Exponential Decay of Semigroups in Hilbert Space

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Abstract

We give simple bounds for the exponential decay of a strongly continuous semigroup $V(t) = exp(At)$ in a Hilbert space. The bounds are expressed by means of the solution X of the Ljapunov equation $A^*X + XA = -I$ and the quantity $\sup_{0\le t\le h}\|V(t)\|$ for small h. Our main bound is attained on any normal semigroup and can therefore not be improved in general.

Let $V(t)$ be a strongly continuous semigroup in a complex Hilbert space $\mathcal{H}$. Then it is exponentially bounded i.e.

$$\|V(t)\| \le Me^{\alpha t} \tag{1}$$

for some $M \ge 1$ and some real α. The set of all such semigroups is denoted by $\mathcal{G}(M, \alpha)$.[1] We say that $V(t)$ *decays exponentially*, if (1) holds with a negative α.

The aim of this paper is to give sharp bounds for the exponential decay and in particular for the number (*type* of the semigroup)

$$\alpha(V) = \inf\{\alpha;\ V(\cdot) \in \mathcal{G}(M, \alpha)\}\ . \tag{2}$$

A first result connecting exponential decay with the Ljapunov equation in the Hilbert space was provided by Datko [1] which we partly resume for convenience:[2]

Theorem 1 *(Datko) The following are equivalent: (i) $V(\cdot)$ decays exponentially, (ii) The strong integral*

$$X = \int_0^\infty V^*(t)V(t)dt \tag{3}$$

represents a bounded selfadjoint operator in $\mathcal{H}$, (iii) There is a bounded selfadjoint operator Y such that for every $\phi, \psi \in \mathcal{D}(A)$, A the infinitesimal generator of $V(\cdot)$, we have

$$2\Re(YA\phi, \psi) = -(\phi, \psi)\ . \tag{4}$$

In this case $Y = X$ and

$$V(\cdot) \in \mathcal{G}\left(M_0,\ -\frac{1}{2e^2\ max\{M_1, M_2\}^3}\right) \tag{5}$$

for some M_0 and

$$M_1 = \sup_{t\ge 0}\|V(t)\|\ ,\ M_2 = \|X\|^2\ . \tag{6}$$

The estimates (5,6) are not contained in the statements of the theorems in [1] but can be read off from the proofs. [1] does not seem to contain any bound for M_0 in (5).

Our main result is

$$V(\cdot) \in \mathcal{G}\left(v_h^{1/2}\frac{v_h\|X\| + h}{\sqrt{\frac{h}{v_h}\|X\| - h^2}},\ -\frac{1}{2h}\ln\left(\frac{h}{v_h\|X\|} + 1\right)\right)\ , \tag{7}$$

[1] See [3], whose notation and terminology we follow here.

[2] For further generalisations of this result, including those for Banach spaces, see [4, 5].

with sufficiently small $h > 0$ and

$$v_h \;=\; \sup_{0\le\tau\le h} \|V(\tau)\|^2 \,. \tag{8}$$

In particular

$$\alpha(V) \le -\frac{1}{2v_0\|X\|} \;. \tag{9}$$

Here

$$v_0 = \lim_{h\to 0} v_h \tag{10}$$

and X satisfies the Ljapunov equation (4) which can be written formally as

$$A^*X + XA = -I \;, \tag{11}$$

where A is the infinitesimal generator of the semigroup $V(\cdot)$. Our exponent bound is shown to improve (5), moreover, the bound (9) is attained on any normal semigroup and thus cannot be improved in general. Our technical tools are similar to those of [1] the main difference lies in our systematical use of form inequalities and monotonicities. Also, instead of a global constant M_1 from (6) we use a local quantity v_h. Being explicit, our bounds can be used for uniform estimates of families of semigroups, an issue which has recently attracted some attention ([6]) and which we will address as well. Note that our main results seem to be new even in the finite dimensional case.

We begin with a simple estimate which is already known for finite matrices ([2]).

Theorem 2 *Let the operator X from (3), together with its inverse, be everywhere defined and bounded. Then*

$$\|V(t)\| \le \sqrt{\|X\|\|X^{-1}\|}e^{-\frac{t}{2\|X\|}} \;. \tag{12}$$

Proof. For $\phi \in \mathcal{D}(A)$ using (4) we obtain

$$\frac{d}{dt}(XV(t)\phi, V(t)\phi) = -(V(t)\phi, V(t)\phi) \le -(XV(t)\phi, V(t)\phi)/\|X\| \;.$$

By integrating this inequality we obtain

$$(XV(t)\phi, V(t)\phi) \le (X\phi,\phi)e^{-\frac{t}{\|X\|}} \;,$$

in other words

$$\|X^{1/2}V(t)\| \le \|X^{1/2}\|e^{-\frac{t}{2\|X\|}} \;. \tag{13}$$

Note that thus far the bounded invertibility of X was not used. Using it we obtain (12). QED.

Although this simple result already produces the proper exponent (note that the quantity v_0 from (10), which is never less than one, does not enter here) it will not be directly used in what follows. More useful is its discrete analogue:

Theorem 3 *Let T be a bounded linear operator in a Hilbert space $\mathcal{H}$, and let the strong sum*

$$Y \;=\; \sum_{n=0}^{\infty} T^{*n}T^n \tag{14}$$

be a bounded operator on $\mathcal{H}$. [3] *Then*

$$\|T^n\| \;\le\; \left(1 - \frac{1}{\|Y\|}\right)^{\frac{n}{2}} \|Y\|^{1/2} \tag{15}$$

and

$$0 \le 1 - \frac{1}{\|Y\|} < 1 \,. \tag{16}$$

[3] This is equivalent to the fact that the form $\eta(\psi,\psi) = \sum_n \|T^n\psi\|^2$ is finite for every $\psi \in \mathcal{H}$.

Proof. By definition of Y we have

$$Y^{-1} \leq I \leq Y \quad [4] \tag{17}$$

Furthermore, Y satisfies the recursion

$$T^{*n}YT^n - T^{*n-1}YT^{n-1} \; = \; -T^{*n-1}T^{n-1} \, , \, n = 1, 2, \ldots \tag{18}$$

From

$$\|Y^{-1}\| T^{*n-1}YT^{n-1} \; \geq \; T^{*n-1}T^{n-1} \; \geq \; \frac{1}{\|Y\|} T^{*n-1}YT^{n-1}$$

it follows

$$T^{*n}YT^n \quad \leq \quad \left(1 - \frac{1}{\|Y\|}\right) T^{*n-1}YT^{n-1} \; , \; n = 1, 2, \ldots$$

By iterating this and using (17) we obtain

$$T^{*n}T^n \; \leq T^{*n}YT^n \; \leq \; \left(1 - \frac{1}{\|Y\|}\right)^n Y \tag{19}$$

Thus, (15) follows. QED.

Theorem 4 *Let the strong integral*

$$X \; = \; \int_0^\infty V(t)^* V(t) dt \tag{20}$$

produce a bounded operator defined on whole $\mathcal{H}$. *Then* X *is selfadjoint and strictly positive. Set* v_h *as in (8). Then there is* $h_0 > 0$ *such that* $[0, h_0) = \{h; hv_h < \|X\|\}$ *and for* $0 < h < h_0$ *we have*

$$\|V(t)\| \leq M_h e^{-\beta_h t} \tag{21}$$

with

$$M_h \; = \; v_h^{1/2} \frac{v_h \|X\| + h}{\sqrt{\frac{h}{v_h}\|X\| - h^2}} \; , \; \beta_h \; = \; \frac{1}{2h} \ln \left(\frac{h}{v_h \|X\|} + 1 \right) \tag{22}$$

Proof. The strict positivity of X is an immediate consequence of the fact that the strong limit of $V(t)^* V(t)$ at zero is the identitity operator. The existence of $h_0 > 0$ follows from the fact that v_h is a positive non-decreasing function of h. For any $h > 0$ set $T_h = V(h)$. For any $t > 0$ we can write

$$t \; = \; nh + \tau \; , \; n \text{ integer}, \; 0 \leq \tau \leq h \; .$$

For any t we have

$$\|T_h^{n+1} x\| \; = \; \|V(h - \tau) V(t) x\| \; \leq \; v_h^{1/2} \|V(t) x\| \; . \tag{23}$$

Therefore

$$X = \int_0^\infty V(t)^* V(t) dt \; = \; \sum_{n=0}^\infty \int_{nh}^{(n+1)h} V(t)^* V(t) \, dt \; \geq$$

$$\frac{h}{v_h} \sum_{n=0}^\infty T_h^{*n+1} T_h^{n+1} \; = \frac{h}{v_h} \sum_{n=1}^\infty T_h^{*n} T_h^n \; ,$$

[4] Here and in the following such inequalities are understood in the sense of quadratic forms.

where we have used (23). Thus,

$$Y_h \;=\; \sum_{n=0}^{\infty} T_h^{*n} T_h^n$$

is bounded and

$$Y_h \;\leq\; I + \frac{v_h}{h} X$$

Similarly, from $\|V(t)\| = \|T_h^n V(\tau)\| \leq v_h^{1/2} \|T_h^n\|$ we obtain

$$X \leq h v_h \sum_{n=0}^{\infty} T_h^{*n} T_h^n = h v_h Y_h$$

Altogether

$$\frac{1}{h v_h} X \;\leq\; Y_h \;\leq\; I + \frac{v_h}{h} X\,. \tag{24}$$

Note that v_h is a non-decreasing function of h. For contractive $V(t)$ we have $v_h \equiv 1$. Now use Th. 2:

$$\|V(t)\| \;=\; \|V(\tau) T_h^n\| \;\leq\; v_h^{1/2} \|T_h^n\| \;\leq\; v_h^{1/2} (1 - \frac{1}{\|Y_h\|})^{\frac{n}{2}} \|Y_h\|^{1/2}$$

$$=\; v_h^{1/2} \left(1 - \frac{1}{\|Y_h\|}\right)^{\frac{1}{2h}(nh+\tau-\tau)} \|Y_h\|^{1/2} \;\leq\; v_h^{1/2} \left(1 - \frac{1}{\|Y_h\|}\right)^{\frac{t}{2h}} \left(1 - \frac{1}{\|Y_h\|}\right)^{-\frac{\tau}{2h}} \|Y_h\|^{1/2}$$

We estimate

$$\left(1 - \frac{1}{\|Y_h\|}\right)^{-\frac{\tau}{2h}} \|Y_h\|^{1/2} \leq \left(1 - \frac{1}{\|Y_h\|}\right)^{-1/2} \|Y_h\|^{1/2} \;=\; \frac{\|Y_h\|}{\sqrt{\|Y_h\| - 1}}\,.$$

Using both sides of (24) we obtain

$$v_h^{1/2} \frac{\|Y_h\|}{\sqrt{\|Y_h\| - 1}} \leq M_h = v_h^{1/2} \frac{\frac{v_h}{h}\|X\| + 1}{\sqrt{\frac{1}{h v_h}\|X\| - 1}} \;=\; v_h^{1/2} \frac{v_h \|X\| + h}{\sqrt{\frac{h}{v_h}\|X\| - h^2}}\,.$$

By

$$\left(1 - \frac{1}{\|Y_h\|}\right)^{\frac{t}{2h}} \;=\; e^{-\beta t}$$

and using (24) together with the fact that $\log \frac{x}{x-1}$ is decreasing in $x > 1$ we obtain

$$\beta \;=\; \frac{1}{2h} \ln \frac{\|Y_h\|}{\|Y_h\| - 1} \;\geq \beta_h = \frac{1}{2h} \ln \left(\frac{h}{v_h \|X\|} + 1 \right)\;.$$

QED.

Corollary 5 *If the semigroup is contractive then* $v_h \equiv 1$ *and for* $h < \|X\|$ *we have*

$$M_h \;=\; \frac{\|X\| + h}{\sqrt{h\|X\| - h^2}}\;, \quad \beta_h \;=\; \frac{1}{2h} \ln \left(\frac{h}{\|X\|} + 1 \right)\;. \tag{25}$$

Since $\frac{1}{2h} \ln(\frac{h}{\|X\|} + 1)$ is decreasing with h the optimal exponent β_h at $h = 0$ is

$$\beta_{opt} \;=\; \beta_0 \;=\; \frac{1}{2\|X\|} \tag{26}$$

which is reached only asymptotically, since our M becomes infinite for $h \to 0$.

Note that the asymptotic behaviour of β for $h \to 0$ can be given in the general case (22) as well, due to the fact that v_h is non-decreasing and not less than 1. Now, $\frac{1}{v_h} \leq 1$ is non-increasing in h. Thus,

$$\begin{aligned} \beta_h &= \tfrac{1}{2h} \ln \left(\tfrac{h}{v_h \|X\|} + 1 \right) = \tfrac{1}{v_h} \left(\tfrac{1}{2\|X\|} + \mathcal{O}(\tfrac{h}{v_h^2 \|X\|^2}) \right) = \\ & \tfrac{1}{v_h} \left(\tfrac{1}{2\|X\|} + \mathcal{O}(\tfrac{h}{\|X\|^2}) \right) \end{aligned} \tag{27}$$

Hence,

$$\beta_{opt} = \lim_{h \to 0} \beta_h = \frac{1}{2v_0 \|X\|} \, , \; v_0 = \lim_{h \to 0} v_h \geq 1 \, . \tag{28}$$

Explicit estimates automatically allow uniform control of families of semigroups. So, our theorem above is related to a recent result by Triggiani [6] which tells that any family V_α of semigroups in a Banach space belonging to the same class $G(M_1, \beta_1)$ and satisfying the uniform estimate

$$\int_0^\infty \|V_\alpha(t)\psi\|^p \leq \xi \|\psi\|^p$$

satisfies a uniform estimate

$$\|V_\alpha(t)\| \leq M e^{-\gamma t}$$

with $M, \gamma > 0$ independent of α .

For the case of a Hilbert space and $p = 2$ we recover this result since then

$$v_h \leq M_1 e^{\beta_1 h}$$

and our estimates (21), (22) are uniform in α. Note that our results make no extra use of uniform boundedness principle.

Since the constant M_1 from (6) dominates v_h our estimate for $\alpha(V)$ clearly improves (5). What is more important, our exponent factor $\beta = \frac{1}{2v_0\|X\|}$ cannot be improved in general. More precisely, it is reached on any normal semigroup. Indeed, in this case we have ([7], Ch. 9)

$$V(t) = e^{At} = \int e^{f(\lambda)t} dE_\lambda \, , \; A = \int f(\lambda) dE_\lambda \, ,$$

where f is a complex-valued measurable function and E_λ is a spectral family on R ([3], Ch. VI). Obviously X is bounded, if and only if

$$ess \; sup \; \Re f = sup \; \Re \sigma(A) < 0 \, .$$

In this case V is contractive (hence $v_h \equiv 1$) and using the functional calculus we obtain

$$X = -(\widetilde{A + A^*})^{-1} = -\int \frac{dE_\lambda}{2\Re f(\lambda)}$$

and

$$\|X\| = -\frac{1}{2 \sup \Re \sigma(A)}$$

.

On the other side

$$\|V(t)\| = \|e^{At}\| = e^{\sup \Re \sigma(A) t} = e^{-\frac{t}{2\|X\|}}$$

thus reaching the bound.

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