove (ii). We have

$$\int_{-n}^{n} |p_j(\lambda,t)|\mathrm{d}\lambda = \int_{\Upsilon_n} |p_j(\lambda,t)|\mathrm{d}\lambda + \int_{\Gamma_n} |p_j(\lambda,t)|\mathrm{d}\lambda, \tag{4.4.3}$$

where Γ_n is lower semi–circle connecting $-n$ and n, and Υ_n is the contour consisting of the segment $[-n,n]$ and the curve Γ_n. The first integral can be calculated by the use of residue theorem. The poles of the functions $\lambda \mapsto p_j(\lambda,t)$ are the zeros of the polynomial g. We calculate the zeros of g. We have

$$\lambda_{1,2} = \pm\frac{\sqrt{4\alpha(t)-\beta(t)^2}}{2\alpha(t)} + i\frac{2\varepsilon\alpha(t)+\beta(t)}{2\alpha(t)}, \tag{4.4.4}$$

in the case $4\alpha(t) \geq \beta(t)^2$, and

$$\lambda_{1,2} = i\frac{2\varepsilon\alpha(t)+\beta(t) \pm \sqrt{\beta(t)^2-4\alpha(t)}}{2\alpha(t)}, \tag{4.4.5}$$

in the case $4\alpha(t) < \beta(t)^2$. Hence the first integral on the right hand side in (4.4.3) is

$$\int_{\Upsilon_n} |p_j(\lambda,t)|\mathrm{d}\lambda = 0,\ j=1,2,3. \tag{4.4.6}$$

To estimate the second integral on the right hand side in (4.4.3) we proceed as follows.

$$\int_{\Gamma_n^-} |p_j(\lambda,t)|\mathrm{d}\lambda = n\int_{\pi}^{2\pi} |p_j(ne^{i\varphi},t)|\mathrm{d}\varphi \leq cn^2\int_{\pi}^{2\pi}\frac{\mathrm{d}\varphi}{|g(ne^{i\varphi},t)|},$$

where $c = \varepsilon + \sup\alpha(t) + \sup\beta(t) + 1,\ j=1,2,3$. We have

$$g(ne^{i\varphi},t) = -n^2e^{2i\varphi}\alpha(t) + ine^{i\varphi}\widetilde{\beta}(t) + \widetilde{\gamma}(t) = -n^2e^{2i\varphi}\alpha(t)\cos 2\varphi -$$
$$- ne^{i\varphi}\widetilde{\beta}(t)\sin\varphi + \widetilde{\gamma}(t) + i(-n^2\alpha(t)\sin 2\varphi + n\widetilde{\beta}(t)\cos\varphi),$$

where $\widetilde{\beta}(t) = 2\varepsilon + \beta(t)$, $\widetilde{\gamma}(t) = \varepsilon^2\alpha(t) + \varepsilon\beta(t) + 1$. Hence

$$|g(ne^{i\varphi},t)|^2 =$$
$$n^4\alpha(t)^2+n^2\widetilde{\beta}(t)^2+\widetilde{\gamma}(t)^2-2n^3\alpha(t)\widetilde{\beta}(t)\sin\varphi-2n^2\alpha(t)\widetilde{\gamma}(t)\cos 2\varphi-2n\widetilde{\beta}(t)\widetilde{\gamma}(t)\sin\varphi.$$

Since $\sin\varphi \le 0$ for $\pi \le \varphi \le 2\pi$ and $\widetilde{\beta}(t) \ge 0$, $\widetilde{\gamma}(t) \ge 1$, we obtain an estimate

$$|g(ne^{i\varphi},t)|^2 \ge n^2\alpha(t). \tag{4.4.7}$$

So, we have obtained

$$\int_{-n}^{n} |p_j(\lambda,t)|\mathrm{d}\lambda \le c\alpha(t)^{-1/2},$$

where c does not depend on t. Hence the integrals in (ii) exist for all $x \in \mathcal{R}(M^{1/2})$. We are now in position to use Fubini theorem on (4.4.2), which leads to

$$\int_{-n}^{n}\int_{0}^{\Xi} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}(E(t)x,x)\mathrm{d}\lambda = \int_{0}^{\Xi}\left(\int_{-n}^{n} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda\right)\mathrm{d}(E(t)x,x).$$

Set $f_n^j(t) = \int_{-n}^{n} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda$, $j = 1,2,3$. Then

$$f_n^j(t) = \int_{\Upsilon_n} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda + \int_{\Gamma_n} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda, \tag{4.4.8}$$

where Υ_n and Γ_n are as in (4.4.3). From (4.4.4) and (4.4.5) we obtain

$$\int_{\Upsilon_n} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda = 0,\ j = 1,2,3, n \in \mathbb{N}.$$

To estimate the second integral in (4.4.8) we use the well–known Jordan lemma [Gon92, Lemma 9.2] which implies

$$\left|\int_{\Gamma_n} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda\right| \le c\max\{|p_j(\lambda,t)| : \lambda \in \Gamma_n\}.$$

Now (4.4.7) implies

$$\max\{|p_j(\lambda,t)| : \lambda\in\Gamma_n\} \le (1+\varepsilon)\alpha(t)^{1/2} + \frac{\beta(t)}{\alpha(t)^{1/2}},\ n\in\mathbb{N},$$

hence $|f_n^j(t)| \le f(t)$, where

$$f(t) = (1+\varepsilon)\alpha(t)^{1/2} + \frac{\beta(t)}{\alpha(t)^{1/2}}.$$

Since f is integrable for all Stieltjes measures generated by $x\in\mathcal{R}(M^{1/2})$, we can use Lebesgue dominate convergence theorem to obtain

$$\lim_{n\to\infty}\int_0^{\Xi}\left(\int_{-n}^{n} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda\right)\mathrm{d}(E(t)x,x) = \int_0^{\Xi}\left(\int_{-\infty}^{\infty} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda\right)\mathrm{d}(E(t)x,x),$$

for all $x\in\mathcal{R}(M^{1/2})$, in the sense of the principal value integral.

Since $(E(t)x,y)$ can be expressed by the polarization formula

$$(E(t)x,y) = \frac{1}{4}(E(t)(x+y,x+y) - (E(t)(x-y),x-y) + i(E(t)(x+iy),x+iy) \\ - i(E(t)(x-iy),x-iy)),$$

we obtain

$$(T(s)\left(\begin{smallmatrix}x_1\\y_1\end{smallmatrix}\right),\left(\begin{smallmatrix}x_2\\y_2\end{smallmatrix}\right)) = -e^{\varepsilon s}\frac{1}{2\pi}\Bigg[\int_0^{\Xi}\left(\int_{-\infty}^{\infty} e^{i\lambda s}p_1(\lambda,t)\mathrm{d}\lambda\right)\mathrm{d}(E(t)x_1,x_2) - \\ \int_0^{\Xi}\left(\int_{-\infty}^{\infty} e^{i\lambda s}p_2(\lambda,t)\mathrm{d}\lambda\right)\mathrm{d}(E(t)y_1,x_2) + \int_0^{\Xi}\left(\int_{-\infty}^{\infty} e^{i\lambda s}p_2(\lambda,t)\mathrm{d}\lambda\right)\mathrm{d}(E(t)x_1,y_2) + \\ \int_0^{\Xi}\left(\int_{-\infty}^{\infty} e^{i\lambda s}p_3(\lambda,t)\mathrm{d}\lambda\right)\mathrm{d}\left(E(t)y_1,y_2\right)\Bigg],$$

for all $x_1,y_1,x_2,y_2\in\mathcal{R}(M^{1/2})$.

Hence we can write

$$T(s)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = -\frac{1}{2\pi}e^{\varepsilon s}\begin{bmatrix} \int\limits_0^{\Xi}\int\limits_{-\infty}^{\infty} e^{i\lambda s}p_1(\lambda,t)\mathrm{d}\lambda\mathrm{d}E(t)x - \int\limits_0^{\Xi}\int\limits_{-\infty}^{\infty} e^{i\lambda s}p_2(\lambda,t)\mathrm{d}\lambda\mathrm{d}E(t)y \\ \int\limits_0^{\Xi}\int\limits_{-\infty}^{\infty} e^{i\lambda s}p_2(\lambda,t)\mathrm{d}\lambda\mathrm{d}E(t)x + \int\limits_0^{\Xi}\int\limits_{-\infty}^{\infty} e^{i\lambda s}p_3(\lambda,t)\mathrm{d}\lambda\mathrm{d}E(t)y \end{bmatrix} \tag{4.4.9}$$

in the sense of Pettis integral (for the definition and the basic properties see, for example, [HP57, Chapter 3]). Moreover, the formula (4.4.9) holds for all $x, y \in \mathcal{Y}$, which easily follows from the fact that $T(s)$ is a bounded operator.

Our next aim is to compute the integrals $\int\limits_{-\infty}^{\infty} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda$, $j = 1,2,3$ and hence to obtain an integral representation of $T(s)$ in terms of the spectral function $E(t)$. Since $p_j(\cdot,t)$ are rational functions such that the degree of the denominator is greater of the degree of the nominator, the standard result from the calculus (see, for example [Gon92]) implies that

$$\int\limits_{-\infty}^{\infty} e^{i\lambda s}p_j(\lambda,t)\mathrm{d}\lambda = 2\pi i\left(\sum_{\lambda\in S_+}\mathrm{Res}(e^{is\cdot}p_j(\cdot,t);\lambda) + \frac{1}{2}\sum_{\lambda\in S_0}\mathrm{Res}(e^{is\cdot}p_j(\cdot,t);\lambda)\right),$$

where S_+ is the set of all poles of the function $e^{is\cdot}p_j(\cdot,t)$ in the upper half plane, and S_0 is the set of all real poles of the function $e^{is\cdot}p_j(\cdot,t)$. The poles of the functions $e^{is\cdot}p_j(\cdot,t)$ are exactly the zeros of the function $g(\cdot,t)$ which are calculated in (4.4.4) and (4.4.5).

From the straightforward calculation we obtain:

(i) in the case $4\alpha(t) > \beta(t)^2$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda, t) \mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cos(\varrho(t)s) + \frac{\beta(t)}{\sqrt{4\alpha(t) - \beta(t)^2}} \sin(\varrho(t)s) \right), \\
\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda, t) \mathrm{d}\lambda =& 4\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \frac{\sqrt{\alpha(t)}}{\sqrt{4\alpha(t) - \beta(t)^2}} \sin(\varrho(t)s), \\
\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda, t) \mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cos(\varrho(t)s) - \frac{\beta(t)}{\sqrt{4\alpha(t) - \beta(t)^2}} \sin(\varrho(t)s) \right),
\end{aligned}$$

where $\varrho(t) = \frac{\sqrt{4\alpha(t) - \beta(t)^2}}{2\alpha(t)}$,

(ii) in the case $4\alpha(t) < \beta(t)^2$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda, t) \mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cosh(\widetilde{\varrho}(t)s) + \frac{\beta(t)}{\sqrt{\beta(t)^2 - 4\alpha(t)}} \sinh(\widetilde{\varrho}(t)s) \right), \\
\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda, t) \mathrm{d}\lambda =& 4\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \frac{\sqrt{\alpha(t)}}{\sqrt{\beta(t)^2 - 4\alpha(t)}} \sinh(\widetilde{\varrho}(t)s), \\
\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda, t) \mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\cosh(\widetilde{\varrho}(t)s) - \frac{\beta(t)}{\sqrt{\beta(t)^2 - 4\alpha(t)}} \sinh(\widetilde{\varrho}(t)s) \right),
\end{aligned}$$

where $\widetilde{\varrho}(t) = \frac{\sqrt{\beta(t)^2 - 4\alpha(t)}}{2\alpha(t)}$,

(iii) and in the (limit) case $4\alpha(t) = \beta(t)^2$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda, t) \mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(1 + s\frac{\beta(t)}{2\alpha(t)} \right), \\
\int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda, t) \mathrm{d}\lambda =& 2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} s\alpha(t)^{-1/2}, \\
\int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda, t) \mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(1 - s\frac{\beta(t)}{2\alpha(t)} \right).
\end{aligned}$$

Let us define functions

$$\widetilde{\sin}(t,s) = \begin{cases} \frac{\sin(\varrho(t)s)}{\sqrt{4\alpha(t)-\beta(t)^2}}, & \varrho(t) \in \mathbb{R}\setminus\{0\}, \\ \frac{s}{2\alpha(t)}, & \varrho(t) = 0, \\ \frac{\sinh(\widetilde{\varrho}(t)s)}{\sqrt{\beta(t)^2-4\alpha(t)}}, & \widetilde{\varrho}(t) \in \mathbb{R}\setminus\{0\}, \end{cases}$$

and

$$\widetilde{\cos}(t,s) = \begin{cases} \cos(\varrho(t)s), & \varrho(t) \in \mathbb{R}, \\ \cosh(\widetilde{\varrho}(t)s), & \widetilde{\varrho}(t) \in \mathbb{R}, \end{cases}.$$

Then we can write

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\lambda s} p_1(\lambda,t)\mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) + \beta(t)\widetilde{\sin}(t,s)\right), \\ \int_{-\infty}^{\infty} e^{i\lambda s} p_2(\lambda,t)\mathrm{d}\lambda =& 4\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \sqrt{\alpha(t)}\,\widetilde{\sin}(t,s), \\ \int_{-\infty}^{\infty} e^{i\lambda s} p_3(\lambda,t)\mathrm{d}\lambda =& -2\pi e^{-s\varepsilon} e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) - \beta(t)\widetilde{\sin}(t,s)\right), \end{aligned}$$

in all three cases.

Hence we have obtained

$$T(s) = \int_0^{\Xi} e^{-s\frac{\beta(t)}{2\alpha(t)}} \begin{bmatrix} \widetilde{\cos}(t,s) + \beta(t)\widetilde{\sin}(t,s) & 2\sqrt{\alpha(t)}\,\widetilde{\sin}(t,s) \\ -2\sqrt{\alpha(t)}\,\widetilde{\sin}(t,s) & \widetilde{\cos}(t,s) - \beta(t)\widetilde{\sin}(t,s) \end{bmatrix} \mathrm{d}E(t).$$

Now we are in position to use the formula (4.1.3) in order to calculate the operator X. Set

$$\begin{aligned} q_1(t,s) &= e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) + \beta(t)\widetilde{\sin}(t,s)\right), \\ q_2(t,s) &= 2e^{-s\frac{\beta(t)}{2\alpha(t)}} \sqrt{\alpha(t)}\,\widetilde{\sin}(t,s), \\ q_3(t,s) &= e^{-s\frac{\beta(t)}{2\alpha(t)}} \left(\widetilde{\cos}(t,s) - \beta(t)\widetilde{\sin}(t,s)\right). \end{aligned}$$

Then

$$(X \left(\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right)) = \int_0^\infty \Bigg(\int_0^\Xi (q_1(t,s)^2 + q_2(t,s)^2) \mathrm{d}(E(t)x_1, x_2) + $$
$$+ \int_0^\Xi (q_1(t,s) - q_3(t,s)) q_2(t,s) \mathrm{d}(E(t)x_1, y_2)$$
$$+ \int_0^\Xi (q_1(t,s) - q_3(t,s)) q_2(t,s) \mathrm{d}(E(t)y_1, x_2) +$$
$$+ \int_0^\Xi (q_3(t,s)^2 + q_2(t,s)^2) \mathrm{d}(E(t)y_1, y_2) \Bigg) \mathrm{d}s. \quad (4.4.10)$$

As before, we would like to change the order of integration in the previous formula. To do that, it is sufficient to prove that conditions (i) and (ii) from page 104 are satisfied. The condition (i) is obviously satisfied.

Note that $(q_1(t,s) - q_3(t,s))q_2(t,s) \geq 0$ for all $t, s > 0$, hence all functions in (4.4.10) are positive. By the use of the standard integration formulae one obtains

$$\int_0^\infty (q_1(t,s)^2 + q_2(t,s)^2) \mathrm{d}s = \frac{1}{2}\beta(t) + \frac{\alpha(t)}{\beta(t)},$$
$$\int_0^\infty (q_1(t,s) - q_3(t,s)) q_2(t,s) \mathrm{d}s = \frac{1}{2}\sqrt{\alpha(t)},$$
$$\int_0^\infty (q_3(t,s)^2 + q_2(t,s)^2 \mathrm{d}s = \frac{\alpha(t)}{\beta(t)}.$$

Hence to be able to change the order of integration we must assume that $\beta(t)^{-1}$ is integrable, which implies that the operator C is boundedly invertible. This implies that a modally damped system decays exponentially if and only if C is boundedly invertible. Therefore, we make this assumption. Then we can write

$$X = \int_0^{\Xi} \begin{bmatrix} \frac{1}{2}\beta(t) + \frac{\alpha(t)}{\beta(t)} & \frac{1}{2}\sqrt{\alpha(t)} \\ \frac{1}{2}\sqrt{\alpha(t)} & \frac{\alpha(t)}{\beta(t)} \end{bmatrix} \mathrm{d}E(t).$$

Note that this formula is a direct generalization of the formula in the matrix case given in [Cox98b] (see also [Cox98a], [VBD01]).

Now take any Gaussian measure in $\mathcal{Y}$ (in the sense of Section 4.1). Then the optimal energy decay on the set of boundedly invertible damping operators C which commute with M is attained for the operator C_{opt} which has a spectral function β_{opt} such that $\frac{1}{2}\beta(t) + 2\frac{\alpha(t)}{\beta(t)} \to \min$ for all $t > 0$. One can easily see that $\beta_{\mathrm{opt}}(t) = 2\sqrt{\alpha(t)}$, i.e.

$$C_{\mathrm{opt}} = 2M^{1/2}. \tag{4.4.11}$$

Note that in the finite–dimensional case (4.4.11) reads $C_{\mathrm{opt}} = 2\Omega$ in the notation of Chapter 2, where we had shown that in $C_{\mathrm{opt}} = 2\Omega$ a function $C \mapsto \mathrm{tr}(X(C)Z)$ attains its global minimum.

Example 4.4.1. We consider Euler–Bernoulli beam in the presence of viscous damping term $\alpha\frac{\partial^3}{\partial x^2 \partial t}$:

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha\frac{\partial^3 u}{\partial x^2 \partial t}(x,t) + \frac{\partial^4 u}{\partial t^4}(x,t) = 0,\ 0 < x < 1,\ t > 0,$$
$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = u(1,t) = \frac{\partial u}{\partial x}(1,t) = 0,$$
$$u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}u(x,0) = u_1(x),$$

where $\alpha > 0$. Multiplying the above differential equation by the smooth test function v such that $v(0) = v(1) = v'(0) = v'(1) = 0$, and by partial integration we obtain

$$\int_0^1 \frac{\partial^2 u}{\partial t^2}(x,t)v(x)\mathrm{d}x - \int_0^l \frac{\partial^2 u}{\partial x \partial t}(x,t)v'(x)\mathrm{d}x + \int_0^l \frac{\partial^2 u}{\partial x^2}(x,t)v''(x)\mathrm{d}x = 0.$$

Hence, the system given above can be written as

$$\mu(\ddot{u}, v) + \gamma(\dot{u}, v) + \kappa(u, v) = 0, \text{ for all } v \in \mathcal{Y},$$

where

$$\mu(u, v) = \int_0^l u(x)\overline{v(x)}\mathrm{d}x,$$
$$\gamma(u, v) = \alpha \int_0^1 u'(x)\overline{v'(x)}\mathrm{d}x,$$
$$\kappa(u, v) = \int_0^1 u''(x)\overline{v''(x)}\mathrm{d}x,$$

and

$$\mathcal{Y} = \mathcal{H}_0^2([0,1]) = \{u \in \mathrm{L}^2([0,l]) : u', u'' \in \mathrm{L}^2([0,l]), u(0) = u'(0) = u(1) = u'(1) = 0\}.$$

One can easily see that μ and γ are dominated by κ and that $\kappa > 0$, hence we are in position to use results from Chapter 3. It is clear that M is compact. The eigenvalues λ_n of M are the solutions of the equation

$$\cosh\lambda \cos\lambda = 1,$$

and the corresponding eigenvectors are

$$u_n(x) = \cos\lambda_n x - \cos\lambda_n x - \beta_n(\sinh\lambda_n x - \sin\lambda_n x),$$

where

$$\beta_n = \frac{\cosh\lambda_n - \cos\lambda_n}{\sinh\lambda_n - \sin\lambda_n}.$$

One can check that

$$\inf \frac{\gamma(u_n, u_n)}{\lambda_n \kappa(u_n, u_n)} > 0,$$

hence, in the light of Remark 3.3.8, the energy of the system has uniform exponential decay. Since $C = \alpha M^{1/2}$, the results of this section imply that the optimal damping is obtained for $\alpha = 2$.

4.5 Cutting–off in the frequency domain

When we minimize the average total energy, we take into account physical properties of damping by choosing appropriate Gaussian measure. If we choose to minimize the maximal total energy, the natural choice is to substitute the identity operator on the right hand side of the Lyapunov equation (4.1.2) with some other positive semi–definite operator Q, i.e. instead of X given by (4.1.2) we can take X which is the solution of the Lyapunov equation

$$A^*X + XA = -Q, \tag{4.5.1}$$

where Q is chosen in such a way to take into account the physical properties of the system. Also, in the case when we minimize the average total energy, we can instead of (4.1.2) use (4.5.1).

In this section we will give one possible construction of Q in such a way that it corresponds to smoothing or cutting off in the frequency domain.

We start with the following result which has its own interest. This is a generalization of the well–known "frequency domain formula" (2.1.11) to the infinite–dimensional case.

Proposition 4.5.1. *Let A be a generator of an uniformly exponentially stable semigroup. Then the solution X of the Lyapunov equation* (4.1.2) *satisfies*

$$Xy = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(i\eta, A)^* R(i\eta, A) y \mathrm{d}\eta, \; y \in D(A), \tag{4.5.2}$$

where the Lebesgue integral above converges strongly in the sense of the principal value, i.e.

$$Xy = \frac{1}{2\pi} \lim_{\Xi \to \infty} \int_{-\Xi}^{\Xi} R(i\eta, A)^* R(i\eta, A) y \mathrm{d}\eta.$$

Remark 4.5.1. The term "frequency domain" is used since (4.5.2) could be understood as integration over amplitudes f of the steady–state solution

$$xe^{iwt}, \; x = (iw - A)^{-1} f,$$

which are the answer to the harmonic load fe^{iwt}. The formula (4.1.3) is sometimes refereed as a "time domain formula".

Moreover, one can see that the maximal dissipative operator A generates an uniformly exponentially stable semigroup if and only if the integral on the right hand side in (4.5.2) converges for all $y \in D(A)$.

Proof of Proposition 4.5.1. We start with the Lyapunov equation (4.1.2). We have

$$-A^* Xx - i\eta Xx + i\eta Xx - XAx = x, \; \text{for all } x \in D(A).$$

Let us multiply this equation by $R(i\eta, A)^*$ on the left and set $y = (i\eta - A)x$. We obtain

$$XR(i\eta, A)y + R(i\eta, A)^* Xy = R(i\eta, A)^* R(i\eta, A)y, \; \text{for all } y \in \mathcal{Y}. \qquad (4.5.3)$$

We have

$$\int_{-\infty}^{\infty} R(i\eta, A)x \mathrm{d}\eta = \pi x, \; \text{for all } x \in D(A),$$

in the sense of the principal value. From the previous relation, (4.5.3) and the fact that X is bounded, (4.5.2) follows. □

Now we modify (4.5.2) by putting into the integral in (4.5.2) a smoothing function g with the following properties:

(i) it is meromorphic in the neighborhood of $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im} z \leq 0\}$, and has there finitely many poles,

(ii) it does not have poles in $\mathbb{R}$ and $g(z) \geq 0$ for all $z \in \mathbb{R}$, and

(iii) $g|_{\mathbb{R}} \in L_1(\mathbb{R})$ and $zg(z) \to 0$ as $R \to \infty$ for $z = Re^{i\varphi}$, $\mathrm{Im} z \leq 0$, uniformly in φ.

We start with the relation (4.5.3) which we multiply by $\frac{1}{2\pi} g(\eta)$, and we integrate this relation from $-\infty$ to ∞. We obtain

$$X \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\eta) R(i\eta, A) x \mathrm{d}\eta + \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\eta) R(i\eta, A)^* x \mathrm{d}\eta \, X = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(i\eta, A)^* g(\eta) R(i\eta, A) x \mathrm{d}\eta.$$

Since $\sup\{\|R(\lambda, A)\| : \lambda \in i\mathbb{R}\} < \infty$ and $g|_{\mathbb{R}} \in L_1(\mathbb{R})$, the previous formula is correct. Set

$$\widehat{X}_g = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(i\eta, A)^* g(\eta) R(i\eta, A) \mathrm{d}\eta, \tag{4.5.4}$$

$$\widehat{g}(A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\eta) R(i\eta, A) \mathrm{d}\eta. \tag{4.5.5}$$

The operator $\widehat{X}_g$ can be seen as a total energy operator smoothed by the function g. Obviously, one has

$$\widehat{X}_g = X\widehat{g}(A) + \widehat{g}(A)^* X.$$

But operator $\widehat{X}_g$ can also be obtained as a solution of a Lyapunov equation. Indeed, from (4.1.3) follows

$$A^*\widehat{X}_g + \widehat{X}_g A = -(\widehat{g}(A) + \widehat{g}(A)^*). \tag{4.5.6}$$

This equation has an unique bounded solution since $\widehat{g}(A)$ is a bounded operator. Since $\widehat{g}(A) + \widehat{g}(A)^*$ is a selfadjoint operator, so it is $\widehat{X}_g$. To give any practical meaning to the formula (4.5.6), the right hand side in (4.5.6) has to be explicitly solved. Thanks to the property (iii) of the function g, this can be done.

We have

$$\int_{-n}^{n} g(\eta)R(i\eta, A)\mathrm{d}\eta = \int_{\Upsilon_n} g(\eta)R(i\eta, A)\mathrm{d}\eta + \int_{\Gamma_n} g(\eta)R(i\eta, A)\mathrm{d}\eta,$$

where Γ_n is lower semi–circle connecting $-n$ and n, and Υ_n is the contour consisting of the segment $[-n, n]$ and the curve Γ_n. The first integral on the right hand side can be calculated by the use of residue theorem.

Let g have poles in points $z_1, \ldots, z_k \in \mathbb{C}_-$ with multiplicities $n_1, \ldots, n_k$. Then we can develop function g in the neighborhood of z_i, $i \in \{1, \ldots, k\}$ into Laurent series

$$g(\eta) = \beta_{i,-n_i}(\eta - z_i)^{-n_i} + \cdots \beta_{i,-1}(\eta - z_i)^{-1} + \beta_{i,0} + \beta_{i,1}(\eta - z_i) + \cdots$$

On the other hand, we have

$$R(i\eta, A) = \sum_{j=0}^{\infty} (-i)^j (\eta - z_i)^j R(iz_i, A)^{j+1},$$

for all $|\eta - z_i| \le \|R(iz_i, A)\|^{-1}$. Hence the coefficient of the term $(\eta - z_i)^{-1}$ in the development of the function $g(\eta)R(i\eta, A)$ in the neighborhood of z_i is given by

$$\sum_{j=1}^{n_i} (-i)^{j-1} \beta_{i,-j} R(iz_i, A)^j.$$

Hence for n big enough, we have

$$\int\limits_{\Upsilon_n} g(\eta)R(i\eta, A)\mathrm{d}\eta = 2\pi i \sum_{i=1}^{k}\sum_{j=1}^{n_i}(-i)^{j-1}\beta_{i,-j}R(iz_i, A)^j.$$

From the Jordan lemma [Gon92, Lemma 9.3] and property (iii) of the function g it follows

$$\lim_{n\to\infty}\int\limits_{\Gamma_n} g(\eta)R(i\eta, A)\mathrm{d}\eta = 0,$$

hence we have obtained

$$\widehat{g}(A) = 2\pi i \sum_{i=1}^{k}\sum_{j=1}^{n_i}(-1)^{j-1}\beta_{i,-j}R(iz_i, A)^j. \tag{4.5.7}$$

From (4.5.7) it readily follows that $\widehat{g}(A)$ is a rational function of the operator A. Note also that $\widehat{X}_g$ is always positive definite.

Example 4.5.1. Let $g_1(\eta) = \frac{1}{1+0.01\eta^2}$. The graph of this function is given in Figure 4.3. Then $\widehat{g}_1(A) = 10\pi R(10, A)$.

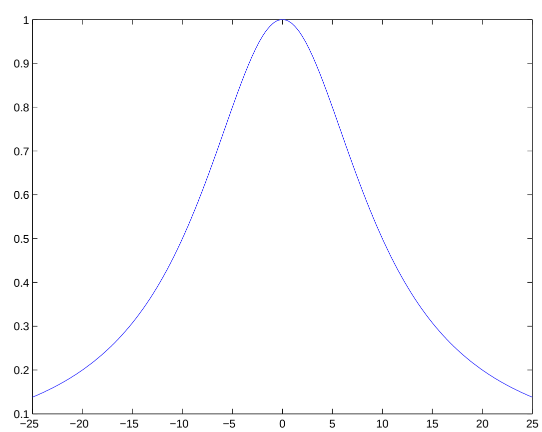


Figure 4.3: The graph of the function g_1

We apply weight function g_1 in the calculation of the optimal damping from Example 4.3.3. The function $\varepsilon \mapsto \widehat{X}_{g_1}(\varepsilon)$ is given on the Figure 4.4. Here the optimal viscosity is $\varepsilon = 0.3$.

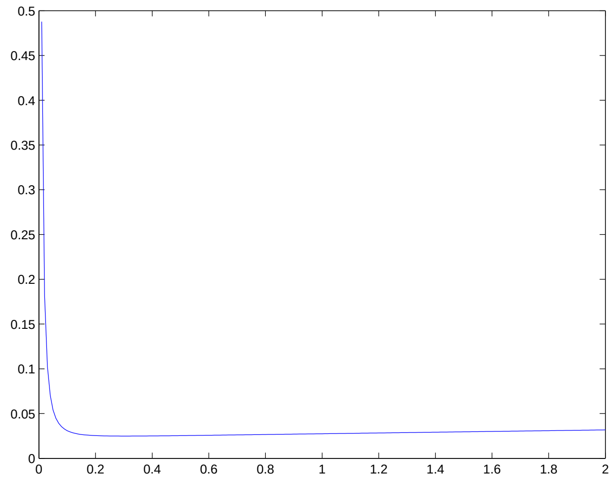


Figure 4.4: The function $\varepsilon \mapsto \widehat{X}_{g_1}(\varepsilon)$

Example 4.5.2. Let $g_2(\eta) = \frac{0.1\eta^2}{(\eta^2+4)(\eta^2+9)}$. The graph of this function is given in Figure 4.5. Then $\widehat{g}_2(A) = \frac{\pi}{45}R(2, A) + \frac{3\pi}{40}R(3, A)$.

We apply weight function g_2 in the calculation of the optimal damping from Example 4.3.3. The function $\varepsilon \mapsto \widehat{X}_{g_2}(\varepsilon)$ is given on the Figure 4.6. Here the optimal viscosity is $\varepsilon = 0.31$.

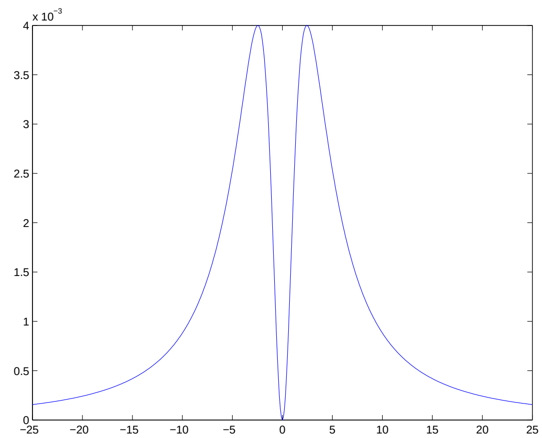


Figure 4.5: The graph of the function g_2

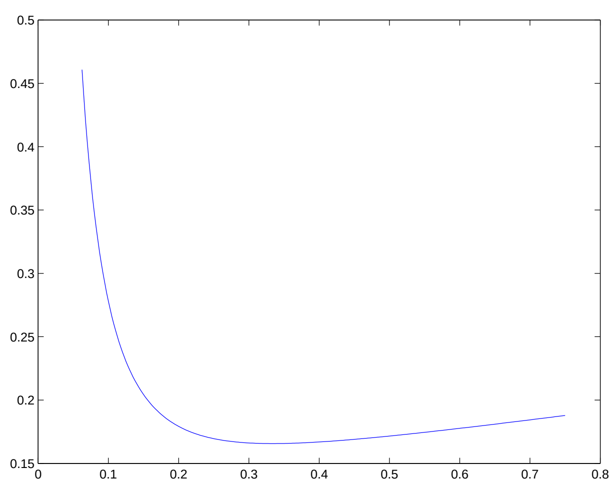


Figure 4.6: The function $\varepsilon \mapsto \widehat{X}_{g_2}(\varepsilon)$

Chapter 5

Applications

In this chapter we show how the theory developed in Chapters 3 and 4 can be applied to the various kinds of damping problems. The applications are grouped into those which are described by the one–dimensional models, and those which are described by the multidimensional problems.

In the case of the one–dimensional models, a more complete analysis of the problems can be given.

In the case of the multidimensional models, the analysis heavily depends on the geometry of the problems, and we only give more abstract results on how these problems fit into our theory.

5.1 General remarks

An application of our theory to concrete problems described by the partial differential equation consists of the following steps.

1. Multiplying by the test function, integrate and then use the boundary conditions to obtain a sesquilinear form representation of the original system of differential equations, i.e. calculate μ, γ and κ.

2. Check that $\kappa > 0$ (the condition $\kappa(x,x) > 0$ can sometimes be bypassed by taking the orthogonal complement of $\mathcal{N}(\kappa)$, as is done in Section 5.3.2). Show that quadratic forms μ, γ and κ do generate bounded operators M and C in some Hilbert space $\mathcal{Y}$. To do this it is enough to check that κ dominates μ and γ, i.e. that there exists $\Delta > 0$ such that $\mu(u,u) \leq \Delta\kappa(u,u)$ and $\gamma(u,u) \leq \Delta\kappa(u,u)$ for all $u \in \mathcal{Y}\setminus\{0\}$. To obtain these inequalities, it usually suffices to apply some Poincaré–type inequality. A brief survey of these inequalities in the connection with the vibrational systems can be found in [CZ93a].

3. Find necessary (and sufficient, if possible) conditions under which the corresponding operator A generates an uniformly exponentially stable semigroup. If one uses Theorem 3.3.9 one can find the spectrum and the eigenfunctions of M by solving the system of (partial) differential equations obtained in the following way:
from the original system of differential equations throw the damping terms out, and if there are damping terms in the boundary conditions, replace these boundary conditions with ones for the free end.

In the light of Remark 3.3.8, one can substitute the term $\|C\phi_n\|$ by $\gamma(\phi_n, \phi_n)$ in (3.3.40). Sometimes, although C cannot be explicitly calculated, one can calculate $\|Cx\|$, for specific x.

4. Choose some appropriate trace class operator K in $\mathcal{Y}_{\mathbb{R}}$ (in the sense of Section 4.1). Choose some appropriate subspace sequence $\mathcal{Y}_n \subset \mathcal{Y}$ such that the corresponding operators A_n satisfy (4.2.3) and (4.2.3), and such that $\widehat{A}_n$ is of the form (4.2.6). Calculate Z_n either by Monte–Carlo method or by the formula given in Section 4.3. By the use of some numerical procedure calculate optimal damping matrices C_n in $\mathcal{Y}$. This gives us a sequence C_n which approximates

an optimal damping operator C in $\mathcal{Y}$ (and hence an optimal damping form γ).

When it comes to the numerical procedures for approximating the continuous systems, we propose the spectral methods (for the basic introduction to these methods, see [Tre00]). Experiments had shown ([DT96], [Tre97]) that the finite difference and finite element methods frequently behave poorly on problems for vibrational systems, in contrast with the spectral methods. Only in the cases of the complex geometry of the problem, the finite element methods should be used.

5.2 One–dimensional problems

5.2.1 Cable with a tip mass

We consider a vertical cable which is pinched at the upper end and with a tip mass attached at the lower end, where a control force linear feedback depending on the velocity is applied.

The dynamics is described by the following system [MRC94] (see also [BX00])

$$
\begin{gathered}
\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0,\ 0 < x < 1,\ t > 0, \\
u(0,t) = 0, \\
\frac{\partial u}{\partial x}(1,t) + m\frac{\partial^2 u}{\partial t^2}(1,t) + \alpha\frac{\partial u}{\partial t}(1,t) = 0, \\
u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}u(x,0) = u_1(x),
\end{gathered}
$$

where we assume $m, \alpha > 0$.

Multiplying the above differential equation by the smooth test function v

such that $v(0) = 0$ and by partial integration we obtain

$$\int_0^1 \frac{\partial^2 u}{\partial t^2}(x,t)v(x)\mathrm{d}x + m\frac{\partial^2 u}{\partial t^2}(1,t)v(1) + \alpha\frac{\partial u}{\partial t}(1,t)v(1) + \int_0^1 \frac{\partial u}{\partial x}(x,t)v'(x)\mathrm{d}x = 0.$$

Hence, the system given above can be written as

$$\mu(\ddot{u}, v) + \gamma(\dot{u}, v) + \kappa(u, v) = 0, \text{ for all } v \in \mathcal{Y}, \tag{5.2.1}$$

where

$$\mu(u,v) = \int_0^1 u(x)\overline{v(x)}\mathrm{d}x + mu(1)\overline{v(1)},$$
$$\gamma(u,v) = \alpha u(1)\overline{v(1)},$$
$$\kappa(u,v) = \int_0^1 u'(x)\overline{v'(x)}\mathrm{d}x,$$

and

$$\mathcal{Y} = \{u \in \mathrm{L}^2([0,1]) : u' \in \mathrm{L}^2([0,1]), u(0) = 0\}.$$

Now from Examples 3.1.1 and 3.1.2 immediately follows that the forms μ, γ and κ give rise to the bounded operators M and C given by

$$(Mu)(x) = \int_0^1 G(x,\xi)u(\xi)\mathrm{d}\xi + mu(1)x,$$
$$(Cu)(x) = \alpha u(1)x,$$

where

$$G(x,\xi) = \begin{cases} x, & x \le \xi, \\ \xi, & x \ge \xi \end{cases}. \tag{5.2.2}$$

The operator M is compact and one can calculate its eigenvalues and eigenfunctions. We obtain that the eigenvalues λ_n are the solutions of the equation

$$m\lambda^{-1/2}\tan\lambda^{-1/2} = 1, \tag{5.2.3}$$

and the corresponding eigenfunctions are

$$u_n(x) = \sin \lambda_n^{-1/2} x. \tag{5.2.4}$$

By a straightforward, but tedious computation we obtain

$$\inf \frac{\|Cu_n\|}{\|Mu_n\|} = 0,$$

hence the energy of the system does not have an uniform exponential decay, which is a well-known fact [LM88].

5.2.2 Vibrating string with viscous and boundary damping

We consider the vibrating string with viscous and boundary damping given by:

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2}(x,t) + \alpha \frac{\partial u}{\partial t}(x,t) - \beta \frac{\partial^3 u}{\partial x^2 \partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0,\ 0 < x < \pi,\ t > 0, \\
u(0,t) = 0, \\
\frac{\partial u}{\partial x}(1,t) + \varepsilon \frac{\partial u}{\partial t}(1,t) = 0, \\
u(x,0) = u_0(x),\ \frac{\partial u}{\partial t} u(x,0) = u_1(x),
\end{gathered} \tag{5.2.5}$$

where we assume $\alpha, \beta, \varepsilon \geq 0$.

Multiplying the above differential equation by the smooth test function v such that $v(0) = 0$, by partial integration, and by differentiating (5.2.5) we obtain

$$\int_0^\pi \frac{\partial^2 u}{\partial t^2}(x,t) v(x) \mathrm{d}x + \alpha \int_0^\pi \frac{\partial u}{\partial t}(x,t) v(x) \mathrm{d}x + \beta \int_0^\pi \frac{\partial^2 u}{\partial x \partial t}(x,t) v'(x) \mathrm{d}x +$$
$$\int_0^\pi \frac{\partial u}{\partial x}(x,t) v'(x) \mathrm{d}x + \varepsilon \frac{\partial u}{\partial t}(\pi,t) v(\pi) + \varepsilon \frac{\partial^2 u}{\partial t^2}(\pi,t) v(\pi) = 0.$$

Hence, the system given above can be written as (5.2.1), where

$$\mu(u,v) = \int_0^\pi u(x)\overline{v(x)}\mathrm{d}x + \varepsilon u(\pi)\overline{v(\pi)},$$

$$\gamma(u,v) = \alpha \int_0^\pi u(x)\overline{v(x)}\mathrm{d}x + \beta \int_0^\pi u'(x)\overline{v'(x)}\mathrm{d}x + \varepsilon u(\pi)\overline{v(\pi)},$$

$$\kappa(u,v) = \int_0^\pi u'(x)\overline{v'(x)}\mathrm{d}x,$$

and

$$\mathcal{Y} = \{u \in \mathrm{L}^2([0,1]) : u' \in \mathrm{L}^2([0,1]), u(0) = 0\}.$$

Now from Examples 3.1.1 and 3.1.2 immediately follows that the forms μ, γ and κ give rise to the bounded operators M and C given by

$$(Mu)(x) = \int_0^1 G(x,\xi)u(\xi)\mathrm{d}\xi + \varepsilon u(1)x,$$

$$(Cu)(x) = \alpha \int_0^1 G(x,\xi)u(\xi)\mathrm{d}\xi + \varepsilon u(1)x + \beta u(x),$$

where the function G is given by (5.2.2).

If $\alpha > 0$ or $\beta > 0$ then the system obviously has uniform decay of energy. In the case $\alpha = 0$, $\beta = 0$ and $\varepsilon > 0$ we are in position of Example 3.1.1, hence we also have an uniform exponential decay of energy.

From the previous subsection we know that the eigenvalues λ_n of M are the solutions of the equation (5.2.3) where instead of m stands ε, and the corresponding eigenfunctions are given by (5.2.4).

Now we will find an approximation of the optimal parameters. We take $\varepsilon = \frac{1}{2}$, and optimize over α and β. We take $N = 30$, $\mathcal{Y}_N = \mathrm{span}\{u_1, \ldots, u_N\}$, where u_i are the eigenfunctions of M. We also choose covariance operator

K such that its (infinite–dimensional) matrix in the basis consisting of the eigenfunctions of M has the form $\mathrm{diag}(K_1, 0, 0, \ldots)$, where K_1 is from Example 4.3.2. It is easy to see that the assumptions from Section 4.2 are satisfied. We calculate

$$A_N(\varepsilon) = \begin{bmatrix} 0 & \Omega_N \\ -\Omega_N & -C_N(\alpha, \beta) \end{bmatrix},$$

where

$$\Omega_N = \mathrm{diag}\left(\tau_1, \ldots, \tau_N\right),$$

with $\tau_1 < \ldots < \tau_N$ positive solutions of the equation

$$\tau \tan \tau - 2 = 0,$$

and

$$C_N(\alpha, \beta) = \alpha \Omega_N^{-2} + \beta I + \frac{1}{2}(1-\alpha) C_0,$$

where

$$(C_0)_{ij} = \frac{\sin \tau_i \pi \sin \tau_j \pi}{\tau_i \tau_j \sqrt{(\frac{1}{2}\pi + \frac{1}{4\tau_i} \sin 2\tau_i \pi)(\frac{1}{2}\pi + \frac{1}{4\tau_j} \sin 2\tau_j \pi)}}.$$

The matrix $A_N(\alpha, \beta)$ is clearly of the form (4.2.6). The function $(\alpha, \beta) \mapsto \mathrm{tr}(X_N(\alpha, \beta) Z_N)$, where $X_N(\alpha, \beta)$ is the solution of the Lyapunov equation

$$A_N(\alpha, \beta)^* X + X A_N(\alpha, \beta) = -I,$$

is plotted on the Figure 5.1.

The optimal damping is attained for $\alpha = 0.5$, $\beta = 0.7$.

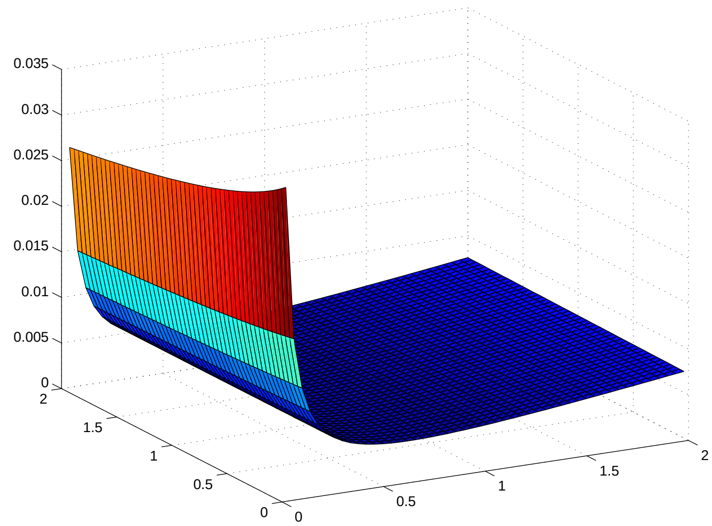


Figure 5.1: The function $(\alpha,\beta)\mapsto \mathrm{tr}(X_N(\alpha,\beta)Z_N)$

5.2.3 Clamped Rayleigh beam with viscous damping

We consider a clamped Rayleigh beam in the presence of viscous damping term $2\frac{\partial}{\partial x}\left(a\frac{\partial^2 u}{\partial t\partial x}\right)$. The corresponding system is given by [Rao97] (see also [Rus86])

$$\frac{\partial^2 u}{\partial t^2}(x,t)-\alpha^2\frac{\partial^4 u}{\partial x^2\partial t^2}(x,t)+\frac{\partial^4 u}{\partial t^4}(x,t)-2\frac{\partial}{\partial x}\left(a\frac{\partial^2 u}{\partial t\partial x}\right)(x,t)=0,\ 0<x<1,\ t>0,$$
$$u(0,t)=\frac{\partial u}{\partial x}(0,t)=u(1,t)=\frac{\partial u}{\partial x}(1,t)=0,$$
$$u(x,0)=u_0(x),\ \frac{\partial u}{\partial t}u(x,0)=u_1(x),$$

where $\alpha^2>0$ is the coefficient of the moment of inertia, and where potential $a\in \mathrm{L}^\infty([0,1])$ is a positive function.

Multiplying the above differential equation by the smooth test function v

such that $v(0) = v(1) = v'(0) = v'(1) = 0$ and by partial integration we obtain

$$\int_0^l \frac{\partial^2 u}{\partial t^2}(x,t)v(x)\mathrm{d}x + \alpha^2 \int_0^1 \frac{\partial^3 u}{\partial t^2 \partial x}(x,t)v'(x)\mathrm{d}x +$$
$$\int_0^1 \frac{\partial^2 u}{\partial x^2}(x,t)v''(x)\mathrm{d}x + 2\int_0^1 a(x)\frac{\partial^2 u}{\partial t \partial x}(x,t)v'(x)\mathrm{d}x = 0.$$

Hence, the system given above can be written as (5.2.1), where

$$\mu(u,v) = \int_0^1 u(x)\overline{v(x)}\mathrm{d}x + \alpha^2 \int_0^1 u'(x)\overline{v'(x)}\mathrm{d}x,$$
$$\gamma(u,v) = 2\int_0^1 a(x)u'(x)\overline{v'(x)}\mathrm{d}x,$$
$$\kappa(u,v) = \int_0^1 u''(x)\overline{v''(x)}\mathrm{d}x,$$

and

$$\mathcal{Y} = \{u \in \mathrm{L}^2([0,1]) : u', u'' \in \mathrm{L}^2([0,1]), u(0) = u'(0) = u(1) = u'(1) = 0\}.$$

Then one can easily see that the forms μ and γ are dominated by κ. $\kappa > 0$ is obvious. The operator M is compact [Rao01, Proposition 2.1].

In the case $a(x) \geq \delta > 0$ the operator C is uniformly positive, and the system has uniform exponential decay of energy.

The eigenvalues λ_n of M are the solutions of the equation

$$\cosh p_\lambda^+ \cos p_\lambda^- \frac{\alpha^2}{2\sqrt{\lambda}} \sinh p_\lambda^+ \sin p_\lambda^- = 1,$$

where

$$p_\lambda^+ = \sqrt{\frac{\alpha^2 + \sqrt{\alpha^4 + 4\lambda}}{2}}, \qquad p_\lambda^- = \sqrt{\frac{\sqrt{\alpha^4 + 4\lambda} - \alpha^2}{2}}.$$

The corresponding eigenfunctions are

$$u_n(x) = \cosh p^+_{\lambda_n} x - \cos p^-_{\lambda_n} x - \frac{\cosh p^+_{\lambda_n} - \cos p^-_{\lambda_n}}{\sinh p^+_{\lambda_n} - \frac{p^+_{\lambda_n}}{p^-_{\lambda_n}} \sin p^-_{\lambda_n}} \left(\sinh p^+_{\lambda_n} x - \frac{\alpha^2}{\sqrt{\lambda_n}} \sin p^-_{\lambda_n} \right).$$

So, in the general case, the system has uniform exponential decay of energy if a is such that

$$\inf \frac{\int\limits_0^1 a(x) u_n'(x)^2 \mathrm{d}x}{\lambda_n \int\limits_0^1 u_n''(x)^2 \mathrm{d}x} > 0,$$

where λ_n and u_n are given above.

5.2.4 Euler–Bernoulli beam with boundary damping

We consider a Euler–Bernoulli beam which is pinched at $x = 0$ and has a damper and the spring attached at $x = 1$. The corresponding equations are

$$\begin{gathered}
\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 u}{\partial x^2} \right)(x,t) = 0, \ t > 0, \ x \in (0,1), \\
u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0, \\
\frac{\partial}{\partial x}\left(EI(x) \frac{\partial^2 u}{\partial x^2} \right)(1,t) = ku(1,t) + c\frac{\partial u}{\partial t}(1,t), \\
\frac{\partial^2 u}{\partial x^2}(1,t) = 0, \\
u(x,0) = u_0(x), \ \frac{\partial u}{\partial t} u(x,0) = u_1(x),
\end{gathered}$$

where $c > 0$ and $k > 0$ are damping and spring coefficient, respectively.

Multiplying the above differential equation by the smooth test function v such that $v(0) = v'(0) = 0$ and by partial integration we obtain

$$\int\limits_0^1 \frac{\partial^2 u}{\partial t^2}(x,t) v(x) \mathrm{d}x + \int\limits_0^1 EI(x) \frac{\partial^2 u}{\partial x^2}(x,t) v''(x) \mathrm{d}x + ku(1,t)v(1) + c\frac{\partial u}{\partial t}(1,t)v(1) = 0.$$

Hence, the system given above can be written as (5.2.1), where

$$\mu(u,v) = \int_0^1 u(x)\overline{v(x)}\mathrm{d}x,$$
$$\gamma(u,v) = cu(1)\overline{v(1)},$$
$$\kappa(u,v) = \int_0^1 EI(x)u''(x)\overline{v''(x)}\mathrm{d}x + ku(1)\overline{v(1)},$$

and

$$\mathcal{Y} = \{u \in \mathrm{L}^2([0,1]) : u', u'' \in \mathrm{L}^2([0,1]), u(0) = u'(0) = 0\}.$$

We assume $EI(x) \geq \delta > 0$. Then one can easily see that the forms μ and γ are dominated by κ. $\kappa > 0$ is obvious. Since M is compact, the eigenvalues λ_n tend to zero for $n \to \infty$. Let us denote the corresponding eigenvalues by u_n. Then

$$\frac{\gamma(u_n,u_n)}{\lambda_n \kappa(u_n,u_n)} = \frac{cu_n(1)^2}{\lambda_n\left(\int_0^1 EI(x)u_n''(x)^2\mathrm{d}x + ku_n(1)^2\right)} \geq \frac{c}{k\lambda_n}.$$

This implies that the system has a uniform exponential decay of energy if $c > 0$.

5.2.5 Euler–Bernoulli beam with Kelvin–Voigt damping

We assume that the Euler–Bernoulli beam is clamped at $x = 0$, and free at $x = l$. The dynamics of transverse vibration is described by the following system [IN97]:

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 u}{\partial x^2} + DI\frac{\partial^3 u}{\partial x^2 \partial t}\right)(x,t) = 0,\ t > 0,\ x \in (0,l),$$

$$u(0,t) = \frac{\partial u}{\partial x}(0,t) = 0,$$

$$EI(x)\frac{\partial^2 u}{\partial x^2}(x,t) + DI(x)\frac{\partial^3 u}{\partial x^2 \partial t}(x,t)\bigg|_{x=l} = 0,$$

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 u}{\partial x^2} + DI\frac{\partial^3 u}{\partial x^2 \partial t}\right)(x,t)\bigg|_{x=l} = 0,$$

$$u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}u(x,0) = u_1(x).$$

In the above system EI is the stiffness coefficient and DI is the damping coefficient of the beam material.

Multiplying the above differential equation by the smooth test function v such that $v(0) = v'(0) = 0$ and by partial integration we obtain

$$\int_0^l \frac{\partial^2 u}{\partial t^2}(x,t)v(x)\mathrm{d}x + \int_0^l DI(x)\frac{\partial^3 u}{\partial x^2 \partial t}(x,t)v''(x)\mathrm{d}x + \int_0^l EI(x)\frac{\partial^2 u}{\partial x^2}(x,t)v''(x)\mathrm{d}x = 0.$$

Hence, the system given above can be written as (5.2.1), where

$$\mu(u,v) = \int_0^l u(x)\overline{v(x)}\mathrm{d}x,$$

$$\gamma(u,v) = \int_0^l DI(x)u''(x)\overline{v''(x)}\mathrm{d}x,$$

$$\kappa(u,v) = \int_0^l EI(x)u''(x)\overline{v''(x)}\mathrm{d}x,$$

and

$$\mathcal{Y} = \{u \in \mathrm{L}^2([0,l]) : u', u'' \in \mathrm{L}^2([0,l]), u(0) = u'(0) = 0\}.$$

Let us assume that the stiffness and damping coefficients satisfy

$$EI(x) \geq \delta > 0 \text{ and } EI \in \mathrm{L}^\infty([0,l]) \text{ and } DI \in \mathrm{L}^\infty([0,l]).$$

Then one can easily see that the forms μ and γ are dominated by κ. $\kappa > 0$ is obvious. Hence we are in position to apply results from the Section 3.3. One can easily see that

$$\|Cu\|^2 = \int_0^l \frac{DI(x)^2}{EI(x)} |u''(x)|^2 \mathrm{d}x.$$

Also, M is obviously a compact operator. Let us denote by λ_n the eigenvalues of M. In the general case, the system need not have a uniform exponential decay of the energy.

In the simplest case $EI(x) = 1$, the eigenvalues λ_n are the solutions of the following equation:

$$(\cosh \lambda l + \cos \lambda l)^2 = 1 + \cosh \lambda l \cos \lambda l,$$

and the eigenfunctions u_n are given by

$$u_n(x) = \cosh \lambda_n x - \cos \lambda_n x + \beta_n (\sinh \lambda_n x - \sin \lambda_n x),$$

where

$$\beta_n = \frac{\sinh \lambda_n l - \sin \lambda_n l}{\cosh \lambda_n l + \cos \lambda_n l}.$$

Then a sufficient condition for the uniform exponential decay of the energy is $DI \geq p > 0$, which easily follows from the Remark 3.3.8.

5.3 Multidimensional problems

5.3.1 Higher dimensional hyperbolic systems

The hyperbolic system which has received the most attention from the controllability viewpoint is the generalized wave equation

$$\rho(x)\frac{\partial^2 u}{\partial t^2}(x,t) - \nabla(A(x)\nabla u)(x,t) + q(x)u(x,t) = 0 \qquad \text{for } x \in \Omega, t \geq 0, \quad (5.3.1)$$

where real valued coefficients in (5.3.1) satisfy the following: $p, q \in \mathrm{L}^\infty(\overline{\Omega})$ and $\rho(x) \geq \rho_0 > 0$, $q(x) \geq 0$ in Ω; $A(x) = (a_{ij}(x))_1^N$ satisfies uniform elipticity condition, its entries have Lipschitz continuous second derivatives, and Ω is a bounded open connected subset in $\mathbb{R}^n$ with the Lipschitz boundary $\partial\Omega$.

We divide the boundary $\partial\Omega$ in two parts, Γ_0 and Γ_1, with $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$ and Γ_0 relatively open in $\partial\Omega$.

We impose the following boundary conditions:

$$u(x,t) = 0,\ x \in \Gamma_0,$$
$$\beta(x)\frac{\partial u}{\partial t}(x,t) + A(x)\nabla u(x,t) \cdot \nu(x) = 0,\ x \in \Gamma_1,$$

where $\beta(x) \geq \beta_0 > 0$ and ν is the unit normal of $\partial\Omega$ pointing towards the exterior of Ω. The problem of decay of the solutions and the problem of finding the optimal β has caught the attention of many researches (see, for example, [Rus78], [Lag83], [Tri89]). By the use of partial integration and divergence theorem, this system can be written in the form (5.2.1) where

$$\mu(u,v) = \int_\Omega u(x)\overline{v(x)}\mathrm{d}x,$$
$$\gamma(u,v) = \int_{\Gamma_1} \beta(x)u(x)\overline{v(x)}\mathrm{d}x,$$
$$\kappa(u,v) = \int_\Omega (\nabla v(x))^* A(x)\nabla u(x)\mathrm{d}x + \int_\Omega q(x)u(x)\overline{v(x)}\mathrm{d}x,$$

and

$$\mathcal{Y} = \{u \in \mathcal{H}^1(\Omega) : u(0) = 0 \text{ on } \Gamma_0\}.$$

It can be shown ([Lag83, Proof of the Theorem 1], [CZ93a, Example 3.5.1] and [Rus78, pp. 682]) that under above assumptions forms μ and γ are dominated by κ. $\kappa > 0$ is obvious. Hence, the results from Chapter 3 are applicable.

Even in the simplest case $A(x) = 1$, $\rho(x) = 1$ and $q(x) = 0$, the exact calculation of the eigenvalues of M is not possible except in the cases when Ω has particulary simple geometry, but even in this cases the computation is usually lengthy and tedious (see, for example, [CFNS91, Section 3.] and [CZ93b, Section 2.6]).

5.3.2 A problem in dissipative acoustics

In this section we are studying linear oscillations of an acoustic (i.e. inviscid, compressible, barotropic) fluid contained in a rigid cavity, with some or all of its walls covered by a thin layer of viscoelastic material able to absorb part of the acoustic energy of the fluid.

In recent years large attention has been paid to this kind of problem, mainly related to the goal of decreasing the level of noise in aircraft or cars (for example, a typical problem in aeronautical engineering is the problem of reducing the noise produced by propellers inside an aircraft by means of thin layers of viscoelastic material). A typical acoustic insulating material is glasswool.

We denote by $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$ the domain occupied by the fluid, which we suppose polyhedral, with boundary $\partial\Omega = \Gamma_A \cup \Gamma_R$, $\Gamma_A = \bigcup_{j=1}^{J} \Gamma_j$, with Γ_j being all the different faces of Ω covered by the damping material, is called the "absorbing boundary". Γ_R is the union of the remaining faces and we call it the "rigid boundary". We assume that Γ_A is not empty. The unit outer normal vector along $\partial\Omega$ is denoted by ν. The equations for our problem are [BDRS00]

$$\rho\frac{\partial^2\mathbf{U}}{\partial t^2}+\nabla P=0 \quad \text{in } \Omega, \tag{5.3.2}$$

$$P=-\rho c^2\operatorname{div}\mathbf{U} \quad \text{in } \Omega, \tag{5.3.3}$$

$$P=\alpha\mathbf{U}\cdot\nu+\beta\frac{\partial\mathbf{U}}{\partial t}\cdot\nu \quad \text{on } \Gamma_A, \tag{5.3.4}$$

$$\mathbf{U}\cdot\nu=0 \quad \text{on } \Gamma_R, \tag{5.3.5}$$

where $\mathbf{U}$ is the displacement vector, P is the fluid pressure, ρ the fluid density, and c the acoustic speed.

The equation (5.3.4) models the effect of viscoelastic material: the fluid pressure on the boundary is in equilibrium with the response of the absorbing walls. This response consists of two terms: the first one is proportional to the normal component of the displacements and acoustics for the elastic behavior of the material, whereas the second one is proportional to the normal velocity and models the viscous damping.

The damped vibration modes of the fluid are complex solutions of (5.3.2)–(5.3.5) of the form $\mathbf{U}(\mathbf{x},t)=e^{\lambda t}\mathbf{u}(\mathbf{x})$ and $P(\mathbf{x},t)=e^{\lambda t}p(\mathbf{x})$. They can be found by solving the following quadratic problem:

$$\int_\Omega \rho c^2\operatorname{div}\mathbf{u}\,\operatorname{div}\overline{\phi}+\int_{\Gamma_A}\alpha\,\mathbf{u}\cdot\nu\,\overline{\phi}\cdot\nu+\lambda\int_{\Gamma_A}\beta\,\mathbf{u}\cdot\nu\,\overline{\phi}\cdot\nu+\lambda^2\int_\Omega\rho\,\mathbf{u}\,\overline{\phi}=0, \qquad \text{for all } \phi\in\mathcal{V}, \tag{5.3.6}$$

where

$$\mathcal{V}=\left\{\phi\in \mathrm{H}(\operatorname{div},\Omega):\phi\cdot\nu\in \mathrm{L}^2(\partial\Omega) \text{ and } \phi\cdot\nu=0 \text{ on } \Gamma_R\right\},$$

endowed with its natural norm

$$\|\phi\|_{\mathcal{V}}=\left(\|\phi\|^2_{\operatorname{div},\Omega}+\|\phi\|^2_{\Gamma_A}\right)^{1/2}.$$

Here

$$\mathrm{H}(\mathrm{div},\Omega)=\{\phi\in\mathrm{L}^2(\Omega):\ \mathrm{div}\phi\in\mathrm{L}^2(\Omega)\}$$

is a Hilbert space (see, for example [GR86]) with the norm

$$\|\phi\|_{\mathrm{div},\Omega}=\left(\|\phi\|_\Omega^2+\|\mathrm{div}\,\phi\|_\Omega^2\right)^{1/2}.$$

Let us define three (symmetric) sesquilinear forms in $\mathcal{V}$

$$\mu(\mathbf{u},\mathbf{v})=\int_\Omega \rho\,\mathbf{u}\,\overline{\mathbf{v}},$$

$$\gamma(\mathbf{u},\mathbf{v})=\int_{\Gamma_A} \beta\,\mathbf{u}\cdot\nu\,\overline{\mathbf{v}}\cdot\nu,$$

$$\kappa(\mathbf{u},\mathbf{v})=\int_\Omega \rho c^2 \mathrm{div}\mathbf{u}\,\mathrm{div}\overline{\mathbf{v}}+\int_{\Gamma_A}\alpha\,\mathbf{u}\cdot\nu\,\overline{\mathbf{v}}\cdot\nu.$$

Then (5.3.6) becomes

$$\lambda^2\mu(\mathbf{u},\mathbf{v})+\lambda\gamma(\mathbf{u},\mathbf{v})+\kappa(\mathbf{u},\mathbf{v})=0,\qquad \text{for all } \mathbf{v}\in\mathcal{V}. \tag{5.3.7}$$

Obviously, $\lambda=0$ is an eigenvalue, with corresponding eigenspace

$$\mathcal{K}=\{\mathbf{u}\in\mathcal{V}:\ \mathrm{div}\,\mathbf{u}=0 \text{ on } \Omega \text{ and } \mathbf{u}\cdot\nu=0 \text{ on } \partial\Omega\},$$

so as the space in which we will operate we take [GR86]

$$\mathcal{Y}:=\mathcal{V}\ominus\mathcal{K}=\{\mathbf{u}\in\mathcal{V}:\ \mathbf{u}=\nabla\varphi \text{ for } \varphi\in\mathrm{H}^1(\Omega)\}.$$

From [BDRS00, Lemma 2.2] follows that the quadratic form κ generates a norm in $\mathcal{Y}$ which is equivalent to $\|\cdot\|_{\mathcal{V}}$. Then clearly the forms μ and γ are dominated by κ, hence the corresponding operators M and C constructed in Section 3.1 are bounded. Moreover, from [BDRS00, Lemma 3.1] follows that M is compact.

Hence we can apply results from Section 3.3 to this problem. In the special cases when the geometry of Ω is simple, the eigenvalues and eigenvectors of

M can be found by the separation of variables technique (for the case when $n = 2$ and Ω is rectangular see [BR99], and for the case when $n = 3$ and Ω is rectangular box see [BHNR01]).

But even in these simple cases, the computation of the operator C is heavily involved, since the operator C cannot be written in a closed form. To gain a little more insight in the structure of C, observe that $\frac{\beta}{\alpha}$ is an eigenvalue of C with the corresponding eigenspace $\mathcal{F} = \{\mathbf{u} \in \mathcal{Y} : \operatorname{div} \mathbf{u} = 0\}$. Also, one can check that $\mathcal{F}^{\perp} = \{\mathbf{u} \in \mathcal{Y} : \mathbf{u} \cdot \nu \text{ is constant on } \Gamma_A\}$, and since $C\mathbf{u}$ depends only on $\mathbf{u} \cdot \nu$ in $\mathcal{F}^{\perp}$, $C_{|\mathcal{F}^{\perp}}$ has one–dimensional range.

Appendix A

Semigroup theory

In the appendix we introduce the basic concepts and results of the semigroup theory which we use in this thesis.

The classic reference on semigroup theory is [HP57]. Standard references are also [Paz83], [EN00], [Gol85] and [BM79].

Let $\mathcal{X}$ be a Hilbert space. The family of bounded linear operators $T(t)$, $t \geq 0$ in $\mathcal{X}$ is said to be a semigroup of operators in $\mathcal{X}$ if

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$.

The semigroup $T(t)$, $t \geq 0$ is said to be strongly continuous if it is continuous in the strong operator topology. Due to the property (ii) this is equivalent to

(iii) $\lim_{t \searrow 0} \|T(t)x - x\| = 0$ for each $x \in \mathcal{X}$.

A strongly continuous semigroup is sometimes called as C_0 semigroup[1].

The infinitesimal generator of $T(t)$, or briefly the generator, is the linear

[1] The symbol C_0 abbreviates "Cesàro summable of order 0".

operator A with domain $D(A)$ defined by

$$D(A) = \{x \in \mathcal{X} : \lim_{t \searrow 0} \frac{1}{t}(T(t)x - x) \text{ exists}\},$$
$$Ax = \lim_{t \searrow 0} \frac{1}{t}(T(t)x - x), \; x \in D(A).$$

The generator is always a closed and densely defined operator. The generator uniquely determines the semigroup.

Proposition A.1 ([EN00]). *For every strongly continuous semigroup $T(t)$, there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \textit{ for all } t \geq 0. \tag{A.1}$$

The infimum of all exponents ω for which an estimate of the form (A.1) holds for a given strongly continuous semigroup plays an important role in the semigroup theory. For a strongly continuous semigroup $T(t)$ generated by A we call

$$\omega(A) = \inf\{\omega : \exists M \geq 1 \text{ such that } \|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0\}$$

its growth bound (or type).

Moreover, a semigroup is called *contractive* if $\omega = 0$ and $M = 1$ is possible, and *uniformly exponentially stable* if its growth bound is negative.

Remark A.1. Let A be generator of strongly continuous semigroup $T(t)$. The number

$$s(A) = \sup\{\text{Re}\lambda : \lambda \in \sigma(A)\}$$

is called the spectral bound of the semigroup $T(t)$. In the finite–dimensional case $s(A) = \omega(A)$ holds, but in general we have only $s(A) \leq \omega(A)$.

In the sequel we will give characterizations of the generators of contractive and uniformly exponentially stable semigroups.

Let A be a linear operator with the dense domain. The operator A is called *dissipative* operator if $\mathrm{Re}(Ax, x) \leq 0$ for all $x \in D(A)$. A dissipative operator which extends a dissipative operator A is called a dissipative extension of A. An operator A is said to be *maximal dissipative* if its only dissipative extension is A itself.

Proposition A.2. *If A is a dissipative operator and $\mathcal{R}(A - \lambda) = \mathcal{X}$ for some λ, $\mathrm{Re}\lambda > 0$, then A is maximal dissipative.*

Theorem A.3 (Lumer–Phillips). *Let A be a linear operator with dense domain. Then A generates a contractive semigroup if and only if A is maximal dissipative.*

Theorem A.4 ([EN00], pp. 302). *A strongly continuous semigroup $T(t)$ in a Hilbert space is uniformly exponentially stable if and only if the half plane $\{\lambda \in \mathbb{C} : \mathrm{Re}\lambda > 0\}$ is contained in the resolvent set $\rho(A)$ of the generator A with the resolvent satisfying*

$$\sup_{\mathrm{Re}\lambda > 0} \|R(\lambda)\| < \infty.$$

From the previous theorem one can obtain

Corollary A.5 ([CHA$^+$87], Theorem 9.6). *If $T(t)$ is a strongly continuous semigroup in a Hilbert space with generator A, then its growth bound is given by*

$$\omega(A) = \inf\{\beta \in \mathbb{R} : \sup_{s \in \mathbb{R}} \|R(\beta + is)\| < \infty\}.$$

The following two results are also needed.

Proposition A.6 ([Gol85]). *Let A generate strongly continuous semigroup $T(t)$. Then $T(t) = 0$ for all $t \geq t_0$, where $t_0 > 0$, if and only if $\sigma(A) = \emptyset$ and there exists a constant M such that*

$$\|R(\alpha + i\beta, A)\| \leq M \max\{1, e^{-\alpha t_0}\}$$

for all $\alpha + i\beta \in \mathbb{C}$.

The following Lemma is a well-known result, but we give the proof because we were not able to find an appropriate reference.

Lemma A.7. *Let $T(t)$ ba a uniformly exponentially stable semigroup with generator A. Assume that*

$$\|R(i\beta)\| \leq M, \quad \textit{for all } \beta \in \mathbb{R},$$

for some $M > 0$. Then $\omega(A) \leq -\frac{1}{M}$.

Proof. From [Wei76, Satz 5.14] follows $\lambda \in \rho(A)$ if $|\mathrm{Re}\lambda| < \frac{1}{M}$ and

$$(\lambda - A)^{-1} = \sum_{n=0}^{\infty} (-1)^n (\mathrm{Re}\lambda)^n (i\mathrm{Im}\lambda - A)^{-n-1}, \text{ for } |\mathrm{Re}\lambda| < \frac{1}{M}.$$

Hence

$$\|(\lambda - A)^{-1}\| \leq \sum_{n=0}^{\infty} |\mathrm{Re}\lambda|^n M^{n+1} = \frac{M}{1 - |\mathrm{Re}\lambda| M}, \text{ for } |\mathrm{Re}\lambda| < \frac{1}{M}.$$

Now Corollary A.5 implies $\omega(A) \leq -\frac{1}{M}$. □

Let A be a linear operator, and let $u_0 \in \mathcal{X}$. Consider the differential equation given by

$$\begin{aligned} \dot{u}(t) &= Au(t) \quad \text{for } t \geq 0, \\ u(0) &= u_0. \end{aligned} \tag{A.2}$$

A function $u : [0, \infty) \to \mathcal{X}$ is called a classical solution of (A.2) if u is continuously differentiable with respect to $\mathcal{X}$, $u(t) \in D(A)$ for all $t \geq 0$ and (A.2) holds.

A continuous function $u : [0, \infty) \to \mathcal{X}$ is called a mild solution of (A.2) if $\int_0^t u(s)\mathrm{d}s \in D(A)$ for all $t \geq 0$ and

$$u(t) = A \int_0^t u(s)\mathrm{d}s + u_0.$$

The following theorem deals with the question of existence and uniqueness of the solution of the problem (A.2), called the abstract Cauchy problem associated to A with the initial value u_0.

Theorem A.8 ([EN00], Propositions 6.2 and 6.4). *Let A be the generator of the strongly continuous semigroup $T(t)$. Then, for every $u_0 \in \mathcal{X}$, the function*

$$u : t \mapsto u(t) := T(t)u_0 \tag{A.3}$$

*is the unique mild solution of (*A.2*).*

*Moreover, for every $u_0 \in D(A)$, the function (*A.3*) is the unique classical solution of (*A.2*).*

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Lebenslauf

Name		Ivica Nakić
Geburt		6.11.1972 in Šibenik, Kroatien
Schulbildung	09/79 - 06/87	Grundschule in Šibenik
	09/87 - 06/91	Gymnasium in Šibenik
Schulabschluß	06/91	Abitur
Studium	10/91 - 12/95	Studium der Mathematik an der Universität Zagreb, Kroatien
Studienabschluß	12/95	Diplom in Mathematik
Nachdiplomstudium	12/96 - 07/98	Nachdiplomstudium an der Universität Zagreb
Nachdiplomstudium_ abschluß	07/98	Magister Diplom in Mathematik
Tätigkeit	03/96 - 10/00	Wissenschaftlicher Mitarbeiter am Lehrgebiet Angewandte Mathematik der Universität Zagreb
	10/00 - 01/02	Wissenschaftlicher Mitarbeiter am Lehrgebiet Mathematische Physik der Fernuniversität Hagen
	seit 01/02	Wissenschaftlicher Mitarbeiter am Lehrgebiet Numerische Mathematik der Universität Zagreb

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