Wielandt and Ky–Fan theorem for matrix pairs

Ivica Nakić [1,2]

Department of Mathematics,
University of Zagreb,
P.O. Box 335,
10000 Zagreb, Croatia

Krešimir Veselić [3]

LG Mathematische Physik,
Fernuniversität Hagen,
Germany

Abstract

The generalization of Wielandt and Ky–Fan theorem is given for Hermitian matrix pairs, and some new eigenvalue perturbation estimates are obtained. The application is made on a class of quadratic matrix pencils.

1 Introduction

The classical Wielandt theorem states that

$$\sup_{\substack{\mathcal{M}_1 \supset \ldots \supset \mathcal{M}_k \\ \dim \mathcal{M}_j = i_j}} \inf_{\substack{x_j \in \mathcal{M}_j \\ x_j^* x_j = \delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j$$

equals the sum $\lambda_{i_1} + \cdots + \lambda_{i_k}$, where $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of the Hermitian matrix $A \in \mathbb{C}^{n \times n}$, and $1 \leq i_1 < \ldots < i_k \leq n$ (see, for instance [18] and [2]).

[1] E–mail address: nakic@math.hr
[2] On leave in LG Mathematische Physik, Fernuniversität Hagen, Germany
[3] E–mail address: kresimir.veselic@fernuni-hagen.de

The Wielandt theorem above was generalized to some specific matrix pairs. In fact, Svarzman in [20] obtained a similar result for the class of Hermitian matrix pairs (A, J), where $J^{-1} = J$ and JA is a positive definite matrix. He obtained a three–part variational characterization of the form "inf sup inf" and "sup inf sup".

Recently, C.-K. Li and R. Mathias in [13] proved a Wielandt–type theorem for the definite matrix pairs (A, B), (i.e. pairs for which there exist numbers $\alpha, \beta \in \mathbb{R}$ such that $\alpha A + \beta B$ is a positive definite matrix), where, instead of eigenvalues, they used angles $\theta = \sphericalangle(Ax, Bx)$, where x is an eigenvector of the pair (A, B).

The classical Ky–Fan theorem ([18], [2]) states that

$$\min_{\substack{X \in \mathbb{C}^{n\times p} \\ X^*X = I_p}} \mathrm{Tr}(X^*AX)$$

equals the sum of the p smallest eigenvalues of the Hermitian matrix $A \in \mathbb{C}^{n\times n}$.

The generalization of this result to the class of Hermitian matrix pairs (A, B), where B is positive definite, is already known ([16]). In [10], Striko-Kovač and Veselić obtained a Ky–Fan–type theorem for the definite matrix pairs.

The classical Wielandt and Ky–Fan theorems are widely used in matrix theory. Using these theorems one can obtain various eigenvalue perturbation bounds, for instance, the famous Cauchy inequality (see [18])

$$\lambda_i(A) + \lambda_{\min}(E) \leq \lambda_i(A + E) \leq \lambda_i(A) + \lambda_{\max}(E)$$

can be derived from the Wielandt theorem. For other perturbation bounds derived from Wielandt and Ky–Fan theorems see, for example, [2] and [13].

Our aim is to give a generalization of the Wielandt and Ky–Fan theorems for an arbitrary Hermitian matrix pair (A, B), the only assumption being the non–singularity of B.

The main tool in this paper will be the variational characterization of eigenvalues of Hermitian matrix pairs given in [7] and [6] (for a variational formulation in an infinite dimensional case see [4]). In [7] the so called "cancellation algorithm" is introduced.

We show that we can variationally characterize, in the form "sup inf", the sum of those non–cancelled eigenvalues which are of the same type, and which do not "jump over" the cancelled pairs of eigenvalues. We first prove the theorem in the case when all real eigenvalues are semi–simple, the general case being a corollary of the semi–simple case [4] . In our Ky–Fan–like theorem, we show that the sum of the p smallest eigenvalues of the same type can be variationally characterized. In both theorems, the matrix B induces a geometry in which we operate.

In the last section we apply both results to a "partly overdamped" quadratic eigenvalue problem.

2 Preliminaries

Let A, B be Hermitian matrices. We say that λ is an eigenvalue of the pair (A, B) if there exists $x \neq 0$ such that $Ax = \lambda Bx$. The set of all eigenvalues of (A, B) is denoted by $\sigma(A, B)$.

In the rest of the paper we treat the case of a Hermitian matrix pair (A, B) from $\mathbb{C}^{N\times N}$, where B is nonsingular.

For indefinite and nonsingular B, it is often appropriate to work in the space with an indefinite inner product induced by B; see [9] for such an approach. In this way, a classification of real eigenvalues can be introduced. Specifically, a real eigenvalue λ is said to be of positive (resp. negative) type if $x^*Bx > 0$ (resp. < 0) for some eigenvector x associated with λ. Note that this definition differs from the usual one, where an eigenvalue λ is of positive (resp. negative) type if $x^*Bx > 0$ (resp. < 0) for every eigenvector x associated with λ. It is possible that some eigenvalue is both of the positive and the negative type, but in this context we consider an eigenvalue with geometric multiplicity k as k eigenvalues, then an eigenvalue is either of positive or negative type. The set of eigenvalues of positive (resp. negative) type of the matrix pair (A, B) we denote by $\sigma^+(A, B)$ (resp. $\sigma^-(A, B)$). With $n^+(A, B)$, $n^-(A, B)$ we denote the number of eigenvalues of positive, (resp. negative) type of pair (A, B), counting multiplicities.

Let $\mathrm{In}(A) = \{\pi(A), \nu(A), \delta(A)\}$ denotes the inertia of a Hermitian matrix A, i.e. π, ν, and δ are the number of positive, negative and zero eigenvalues of A, respectively.

With $\mathcal{S}_n$ we denote the sum of all eigenspaces corresponding to non–real eigenvalues.

For the sake of brevity, we introduce the following notation:
$a(x) = x^*Ax$, $b(x) = x^*Bx$, and $a \gtrless 0$ ($b \gtrless 0$) on some set $\mathcal{S}$ means $a(x) \gtrless 0$

[4] In the first draft of the paper we only proved the semi–simple case. The referee's comments encouraged us to try to prove the general case, in which we succeeded.

($b(x) \gtrless 0$) for all non–zero vectors from $\mathcal{S}$.

We say that the vector x is B–normalized if $b(x) = \pm 1$. We say that the vectors $\{x_1, \ldots, x_k\}$ form an B–orthonormal set if $x_i^* B x_j = \pm\delta_{ij}$, where δ_{ij} denotes the Kronecker symbol. We say that the set $\{x_1, \ldots, x_k\}$ is a B–orthonormal basis for the subspace $\mathcal{W}$ if it is a basis for $\mathcal{W}$ and if it is B–orthonormal. With $\mathcal{B}_\pm$ we denote the set of all x such that $x^* B x \gtrless 0$.

3 Main results

First we consider the case when all real eigenvalues are semi–simple, i.e. with corresponding Jordan chains of length one.

Let $\lambda_{n^-}^- \leq \ldots \leq \lambda_1^-$ and $\lambda_1^+ \leq \ldots \leq \lambda_{n^+}^+$ be the eigenvalues of negative and positive type of the pair (A, B), respectively, counted with their multiplicity. If there exist $\lambda^\pm \in \sigma^\pm(A, B)$ such that $\lambda^+ < \lambda^-$, then we define γ_{-1} as smallest λ_j^- greater then some λ_k^+, and γ_1 as greatest $\lambda_j^+ < \gamma_{-1}$. Obviously, $\gamma_{\pm 1}$ are well defined, $(\gamma_1, \gamma_{-1}) \neq \emptyset$ and does not contain eigenvalues of (A, B). Now we repeat this procedure inductively on the set $\sigma(A, B) \setminus \{\gamma_{\pm i}\}$ until there are no $(+, -)$ pairs. The eigenvalues $\gamma_{\pm 1}, \ldots, \gamma_{\pm c}$ are called "cancelled" eigenvalues, and the number $d = c + n$ is called variational shift. Here we took $n = \frac{1}{2}\dim(\mathcal{S}_n)$. This procedure is called cancellation algorithm. With $r^\pm = r^\pm(A, B)$ we denote the number of non–cancelled eigenvalues of positive (resp. negative type). Notice that $N = 2c + 2n + r^+ + r^- = r^+ + r^- + 2d$. The following figure gives an example of the cancellation algorithm.

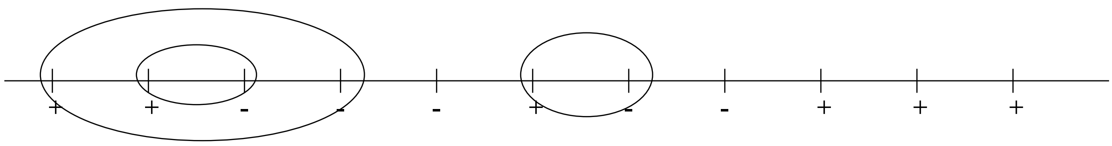

Fig. 1. Cancellation algorithm

With $\mathcal{S}_c$ we denote the sum of all eigenspaces corresponding to cancelled eigenvalues.

The following theorem gives the variational characterization of non-cancelled eigenvalues. Set

$$\sigma_k^\pm = \sup_{\substack{\mathcal{S} \\ \operatorname{codim}\mathcal{S} = k-1}} \inf_{\substack{x \in \mathcal{S} \\ x^* B x \gtrless 0}} \frac{x^* A x}{x^* B x}, \tag{1}$$

with convention that $\inf \emptyset = -\infty$. We will state the theorem only for eigenvalues of positive type. The formulation for eigenvalues of negative type is analogous.

Theorem 1 *Let (A, B) be Hermitian matrix pair such that all real eigenvalues are semi–simple. Let $\mu_1 \leq \ldots \leq \mu_{r^+}$ denote the non-cancelled eigenvalues of positive type. Then we have*

$$\sigma_k^+ = \begin{cases} -\infty & \textit{for } k \leq d, \\ \mu_{k-d} & \textit{for } d < k \leq n^+ + d, \\ +\infty & \textit{for } k > n^+ + d, \end{cases} \tag{2}$$

where d is the variational shift.

The numbers d and $n^+ + d$ can be obtained without prior knowledge of the spectrum. Indeed, it can be seen that $d = \min_{\lambda \in \mathbb{R}} \nu(\lambda)$ and $n^+ + d = \pi(B)$ (see [5]), where $\nu(\lambda) = \nu(A - \lambda B)$.

By simple substitution of A by $-A$ and/or B with $-B$, we get three other "dual" versions of Theorem 1. These "dual" variational formulations are obtained by interchange of inf and sup and/or by change of $\gtreqless$ by $\lesseqgtr$ in (1), along with obvious modifications of the cancellation algorithm. This may lead to different sets of non–cancelled eigenvalues.

3.1 Wielandt–type theorem

We need the following two lemmas from [7].

Lemma 2 *For all real λ there exists a c–dimensional subspace $\mathcal{S}_c(\lambda) \subset \mathcal{S}_c$ on which $b > 0$ and $a - \lambda b < 0$ holds.*

Lemma 3 *For all real λ there exist n–dimensional subspaces $\mathcal{S}_{n0}$, $\mathcal{S}_\pm(\lambda) \subset \mathcal{S}_n$ such that*

$$a = b = 0 \quad \textit{on} \quad \mathcal{S}_{n0},$$

and

$$b > 0, \ \pm(a - \lambda b) > 0 \quad \textit{on} \quad \mathcal{S}_\pm(\lambda).$$

The following Lemma is a simple generalization of Lemma 2.0 from [1].

Lemma 4 *Suppose $\mathcal{W}_1 \supset \cdots \supset \mathcal{W}_k$ are subspaces of $\mathbb{C}^N$ such that $\dim \mathcal{W}_j = k - j + 1$, and $b > 0$ on $\mathcal{W}_j$, $j = 1, \ldots, k$. Let w_j, $j = 1, \ldots, k-1$ be B–orthonormal vectors such that $w_j \in \mathcal{W}_j$, $j = 1, \ldots, k-1$, and let $\mathcal{U} = \operatorname{span}\{w_1, \ldots, w_{k-1}\}$. Then there exists a vector $u \in \mathcal{W}_1 \setminus \mathcal{U}$, $u \neq 0$ such that $\mathcal{U} \dotplus \operatorname{span}\{u\}$ has a B–orthonormal basis $\{v_1, \ldots, v_k\}$, where $v_j \in \mathcal{W}_j$, $j = 1, \ldots, k$.*

The Lemma 4, and Theorem 2.1. and Corollary 2.2. from [1] imply the following result.

Lemma 5 *Let $\mathcal{N}_1 \subset \cdots \subset \mathcal{N}_k$ be subspaces such that $\dim \mathcal{N}_j = d + i_j$, $j = 1, \ldots, k$, $1 \le i_1 < \cdots < i_k \le N - d$, $0 \le d < N$. Suppose also that $b > 0$ on $\mathcal{N}_j$, $j = 1, \ldots, k$. Let $\mathcal{M}_1 \supset \cdots \supset \mathcal{M}_k$ be another set of subspaces such that $\dim \mathcal{M}_j = N - (d + i_j) + 1$, $j = 1, \ldots, k$. Then there exists a B–orthonormal set $\{v_1, \ldots, v_k\}$ where $v_j \in \mathcal{N}_j$, $j = 1, \ldots, k$, and a B–orthonormal set $\{w_1, \ldots, w_k\}$ where $w_j \in \mathcal{M}_j$, $j = 1, \ldots, k$, such that*

$$\operatorname{span}\{v_1, \ldots, v_i\} = \operatorname{span}\{w_1, \ldots, w_i\}, \quad i = 1, \ldots, k.$$

Now we can state our generalization of Wielandt theorem for matrix pairs, in case of semi–simple real eigenvalues. The general case will be obtained as a corollary to this theorem. We will state the result only for eigenvalues of positive type, the version for eigenvalues of negative type is similar.

Theorem 6 *Let (A, B) be a Hermitian matrix pair from $\mathbb{C}^{N\times N}$ with B nonsingular and such that all real eigenvalues are semi–simple. We denote the non–cancelled eigenvalues of negative (resp. positive) type with $\mu^-_{r^-} \le \ldots \le \mu^-_1$ (resp. $\mu^+_1 \le \ldots \le \mu^+_{r^+}$), and cancelled pairs with $\gamma_{\pm j} \in \sigma_\pm(A, B)$, $j = 1, \ldots, c$, $\gamma_1 < \ldots < \gamma_c$.*

Let $1 \le i_1 < \ldots < i_k \le r^+$. Let γ_{-j} be greatest cancelled eigenvalue such that $\gamma_{-j} < \mu^+_{i_1}$. Assume that $\mu^+_{i_k} < \gamma_{j+1}$, i.e. that eigenvalues $\mu^+_{i_1}, \ldots, \mu^+_{i_k}$ do not "jump over" the cancelled pairs (γ_{-j}, γ_j). Then

$$\mu^+_{i_1} + \cdots + \mu^+_{i_k} = \sup_{\substack{\mathcal{N}_1 \supset \cdots \supset \mathcal{N}_k \\ \operatorname{codim}\mathcal{N}_j = i_j + d}} \inf_{\substack{x_j \in \mathcal{N}_j \\ j=1,\ldots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j, \tag{3}$$

where $d = c + n$ is the variational shift.

PROOF. With $u^\pm_j$, $u_{\pm j}$, we denote the corresponding B–orthonormal eigenvectors of $\mu^\pm_j$ and $\gamma_{\pm j}$, respectively.

We define $z_l = u_{\pm l}$ such that $\pm(\gamma_{\pm l} - \mu^+_{i_j}) \ge 0$, $1 \le j \le k$. Set

$$\mathcal{V}_j = \mathcal{S}_n \dotplus \operatorname{span}\{u^+_{i_j}, \ldots, u^+_{r^+}, u^-_1, \ldots, u^-_{r^-}, z_1, \ldots, z_c\}. \tag{4}$$

Obviously, $\mathcal{V}_1 \supset \cdots \supset \mathcal{V}_k$ and $\dim \mathcal{V}_j = N - (d + i_j) + 1$. Notice that we used here "jumping over" assumption. From a straightforward calculation it follows that $(a - \mu^+_{i_j} b)(x) \ge 0$ for all $x \in \mathcal{V}_j$, and $(a - \mu^+_{i_j} b)(u^+_{i_j}) = 0$, hence

$$\min_{\substack{x_j \in \mathcal{V}_j \\ j=1,\ldots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j = \sum_{j=1}^{k} \mu^+_{i_j}.$$

This implies

$$\sup_{\substack{\mathcal{V}_1 \supset \cdots \supset \mathcal{V}_k \\ \mathrm{codim} \mathcal{V}_j = i_j + d}} \inf_{\substack{x_j \in \mathcal{V}_j \\ j=1,\ldots,k \\ x_j^* B x_i = \delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j \geq \sum_{j=1}^{k} \mu_{i_j}^{+}. \tag{5}$$

On the other hand, let $\mathcal{M}_1 \supset \cdots \supset \mathcal{M}_k$ be a subspaces such that $\mathrm{codim} \mathcal{M}_j = d + i_j$. To prove the relation (3) it is sufficient to find vectors $x_j \in \mathcal{M}_j$, $j = 1, \ldots, k$, $x_i^* B x_i = \delta_{ij}$ such that $\sum_{j=1}^{k} a(x_j) \leq \sum_{j=1}^{k} \mu_{i_j}^{+}$. To end this, we define

$$\mathcal{N}_j = \mathcal{S}_c(\mu_{i_1}^{+}) \dotplus \mathcal{S}_{-}(\mu_{i_1}^{+}) \dotplus \mathrm{span}\{u_1^{+}, \ldots, u_{i_j}^{+}\}.$$

Obviously, $\mathcal{N}_1 \subset \cdots \subset \mathcal{N}_k$ and $\dim \mathcal{N}_j = d + i_j$. It is easy to see that $b > 0$ on $\mathcal{N}_j$ and that $a - \mu_{i_j}^{+} b \leq 0$ on $\mathcal{N}_j$. Now, Lemma 5 implies that there exist a B–orthonormal sets $\{x_1, \ldots, x_k\}$, $x_j \in \mathcal{M}_j$, $j = 1, \ldots, k$ and $\{y_1, \ldots, y_k\}$, $y_j \in \mathcal{N}_j$, $j = 1, \ldots, k$, such that

$$\mathrm{span}\{x_1, \ldots, x_k\} = \mathrm{span}\{y_1, \ldots, y_k\} =: \mathcal{W}.$$

Since $\mathcal{W} \subset \mathcal{N}_k$, the subspace $\mathcal{W}$ with the scalar product defined with $[x, y] = y^* B x$ becomes a Hilbert space. With $\mathcal{W}^{[\perp]}$ we denote the orthogonal complement of $\mathcal{W}$ generated by this scalar product, i.e.

$$\mathcal{W}^{[\perp]} = \{x \in \mathbb{C}^N : [x, y] = 0, \forall y \in \mathcal{W}\}.$$

Since $\mathcal{W}$ is a Hilbert space with this scalar product, $\mathcal{W}^{[\perp]}$ is well–defined. With $P_{\mathcal{W}}$ we denote the projector on $\mathcal{W}$ along the subspace $\mathcal{W}^{[\perp]}$. Set $Q_{\mathcal{W}} = P_{\mathcal{W}} B^{-1} A P_{\mathcal{W}}$, for $x, y \in \mathcal{W}$. Then

$$[Q_{\mathcal{W}} x, y] = y^* A y, \quad \text{for all} \quad x, y \in \mathcal{W},$$

hence $Q_{\mathcal{W}}$ is a Hermitian operator on $(\mathcal{W}, [\cdot, \cdot])$. This implies $\mathrm{Tr} Q_{\mathcal{W}} = \sum_{j=1}^{k} \lambda_j$, where $\lambda_1 \leq \ldots \leq \lambda_k$ are the eigenvalues of $Q_{\mathcal{W}}$.

Let v_j be corresponding eigenvector for λ_j, $j = 1, \ldots, k$. Set $\mathcal{U}_j = \mathrm{span}\{x_1, \ldots, x_j\}$, $\mathcal{V}_j = \mathrm{span}\{v_j, \ldots, v_k\}$. Then $\mathcal{U}_j, \mathcal{V}_j \subset \mathcal{W}$ and $\mathcal{U}_j \cap \mathcal{V}_j \neq \{0\}$. Let $u_j \in \mathcal{U}_j \cap \mathcal{V}_j$ be such that $b(u) = 1$. Then

$$\lambda_j \leq [Q_{\mathcal{W}} u, u] = u^* A u \leq \mu_{i_j}^{+}, \quad j = 1, \ldots, k.$$

Hence,

$$\begin{aligned} \sum_{j=1}^{k} a(x_j) &= \sum_{j=1}^{k} [Q_{\mathcal{W}} x_j, x_j] = \mathrm{Tr} Q_{\mathcal{W}} \\ &= \sum_{j=1}^{k} \lambda_j \leq \sum_{j=1}^{k} \mu_{i_j}^{+}, \end{aligned}$$

which, together with (5), finishes the proof. $\square$

The general case will be treated using a perturbation technique. From [6], it always follows the existence of a number $\delta_0 > 0$, and a positive semi–definite matrix P, such that the pair $(A+\delta P, B)$ has all real eigenvalues semi–simple, for all $0 < \delta < \delta_0$. This perturbation also has the following property: the real eigenvalue of the pair (A, B) with Jordan chain of length $m > 1$ and of type $* \in \{+, -\}$ under perturbation becomes

(i) one semi–simple eigenvalue of type $*$ and k non–real eigenvalues, if $m = 2k + 1$,
(ii) two semi–simple eigenvalues, one of the $+$ type, the other of the $-$ type, where eigenvalue of negative type is smaller of the eigenvalue of the positive type, and $k - 1$ non–real eigenvalues, if $m = 2k$, $* = +$,
(iii) k non–real eigenvalues, if $m = 2k$, $* = -$.

Now we introduce the extended cancellation algorithm (see [6]). First we perturb the matrix A by δP, with $0 < \delta < \delta_0$, and then use the cancellation algorithm defined before.

We have ([6])

$$d(\delta) = d, \text{ for all } 0 < \delta < \delta_0,$$

where $d(\delta) = d(A + \delta P, B)$. Now we prove the general result. Again, we only state the result for the eigenvalues of the positive type.

Theorem 7 *Let (A, B) be a Hermitian matrix pair from $\mathbb{C}^{N\times N}$, with B nonsingular. We denote the non–cancelled eigenvalues of negative (resp. positive) type of extended cancellation algorithm with $\mu^-_{r^-} \leq \ldots \leq \mu^-_1$ (resp. $\mu^+_1 \leq \ldots \leq \mu^+_{r^+}$), and cancelled pairs with $\gamma_{\pm j} \in \sigma_\pm(A, B)$, $j = 1, \ldots, c$, $\gamma_1 < \ldots < \gamma_c$.*

Let $d = n + c$, where $2n$ is the dimension of the eigenspace corresponding to all non–real eigenvalues and those eigenvalues which become non–real under the perturbation.

Let $1 \leq i_1 < \ldots < i_k \leq r^+$. Let γ_{-j} be greatest cancelled eigenvalue such that $\gamma_{-j} < \mu^+_{i_1}$. Assume that $\mu^+_{i_k} < \gamma_{j+1}$, i.e. that eigenvalues $\mu^+_{i_1}, \ldots, \mu^+_{i_k}$ do not "jump over" the cancelled pairs (γ_{-j}, γ_j).

Then (3) holds.

PROOF. Let us denote with $\mu^+_i(\delta)$ the corresponding perturbed eigenvalues of the pair $(A + \delta P, B)$, $0 < \delta < \delta_0$. Since the eigenvalues $\mu^+_{i_1}(\delta), \ldots, \mu^+_{i_k}(\delta)$

satisfy the assumptions of the Theorem 6, we have

$$\mu_{i_1}^+(\delta)+\cdots+\mu_{i_k}^+(\delta)=\sup_{\substack{\mathcal{N}_1\supset\cdots\supset\mathcal{N}_k\\ \mathrm{codim}\mathcal{N}_j=i_j+d(\delta)}}\ \inf_{\substack{x_j\in\mathcal{N}_j\\ j=1,\ldots,k\\ x_j^*Bx_i=\delta_{ij}}}\sum_{j=1}^k x_j^*(A+\delta P)x_j.$$

Now, when $\delta\searrow 0$ we obtain (3). $\square$

By simple substitution of A by $-A$ and/or B by $-B$ we get three other "dual" versions of the cancellation algorithm, and, hence, of Lemmas 2, 3 and of Theorem 7. Also, by imitating the proof of [1, Theorem 2.3], we can generalize Theorem 7 by replacing the left hand side of (3) by $\varphi(\mu_{i_1}^+,\ldots,\mu_{i_k}^+)$, and the right hand side by the corresponding relation, where φ is some symmetric function in k variables. We omit the details.

Notice that for definite matrix pairs $d=0$, hence all eigenvalues are non–cancelled.

Also, from the proof of the Theorem 6 it follows that it always holds

$$\mu_{i_1}^+ +\cdots+\mu_{i_k}^+\le\sup_{\substack{\mathcal{N}_1\supset\cdots\supset\mathcal{N}_k\\ \mathrm{codim}\mathcal{N}_j=i_j+d}}\ \inf_{\substack{x_j\in\mathcal{N}_j\\ j=1,\ldots,k\\ x_j^*Bx_i=\delta_{ij}}}\sum_{j=1}^k x_j^*Ax_j,$$

regardless of the "jumping over" condition.

The following example shows that we cannot, in general, drop the assumption of "jumping over".

Example 8 *Let*

$$A=\mathrm{diag}(-1,0,-1,2),\ B=\mathrm{diag}(1,1,-1,1).$$

Using the cancellation algorithm we obtain that $\mu_1^+=-1$, $\mu_2^+=2$, $\gamma_1=0$, $\gamma_{-1}=1$.

Let $\mathcal{V}$ *be an arbitrary three–dimensional subspace. We define* $\mathcal{Y}=\mathrm{span}\{e_1,e_2,e_3\}$. *It is easy to see that for each* $x,y\in\mathcal{Y}$, $x^*Bx>0$, $y^*By>0$ *it holds*

$$\frac{x^*Ax}{x^*Bx}+\frac{y^*Ay}{y^*By}\le 0.$$

Notice that $\dim(\mathcal{Y}\cap\mathcal{V})\ge 2$. *We define* $\mathcal{X}=\mathrm{span}\{e_1,e_2,e_4\}$. *The subspace* $\mathcal{X}$ *is* B*–positive definite subspace. It holds* $\dim(\mathcal{Y}\cap\mathcal{V}\cap\mathcal{X})\ge 1$, *hence there exists* $0\ne x_1\in\mathcal{Y}\cap\mathcal{V}\cap\mathcal{X}$.
Let $0\ne y_1\in\mathcal{Y}\cap\mathcal{V}\dot{-}\mathrm{span}\{x_1\}$ *be arbitrary. If* $y_1^*By_1>0$, *then there exist*

$x, y \in \mathrm{span}\{x_1, y_1\} \subset \mathcal{Y} \cap \mathcal{V}$ such that $x^*Bx > 0$, $y^*By > 0$ and $x^*By = 0$. If $y_1^*By_1 < 0$, there exists $\alpha > 0$ such that for $y_0 = \alpha x_1 + y_1$ holds $y_0^*By_0 > 0$, and again we can find $x, y \in \mathcal{Y} \cap \mathcal{V}$ such that $x^*Bx > 0$, $y^*By > 0$ and $x^*By = 0$.

Hence, for each three–dimensional subspace $\mathcal{V}$ it holds

$$\min_{\substack{x,y\in\mathcal{V}\\ x^*Bx>0, y^*By>0\\ x^*By=0}} \frac{x^*Ax}{x^*Bx} + \frac{y^*Ay}{y^*By} \leq 0,$$

which implies

$$\sup_{\substack{\mathcal{N}_1\supset\mathcal{N}_2\\ \mathrm{codim}\mathcal{N}_j=j+1}} \inf_{\substack{x_j\in\mathcal{N}_j\\ j=1,2\\ x_i^*Bx_j=\delta_{ij}}} \sum_{j=1}^{2} x_j^*Ax_j < -1 + 2 = \mu_1^+ + \mu_2^+.$$

Remark 9 *Let (A, B) be a strongly definitizable matrix pair, i.e. a Hermitian matrix pair with all real eigenvalues semi–simple, and not of mixed type (for more details see [11] and [15]). Let δA be a Hermitian perturbation such that the pair $(\widetilde{A}, B) = (A + \delta A, B)$ has the same eigenvalue structure as the pair (A, B). The sufficient conditions on δA that imply this assumptions can be found in [15].*

Using the technique of [19] (see also [12]) we can obtain an eigenvalue bound. We present only the bound for eigenvalues of positive type, bound for the eigenvalues of negative type is similar.

Let μ_j^+, $1 \leq j \leq r^+$, be the non–cancelled eigenvalue of positive type, where the eigenvalues are ordered as in the Theorem 6, and let $\widetilde{\mu}_j^+$ be corresponding eigenvalue of the pair $(\widetilde{A}, B)$. Let $\mathcal{V}_j$ be defined with (4), and let $\widetilde{\mathcal{V}}_j$ be a corresponding subspace for the pair $(\widetilde{A}, B)$.

We define

$$r_j = \max\{ \sup_{\substack{x\in\mathcal{V}_j\\ b(x)=1}} x^*\delta Ax, \sup_{\substack{x\in\widetilde{\mathcal{V}}_j\\ b(x)=1}} x^*\delta Ax\}.$$

Then

$$|\widetilde{\mu}_j^+ - \mu_j^+| \leq r_j.$$

Obviously, r_j can be bounded with $\eta = \sup_{b(x)=1} x^*\delta Ax$, but, unfortunately, as can be seen from (2), it is possible that $\eta = \infty$.

3.2 Ky–Fan theorem

Now we prove the generalization of the well–known Ky–Fan theorem. The theorem will be formulated only for eigenvalues of positive type, the formulation for the eigenvalues of negative type being analogous.

Theorem 10 *Let (A, B) be a Hermitian matrix pair, where B is non–singular. Let $\mu_j^{\pm}$, $r^{\pm}$, and c be defined as in Theorem 7. Then for $1 \le p \le r^+$ it holds*

$$\min_{\substack{X \in \mathbb{C}^{N\times p} \\ X^*BX = I_p}} \operatorname{Tr}(X^*AX) = \sum_{j=1}^{p} \mu_i^+, \tag{6}$$

where I_p denotes the identity matrix in $\mathbb{C}^{p\times p}$.

PROOF. As before, we first prove the case when all real eigenvalues are semi–simple. Set $X = [x_1 \dots x_p]$, where x_i are eigenvectors corresponding to μ_i^+, $i = 1, \dots, p$, such that $b(x_i) = 1$. Then, obviously,

$$\operatorname{Tr}(X^*AX) = \sum_{i=1}^{p} \mu_i^+ \quad \text{and} \quad X^*BX = I_p.$$

Let $(\mathcal{H}, (\cdot,\cdot))$ be an abstract complex N–dimensional space. Take some basis in $\mathcal{H}$. There exist Hermitian operators such that A and B are their matrices in this basis. Those operators we also denote by A and B, respectively. On $\mathcal{H}$ we introduce a new scalar product defined by $[f, g] = (Bf, g)$ for $f, g \in \mathcal{H}$, and with $[\perp]$ we denote orthogonality in this new scalar product.

Let $X \in \mathbb{C}^{N\times p}$ be such that holds $X^*BX = I_p$. With $\mathcal{X}$ we denote the subspace in $\mathcal{H}$ corresponding to X, i.e the columns of X are coordinate vectors of some B–orthonormal basis for $\mathcal{X}$. From $X^*BX = I_p$ it follows that $\mathcal{X}$ is a positive definite subspace in $(\mathcal{H}, [\cdot,\cdot])$, i.e. $[f, f] > 0$ for $f \in \mathcal{X} \setminus \{0\}$, hence there exists a B–orthonormal projector $P_{\mathcal{X}} : \mathcal{H} \to \mathcal{X}$. Set $S = B^{-1}A$ and $\widehat{S} = P_{\mathcal{X}} S P_{\mathcal{X}}$. The operator $\widehat{S}$ is obviously Hermitian operator on Hilbert space $(\mathcal{X}, [\cdot,\cdot])$. Let $f, g \in \mathcal{X}$. Then

$$[\widehat{S}f, g] = [SP_{\mathcal{X}}f, P_{\mathcal{X}}g] = [Sf, g] = (Ax, y),$$

hence the matrix $X^*AX \in \mathbb{C}^{p\times p}$ is the corresponding matrix of the operator $\widehat{S}$.

With $\lambda_1 \le \dots \le \lambda_p$ we denote the eigenvalues of the operator $\widehat{S}$, e.g. of matrix X^*AX. From previous considerations and variational principle given in Theorem 1, for $1 \le i \le p$ it follows

$$\begin{aligned}
\lambda_i &= \max_{\substack{\mathcal{S}\subset\mathcal{X}\\ \dim S=i-1}} \min_{\substack{f\in\mathcal{X}\\ f[\perp]\mathcal{S}}} \frac{[\widehat{S}f,f]}{[f,f]} \\
&= \max_{g_1,\dots,g_{i-1}\in\mathcal{X}} \min_{\substack{f\in\mathcal{X}\\ f\in\operatorname{span}\{g_1,\dots,g_{i-1}\}^{[\perp]}}} \frac{[\widehat{S}f,f]}{[f,f]} \\
&= \max_{g_1,\dots,g_{i-1}\in\mathcal{H}} \min_{\substack{f\in\mathcal{X}\\ f\in\operatorname{span}\{P_{\mathcal{X}}g_1,\dots,P_{\mathcal{X}}g_{i-1}\}^{[\perp]}}} \frac{[\widehat{S}f,f]}{[f,f]} \\
&= \max_{g_1,\dots,g_{i-1}\in\mathcal{H}} \min_{\substack{f\in\mathcal{X}\\ f\in\operatorname{span}\{g_1,\dots,g_{i-1}\}^{[\perp]}}} \frac{[\widehat{S}f,f]}{[f,f]} \ge \\
&\ge \sup_{g_1,\dots,g_{i-1}\in\mathcal{H}} \inf_{\substack{f\in\mathcal{H}\\ f\in\operatorname{span}\{g_1,\dots,g_{i-1}\}^{[\perp]}\\ (Bf,f)>0}} \frac{(Af,f)}{(Bf,f)} \\
&= \sup_{\substack{\mathcal{M}\subset\mathcal{H}\\ \operatorname{codim}\mathcal{M}=i-1}} \inf_{\substack{f\in\mathcal{M}\\ (Bf,f)>0}} \frac{(Af,f)}{(Bf,f)} = \mu_i^+.
\end{aligned}$$

The second equality in the preceding formula is obtained using the fact that

$$\begin{aligned}
&\max_{g_1,\dots,g_{i-1}\in\mathcal{X}} \min_{\substack{f\in\mathcal{X}\\ f\in\operatorname{span}\{g_1,\dots,g_{i-1}\}^{[\perp]}}} \frac{[\widehat{S}f,f]}{[f,f]} \\
&= \max\left\{ \max_{\substack{\mathcal{S}\subset\mathcal{X}\\ \dim\mathcal{S}=j}} \min_{\substack{f\in\mathcal{X}\\ f[\perp]\mathcal{S}}} \frac{[\widehat{S}f,f]}{[f,f]} : j=1,\dots,i-1 \right\} \\
&= \max\{\lambda_i, \lambda_{i-1},\dots,\lambda_1\}.
\end{aligned}$$

Similarly is obtained the last equality. Since $\lambda_1+\cdots+\lambda_p = \mathrm{Tr}\widehat{S} = \mathrm{Tr}X^*AX$, it follows

$$\mathrm{Tr}X^*AX \ge \sum_{j=1}^{p} \mu_j^+.$$

As before, by use of perturbation technique, we easily prove the general case. $\square$

4 Application

Let C, K be symmetric matrices such that K is positive definite and C is positive semi–definite. We will investigate the spectrum of the matrix pencil

$$L(\lambda) = \lambda^2 I + \lambda C + K.$$

The spectrum of the matrix pencil $L(\lambda)$ is the set of complex numbers λ such that there is a non–zero vector x such that $L(\lambda)x = 0$. It is easy to see that the spectrum of $L(\lambda)$ is situated in a left half plane.

We assume that matrices C and K can be decomposed in the following way:

$$C = \begin{bmatrix} 0 & 0 \\ 0 & C_{22} \end{bmatrix}, K = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix}, \tag{7}$$

where C_{22} is positive definite. We assume also that the operator pencil $L_2(\lambda) = \lambda^2 I + \lambda C_{22} + K_{22}$ is overdamped. A quadratic pencil $L(\lambda) = \lambda^2 + \lambda C + K$ is overdamped if $x^*Cx^2 > 4x^*Kx$ holds for all $\|x\| = 1$. Let $x^* = (u \;\; v)$. Then

$$(\lambda^2 I + \lambda C + K)x = 0$$

can be written as

$$\lambda^2 \begin{bmatrix} u \\ v \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & C_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^* & K_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

From this it follows

$$\lambda^2 u + K_{11}u + K_{12}v = 0,$$
$$\lambda^2 v + \lambda C_{22}v + K_{12}^* u + K_{22}v = 0.$$

Substituting u in the second equation from the first we get

$$\lambda^2 v + \lambda C_{22}v + (K_{22} - K_{12}^*(K_{11} + \lambda^2)^{-1}K_{12})v = 0. \tag{8}$$

We introduce the matrix function $K(\lambda) = K_{22} - K_{12}^*(K_{11} + \lambda^2)^{-1}K_{12}$, and the (nonlinear) functionals

$$p_\pm(v, \lambda) = \frac{-v^*C_{22}v \pm \sqrt{v^*C_{22}v - 4v^*vv^*K(\lambda)v}}{2v^*v}. \tag{9}$$

Obviously, $p_\pm(v, \lambda) = \lambda$, if and only if $\lambda^2 v^*v + \lambda v^*C_{22}v + v^*K(\lambda)v = 0$. Hence, if λ is an eigenvalue of $L(\lambda)$ with corresponding eigenvector $x^* = (u \;\; v)$, then we have $p_+(v, \lambda) = \lambda$ or $p_-(v, \lambda) = \lambda$ or both. Since $K(\lambda) \le K_{22}$, $p_+(v, \lambda) \ge p'_+(v)$ and $p_-(v, \lambda) \le p'_-(v)$, where $p'_\pm$ is the corresponding functional for the matrix pencil $L_2(\lambda)$. Since the matrix pencil $L_2(\lambda)$ is overdamped, we have $\sup p'_- < \inf p'_+$ [8], hence it follows that there is a gap in the real part of the spectrum of the pencil $L(\lambda)$.

Next we introduce the linearization matrix of matrix pencil $L(\lambda)$:

$$A = \begin{bmatrix} 0 & K^{1/2} \\ -K^{1/2} & -C \end{bmatrix}.$$

Obviously, A is J–Hermitian matrix, where $J = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. Also, it is easy to see that λ is an eigenvalue of the matrix pencil $L(\lambda)$ if and only if λ is an eigenvalue of the Hermitian matrix pair (JA, J). Hence, we can split the eigenvalues of matrix pencil $L(\lambda)$ into those of positive and negative type, as in Section 2.

Let $p_{\pm}(x)$ be corresponding functional for the matrix pencil $L(\lambda)$, constructed analogously as in (9). We will show that λ is an eigenvalue of positive (negative) type, if and only if $p_+(x) = \lambda$ ($p_-(x) = \lambda$) for some x. To this end, let (λ, x) be an eigenpair for matrix A, with $x^*Jx > 0$, $x^* = (x_1 \quad x_2)$. From the construction of A it follows that

$$\begin{aligned} \lambda x_1 &= K^{1/2} x_2, \\ \lambda^2 x_2 + \lambda C x_2 + K x_2 &= 0. \end{aligned}$$

The relation $x^*Jx > 0$ implies $x_1^* x_1 > x_2^* x_2$. Hence, $\lambda^2 x_1^* x_1 = x_2^* K x_2$ implies $\lambda^2 x_2^* x_2 < \lambda^2 x_1^* x_1 = x_2^* K x_2$. From

$$0 = \lambda^2 x_2^* x_2 + \lambda x_2^* C x_2 + x_2^* K x_2 > 2\lambda^2 x_2^* x_2 + \lambda x_2^* C x_2,$$

it follows

$$p_-(x_2) < \frac{-x_2^* C x_2}{2 x_2^* x_2} < \lambda,$$

hence $\lambda = p_+(x_2)$, and conversely, $p_-(v) = \lambda$ implies $x^*Jx < 0$. The other case is similar.

Also, it can easily be seen that an eigenvalue λ of matrix A has a Jordan chain $\{x^0, x^1\}$, i.e. $Ax^1 = \lambda x^1 + x^0$, $Ax^0 = \lambda x^0$, if and only if

$$L(\lambda) x_2^1 + L'(\lambda) x_2^0 = 0, \tag{10}$$
$$L(\lambda) x_2^0 = 0, \tag{11}$$

where $x^0 = \left(\begin{smallmatrix} x_1^0 \\ x_2^0 \end{smallmatrix}\right)$, $x^1 = \left(\begin{smallmatrix} x_1^1 \\ x_2^1 \end{smallmatrix}\right)$.

Let (λ, x) be an eigenpair of the matrix pencil $L(\lambda)$, $x^* = (u \quad v)$, and let $p_-(v, \lambda) = \lambda$, i.e.

$$\lambda = \frac{-v^* C_{22} v - \sqrt{v^* C_{22} v - 4 v^* v v^* K(\lambda) v}}{2 v^* v}.$$

Then, it follows that $2\lambda v^*v + v^*C_{22}v < 0$, and since $v^*v \le x^*x$ and $v^*C_{22}v = x^*Cx$, it follows

$$\lambda < -\frac{x^*Cx}{2x^*x} \le p_+(x),$$

hence, $\lambda = p_-(x)$. So, all eigenvalues from the left side of the gap are of negative type. This also implies that all eigenvalues on the left side of the gap are semi–simple, which follows from the fact that the eigenspace corresponding to these eigenvalues is definite.

Unfortunately, the eigenvalues on the right side of the gap need not be of positive type. This can be readily seen from the Figure 2. The picture on Figure 2. is obtained by plotting the eigenvalues of random matrix pencil of order three, which satisfy our conditions, while keeping the smallest eigenvalue of K to one.

An concrete example is the matrix pencil with the following entries:

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 301.822 \end{bmatrix}, K = \begin{bmatrix} 136.63382 & 113.58142 & 1583.7435 \\ 113.58142 & 97.472318 & 1343.3431 \\ 1583.7435 & 1343.3431 & 18709.024 \end{bmatrix}.$$

The eigenvalues of this pencil are $\lambda_1 = -215.39099, \lambda_2 = -82.517913, \lambda_3 = -3.3700361, \lambda_{45} = -0.04725 \pm 1.34i, \lambda_6 = -0.44885985$. The eigenvalues λ_1 and λ_3 are of negative type, other (real) eigenvalues are of positive type. The gap is $[-214.6697, -87.152]$, hence the eigenvalue λ_3 is of negative type, and it is on the right hand side of the gap.

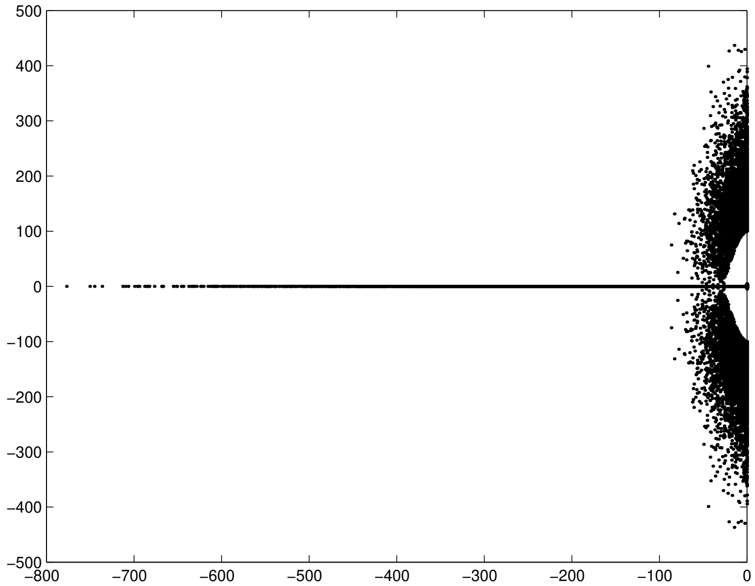

Fig. 2. Distribution of the eigenvalues

It can be seen that the number of eigenvalues on the left side of the gap, counted by multiplicity, equals the range of the matrix C.

Moreover, inside of the circle with the center in the origin and the radius $\lambda_{\min}^{1/2}(K)$ there are no non-real eigenvalues. Indeed, let us assume that $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is an eigenvalue of $L(\lambda)$ with the corresponding eigenvector x. Then it holds

$$\lambda = \frac{-x^*Cx \pm \sqrt{x^*Cx^2 - 4x^*xx^*Kx}}{2x^*x}.$$

Since λ is non–real, this can be written as

$$\lambda = \frac{-x^*Cx}{2x^*x} \pm i\frac{\sqrt{4x^*xx^*Kx - x^*Cx^2}}{2x^*x},$$

hence

$$|\lambda|^2 = \frac{x^*Kx}{x^*x}.$$

We can summarize results of this Section in

Theorem 11 *Let $L(\lambda) = \lambda^2 I + \lambda C + K$ be a Hermitian quadratic pencil with C and K positive semi–definite. Assume that*

(1) matrices C and K can be decomposed as in (7), and
(2) operator pencil $L_2(\lambda) = \lambda^2 I + \lambda C_{22} + K_{22}$ is overdamped.

Let r be the size of C_{22}, and denote with $\mu_r^- \leq \ldots \leq \mu_1^-$ the eigenvalues from the left side of the gap $[\sup p'_-, \inf p'_+]$. Then the following holds.

(1) Let $1 \leq i_1 < \ldots < i_k \leq r$. Then

$$\mu_{i_1}^- + \cdots + \mu_{i_k}^- = \sup_{\substack{\mathcal{N}_1 \supset \cdots \supset \mathcal{N}_k \\ \mathrm{codim}\mathcal{N}_j = i_j + d}} \inf_{\substack{x_j \in \mathcal{N}_j \\ j=1,\ldots,k \\ x_j^* J x_i = -\delta_{ij}}} \sum_{j=1}^{k} x_j^* A x_j.$$

(2) The following holds

$$\min_{\substack{X \in \mathbb{C}^{N\times r} \\ X^*JX = -I_r}} \mathrm{Tr}(X^*AX) = \sum_{j=1}^{r} \mu_i^-.$$

Using the variational characterization from [3], one can also obtain the eigenvalues from the left hand side of the gap. More precisely, for the eigenvalues on the left hand side of the gap [3] yields

$$\mu_j^- = \min_{\substack{S \\ \dim S = j \\ S \subset \mathcal{M}}} \max_{\|x\|=1} p_-(x),$$

where $\mathcal{M} = \{x : x^*Cx \geq x^*Kxx^*x\}$. The advantage of our approach is that we can also characterize the sum of eigenvalues.

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