MULTIPLICATIVE PERTURBATIONS OF POSITIVE OPERATORS IN KREIN SPACES

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Introduction

Let A be a positive operator with a nonempty resolvent set in a Krein space $\mathcal{K}$. Then A has spectral function with 0 and ∞ being the only possible critical points, see [9]; if neither of these points is a singular critical point then A is similar to a Hilbert space selfadjoint operator, that is, it is a scalar operator with real spectrum (see [9] for the definition and properties of the Krein space operators).

The problem of persistence of nonsingularity of critical points has been started by one of the authors of the present note ([13, 14]) and later continued by a number of authors ([6, 2, 3, 10] etc.). All these references deal with additive perturbations $A+V$ of A; such results are insufficient in some cases, e.g. in the case of elliptic operators with indefinite weights, where

$$A = \frac{1}{\rho}L,$$

L is an elliptic operator and ρ is a real valued function which is not of constant sign. If the weight ρ is perturbed into $\tilde{\rho}$ with

$$|\tilde{\rho} - \rho| \leq \varepsilon|\rho|, \ \varepsilon < 1$$

we have to investigate multiplicative perturbation $(I+V)A$ of A. We prove a result which ensures the persistence of nonsingularity of critical points if the perturbation is sufficiently small as well as the analyticity of certain operators associated with the analytic family $(I + \varepsilon X)A$. The proof goes via the construction of the signum operator, similar to that in [13, 14], our situation is much more singular so that the above estimate for the perturbed weight has to be completed by an additional one for the derivative.

We mention the recent note [5] where the persistence of nonsingularity of the critical point ∞ under multiplicative perturbations is considered. This is in contrast with the present work where we prove the regularity of both critical points 0 and ∞.

The results

Let $(\mathcal{K}, [\cdot|\cdot])$ be a Krein space, J a fundamental symmetry in $\mathcal{K}$. Then $\mathcal{H} = (\mathcal{K}, [J\cdot|\cdot])$ is a Hilbert space; let $(u|v) = [Ju|v]$ be the corresponding scalar product and $\|u\|$ the corresponding norm. Let A be a selfadjoint operator in $\mathcal{K}$ such that the following assumption is satisfied:

(A1) A is a strictly positive operator in $\mathcal{K}$ with a nonempty resolvent set.

It follows from (A1) that the form $a = [A\cdot|\cdot]$ defined on $\mathcal{D}(A) \times \mathcal{D}(A)$ has a closure a in $\mathcal{K}$; its domain $\mathcal{D}(a)$ coincides with the domain $\mathcal{D}((JA)^{1/2})$ of

$$M := (JA)^{1/2} \tag{1}$$

in $\mathcal{H}$. As noted in the Introduction, from (A1) it follows that A has spectral function E in the sense of [9] with the only possible critical points being 0 and ∞. Next assumption excludes the possibility that either of these points is a singular critical point:

(A2) 0 and ∞ are not singular critical points of A.

It follows from (A2) that the projections $E[0,\infty)$ and $E(-\infty,0)$ are well defined, and hence also

$$P = sgnA = E[0,\infty) - E(-\infty,0).$$

Then (see [13] or [6])

$$P = -\frac{1}{i\pi}\, w - \lim_{\substack{r \to 0 \\ R \to \infty}} \left(\int_r^R dt + \int_{-R}^{-r} dt\right)(it - A)^{-1}. \tag{2}$$

Moreover, P commutes with A and it is a uniformly positive bounded operator, i.e. the space $(\mathcal{K}, < \cdot|\cdot >)$ with

$$< u|v >= [Pu|v]$$

is a Hilbert space with the norm $|||\cdot|||$, equivalent to $\|\cdot\|$. Then A is selfadjoint in $(\mathcal{K}, <\cdot|\cdot>)$; the operator

$$\hat{A} = (JP)^{1/2} A (JP)^{-1/2}$$

is selfadjoint in $\mathcal{H}$. Note also that by $P^2 = 1$ we have $\|P\| \geq 1$.

Lemma 1 *For any bounded measurable function f we have*

$$\|f(A)\| \leq \|f\|_\infty \|P\|$$

Proof. Since $\tilde{A}$ is selfadjoint in $\mathcal{H}$ we have

$$\|f(A)\| = \|(JP)^{-1/2} f(\tilde{A})(JP)^{1/2}\| \leq \|(JP)^{-1/2}\| \|f(\tilde{A})\| \|(JP)^{1/2}\|$$

$$\leq \|f\|_\infty \|JP\|^{1/2} \|P^{-1}J\|^{1/2} \leq \|f\|_\infty \|P\|^{1/2} \|P^{-1}\|^{1/2} = \|f\|_\infty \|P\|$$

where we have used the unitarity of J and the fact that $P^{-1} = P$. QED.

Let X be an operator in $\mathcal{K}$ with the properties

(X1) X is selfadjoint and bounded in $\mathcal{K}$

(X2) $JX = XJ$.

hold. From (X1) and (X2) it follows that X is selfadjoint also in $\mathcal{H}$. Next assumptions connect A, M (from (1)) and X :

(AX1) $X\mathcal{D}(M) \subset \mathcal{D}(M)$

(AX2) There exists $C > 0$ such that

$$\|MXu - XMu\| \leq C\|Mu\| \qquad (u \in \mathcal{D}(M)).$$

Since A is positive, M is injective and (AX2) is implied by (X1) and

Proposition 2 *Let*

$$A_1 = (1 + X)A$$

where A, X are as above and

$$\|X\| < 1/\|P\|$$

Then A_1 has a non-void resolvent set and is a positive selfadjoint operator in the Krein space $(\mathcal{K}, [\cdot|\cdot]_1)$ where

$$[\cdot|\cdot]_1 = [(1 + X)^{-1} \cdot |\cdot] = (J(1 + X)^{-1} \cdot |\cdot)$$

Proof. $1+X$ is bicontinuous in $\mathcal{H}$ and thus A_1 is selfadjoint and positive in $(\mathcal{K}, [\cdot|\cdot]_1)$. Also, by Lemma 1. for any real $\eta \neq 0$ we have

$$\|X(A(i\eta - A)^{-1}\| \leq \|X\| \sup_{t\in\mathbf{R}} |\frac{t}{i\eta - t}| = \|X\|\|P\| < 1$$

and $i\eta$ belongs to the resolvent set of A for all $\eta \in \mathcal{R}\backslash\{0\}$. QED.

Our results are summarized in the following theorem.

Theorem 3 *Assume (A1), (A2),(X1),(X2),(AX1) and (AX2). Let also* $\|X\| + C < 1$. *Then for* $|\varepsilon| \leq 1$ *the operator* $1 + \varepsilon X$ *has a bounded inverse, the operator* A_1 *is a positive selfadjoint operator in* $\mathcal{K}_1$ *with a nonempty resolvent set. Neither* 0 *nor* ∞ *is a singular critical point of* A_1, *the operator* A_1 *is similar to a selfadjoint operator in* $\mathcal{H}$. *The operator*

$$P_1 = -\frac{1}{i\pi}\ w - \lim_{\substack{r\to 0 \\ R\to\infty}} (\int_r^R dt + \int_{-R}^{-r} dt)(it - A_1)^{-1} \tag{3}$$

is a positive definite operator in $\mathcal{K}_1$ *and* $P_1^2 = I$. *The space* $(\mathcal{K}, <\cdot|\cdot>_1)$, *where*

$$<u|v>_1 = [P_1 u|v]_1,$$

is a Hilbert space. The operator A_1 *is selfadjoint in* $(\mathcal{K}, <\cdot|\cdot>_1)$. *The operator* $J(I+X)^{-1}P_1$ *is boundedly invertible and positive definite in* $\mathcal{H}$. *The operator* $(J(I+X)^{-1}P_1)^{1/2}A_1(J(I+X)^{-1}P_1)^{-1/2}$ *is selfadjoint in* $\mathcal{H}$.

Proof. We embed the operator A_1 in the family

$$A(\varepsilon) = (I + \varepsilon X)A$$

and accordingly define $\mathcal{K}_\varepsilon$, $[\cdot|\cdot]_\varepsilon$, $<\cdot|\cdot>_\varepsilon$ and $P(\varepsilon)$. For $\eta \in \mathbf{R}\backslash\{0\}$ set .

$$F(\eta, \varepsilon) := (i\eta - \varepsilon A)^{-1} - (i\eta A)^{-1} = \sum_{k=0}^{\infty} \varepsilon^k F_k(\eta, \varepsilon) \tag{4}$$

$$F_k(\eta) := (i\eta - A)^{-1}[XA(i\eta - A)^{-1}]^k \quad (\eta \in \mathbf{C}\backslash\mathbf{R}). \tag{5}$$

This series converges whenever

$$|\varepsilon|\|XA(i\eta - A)^{-1}\| < 1, \eta \in \mathbf{R}\backslash\{0\} \tag{6}$$

Moreover, it follows from (6) and (4) that F is analytic in η and ε for $\eta \in \mathbf{C}\backslash\mathbf{R}$ and ε such that (4) holds. From the selfadjointness of A and (X2) it follows

$$\begin{aligned}(F_{k+1}(\eta)x|y) &= ((i\eta - A)^{-1}X[A(i\eta - A)^{-1}X]^k A(i\eta - A)^{-1}x|y) \\ &= (X[A(i\eta - A)^{-1}X]^k A(i\eta - A)^{-1}x|J(-i\eta - A)^{-1}Jy).\end{aligned}$$

From $A = JM^2$ we find $[A(i\eta - A)^{-1}X]^k = J[M^2(i\eta - JM^2)^{-1}XJ]^k J$ and therefore by (X2)

$$\begin{aligned}&(F_{k+1}(\eta)x|y) = \\ &(X[M^2(i\eta - JM^2)^{-1}XJ]^k JA(i\eta - A)^{-1}x|(-i\eta - A^{-1})Jy) \\ &= ([M^2(i\eta - JM^2)^{-1}XJ]^k M^2(i\eta - A)^{-1}x|X(-i\eta - A)^{-1}Jy).\end{aligned}$$

From $M\mathcal{D}(A) \subset \mathcal{D}(M)$ it follows that $\mathcal{R}(M(i\eta - A)^{-1}) \subset \mathcal{D}(M)$ and consequently

$$\begin{aligned}&[M^2(i\eta - JM^2)^{-1}XJ]^k M^2(i\eta - A)^{-1} = \\ &M[M(i\eta - JM^2)^{-1}XJM]^k M(i\eta - A)^{-1}.\end{aligned}$$

This implies

$$\begin{aligned}&(F_{k+1}(\eta)x|y) = \\ &(G(\eta)^k M(i\eta - A)^{-1}x|(MX(-i\eta - A)^{-1}Jy).\end{aligned} \tag{7}$$

with

$$G(\eta) = M(i\eta - JM^2)^{-1}JXM.$$

Note that $G(\eta)$ is defined on $\mathcal{D}(M)$ and leaves that space invariant. Our goal is to estimate the norm of $G(\eta)$.

Lemma 4 *For every $x \in \mathcal{D}(M)$ and every $\eta \in \mathbf{R}\backslash\{0\}$ we have*

$$\|MJ(i\eta - M^2J)^{-1}Mx\| \leq \|x\| \tag{8}$$

Proof. Set $M_\lambda = MJ(i\eta - M^2J)^{-1}M$. If M is bounded, then $M_\lambda = MJM(i\eta - MJM)^{-1}$ and the assertion follows immediately from the spectral calculus of the selfadjoint operator MJM. Our proof will follow the same pattern and will, in fact, construct a "selfadjoint realization" of the formal expression MJM. We thus consider the operator

$$R_{i\eta} = M(i\eta - JM^2)^{-1}M^{-1} \;:\; \mathcal{D}(M^{-1}) \to \mathcal{D}(M^{-1}),\; \eta \in \mathbf{R}\backslash\{0\}$$

It is immediately verified that $R_{i\eta}$ satisfies the resolvent equation. For $x \in \mathcal{D}(M^{-1})$:

$$(R_{i\eta} - R_{i\eta'})x = i(\eta - \eta')R_{i\eta}R_{i\eta'}x \tag{9}$$

Furthermore, for $x \in \mathcal{D}(M^{-1})$, $y \in \mathcal{D} = \mathcal{D}(M) \cap \mathcal{D}(M^{-1})$ and $y' = M^{-1}y$ we have

$$\begin{gathered}(R_{i\eta}x|y) = (M^{-1}x|(-i\eta - M^2J)^{-1}My) = \\ (M^{-1}x|(-i\eta - M^2J)^{-1}M^2y') = \\ (M^{-1}x|J(-i\eta - JM^2)^{-1}JM^2y') = \\ (M^{-1}x|M^2(-i\eta - JM^2)^{-1}y') = (x|R_{-i\eta}y)\end{gathered} \tag{10}$$

In particular, all $R_{i\eta}$ leave $\mathcal{D}$ invariant and commute there. We can set

$$A_\eta = \frac{R_{i\eta} + R_{-i\eta}}{2},\ B_\eta = \frac{R_{i\eta} - R_{-i\eta}}{2i},$$

Obviously these are commuting symmetric operators in $\mathcal{H}$, defined on the dense subspace $\mathcal{D}$ and leaving this subspace invariant. For $x \in \mathcal{D}$ the resolvent equation (9) gives

$$B_\eta x = \eta(A_\eta - iB_\eta)(A_\eta + iB_\eta)x = \eta(A_\eta^2 + B_\eta^2)x \tag{11}$$

Taking e.g. $\eta > 0$ and applying the Schwarz inequality this gives

$$\eta(B_\eta x|B_\eta x) \leq (B_\eta x|x) \leq \|B_\eta x\|^2$$

we see that A_η is bounded, and similarly for $\eta < 0$. By (11) the same follows for B_η and then also for $R_\eta|_{\mathcal{D}}$ whose closure $\tilde{R}_{i\eta}$ is a pseudoresolvent and its null space $\mathcal{N}$ is known to be independent of η By (10) we have $\tilde{R}^*_{i\eta} = \tilde{R}_{-i\eta}$ and thus $\tilde{R}_{i\eta}$ leaves invariant both $\mathcal{N}$ and $\mathcal{N}^\perp$. Thus, there is a unique selfadjoint operator H_0 in the Hilbert space $\mathcal{N}^\perp \subset \mathcal{H}$ such that

$$R_{i\eta} = (i\eta - H_0)^{-1}P \tag{12}$$

where P is the orthogonal projection onto $\mathcal{N}^\perp$.

We now connect $R_{i\eta}$ with M_η. For $x, y \in \mathcal{D}(M)$ we have

$$(M_\eta x|y) = (x|M_{-\eta}y)$$

thus, M_η is closable on $\mathcal{D}(M)$. For $x \in \mathcal{D}$ we have

$$M_\eta x = MJ(i\eta - M^2J)^{-1}x - i\eta M(i\eta - JM^2)^{-1}M^{-1}x + i\eta R_{i\eta}x$$

$$= M(i\eta - jM^2)^{-1}(JMx - i\eta M^{-1}x) + i\eta R_{-i\eta}x$$

By setting $x = Mx'$ we have $x' \in \mathcal{D}(M^2)$ and

$$M_\eta = M(i\eta - JM^2)^{-1}(JM^2 - i\eta)x' i\eta R_{-i\eta}x =$$

$$(-1 + i\eta\tilde{R}_{i\eta})x = (-1 + i\eta(i\eta - H_0^{-1})P)x$$

Thus,

$$\|M_\eta x\|^2 = \|(1-P)x\|^2 + \|[1 - i\eta(i\eta - H_0)^{-1}]\|^2$$
$$= \|(1-P)x\|^2 + \|H_0(i\eta - H_0)^{-1}Px\|^2$$
$$\leq \|(1-P)x\|^2 + \|Px\|^2 = \|x\|^2$$

Since M_η is closable on $\mathcal{D}(M)$ and $\mathcal{D} \subset \mathcal{D}(M)$ is dense in $\mathcal{H}$ the inequality above extends to all $x \in \mathcal{D}(M)$. QED.

Lemma 5 *For all $\eta \in \mathbf{R}\backslash\{0\}$ and all $x \in \mathcal{D}(M)$ we have*

$$\|G(\eta)x\| \leq (\|X\| + C)\|x\|$$

Proof. It is obviously enough to prove the same identity for the formal adjoint

$$\overline{G}(\eta) = MXJ(-i\eta - M^2J)^{-1}M \;:\; \mathcal{D}(M) \to \mathcal{D}(M)$$

which has the property

$$(G(\eta)x|y) = (x|\overline{G}(\eta)y),\; x, y \in \mathcal{D}(M).$$

We have by Lemma 4 and (AX2)

$$\|\overline{G}(\eta)\| \leq \|(MX - XM)J(-i\eta - M^2J)^{-1}My\| +$$

$$\|XMJ(-i\eta - M^2J)^{-1}My\| \leq (C + \|X\|)\|y\|$$

QED.

We now continue with the proof of our theorem. From (5) and Lemma 5 it follows

$$\begin{aligned} |(F_{k+1}(\eta)x|y)| &\leq \|G(\eta)^k\|\; \|M(i\eta - A)^{-1}x\|\; \|MX(-i\eta - A)^{-1}Jy\| \\ &\leq \gamma^k\|M(i\eta - A)^{-1}x\|\; \|MX(-i\eta - A)^{-1}Jy\| \end{aligned}$$

with $\gamma = \|X\| + C$. Therefore

$$|\int (F_{k+1}(\eta)x|y)d\eta|^2 \leq$$

$$\gamma^{2k}\int\|M(i\eta-A)^{-1}x\|^2d\eta\int\|MX(-i\eta-A)^{-1}Jy\|^2d\eta$$

To evaluate the first integral on the RHS, we note the identity

$$\int\|M(i\eta-A)^{-1}x\|^2d\eta=\int((-i\eta-A)^{-1}M^2(i\eta-A)^{-1}x|x)d\eta$$

$$=\int[A(i\eta-A)^{-1}x|(i\eta-A)^{-1}x]d\eta,$$

i.e.

$$\int\|M(i\eta-A)^{-1}x\|^2d\eta=$$

$$\int\|M(-i\eta-A)^{-1}x\|^2d\eta=\int[A(\eta^2+A^2)^{-1}x|x]d\eta.$$

The assumption (AX2) yields

$$\|MX(-i\eta-A)^{-1}Jy\|\leq(\|X\|+C)\|M(-i\eta-A)^{-1}Jy\|,$$

hence

$$\int\|MX(-i\eta-A)^{-1}Jy\|^2d\eta$$

$$\leq(\|X\|+C)^2\int\|M(-i\eta-A)^{-1}Jy\|^2d\eta$$

$$=(\|X\|+C)^2\int[A(\eta^2+A^2)^{-1}Jy|Jy]d\eta.$$

For a measurable set $S\subset\mathbf{R}$, $z\in\mathcal{H}$, we set $I_s(z)=\int_S[A(y^2+A^2)^{-1}z|z]d\eta$. We have proved

$$|\int_S(F_{k+1}(\eta)x|y)d\eta|^2\leq\gamma^{2k}(\|X\|+C)^2I_S(X)I_S(Jy),$$

hence

$$|\int_S(F_{k+1}(y)x|y)d\eta|\leq\gamma^k(\|X\|+C)(I_S(x)+I_S(Jy)). \tag{13}$$

Inserting this into (7), we obtain

$$|\int_S(F(\eta,\varepsilon)x|y)d\eta|\leq\sum_{k=1}^{\infty}\varepsilon^k\gamma^{k-1}(\|X\|+C)(I_S(x)+I_S(Jy))$$

$$=\frac{\varepsilon}{1-\varepsilon\gamma}(\|X\|+C)(I_S(x)+I_s(Jy))$$

as soon as $|\varepsilon| < \dfrac{1}{K(\|X\| + C)}$ and in particular for $|\varepsilon| \leq 1$.

We use this estimate for $S = [-R, -r] \cup [r, R]$. Spectral calculations then yield

$$I_S(z) = [f(r, R; A)z|z]$$

where

$$f(r, R; t) = 2(arctg\,\frac{R}{|t|} - arctg\,\frac{r}{|t|})sgn\,t$$

It follows

$$|\lim_{\substack{r \to 0 \\ R \to \infty}} (\int_r^R d\eta + \int_{-R}^{-r} d\eta)([(i\eta - A(\varepsilon))^{-1} - (i\eta - A)^{-1}]x|y)|$$

$$\leq C\varepsilon(\|x\|^2 + \|y\|^2).$$

This proves

1. the existence of the limit (3) and its analyticity (as a bounded operator) in ε for $|\varepsilon| \leq \gamma$
2. $P(\varepsilon)$ is selfadjoint in $\mathcal{K}_\varepsilon$, $[\cdot|\cdot]_\varepsilon$ for ε real and $|\varepsilon| \leq \gamma$.
3. By continuity $(J(1 + \varepsilon X)P(\varepsilon)$ is positive definite in $\mathcal{H}$ for ε real and $|\varepsilon|$ small enough. For such ε the operator

   $$(J(I + X)^{-1}P_1)^{1/2}A_1(J(I + X)^{-1}P_1)^{-1/2}$$

   is selfadjoint in $\mathcal{H}$ and thus $P(\varepsilon) = 1$.
4. By the analyticity $P(\varepsilon) = 1$ extends to all ε for $|\varepsilon| \leq \gamma$. The positive definiteness of $(J(1 + \varepsilon X)P(\varepsilon)$ thus extends to all such real ε. Those include $\varepsilon = 1$.

QED.

Application

Let $\mathcal{H} = L^2(\mathbf{R}^n), \mathcal{D}(A) = H^2(\mathbf{R}^n)$,

$$Au = -sgn\, x_n\, \Delta u\, (u \in H^2(\mathbf{R}^n)),$$

Set $Ju = (sgn\, x_n)u$, $[u|v] = \int_{\mathbf{R}^n} u\bar{v} sgn\, x_n dx$. Then A is similar to a selfadjoint operator and (A1), (A2) are satisfied, see [4]. Let X be the operator of multiplication by a measurable real valued function ρ on $\mathbf{R}^n$ such that

$$\rho \in L^\infty(\mathbf{R}^n), \tag{14}$$

Then (X1) and (X2) are satisfied. Note that $\mathcal{D}(M) = \Delta)^{1/2}) = \{u \in L^2 : \text{grad } u \in L^2\} = H^1(\mathbf{R}^n)$. If

$$\partial_j \rho \in L^n(\mathbf{R}^n), \quad j = 1, 2, 3, \tag{15}$$

then

$$\|\partial_j(\rho u)\|_2 \leq \|\rho\|_\infty \|\partial_j u\|_2 + \|u(\partial_j \rho)\|_2 \leq$$
$$\|\rho\|_\infty \|\partial_j u\|_2 + \|\partial_j \rho\|_n \|u\|_{2n/(n-2)}.$$

Since $\|u\|_{2n/(n-2)} \leq K\|\text{grad } u\|_{L^2}$ (see [1], we obtain

$$\|\partial_j(\rho u)\|_2 \leq (\|\rho\|_\infty + \|\partial_j \rho\|_n)\|\text{grad } u\|_2.$$

From $\frac{1}{K}\|Mu\| \leq \|\text{grad } u\| \leq K\|Mu\|$ and from (14), (15) it follows that $XH^1(\mathbf{R}^n) \subset H^1(\mathbf{R}^n)$ and $\|MXu\| \leq K(\|\rho\|_{L^\infty} + \|\text{grad } \rho\|_n)\|Mu\|$, hence (AX1) and (AX2) are also satisfied.

Hence for ε real, $|\varepsilon|$ sufficiently small, the operator

$$A(\varepsilon) = -(1 + \varepsilon\rho)\, \text{sgn}\, x_n\, \Delta$$

is similar to a selfadjoint operator in $L^2(\mathbf{R}^n)$; moreover the similarity operator can be chosen to be analytic in ε.
If $n = 3$, instead of (15) one can assume

$$|\partial_j \rho(x)| \leq \frac{K}{|x|}, \quad j = 1, 2, 3, \quad x \in \mathbf{R}^n \backslash \{0\}, \tag{16}$$

since the estimate (4.6) in [7, ∮ VI. 4] (or [11], the lemma after T. X.18) implies $\|u(\partial_j \rho)\|_2 \leq K\|u\|_2$.

Further sufficient conditions for (AX1) and (AX2) can be deduced from [8, pp. 275-277]. In fact, if (16) holds then (AX1) and (AX2) are satisfied if $\|(\partial_j \rho)M^{-1}u\| \leq K\|u\|$ for $M = (-\Delta)^{1/2}$ and all $u \in H^1(\mathbf{R}^n)$. It is sufficient that $\|(\partial_j \rho)M^{-2}(\partial_k \rho)\| \leq K$ for all $j, k \in \{1, \ldots, n\}$. In [8] sufficient conditions on multiplication operators V and W are given in order that $V\Delta^{-1}W$ be a bounded operator in $L^2(\mathbf{R}^n)$.

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