Estimating the operator exponential

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Abstract

We obtain some new spectral norm and trace norm estimates for the decay of the operator exponential by means of the solution of the underlying Ljapunov equation.

The aim of this note is to give some new estimates for the exponential function

$$V(t) = e^{At}\,, t > 0 \tag{1}$$

where A is a square matrix or, more generally, the generator of a strongly continuous semigroup in a Hilbert space. Such estimates are of importance in the control of linear systems. Recently in [3] the estimate

$$Tr\, e^{A^* t} e^{At} \le n\|X\|\|X^{-1}\| e^{\frac{t}{\|X\|}} \tag{2}$$

was derived, where $n < \infty$ is the space dimension[1] and

$$X = \int_0^\infty e^{A^* t} e^{At} dt \tag{3}$$

is the unique solution to the Ljapunov equation

$$A^* X + XA = -I\ , \tag{4}$$

under the condition that A be asymptotically stable (this is equivalent to say that all eigenvalues of A have negative real parts). Note that (2) is an immediate consequence of the norm estimate

$$\|e^{A^* t} e^{At}\| \le \|X\|\|X^{-1}\| e^{\frac{t}{\|X\|}} \tag{5}$$

obtained in [1]. This estimate holds in the infinite dimensional space as well ([2]).[2]

In this note we will also be bounding $Tr\, e^{A^* t} e^{At}$, our main improvements with respect to (2) will be

- to get rid of the dimension factor n
- to get rid of the factor $\|X^{-1}\|$

The fact that the factor $\|X^{-1}\|$ can be dispensed with was established in [2] for norm estimates. This factor is much more nefarious when we work with traces: if X is trace class then it cannot have bounded inverse, unless the space is finite dimensional. Our results, contained in the following two theorems, are formulated in terms of form inequalities, which also allow a new operator norm estimate improving the one from [2].

Theorem 1 *Let $V(t)$ be any strongly continuous semigroup such that the strong integral*

$$X \;=\; \int_0^\infty V(t)^* V(t) dt \tag{6}$$

represents an everywhere defined - and therefore bounded selfadjoint - operator. If X has a bounded inverse then

$$V(t)^* V(t) \le \|X^{-1}\| X e^{-\frac{t}{\|X\|}} \quad ^3 \tag{7}$$

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[1] Here and in the following $\|\cdot\|$ denotes the spectral norm or, equivalently the Hilbert space operator norm.

[2] Throughout this paper we will assume certain acquaintance with [2].

[3] Here "$\le$" denotes the usual quadratic form ordering of selfadjoint operators.

in particular, if Z is a Hilbert-Schmidt operator,

$$Tr\, Z^*V(t)^*V(t)Z \le \|X^{-1}\| Tr\, Z^*XZe^{-\frac{t}{\|X\|}} \tag{8}$$

Note that in a finite dimensional space we may set $Z = I$ thus obtaining

$$Tr\, V(t)^*V(t) \le \|X^{-1}\| Tr\, Xe^{-\frac{t}{\|X\|}} \tag{9}$$

which improves (2).

Theorem 2 *Let V and X be as above and let*

$$v_h = \sup_{0\le\tau\le h} \|V(\tau)\|^2 \tag{10}$$

Then for any $h > 0$ and $t > h$

$$V(t)^*V(t) \le \frac{v_h^2}{h} Xe^{-\beta_h(t-2h)} \equiv \frac{v_h^2}{h} X \left(1 + \frac{h}{v_h\|X\|}\right)^2 e^{-\beta_h t} \tag{11}$$

with

$$\beta_h = \frac{1}{h}\log\left(1 + \frac{h}{v_h\|X\|}\right) \tag{12}$$

Corollary 3 *Under the conditions of the theorem above we have*

$$\|V(t)\|^2 \le \frac{v_h^2}{h}\|X\|e^{-\beta_h t}\left(1 + \frac{h}{v_h\|X\|}\right)^2 \tag{13}$$

If, in addition, X is trace class, then

$$Tr\, V(t)^*V(t) \le \frac{v_h^2}{h} Tr\, X \left(1 + \frac{h}{v_h\|X\|}\right)^2 e^{-\beta_h t} \tag{14}$$

Note that by $v_h \ge 1$ our exponent factor β_h is generally less sharp than that in (8). In the limit case $h \to 0$ it approaches the infimum value $1/(v_0\|X\|)$ with $v_0 = \lim_{h\to 0} v_h \ge 1$. If the semigroup $V(t)$ is contractive, then $v_h \equiv 1$ and the limit exponent is again $1/\|X\|$. As in [2] one can easily show that the exponent $1/\|X\|$ (a bound for the type of the semigroup) cannot be improved in general, in fact, it is attained on any normal semigroup. On the other hand, the estimate (13) is an improvement over the corresponding estimate in [2], Th. 4.

Proof of Theorem 1. According to [2], proof of Th. 2., we have

$$V(t)^*XV(t) \le Xe^{-\frac{t}{\|X\|}} \tag{15}$$

from which (7) and (8) immediately follow. Q.E.D.

Proof of Theorem 2. The boundedness of X implies the same for

$$Y_h = \sum_{n=0}^{\infty} T_h^{*n}T_h^n, \quad T_h = V(h) \tag{16}$$

as well as the inequality (cf. [2], formula (24))

$$I \le Y_h \le I + \frac{v_h}{h}X \tag{17}$$

From (6) and (16) it immediately follows

$$X = \int_0^h V(\tau)^*Y_hV(\tau)d\tau \tag{18}$$

Also, according to [2], formula (19)

$$T_h^{*n} T_h^n \leq \left(1 - \frac{1}{\|Y_h\|}\right)^n Y_h \tag{19}$$

For any $t > h$ we can write

$$t = nh + \tau \ , \ n \text{ integer}, \ 0 \leq \tau < h \ .$$

Hence

$$V(t)^* V(t) = T_h^{*n} V(\tau)^* V(\tau) T_h^n \leq v_h T_h^{*n} T_h^n \leq v_h^2 V(nh - \tau')^* V(nh - \tau')$$

for any τ' between 0 and 1. Hence for $t > h$

$$V(t)^* V(t) \leq \frac{v_h^2}{h} \int_0^h V(nh - \tau')^* V(nh - \tau') d\tau'$$

$$= \frac{v_h^2}{h} \int_0^h V(\tau')^* T_h^{*n-1} T_h^{n-1} V(\tau') d\tau'$$

$$\leq \frac{v_h^2}{h} \left(1 - \frac{1}{\|Y_h\|}\right)^{n-1} \int_0^h V(\tau')^* Y_h V(\tau') d\tau'$$

$$= \frac{v_h^2}{h} \left(1 - \frac{1}{\|Y_h\|}\right)^{n-1} X$$

where we have used (18) and (19). By (17) we have

$$\left(1 - \frac{1}{\|Y_h\|}\right)^{n-1} \leq \left(1 - \frac{1}{1 + \frac{v_h}{h}\|X\|}\right)^{n-1} \left(1 + \frac{h}{v_h \|X\|}\right)^{\frac{t}{h} + \frac{\tau + h}{h}}$$

$$= e^{-\beta_h (t - \tau - h)} \leq e^{-\beta_h (t - 2h)}$$

Altogether we obtain (11). Q.E.D.

Th. 2 has an immediate consequence: *if X is trace class then $V(t)^* V(t)$ is also trace class for all $t > 0$.*

References

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