THE TAN 2Θ THEOREM FOR INDEFINITE QUADRATIC FORMS

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ABSTRACT. A version of the Davis-Kahan Tan 2Θ theorem [SIAM J. Numer. Anal. **7** (1970), 1 – 46] for not necessarily semibounded linear operators defined by quadratic forms is proven. This theorem generalizes a recent result by Motovilov and Selin [Integr. Equat. Oper. Theory **56** (2006), 511 – 542].

1. INTRODUCTION

In the 1970 paper [3] Davis and Kahan studied the rotation of spectral subspaces for 2×2 operator matrices under off-diagonal perturbations. In particular, they proved the following result, the celebrated "Tan 2Θ theorem": Let $A_\pm$ be strictly positive bounded operators in Hilbert spaces $\mathfrak{H}_\pm$, respectively, and W a bounded operator from $\mathfrak{H}_-$ to $\mathfrak{H}_+$. Denote by

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} \quad \text{and} \quad B = A + V = \begin{pmatrix} A_+ & W \\ W^* & -A_- \end{pmatrix}$$

the block operator matrices with respect to the orthogonal decomposition of the Hilbert space $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. Then

tan2 (1.1) $$\|\tan 2\Theta\| \le \frac{2\|V\|}{d}, \qquad \mathrm{spec}(\Theta) \subset [0, \pi/4),$$

where Θ is the operator angle between the subspaces $\mathrm{Ran}\,\mathsf{E}_A(\mathbb{R}_+)$ and $\mathrm{Ran}\,\mathsf{E}_B(\mathbb{R}_+)$ and

$$d = \mathrm{dist}(\mathrm{spec}(A_+), \mathrm{spec}(-A_-))$$

(see, e.g., [8]).

Estimate (1.1) can equivalently be expressed as the following inequality for the norm of the difference of the orthogonal projections $P = \mathsf{E}_A(\mathbb{R}_+)$ and $Q = \mathsf{E}_B(\mathbb{R}_+)$:

tan:2:Theta (1.2) $$\|P - Q\| \le \sin\left(\frac{1}{2} \arctan \frac{2\|V\|}{d}\right),$$

which, in particular, implies the estimate

a1 (1.3) $$\|P - Q\| < \frac{\sqrt{2}}{2}.$$

Independently of the work of Davis and Kahan, inequality (1.3) has been proven by Adamyan and Langer in [1], where the operators $A_\pm$ were allowed to be semibounded. The case $d = 0$ has been considered in the work [9] by Kostrykin, Makarov, and Motovilov. In particular, it was proven that there is a unique orthogonal projection Q from the operator interval $[\mathsf{E}_B((0,\infty)), \mathsf{E}_B([0,\infty))]$ such that

$$\|P - Q\| \le \frac{\sqrt{2}}{2},$$

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where $P \in [\mathsf{E}_A((0,\infty)), \mathsf{E}_A([0,\infty))]$ is the orthogonal projection onto the invariant (not necessary spectral) subspace $\mathcal{H}_+ \subset \mathcal{H}$ of the operator A. A particular case of this result has been obtained earlier by Adamyan, Langer, and Tretter, in [2]. Recently, a version of the Tan 2Θ Theorem for off-diagonal perturbations V that are relatively bounded with respect to the diagonal operator A has been proven by Motovilov and Selin in [11].

In the present work we obtain several generalizations of the aforementioned results assuming that the perturbation is given by an off-diagonal symmetric form.

Given a sesquilinear symmetric form $\mathfrak{a}$ and a self-adjoint involution J such that the form $\mathfrak{a}_J[x,y] := \mathfrak{a}[x,Jy]$ is a positive definite and

$$\mathfrak{a}[x,Jy] = \mathfrak{a}[Jx,y],$$

we call a symmetric sesquilinear form $\mathfrak{v}$ off-diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ with $\mathfrak{H}_\pm = \mathrm{Ran}(I \pm J)$ if

$$\mathfrak{v}[Jx,y] = -\mathfrak{v}[x,Jy].$$

Based on a close relationship between the symmetric form $\mathfrak{a}[x,y] + \mathfrak{v}[x,y]$ and the sectorial sesquilinear form $\mathfrak{a}[x,Jy] + \mathrm{i}\mathfrak{v}[x,Jy]$ (cf. [11], [13]), under the assumption that the off-diagonal form $\mathfrak{v}$ is relatively bounded with respect to the form $\mathfrak{a}_J$, we prove

(i) an analogue the First Representation Theorem for block operator matrices defined as not necessarily semibounded quadratic forms,
(ii) a relative version of the Tan 2Θ Theorem.

We also provide several versions of the relative Tan 2Θ Theorem in the case where the form $\mathfrak{a}$ is semibounded.

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sec:off 2. THE FIRST REPRESENTATION THEOREM FOR OFF-DIAGONAL FORM PERTURBATIONS

To introduce the notation, it is convenient to assume the following hypothesis.

hh1 **Hypothesis 2.1.** *Let $\mathfrak{a}$ be a symmetric sesquilinear form on $\mathrm{Dom}[\mathfrak{a}]$ in a Hilbert space $\mathfrak{H}$. Assume that J is a self-adjoint involution such that*

$$J\,\mathrm{Dom}[\mathfrak{a}] = \mathrm{Dom}[\mathfrak{a}].$$

Suppose that

$$\mathfrak{a}[Jx,y] = \mathfrak{a}[x,Jy] \quad \textit{for all} \quad x,y \in \mathrm{Dom}[\mathfrak{a}_J] = \mathrm{Dom}[\mathfrak{a}],$$

and that the form $\mathfrak{a}_J$ given by

$$\mathfrak{a}_J[x,y] = \mathfrak{a}[x,Jy], \quad x,y \in \mathrm{Dom}[\mathfrak{a}_J] = \mathrm{Dom}[\mathfrak{a}].$$

is a positive definite closed form. Denote by $m_\pm$ the greatest lower bound of the form $\mathfrak{a}_J$ restricted to the subspace

$$\mathfrak{H}_\pm = \mathrm{Ran}(I \pm J).$$

Definition 2.2. *Under Hypothesis 2.1, a symmetric sesquilinear form* $\mathfrak{v}$ *on* $\mathrm{Dom}[\mathfrak{v}] \supset \mathrm{Dom}[\mathfrak{a}]$ *is said to be off-diagonal with respect to the orthogonal decomposition*

$$\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$$

if

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy], \quad x, y \in \mathrm{Dom}[\mathfrak{a}].$$

If, in addition,

xyz (2.1)
$$v_0 := \sup_{0 \neq x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_J[x]} < \infty, \tag{2.1}$$

the form $\mathfrak{v}$ *is said to be an* $\mathfrak{a}$*-bounded off-diagonal form.*

rem:4 **Remark 2.3.** *If* $\mathfrak{v}$ *is an off-diagonal symmetric form and* $x = x_+ + x_-$ *is a unique decomposition of an element* $x \in \mathrm{Dom}[\mathfrak{a}]$ *such that* $x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}]$, *then*

tog (2.2)
$$\mathfrak{v}[x] = 2\mathrm{Re}\, \mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}]. \tag{2.2}$$

Moreover, if $v_0 < \infty$, *then*

vad (2.3)
$$|\mathfrak{v}[x]| \leq 2v_0 \sqrt{\mathfrak{a}_J[x_+]\mathfrak{a}_J[x_-]}. \tag{2.3}$$

Proof. To prove (2.2), we use the representation

$$\mathfrak{v}[x] = \mathfrak{v}[x_+ + x_-, x_+ + x_-] = \mathfrak{v}[x_+] + \mathfrak{v}[x_-] + \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+], \quad x \in \mathrm{Dom}[\mathfrak{a}].$$

Since $\mathfrak{v}$ is an off-diagonal form, one obtains that

$$\mathfrak{v}[x_+] = \mathfrak{v}[x_+, x_+] = \mathfrak{v}[Jx_+, Jx_+] = -\mathfrak{v}[x_+, x_+] = -\mathfrak{v}[x_+] = 0,$$

and similarly $\mathfrak{v}[x_-] = 0$. Therefore,

$$\mathfrak{v}[x] = \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+] = 2\mathrm{Re}\, \mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}].$$

To prove (2.3), first one observes that

$$\mathfrak{a}_J[x] = \mathfrak{a}_J[x_+] + \mathfrak{a}_J[x_-]$$

and, hence, combining (2.2) and (2.1), one gets the estimate

$$|2\mathrm{Re}\, \mathfrak{v}[x_+, x_-]| \leq v_0 \mathfrak{a}_J[x] = v_0(\mathfrak{a}_J[x_+] + \mathfrak{a}_J[x_-]) \quad \text{for all} \quad x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}].$$

Hence, for any $t \geq 0$ (and, therefore, for all $t \in \mathbb{R}$) one gets that

$$v_0 \mathfrak{a}_J[x_+]\, t^2 - 2|\mathrm{Re}\, \mathfrak{v}[x_+, x_-]|\, t + v_0 \mathfrak{a}_J[x_-] \geq 0,$$

which together with (2.2) implies the inequality (2.3). □

In this setting we present an analog of the First Representation Theorem in the off-diagonal perturbation theory.

repr **Theorem 2.4.** *Assume Hypothesis 2.1. Suppose that* $\mathfrak{v}$ *is an* $\mathfrak{a}$*-bounded off-diagonal with respect to the orthogonal decomposition* $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$ *symmetric form. On* $\mathrm{Dom}[\mathfrak{b}] = \mathrm{Dom}[\mathfrak{a}]$ *introduce the symmetric form*

$$\mathfrak{b}[x, y] = \mathfrak{a}[x, y] + \mathfrak{v}[x, y], \quad x, y \in \mathrm{Dom}[\mathfrak{b}].$$

Then

(i) *there is a unique self-adjoint operator* B *in* $\mathfrak{H}$ *such that* $\mathrm{Dom}(B) \subset \mathrm{Dom}[\mathfrak{b}]$ *and*

$$\mathfrak{b}[x, y] = \langle x, By \rangle \quad \textit{for all} \quad x \in \mathrm{Dom}[\mathfrak{b}], \quad y \in \mathrm{Dom}(B).$$

(ii) *the operator* B *is boundedly invertible and the open interval* $(-m_-, m_+) \ni 0$ *belongs to its resolvent set.*

Proof. (i). Given $\mu \in (-m_-, m_+)$, on $\mathrm{Dom}[\mathfrak{a}_\mu] = \mathrm{Dom}[\mathfrak{a}]$ introduce the positive closed form $\mathfrak{a}_\mu$ by

$$\mathfrak{a}_\mu[x,y] = \mathfrak{a}[x, Jy] - \mu\langle x, Jy\rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{a}_\mu],$$

and denote by $\mathfrak{H}_{\mathfrak{a}_\mu}$ the Hilbert space $\mathrm{Dom}[\mathfrak{a}_\mu]$ equipped with the inner product $\langle\cdot,\cdot\rangle_\mu = \mathfrak{a}_\mu[\cdot,\cdot]$. We remark that the norms $\|\cdot\|_\mu = \sqrt{\mathfrak{a}_\mu[\cdot]}$ on $\mathfrak{H}_{\mathfrak{a}_\mu} = \mathrm{Dom}[\mathfrak{a}_\mu]$ are obviously equivalent. Since $\mathfrak{v}$ is $\mathfrak{a}$-bounded, one concludes then that

$$v_\mu := \sup_{0\neq x\in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]} < \infty, \quad \text{for all } \mu \in (-m_-, m_+).$$

Along with the off-diagonal form $\mathfrak{v}$, introduce a dual form $\mathfrak{v}'$ by

$$\mathfrak{v}'[x,y] = \mathrm{i}\mathfrak{v}[x, Jy], \quad x, y \in \mathrm{Dom}[\mathfrak{a}].$$

We claim that $\mathfrak{v}'$ is an $\mathfrak{a}$-bounded off-diagonal symmetric form. It suffices to show that

$$v_\mu = v'_\mu < \infty, \quad \mu \in (-m_-, m_+),$$

where

vaumu:bis
$$v'_\mu := \sup_{0\neq x\in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}'[x]|}{\mathfrak{a}_\mu[x]}. \tag{2.4}$$

Indeed, let $x = x_+ + x_-$ be a unique decomposition of an element $x \in \mathrm{Dom}[\mathfrak{a}]$ such that $x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}]$. By Remark 2.3,

$$\mathfrak{v}[x] = \mathfrak{v}[x_+, x_-] + \mathfrak{v}[x_-, x_+] = 2\mathrm{Re}\,\mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}].$$

In a similar way (since the form $\mathfrak{v}'$ is obviously off-diagonal) one gets that

$$\begin{aligned}\mathfrak{v}'[x] &= \mathrm{i}\mathfrak{v}[x_+ + x_-, J(x_+ + x_-)] = \mathrm{i}\mathfrak{v}'[x_+] - \mathrm{i}\mathfrak{v}'[x_-] - \mathrm{i}\mathfrak{v}[x_+, x_-] + \mathrm{i}\mathfrak{v}[x_-, x_+]\\ &= -\mathrm{i}\mathfrak{v}[x_+, x_-] + \mathrm{i}\overline{\mathfrak{v}[x_+, x_-]} = 2\mathrm{Im}\,\mathfrak{v}[x_+, x_-], \quad x \in \mathrm{Dom}[\mathfrak{a}].\end{aligned}$$

Clearly, from (2.4) it follows that

$$v'_\mu = 2 \sup_{0\neq x\in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathrm{Im}\,\mathfrak{v}[x_+, x_-]|}{\mathfrak{a}_\mu[x]} = 2 \sup_{0\neq x\in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathrm{Re}\,\mathfrak{v}[x_+, x_-]|}{\mathfrak{a}_\mu[x]} = v_\mu,$$
$$\mu \in (-m_-, m_+),$$

which completes the proof of the claim.

Next, on $\mathrm{Dom}[\mathfrak{t}_\mu] = \mathrm{Dom}[\mathfrak{a}]$ introduce the sesquilinear form

$$\mathfrak{t}_\mu := \mathfrak{a}_\mu + \mathrm{i}\mathfrak{v}', \quad \mu \in (-m_-, m_+).$$

Since the form $\mathfrak{a}_\mu$ is positive definite and the form $\mathfrak{v}'$ is an $\mathfrak{a}_\mu$-bounded symmetric form, the form $\mathfrak{t}$ is a closed sectorial form with the vertex 0 and semi-angle

tmu
$$\theta_\mu = \arctan(v'_\mu) = \arctan(v_\mu). \tag{2.5}$$

Let T_μ be a unique m-sectorial operator associated with the form $\mathfrak{t}_\mu$. Introduce the operator

$$B_\mu = JT_\mu \quad \text{on} \quad \mathrm{Dom}(B_\mu) = \mathrm{Dom}(T_\mu), \quad \mu \in (-m_-, m_+).$$

One obtains that

bbb
$$\begin{aligned}\langle x, B_\mu y\rangle = \langle x, JT_\mu\rangle = \langle Jx, T_\mu y\rangle &= \mathfrak{a}_\mu[Jx, y] + \mathrm{i}\mathfrak{v}'[Jx, y]\\ &= \mathfrak{a}[x,y] - \mu\langle Jx, Jy\rangle + \mathrm{i}^2\mathfrak{v}[Jx, Jy]\\ &= \mathfrak{a}[x,y] - \mu\langle x, y\rangle + \mathfrak{v}[x,y],\end{aligned} \tag{2.6}$$

for all $x \in \mathrm{Dom}[\mathfrak{a}]$, $y \in \mathrm{Dom}(B_\mu) = \mathrm{Dom}(T_\mu)$. In particular, B_μ is a symmetric operator on $\mathrm{Dom}(B_\mu)$, since the forms $\mathfrak{a}$ and $\mathfrak{v}$ are symmetric, and $\mathrm{Dom}(B_\mu) = \mathrm{Dom}(T_\mu) \subset \mathrm{Dom}[a]$.

For the real part of the form $\mathfrak{t}_\mu$ is positive definite with a positive lower bound, the operator T_μ has a bounded inverse. This implies that the operator $B_\mu = JT_\mu$ has a bounded inverse and, therefore, the symmetric operator B_μ is self-adjoint on $\mathrm{Dom}(B_\mu)$.

As an immediate consequence, one concludes (put $\mu = 0$) that the self-adjoint operator $B := B_0$ is associated with the symmetric form $\mathfrak{b}$ and that $\mathrm{Dom}(B) \subset \mathrm{Dom}[\mathfrak{a}]$.

To prove uniqueness, assume that B' is a self-adjoint operator associated with the form $\mathfrak{b}$. Then for all $x \in \mathrm{Dom}(B)$ and all $y \in \mathrm{Dom}(B')$ one gets that

$$\langle x, B'y\rangle = \mathfrak{b}[x, y] = \overline{\mathfrak{b}[y, x]} = \overline{\langle y, Bx\rangle} = \langle Bx, y\rangle,$$

which means that $B = (B')^* = B'$.

(ii). From (2.6) one concludes that the self-adjoint operator $B_\mu + \mu I$ is associated with the form $\mathfrak{b}$ and, hence, by the uniqueness

$$B_\mu = B - \mu I \quad \text{on} \quad \mathrm{Dom}(B_\mu) = \mathrm{Dom}(B).$$

Since B_μ has a bounded inverse for all $\mu \in (m_-, m_+)$, so does $B - \mu I$ which means that the interval $(-m_-, m_+)$ belongs to the resolvent set of the operator B_0. □

reg **Remark 2.5.** *In the particular case $\mathfrak{v} = 0$, from Theorem 2.4 it follows that there exists a unique self-adjoint operator A associated with the form $\mathfrak{a}$.*

For a different, more constructive proof of Theorem 2.4 as well as for the history of the subject we refer to our work [4].

Remark 2.6. *For the part (i) of Theorem 2.4 to hold it is not necessary to require that the form $\mathfrak{a}_J$ in Hypothesis 2.1 is positive definite. It is sufficient to assume that $\mathfrak{a}_J$ is a semi-bounded from below closed form (see, e.g.,* [12]*).*

3. THE TAN 2Θ THEOREM

sec:tan

The main result of this work provides a sharp upper bound for the angle between the positive spectral subspaces $\mathrm{Ran}\, \mathsf{E}_A(\mathbb{R}_+)$ and $\mathrm{Ran}\, \mathsf{E}_B(\mathbb{R}_+)$ of the operators A and B respectively.

thm:esti **Theorem 3.1.** *Assume Hypothesis 2.1 and suppose that $\mathfrak{v}$ is off-diagonal with respect to the decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$. Let A be a unique self-adjoint operator associated with the form $\mathfrak{a}$ and B the self-adjoint operator associated with the form $\mathfrak{b} = \mathfrak{a} + \mathfrak{v}$ referred to in Theorem 2.4.*

Then the norm of the difference of the spectral projections $P = \mathsf{E}_A(\mathbb{R}_+)$ and $Q = \mathsf{E}_B(\mathbb{R}_+)$ satisfies the estimate

$$\|P - Q\| \le \sin\left(\frac{1}{2}\arctan v\right) < \frac{\sqrt{2}}{2},$$

where

$$v = \inf_{\mu \in (-m_-, m_+)} v_\mu = \inf_{\mu \in (-m_-, m_+)} \sup_{0 \ne x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]},$$

with

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu\langle x, Jy\rangle, \quad x, y \in \mathrm{Dom}[\mathfrak{a}_\mu] = \mathrm{Dom}[\mathfrak{a}].$$

The proof of Theorem 3.1 uses the following result borrowed from [14].

lem:4:2 **Proposition 3.2.** *Let T be an m-sectorial operator of semi-angle $\theta < \pi/2$. Let $T = U|T|$ be its polar decomposition. If U is unitary, then the unitary operator U is sectorial with semi-angle θ.*

Remark 3.3. *We note that for a bounded sectorial operator T with a bounded inverse the statement is quite simple. Due to the equality*

$$\langle x, Tx\rangle = \langle |T|^{-1/2}y, U|T|^{1/2}y\rangle = \langle y, |T|^{-1/2}U|T|^{1/2}y\rangle, \qquad y = |T|^{1/2}x,$$

the operators T and $|T|^{-1/2}U|T|^{1/2}$ are sectorial with the semi-angle θ. The resolvent sets of the operators $|T|^{-1/2}U|T|^{1/2}$ and U coincide. Therefore, since U is unitary, it follows that U is sectorial with semi-angle θ.

Proof of Theorem 3.1. Given $\mu \in (-m_-, m_+)$, let $T_\mu = U_\mu |T_\mu|$ be the polar decomposition of the sectorial operator T_μ with vertex 0 and semi-angle θ_μ, with

ugl (3.1)
$$\theta_\mu = \arctan(v_\mu)$$

(as in the proof of Theorem 2.4 (cf. (2.5)). Since $B_\mu = JT_\mu$, one concludes that

$$|T_\mu| = |B_\mu| \quad \text{and} \quad U_\mu = J^{-1}\operatorname{sign}(B_\mu).$$

Since T_μ is a sectorial operator with sem-iangle θ_μ, by a result in [14] (see Proposition 3.2), the unitary operator U_μ is sectorial with vertex 0 and semi-angle θ_μ as well. Therefore, applying the spectral theorem for the unitary operator U_μ from (3.1) one obtains the estimate

$$\|J - \operatorname{sign}(B_\mu)\| = \|I - J^{-1}\operatorname{sign}(B_\mu)\| = \|I - U_\mu\| \le 2\sin\left(\frac{1}{2}\arctan v_\mu\right).$$

Since the open interval $(-m_-, m_+)$ belongs to the resolvent set of the operator $B = B_0$, the involution $\operatorname{sign}(B_\mu)$ does not depend on $\mu \in (-m_-, m_+)$ and hence one concludes that

$$\operatorname{sign}(B_\mu) = \operatorname{sign}(B_0) = \operatorname{sign}(B), \quad \mu \in (-m_-, m_+).$$

Therefore,

muo (3.2)
$$\|P - Q\| = \frac{1}{2}\|J - \operatorname{sign}(B)\| = \frac{1}{2}\|J - \operatorname{sign}(B_\mu)\| \le \sin\left(\frac{1}{2}\arctan v_\mu\right)$$

and, hence, since $\mu \in (-m_-, m_+)$ has been chosen arbitrarily, from (3.2) it follows that

$$\|P - Q\| \le \inf_{\mu \in (-m_-, m_+)} \sin\left(\frac{1}{2}\arctan v_\mu\right) \le \sin\left(\frac{1}{2}\arctan v\right).$$

The proof is complete. □

As a consequence, we have the following result that can be considered a geometric variant of the Birman-Schwinger principle for the off-diagonal form-perturbations.

cor:pi8 **Corollary 3.4.** *Assume Hypothesis 2.1 and suppose that $\mathfrak{v}$ is off-diagonal. Then the form $\mathfrak{a}_J + \mathfrak{v}$ is positive definite if and only if the a_J-relative bound* (2.1) *of $\mathfrak{v}$ does not exceed one. In this case*

$$\|P - Q\| \le \sin\left(\frac{\pi}{8}\right),$$

where P and Q are the spectral projections referred to in Theorem 3.1.

Proof. Since $\mathfrak{v}$ is an $\mathfrak{a}$-bounded form, one concludes that there exists a self-adjoint bounded operator $\mathcal{V}$ in the Hilbert space $\operatorname{Dom}[\mathfrak{a}]$ such that

$$v[x, y] = \mathfrak{a}_J[x, \mathcal{V}y], \quad x, y \in \operatorname{Dom}[\mathfrak{a}].$$

Since $\mathfrak{v}$ is off-diagonal, the numerical range of $\mathcal{V}$ coincides with the symmetric about the origin interval $[-\|\mathcal{V}\|, \|\mathcal{V}\|]$. Therefore, one can find a sequence $\{x_n\}_{n=1}^\infty$ in $\operatorname{Dom}[\mathfrak{a}]$ such that

$$\lim_{n\to\infty} \frac{\mathfrak{v}[x_n]}{\mathfrak{a}_J[x_n]} = -\|\mathcal{V}\|,$$

which proves that $\|\mathcal{V}\| \le 1$ if and only if the form $\mathfrak{a}_J + \mathfrak{v}$ is positive definite. If it is the case, applying Theorem 3.1, one obtains the inequality

$$\|P - Q\| \le \sin\left(\frac{1}{2}\arctan\left(\|\mathcal{V}\|\right)\right) \le \sin\left(\frac{\pi}{8}\right)$$

which completes the proof. □

Remark 3.5. *We ramark that in accordance with the Birman-Schwinger principle, for the form* $\mathfrak{a}_J + \mathfrak{v}$ *to have negative spectrum it is necessary that the* a_J*-relative bound* $\|\mathcal{V}\|$ *of the perturbation* $\mathfrak{v}$ *is greater than one. As Corollary 3.4 shows, in the off-diagonal perturbation theory this condition is also sufficient.*

sec:semi

4. TWO SHARP ESTIMATES IN THE SEMIBOUNDED CASE

In this section we will be dealing with the case of off-diagonal form-perturbations of a semi-bounded operator.

ppp **Hypothesis 4.1.** *Assume that* A *is a self-adjoint semi-bounded from below operator. Suppose that* A *has a bounded inverse. Assume, in addition, that an open finite interval* (α, β) *belongs to the resolvent set of the operator* A.

We set

$$\Sigma_- = \operatorname{spec}(A) \cap (-\infty, \alpha] \quad \textit{and} \quad \Sigma_+ = \operatorname{spec}(A) \cap [\beta, \infty].$$

Suppose that $\mathfrak{v}$ *is a symmetric form on* $\operatorname{Dom}[\mathfrak{v}] \supset \operatorname{Dom}(|A|^{1/2})$ *such that*

nach
$$v := \sup_{0 \neq x \in \operatorname{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\||A|^{1/2}x\|^2} < \infty. \tag{4.1}$$

Assume, in addition, that $\mathfrak{v}$ *is off-diagonal with respect to the orthogonal decomposition* $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$*, with*

$$\mathfrak{H}_+ = \operatorname{Ran} \mathsf{E}_A((\beta, \infty)) \quad \textit{and} \quad \mathfrak{H}_- = \operatorname{Ran} \mathsf{E}_A((-\infty, \alpha)).$$

That is,

$$\mathfrak{v}[Jx, y] = -\mathfrak{v}[x, Jy], \quad x, y \in \operatorname{Dom}[\mathfrak{a}], \tag{4.2}$$

where the self-adjoint involution J *is given by*

J
$$J = \mathsf{E}_A\left((\beta, \infty)\right) - \mathsf{E}_A\left((-\infty, \alpha)\right). \tag{4.3}$$

Let $\mathfrak{a}$ be the closed form represented by the operator A. A direct application of Theorem 2.4 shows that under Hypothesis 4.1 there is a unique self-adjoint boundedly invertible operator B associated with the form

$$\mathfrak{b} = \mathfrak{a} + \mathfrak{v}.$$

Under Hypothesis 4.1 we distinguish two cases (see Fig. 1 and 2).

Case I. Assume that $\alpha < 0$ and $\beta > 0$. Set

$$d_+ = \text{dist}(\inf(\Sigma_+), 0) \quad \text{and} \quad d_- = \text{dist}(\inf(\Sigma_-), 0)$$

and suppose that $d_+ > d_-$.

Case II. Assume that $\alpha, \beta > 0$. Set

$$d_+ = \operatorname{dist}(\inf(\Sigma_+), 0) \quad \text{and} \quad d_- = \operatorname{dist}(\sup(\Sigma_-), 0).$$

As it follows from the definition of the quantities $d_\pm$, the sum $d_- + d_+$ coincides with the distance between the lower edges of the spectral components Σ_+ and Σ_- in Case I, while in Case II the difference $d_+ - d_-$ is the distance from the lower edge of Σ_+ to the upper edge of the spectral component Σ_-. Therefore, $d_+ - d_-$ coincides with the length of the spectral gap (α, β) of the operator A in latter case.

We remark that the condition $d_+ > d_-$ required in Case I, holds only if the length of the convex hull of negative spectrum Σ_- of A does not exceed the one of the spectral gap $(\alpha, \beta) = (\sup(\Sigma_-), \inf(\Sigma_+))$.

Now we are prepared to state a relative version of the Tan 2Θ Theorem in the case where the unperturbed operator is semi-bounded or even positive.

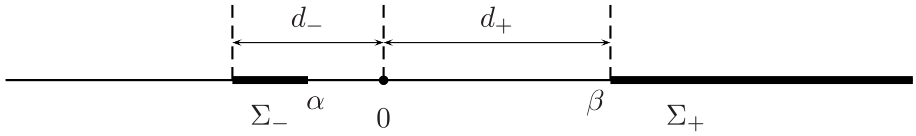

fig:2 FIG. 1. The spectrum of the unperturbed sign-indefinite semibounded invertible operator A in Case I.

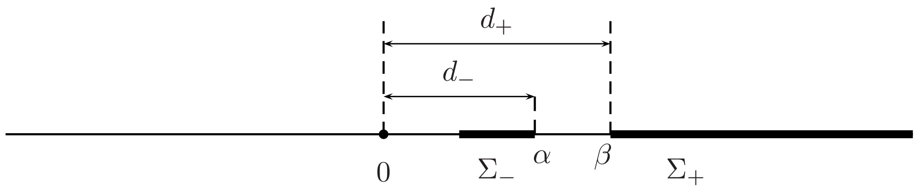

fig:3 FIG. 2. The spectrum of the unperturbed strictly positive operator A with a gap in its spectrum in Case II.

thm:esti:prime **Theorem 4.2.** *In either Cases I or II, introduce the spectral projections*

spp (4.4)
$$P = \mathsf{E}_A((-\infty, \alpha]) \quad \textit{and} \quad Q = \mathsf{E}_B((-\infty, \alpha])$$

of the operators A and B respectively.

Then the norm of the difference of P and Q satisfies the estimate

Davis (4.5)
$$\|P - Q\| \le \sin\left(\frac{1}{2}\arctan\left[2\frac{v}{\delta}\right]\right) < \frac{\sqrt{2}}{2},$$

where

delta (4.6)
$$\delta = \frac{1}{\sqrt{d_+ d_-}}\begin{cases} d_+ + d_- & \textit{in Case I},\\ d_+ - d_- & \textit{in Case II},\end{cases}$$

and v stands for the relative bound of the off-diagonal form $\mathfrak{v}$ (with respect to $\mathfrak{a}$) given by (4.1).

Proof. We start with the remark that the form $\mathfrak{a} - \mu$, where $\mathfrak{a}$ is the form of A, satisfies Hypothesis 2.1 with J given by (4.3). Set

$$\mathfrak{a}_\mu = (\mathfrak{a} - \mu)_J, \quad \mu \in (\alpha, \beta),$$

that is,

$$\mathfrak{a}_\mu[x, y] = \mathfrak{a}[x, Jy] - \mu[x, Jy], \quad x, y \in \mathrm{Dom}[\mathfrak{a}].$$

Notice that $\mathfrak{a}_\mu$ is a strictly positive closed form represented by the operators $JA - J\mu = |A| - \mu J$ and $JA - \mu J = |A - \mu I|$ in Cases I and II, respectively.

Since $\mathfrak{v}$ is off-diagonal, from Theorem 3.1 it follows that

first (4.7)
$$\|\mathsf{E}_{A-\mu I}(\mathbb{R}_+) - \mathsf{E}_{B-\mu I}(\mathbb{R}_+)\| \le \sin\left(\frac{1}{2}\arctan v_\mu\right) \quad \text{for all} \quad \mu \in (\alpha, \beta),$$

with

(4.8)
$$v_\mu =: \sup_{0 \ne x \in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]}.$$

Since $\mathfrak{v}$ is off-diagonal, by Remark 2.3 one gets the estimate

$$|\mathfrak{v}[x]| \le 2v_0\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}, \quad x \in \mathrm{Dom}[\mathfrak{a}],$$

where $x = x_+ + x_-$ is a unique decomposition of the element $x \in \mathrm{Dom}[\mathfrak{a}]$ with

$$x_\pm \in \mathfrak{H}_\pm \cap \mathrm{Dom}[\mathfrak{a}].$$

Thus, in these notations, taking into account that

$$v_0 = v,$$

where v is given by (4.1), one gets the bound

osnb (4.9)
$$v_\mu \le 2v \sup_{0\ne x\in \mathrm{Dom}[\mathfrak{a}]} \frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_\mu[x]}. \tag{4.9}$$

Since $\mathfrak{a}_\mu$ is represented by $JA - J\mu = |A| - \mu J$ and $JA - \mu J = |A - \mu I|$ in Cases I and II, respectively, one observes that

nado (4.10)
$$\mathfrak{a}_\mu[x] = \begin{cases} \mathfrak{a}_0[x_+] - \mu\|x_+\|^2 + \mathfrak{a}_0[x_-] + \mu\|x_-\|^2, & \text{in Case I,} \\ \mathfrak{a}_0[x_+] - \mu\|x_+\|^2 - \mathfrak{a}_0[x_-] + \mu\|x_-\|^2, & \text{in Case II.} \end{cases} \tag{4.10}$$

Introducing the elements $y_\pm \in \mathfrak{H}_\pm$,

$$y_\pm := \begin{cases} (|A| \mp \mu I)^{1/2} x_\pm, & \text{in Case I,} \\ \pm(A - \mu I)^{1/2} x_\pm, & \text{in Case II,} \end{cases}$$

and taking into account (4.10), one obtains the representation

$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_\mu[x]} = \frac{\||A|^{1/2}(|A| - \mu I)^{-1/2} y_+\| \, \||A|^{1/2}(-A + \mu I)^{-1/2} y_-\|}{\|y_+\|^2 + \|y_-\|^2},$$

valid in both Cases I and II. Using the elementary inequality

$$\|y_+\|\,\|y_-\| \le \frac{1}{2}\left(\|y_+\|^2 + \|y_-\|^2\right),$$

one arrives at the following bound

eins (4.11)
$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_\mu[x]} \le \frac{1}{2}\||A|^{1/2}(|A| - \mu I)^{-1/2}|_{\mathfrak{H}_+}\| \cdot \||A|^{1/2}(-A + \mu I)^{-1/2}|_{\mathfrak{H}_-}\|. \tag{4.11}$$

It is easy to see that

zwei (4.12)
$$\||A|^{1/2}(|A| - \mu I)^{-1/2}|_{\mathfrak{H}_+}\| \le \frac{\sqrt{d_+}}{\sqrt{d_+ - \mu}} \quad \mu \in (\alpha, \beta), \quad \text{in Cases I and II,} \tag{4.12}$$

while

drei (4.13)
$$\||A|^{1/2}(-A + \mu I)^{-1/2}|_{\mathfrak{H}_-}\| \le \begin{cases} \frac{\sqrt{d_-}}{\sqrt{d_- + \mu}}, & \mu \in (0, \beta), \quad \text{in Case I,} \\ \frac{\sqrt{d_-}}{\sqrt{\mu - d_-}}, & \mu \in (\alpha, \beta), \quad \text{in Case II.} \end{cases} \tag{4.13}$$

Choosing $\mu = \frac{d_+ - d_-}{2} > 0$ in Case I (recall that $d_+ > d_-$ by the hypothesis) and $\mu = \frac{d_+ + d_-}{2}$ in Case II, and combining (4.11), (4.12), (4.13), one gets the estimates

$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\frac{d_+ - d_-}{2}}[x]} \le \frac{\sqrt{d_+ d_-}}{d_+ + d_-} \quad \text{in Case I}$$

and

$$\frac{\sqrt{\mathfrak{a}_0[x_+]\mathfrak{a}_0[x_-]}}{\mathfrak{a}_{\frac{d_+ + d_-}{2}}[x]} \le \frac{\sqrt{d_+ + d_-}}{d_+ - d_-} \quad \text{in Case II.}$$

Hence, from (4.9) it follows that

$$v_{\frac{d_+-d_-}{2}} \leq 2v\frac{\sqrt{d_+d_-}}{d_+ + d_-} \quad \text{in Case I}$$

and

$$v_{\frac{d_++d_-}{2}} \leq 2v\frac{\sqrt{d_+d_-}}{d_+ - d_-} \quad \text{in Case II.}$$

Applying (4.7), one gets the norm estimates

first1

$$\|\mathsf{E}_{A-\frac{d_+-d_-}{2}I}(\mathbb{R}_+) - \mathsf{E}_{B-\frac{d_+-d_-}{2}I}(\mathbb{R}_+)\| \leq \sin\left(\frac{1}{2}\arctan\left[2\frac{\sqrt{d_+d_-}}{d_+ + d_-}v\right]\right) \tag{4.14}$$

in Case I and

first2

$$\|\mathsf{E}_{A-\frac{d_++d_-}{2}I}(\mathbb{R}_+) - \mathsf{E}_{B-\frac{d_++d_-}{2}I}(\mathbb{R}_+)\| \leq \sin\left(\frac{1}{2}\arctan\left[2\frac{\sqrt{d_+d_-}}{d_+ - d_-}v\right]\right) \tag{4.15}$$

in Case II. In remains to observe that $\|P-Q\|$, where the spectral projections P and Q are given by (4.4), coincides with the left hand side of (4.14) and (4.15) in Case I and Case II, respectively.

The proof is complete. □

Remark 4.3. *We remark that the quantity δ given by* (4.6) *coincides with the* relative distance *(with respect to the origin) between the lower edges of the spectral components Σ_+ and Σ_- in Case I and it has the meaning of the* relative length *(with respect to the origin) of the spectral gap (d_-, d_+) in Case II.*

For the further properties of the relative distance and various relative perturbation bounds we refer to the paper [10] *and references quoted therein.*

We also remark that in Case II, i.e., in the case of a positive operator A, the bound (4.5) *directly improves a result obtained in* [6], the relative $\sin\Theta$ Theorem, *that in the present notations is of the form*

$$\|P - Q\| \leq \frac{v}{\delta}.$$

We conclude our exposition with considering an example of a 2×2 numerical matrix that shows that the main results obtained above are sharp.

exam **Example 4.4.** Let $\mathfrak{H}$ be the two-dimensional Hilbert space $\mathfrak{H} = \mathbb{C}^2$, $\alpha < \beta$ and $w \in \mathbb{C}$.

We set

$$A = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}, \quad V = \begin{pmatrix} 0 & w \\ w^* & 0 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathfrak{v}$ be the symmetric form represented by (the operator) V.

Clearly, the form $\mathfrak{v}$ satisfy Hypothesis 4.1 with the relative bound v given by

$$v = \frac{|w|}{\sqrt{|\alpha\beta|}},$$

provided that $\alpha, \beta \neq 0$. Since $VJ = -JV$, the form $\mathfrak{v}$ is off-diagonal with respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{H}_+ \oplus \mathfrak{H}_-$.

In order to illustrate our results, denote by B the self-adjoint matrix associated with the form $\mathfrak{a} + \mathfrak{v}$, that is,

$$B = A + V = \begin{pmatrix} \beta & w \\ w^* & \alpha \end{pmatrix}.$$

Denote by P the orthogonal projection associated with the eigenvalue α of the matrix A, and by Q the one associated with the lower eigenvalue of the matrix B.

It is well know (and easy to see) that the classical Davis-Kahan Tan 2Θ theorem (1.2) is exact in the case of 2×2 numerical matrices. In particular, the norm of the difference of P and Q can be computed explicitly

sharp
$$\|P - Q\| = \sin\left(\frac{1}{2}\arctan\left[\frac{2|w|}{\beta-\alpha}\right]\right). \tag{4.16}$$

Since, in the case in question,

susu2
$$v_\mu = \sup_{0\neq x\in \mathrm{Dom}[\mathfrak{a}]} \frac{|\mathfrak{v}[x]|}{\mathfrak{a}_\mu[x]} = \frac{|w|}{\sqrt{(\beta-\mu)(\mu-\alpha)}}, \quad \mu\in(\alpha,\beta), \tag{4.17}$$

from (4.17) it follows that

$$\inf_{\mu\in(\alpha,\beta)} v_\mu = \frac{2|w|}{\beta-\alpha}$$

(with the infimum attained at the point $\mu = \frac{\alpha+\beta}{2}$).

Therefore, the result of the relative $\tan 2\Theta$ Theorem 3.1 is sharp.

It is easy to see that if $\alpha < 0 < \beta$ (Case I), then the equality (4.16) can also be rewritten in the form

zz
$$\|P - Q\| = \sin\left(\frac{1}{2}\arctan\left[2\frac{\sqrt{d_+d_-}}{d_+ + d_-}v\right]\right), \tag{4.18}$$

where $d_+ = \beta$, $d_- = -\alpha$ and $v = \frac{|w|}{\sqrt{|\alpha|\beta}}$.

If $0 < \alpha < \beta$ (Case II), the equality (4.16) can be rewritten as

zzz
$$\|P - Q\| = \sin\left(\frac{1}{2}\arctan\left[2\frac{\sqrt{d_+d_-}}{d_+ - d_-}v\right]\right), \tag{4.19}$$

with $d_+ = \beta$, $d_- = \alpha$, and $v = \frac{|w|}{\sqrt{\alpha\beta}}$.

The representations (4.18) and (4.19) show that the estimate (4.5) becomes equality in the case of 2×2 numerical matrices and, therefore, the results of Theorem 4.2 are sharp.

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