Comprehensive focusing analysis of various Fresnel zone plates

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A series-form expression for the individual diffracted field of a general annular ring is derived from the Rayleigh–Sommerfeld diffraction integral. It can be used for the accurate and fast simulation of any diffractive focusing element composed of concentric transparent rings. We present a comprehensive analysis, based on the leading term and the linear superposition principle, of the focusing performances of various Fresnel zone plates. Many problems, such as the equivalent aperture function, the diffraction efficiency, the focal spot pattern, the suppression of higher orders and the appearance of “fractional orders,” and the explanation for the appearance of Fraunhofer diffraction patterns, are analytically investigated in detail. Because of the great similarity between Fresnel zone plates and multilevel diffractive lenses, most of the obtained results are also applicable to multilevel diffractive lenses. © 2004 Optical Society of America


1. INTRODUCTION

The focusing and the imaging of soft x rays and extreme-ultraviolet (EUV) radiation have many applications in physics and the life sciences, such as in high-resolution microscopy, spectroscopy, and nanolithography. Unfortunately, the refractive lens cannot be used for this kind of microscopy, spectroscopy, and nanolithography. Traditionally, the refractive lens cannot be used for this kind of focusing, because all solids are strongly absorbing in the spectral regions of soft x rays and EUV radiation. Traditional Fresnel zone plates (TFZPs) can be used for this kind of focusing. However, the focal spot size of a TFZP is approximately the order of the width of the outermost half-zone, so its spatial resolution is limited in technology by the smallest structure (20–40 nm) that can be fabricated by lithography. This drawback can be partially overcome by a composite FZP in the paraxial case. However, the sidelobes may still exist and blur the image. Recently, a novel diffractive optical element called a photon sieve, which consists of a great number of pinholes properly distributed over the Fresnel zones, was proposed to overcome the above-mentioned drawbacks of a TFZP.

Following the initial work by Kipp et al., we developed the individual far-field model for photon sieves. More recently, stimulated by the success of this simple model for photon sieves, we derived an approximate formula for the individual diffracted fields of the individual open rings of a FZP. This formula is quite simple, each individual diffracted field including only one term that is related to the zero-order Bessel function of the first kind. Basing our approach on this elegant analytical treatment and the linear superposition principle, we propose the modified Fresnel zone plates (MFZPs), which can produce sharp Gaussian focal spots for the focusing and the imaging of soft x rays and EUV radiation. Similar to a photon sieve, such a MFZP can also increase the spatial resolution and suppress the sidelobes. The produced focal spot at the main focal plane is even better than that produced by a photon sieve, because a focused Gaussian beam has the additional advantages of circular symmetry, long focal depth, and excellent beam quality.

In this paper, we shall further improve the analytical treatment developed in Ref. 22 and extend its application range to general diffractive focusing elements composed of concentric transparent rings. In Section 2, we shall present a series-form expression for the diffracted field of an individual annular ring. The leading term is just the approximate formula previously presented in Ref. 22. The new expression can be used for the accurate and fast simulations of various FZPs. In Section 3, we shall investigate the focusing problem of various FZPs with a large number of open rings. In particular, we shall introduce the equivalent aperture function, consider the construction of various desirable focal spot patterns, investigate the diffraction efficiency, discuss the suppression of high order and the appearance of “fractional orders,” and explain the appearance of the Fraunhofer diffraction pattern.

In Section 4, we shall conclude this paper and discuss some additional problems. Because of the great similarity between Fresnel zone plates and multilevel diffractive lenses, most of the following investigations are also valid for DLs.

2. ANALYTICAL FORMULA FOR DIFFRACTED FIELDS OF INDIVIDUAL ANNULAR RINGS

Consider a FZP consisting of N concentric open rings. As shown in Fig. 1, it is perpendicularly illuminated by a plane wave with unit amplitude. It is suitable to use polar coordinates for those transverse planes that are perpendicular to the propagation axis. We denote by (r, θ) the polar coordinates at the aperture plane. Similarly, we denote by (R, φ) the polar coordinates at the focal plane. The focal length is f, and the desired focal point is located at (R = 0, φ = φ0). For convenience, we shall also use the coordinate s = r2 in the remainder of this pa-
per. We denote by $U(R)$ the total diffracted field distribution at the focal plane because it is rotationally symmetric. For the same reason, we denote by $U_n(R)$ the individual diffracted field at the focal plane from the $n$th open ring. According to the linear superposition principle, $U(R)$ is the simple sum of the individual diffracted fields $U_n(R)$, i.e., $U(R) = \sum_{n=1}^{N} U_n(R)$. From the Rayleigh–Sommerfeld diffraction integral,28–30 we know that

$$U_n(R) = \frac{1}{\lambda} \int_{A_n} \frac{f}{p^2} \exp(jk\rho)r dr d\theta,$$  

(1)

where $j$ is the imaginary unit, $k = 2\pi/\lambda$, $\lambda$ is the wavelength, $\rho = [f^2 + R^2 + r^2 - 2Rr \cos(\theta - \phi)]^{1/2}$, $A_n$ is the area of the $n$th open ring, and a constant factor $-j$ has been ignored. Equation (1) is accurate provided that $f \gg \lambda$. We rewrite $\rho$ as $\rho = [f^2 + r^2 + R^2 + (r^2 - r_n^2) - 2Rr \cos(\theta - \phi)]^{1/2}$, where $r_n$ is the characteristic coordinate of the $n$th open ring. We shall further define the parameter $r_n$ below. For sharp focusing, the area of interest at the focal plane is actually the focal area, in which the radial coordinate $R$ is small. Also, the quantity $r^2 - r_n^2$ is far smaller than the quantity $f^2 + r_n^2$ because the width of the $n$th open ring is small. By taking these properties into account, we expand $\rho$ as

$$\rho \approx f_n + \frac{R^2 + (r^2 - r_n^2) - 2Rr \cos(\theta - \phi)}{2f_n},$$  

(2)

where $f_n = (f^2 + r_n^2)^{1/2}$. It is worth mentioning that a similar expansion for $\rho$ was presented in Ref. 30, which studied the off-axis diffraction of circular apertures. We use the approximation (2) for the $\rho$ in the exponential of Eq. (1). For the $\rho$ in the denominator of Eq. (1), the approximation $\rho = f_n$ is already good enough. By substituting these two approximations into Eq. (1) and using the equality $f_n^{3/2} \exp(-ju \cos(\theta - \phi)) d\theta = 2\pi J_0(u)$, where $u = kRr/f_n$, one can obtain

$$U_n(R) = \frac{kf_n}{f_n} \exp\left[jk\left(f_n + \frac{R^2}{2f_n}\right)\right] G(R),$$  

(3)

$$G(R) = \int_{a_n}^{b_n} \exp\left[jk\left(\frac{r^2 - r_n^2}{2f_n}\right)\right] J_0\left(\frac{kRr}{f_n}\right) r dr,$$  

(4)

where $J_0(\cdot)$ is the zero-order Bessel function of the first kind and $a_n$ and $b_n$ are the radii of the lower and upper edges of the $n$th open ring, respectively. It should be mentioned that the result of Eqs. (3) and (4) was previously presented in Ref. 22. However, we here rederive it for readability. It is convenient to describe the $n$th open ring in the $s = r^2$ coordinate. We denote by $s_n$ and $d_n$ the midpoint and the half-width, respectively, of the $n$th open ring in the $s$ coordinate. One can easily find the relations $s_n = (a_n^2 + b_n^2)/2$ and $d_n = (b_n^2 - a_n^2)/2$ corresponding to these definitions. To this end, we clearly define the characteristic coordinate $r_n$ by $r_n = s_n^{1/2}$.

The validity of Eq. (3) depends on the approximation (2). According to the Taylor-series expansion, this approximation is highly valid for the phase factor when $|\varphi_4| = k[\frac{R^2 + (r^2 - r_n^2)}{2} - 2Rr \cos(\theta - \phi)](8f_n^2) < 1$, where $\varphi_4$ is the lowest-order one of those discarded phase terms. We first consider its validity for the field values at the focal point. Simply letting $R = 0$, one can get $|\varphi_4| = k(r^2 - r_n^2)/(8f_n^2) = k d_n^2/(8f_n^2)$. To have an intuitive impression of the quantity of $|\varphi_4|$ in this case, we consider a TFZP. It can be proved that $d_n = \lambda f_n/2$ for those open rings of a TFZP. By use of this property and the relation $f_n \gg f$, we find that $|\varphi_4| \leq \pi \lambda(16f_n^2)$. If one lets $\lambda = 2.4$ nm and $f = 500 \mu$m, which are the typical values for the application in the soft-x-ray spectral region,17,22 one can find that $|\varphi_4| \approx 9.4 \times 10^{-7}$ rad. This value explicitly shows the excellent validity of relation (2) for the field values at the focal point. We now further discuss the validity of relation (2) for the field values in the focal area. In this area, the radial coordinate $R$ is so small that the term $R^2$ in $\varphi_4$ can be ignored. Also, $|\cos(\theta - \phi)| \approx 1$. By taking these properties into account, we get $|\varphi_4| \leq k(r^2 - r_n^2)/2Rr^2/(8f_n^2)$. To have an intuitive impression of the quantity of $|\varphi_4|$ in the focal area, we again consider the above-mentioned TFZP. In this case, we have $|r^2 - r_n^2| = |(r + r_n)(r - r_n)| \approx 2wr - r_nw = wR_n$, where $w$ is the width of the $n$th open ring in the real $r$ coordinate. We then let $R = C_0w$, where $C_0$ is a variable. By use of the relations $\lambda = 2.4$ nm, $f = 500 \mu$m, $R = C_0w$, $r \approx r_n$, $wR_n \approx (b_n - a_n)(b_n + a_n)/2 = d_n = \lambda f_n/2$, and $f_n \gg f$, we get $|\varphi_4| \leq (2C_0 + 1)^2 \pi \lambda(16f_n^2) \approx 9.4 	imes 10^{-7} \times (2C_0 + 1)^2$ rad. This functional relation is shown in Fig. 2. We find that $|\varphi_4| \leq 4.2 \times 10^{-4}$ in the range $0 \leq R \leq 10w$. This result explicitly shows the excellent validity of relation (2) for the range $0 \leq R \leq 10w$. It is well-known that the focal spot size of a TFZP is approximately the order of the width of the outermost open ring. Because the outermost open ring is the narrowest one, the range $0 \leq R \leq 10w$ actually includes a circular region that is far larger (at least ten times larger) than the focal spot size. Outside this region, the intensity value is definitely negligible if the number of open rings is not very few. Therefore one does not need to worry about the validity of Eq. (3) at all if the number of open rings is not extremely few.
We now try to derive an analytical expression for \( G(R) \). For this purpose, we re-express Eq. (4) as

\[
G(R) = \frac{1}{2} \int_{s_n-d_n}^{s_n+d_n} \exp \left( jk - \frac{R}{f_n} s \right) \frac{dJ_m\left( kR/f_n \right)}{s^{1/2}} ds. \tag{5}
\]

We then further expand the integrated function \( J_0(kRs^{1/2}/f_n) \) in Eq. (5) as the following Taylor series:

\[
J_0(\epsilon s^{1/2}) = \sum_{m=0}^{\infty} \frac{1}{m! \left[ \frac{d^m}{ds^m} J_0(\epsilon s^{1/2}) \right]} \left|_{s=s_n} \right. (s-s_n)^m, \tag{6}
\]

where \( \epsilon = kR/f_n \). By use of this Taylor expansion, we finally obtain the series-form expression for \( G(R) \) (see Appendix A):

\[
G(R) = \sum_{m=0}^{\infty} G_m(R), \tag{7}
\]

\[
G_m(R) = \frac{j^m (aR)^m d_n^{m+1}}{m! 12^m \gamma^m} \, h_m J_m(aR), \tag{8}
\]

\[
h_m = \sum_{i=0}^{m} (-1)^i m! \frac{\gamma^m-i}{(m-i)!} \, \text{Im}[(i \gamma)^{m-i} \exp(i \gamma)], \tag{9}
\]

where \( \alpha = kr_n/f_n \), \( \gamma = k d_n/(2 f_n) \), \( \text{Im}(\cdot) \) expresses the imaginary part, and \( J_m(\cdot) \) is the \( m \)-th-order Bessel function of the first kind. In particular, the first two terms are simply given by

\[
G_0(R) = \frac{\sin(\gamma) J_0(aR)}{\gamma}, \tag{10}
\]

\[
G_1(R) = \frac{j d_n^2 aR}{2 \sin^{2}(\gamma)} \left[ \gamma \cos(\gamma) - \sin(\gamma) \right] J_1(aR). \tag{11}
\]

From Eqs. (7)–(9), one can find that all the even-order terms are purely real and all the odd-order terms are purely imaginary. One can also find that all the higher-order terms disappear at the focal point \( R = 0 \). This property implies that one does not need the high-order terms if one intends to consider the field value at only the focal point. If one substitutes the approximation \( G(R) \approx G_0(R) \) and Eq. (10) into Eq. (3), one can get

\[
U_n(R) = \frac{2 f}{f_n} \exp \left[ jk \left( f_n - f + \frac{R^2}{2 f_n} \right) \right] \times \sin \left( \frac{kd_n}{2 f_n} \right) J_0 \left( \frac{kr_n}{f_n} R \right), \tag{12}
\]

where a factor \( \exp(jkf) \) has been ignored. It is worth mentioning that relation (12) was previously presented in Ref. 22. However, we here derive the complete series (7).

The series-form expression given by Eqs. (7)–(9) can be used for the accurate and fast simulations of any diffractive focusing element composed of concentric transparent rings. In the practical simulations, we suggest the use of the recurrence relation (see Appendix A) \( h_m = \text{Im}[(i \gamma)^m \exp(i \gamma)] - mh_{m-1} \) for \( m \gg 1 \), because it is more convenient to use than Eq. (9). Generally speaking, when the focal length \( f \) and the wavelength \( \lambda \) are given, the larger the number of transparent rings of the considered diffractive focusing element is, the smaller the produced focal spot is, and the faster the series converges in the focal area. Usually, the first few terms in Eq. (7) are sufficient if the number of transparent rings of the considered diffractive focusing element is not very few. To understand this statement better and have an intuitive impression of the powerful ability of the series-form expression, we now consider three opaque-center TFZPs. They have 5, 10, and 15 concentric open rings, respectively. The common parameters for these three TFZPs are chosen such that \( f = 500 \ \mu \text{m}, \ \lambda = 2.4 \ \mu \text{m}, \ f_n = f + (n-0.5) \lambda \), \( s_n = f_n^2 - f^2 \), and \( d_n = \lambda f_n/2 \). The ranges of \( n \) are \( 1 \leq n \leq 5 \), \( 1 \leq n \leq 10 \), and \( 1 \leq n \leq 15 \) for the three TFZPs, respectively. For these three TFZPs, as shown in Fig. 3, high accuracies can be obtained, respectively, by use of the first four, the first three, and only the first two terms of Eq. (7) for each open ring. It can be expected that the first two terms of Eq. (7) are enough for highly accurate simulation if the considered diffractive focusing element is composed of a large number of transparent rings. Our simulation method is much faster than the direct two-dimensional numerical integral of Eq. (1), because analytical expressions are used. This advantage is especially important for the high-numerical-aperture (high-NA) FZPs working in the
soft-x-ray region, because they are composed of a great
number of open rings. This advantage is also especially
important for FZPs illuminated by ultrashort pulsed
beams with high bandwidths. By use of our analytical
treatment, much time can be saved in these cases.

3. FOCUSING ANALYSIS OF VARIOUS
FRESNEL ZONE PLATES

We now investigate the focusing problem of various FZPs
that are composed of a large number of open (as well as
opaque) rings. For them, the leading term $G_n(R)$ in Eq.
(7) is a good approximation for $G(R)$. Under this ap-
proximation, the individual diffracted field $U_n(R)$ at the
focal plane can be simply expressed by relation (12). Many
related problems can be analytically discussed based on relation (12).

A. Selection Conditions for Open Rings

From Eq. (12), one can find that the field value $U_n(0)$ at
the focal point is given by

$$U_n(0) = \frac{2f}{f_n} \exp[ jk(f_n - f) \sin \left( \frac{kd_n}{2f_n} \right) ] .$$

To get effective focusing, one should let those individual
diffracted fields have the same argument at the desired
focus. This selection condition can be compactly ex-
pressed as $\arg(U_n(0)) = \text{constant}$ for all those open
rings, where $\arg(\cdot)$ expresses the argument. In principle,
the constant can be arbitrarily chosen. If the constant is
chosen to be 0, the selection condition can be given by

$$f_n = f + m_n \lambda, \quad \sin \left( \frac{kd_n}{2f_n} \right) > 0 ,$$

$$f_n = f + \left( m_n + \frac{1}{2} \right) \lambda, \quad \sin \left( \frac{kd_n}{2f_n} \right) < 0 ,$$

where $m_n$ is a nonnegative integer. Note that the integer
$m_n$ could be different from $n$. The first relation in ex-
pressions (14) (or expressions (15)) is used to determine
the positions of the open rings and the second one to de-
terminetheir widths.

B. Equivalent Aperture Function

We now divide the whole element into $N$ zones and let
each zone include just one open ring. We denote by $D_n$
the width of the $n$th zone in the $s$ coordinate. We denote
by $A$ the total radius of the FZP in the real $r$ coordinate.
Obviously, the sum of those widths $D_n$ is equal to $A^2$ in
the $s$ coordinate. That is to say, $\sum_{n=0}^{N} D_n = A^2$. We then
introduce the equivalent aperture function $W(r)$, whose
values $W(r_n)$ at the sampling points $r_n$ satisfy

$$W(r_n) = \frac{2\lambda F}{\pi D_n f_n} \exp[ jk(f_n - f) \sin \left( \frac{kd_n}{2f_n} \right) ] ,$$

where $F$ is the average of all those $f_n$. The value of $F$
can be approximately given by $F \approx \left( f + \frac{f^2 + A^2}{2} \right)$
if $A^2 \ll f^2$. The aperture function $W(r)$ can be well defined
if $W(r_n)$ changes slowly with $n$ (except at some potential
large discontinuities). Then, if all those $D_n$ values are
far smaller than $A^2$, one can use the relation $D_n / A \approx d_s$
$= 2\pi d r$. If one uses this approximation, Eq. (16), and
relation (12) in the expression $U(R) = \sum_{n=1}^{N} U_n(R)$ and
correspondingly replaces the sum sign $\Sigma$ and the quanti-
ties $r_n$ (including those that implicitly appear in $f_n$) by
the integral sign $\int$ and the variable $r$, respectively, one gets

$$U(R) = \frac{k}{F} \int_0^A \exp \left( \frac{R^2}{2Q} \right) W(r) J_0 \left( \frac{kR}{Q} \right) r dr ,$$

where $Q = (f^2 + \frac{f^2 + A^2}{2})$. To get physical insight from Eq.
(17), we now consider the focusing problem of a circular
aperture with a radius $A$ that is illuminated by a complex
wave field $Q^2 W(r) \exp(-jQk) (F)$ at the aperture plane.
According to the Rayleigh–Sommerfeld diffraction integral,
$\frac{\pi}{2}$–$\frac{\pi}{2}$ the field distribution $U_L(R)$ at the focal plane is given by

$$U_L(R) = \frac{1}{\lambda} \int_0^A \int_0^{2\pi} \frac{Q^2}{R^2} W(r) \exp(-jQk \cos(jk \rho) \exp(jk \rho) d \rho dr,$

where a constant factor $-j$ has been ignored. By the
way, we denote by $U_L(R)$ the field distribution at the focal
plane because the incident field has a factor $\exp(-jQk)$
that is usually regarded as the transfer function of a con-
verging lens. If one uses the approximation $\rho \approx Q$ in
the denominator, the approximation $\rho \approx Q + [R^2$
$- 2R \cos(\theta - \phi) ]/(2Q)$ in the exponential, and the
equality $\int_0^\phi \exp(-jQk \cos(\theta - \phi) \cos(\theta - \phi) \exp(2jQ \sin(\phi))$, one can find that $U_L(R) = U(R)$, whose $u = kR / \phi$. This relation
shows that the focusing at the main focal plane of a
FZP is equivalent to that of a circular aperture with a radi-
us $A$ that is illuminated by a complex wave field
$Q^2 W(r) \exp(-jQk)$ at $F$. As we shall discuss in Subsec-
tion 3.D, this equivalence is especially important for the
evaluation of the first-order diffraction efficiency.

Usually, the relation $A^2 \gg f^2$ is well satisfied. Also, $R$
is very small in the focal area. By taking these proper-
ties into account, one can get $\exp(jkR^2/(2Q)) \approx \exp(jkR^2/(2F))$ and
$J_0(kR / \phi)$ as $J_0(kR / \phi)$. In terms of these approximations, Eq. (17) can further reduce to

$$U(R) = \frac{2\pi}{\lambda F} \int_0^A r W(r) J_0 \left( \frac{2\pi}{\lambda F} R \right) r dr ,$$

where we have ignored the factor $\exp(jkR^2/(2F))$. If one
lets $W(r) = W(r)$ for $0 < r < A$ and $W(r) = 0$ for $r$
$> A$, the upper integral limit $A$ can be replaced by $\infty$.
Then, one can immediately find that, except for a factor
$1/(\lambda F)$, $U(R)$ is the Fourier–Bessel transform of $W(r)$
with a scale factor $\lambda F$. This relation is helpful for un-
derstanding the focusing behaviors of various FZPs.
It is worth mentioning that, for the special cases of a paraxial
FFZP and of a paraxial composite FZP, this relation was
partially mentioned by the authors of Ref. 16. However,
we here present a robust theoretical fundament and the
quantitative expression for general FZPs. In addition, it
is worth mentioning that the equivalent aperture func-
tion $W(r)$ could be a complex function (more strictly, we
mean that the argument of $W(r)$ varies with the coordi-
nate $r$, depending on the concrete situation. For ex-
ample, for an imperfect FZP, its equivalent aperture function \( W(r) \) is usually complex. Also, for a FZP illuminated by an ultrashort pulsed beam\(^{31} \) with a high bandwidth, the equivalent aperture function \( W(r, \nu) \) is frequency dependent. In this case, except for the component \( W(r, \nu_0) \) corresponding to the central frequency \( \nu_0 \), \( W(r, \nu) \) is generally complex, because of chromatic aberration.

C. Construction of Desirable Focal Spot Patterns

Equations (16) and (19) can be used, on the one hand, to evaluate the focal spot patterns of prefigured FZPs and, on the other hand, to construct desirable focal spot patterns. We first discuss the evaluation of the focal spot pattern of a given FZP. For this purpose, one needs two steps: (1) determining the aperture function \( W(r) \) by use of Eq. (16) and (2) determining the focal spot pattern by use of relation (19). For example, for an opaque-center TFZP, one can get \( \sin[kd_n/(2f_n)] - \exp[\pm ikl_n - f] = -1 \), \( D_n = 2\nu f_n \approx 2\nu Q \), and \( \nu f \approx Q \). By use of these relations, one obtains \( W(r) = -FF/(\pi Q^2) \). Substituting this aperture function and the approximation \( Q \sim F \) into relation (19), one can find that \( U(R) \) is just the Airy pattern distribution. This result clearly explains the well-known phenomenon that the Fraunhofer diffraction of a circular aperture with a radius \( A \) appears at the focal plane of a TFZP when the number of open rings becomes large.

We now consider the construction of a desirable focal spot pattern. For this purpose, one also needs two steps: (1) Choose a suitable aperture function \( W(r) \) whose Fourier–Bessel transform [we refer to relation (19)] corresponds to the desirable focal spot pattern. We suggest the choice of a positive-value (or a negative-value) aperture function. (2) Determine the appropriate parameters by use of Eq. (16). If one lets \( \arg[U_n(0)] = 0 \), which corresponds to a positive-value aperture function, one solution of Eq. (16) can be explicitly given by

\[
d_n = \frac{2f_n}{k} \left[ M \pi - \arcsin \left( \frac{\pi D_n f_n}{2\nu F} W(r_n) \right) \right],
\]

where \( M \) is a positive integer. When the selection condition \( f_n = f + m_nA \) is used, \( M \) should be chosen to be an odd integer. When the selection condition \( f_n = f + (m_n + 1/2)A \) is used, \( M \) should be chosen to be an even integer. It is worth mentioning that the solution (20) for a certain aperture function \( W(r) \) is not unique, because one has many parameters that can be used. However, one should carefully choose the parameters to satisfy the relation \( \pi D_n f_n W(r_n)/(2\nu F) \approx 1 \) for each open ring, because Eq. (20) is valid only under this condition. It also should be noted that the produced focal spot pattern is not the aperture function \( W(r) \) itself but its Fourier–Bessel transform. It is interesting that the focal spot pattern can be described by the same kind of function if the aperture function \( W(r) \) is a self-Fourier–Bessel-transform function. One of these functions is the Gaussian function. Actually, Ref. 22 employed this property to construct a sharp Gaussian focal spot. To have an intuitive impression of the construction of desirable focal spot patterns, we now consider the case of super-Gaussian aperture functions. For them, \( W(r_n) \) can be expressed as

\[
W(r_n) = \frac{2\beta_s FF}{\pi D f_1} \exp \left( -\frac{s_n^m - s_1^m}{\sigma^{2m}} \right)
\]

if \( W(r_n) \approx W(r_1) \), where \( \beta \) is a dimensionless constant and \( \sigma \) and \( m \) are the width and the order of the super-Gaussian aperture function. The condition \( W(r_n) \approx W(r_1) \) ensures that the truncation effect can be ignored. The value of \( \beta \) must be in the range \( 0 < \beta \leq 1 \) because it is actually equal to \( |\sin[kd_s/(2f_1)]| \). According to relation (19), the total field distribution \( U(R) \) at the focal plane is the Fraunhofer diffraction of the super-Gaussian aperture function \( W(r) = 2\beta_s FF \exp(-\sigma^{2m} - s_1^m)/(\pi D f_1) \). In particular, for the special case of a Gaussian aperture function, by substituting Eq. (21) into Eq. (20) and letting \( m = 1 \), one can get

\[
d_n = \frac{2f_n}{k} \left[ M \pi - \arcsin \left( \frac{\beta D_n f_n}{D f_1} \exp \left( -\frac{s_n - s_1}{\sigma^2} \right) \right) \right],
\]

which was previously presented in Ref. 22. The design considerations as well as the design process for a FZP with a super-Gaussian aperture function are similar to those for the FZP with a Gaussian aperture function. The latter was investigated in detail in Ref. 22. Therefore, to save space, we do not further discuss this issue here.

The interested reader can consult the concrete example given in Section 4 of Ref. 22. By the way, in steps 7 and 8 of the design considerations of the example and in the paragraph just above Fig. 2 of Ref. 22, all five \( D_n \) should be \( W_n \). They were miswritten.

D. Diffraction Efficiency

We now investigate the diffraction efficiency of a FZP at the first order. Strictly speaking, the concept of diffraction order can be exactly defined in the paraxial approximation for only those FZPs whose transmittance functions are periodic in the \( s = r^2 \) coordinate. However, for convenience, the concept of diffraction order is still approximately used for various nonperiodic FZPs and DLs.

In the nonperiodic case, the focusing at the main focal plane (i.e., the first-order focus) of a FZP is approximately equivalent to that of a circular aperture with a radius \( A \) that is illuminated by a complex wave field \( Q^2 W(r) \exp(-jkQ)/(F) \). Thus the power \( P_1 \) included in the first order is approximately equal to the power carried by the wave field \( Q^2 W(r) \exp(-jkQ)/(F) \) that is truncated by a circular aperture with a radius \( A \). For a scalar field, say \( E(x, y, z) = E_0(x, y, z) \exp[\pm ikL(x, y, z)] \), the exact power density\(^{35-38} \) at the aperture plane is \( -jkL^2E^2(\partial E/\partial z) = |E_0|^2(\partial L/\partial z) \), where \( E_0 \) is the real amplitude, \( L \) is the eikonal, \( \partial \) expresses the real part, and \( x \) and \( y \) are the rectangular coordinates at the aperture plane. From the eikonal equation \( \nabla L \approx 1 \), we know that \( \partial L/\partial z \approx \{1 - (\partial L/\partial x)^2 - (\partial L/\partial y)^2\}^{1/2} \) for a forward-propagating wave field \( E(x, y, z) \). The trans-
verse partial derivatives \(\partial L/\partial x\) and \(\partial L/\partial y\) can be approximately given by \(\partial L/\partial x \approx -\partial Q/\partial x\) and \(\partial L/\partial y \approx -\partial Q/\partial y\), respectively, if the phase of \(W(r)\) changes much more slowly than \(-kQ\). By taking these approximations and the relation \(Q = (f^2 + x^2 + y^2)^{1/2}\) into account, one can get \(\partial L/\partial z \approx f\partial Q/\partial z\). Substituting this result into the expression for power density and integrating in the region \(0 \leq r \leq A\), one can obtain 
\[
P_1 = 2\pi f^{-1}F^{-2}\int_0^A Q^3 |W(r)|^2 \times \pi dr.
\]
Therefore the diffraction efficiency \(\eta\) of the first order can be approximately given by

\[
\eta \approx \eta_1 = \frac{P_1}{P} \approx \frac{2}{f^2A^2} \int_0^A Q^3 |W(r)|^2 \times \pi dr.
\]  

(22)

As a test of relation (22), we consider a TFZP, for which \(|W(r)| = |f|/\pi Q^2\). Substituting this aperture function into relation (22) and using the approximation \(Q \approx F\), one gets \(\eta \approx f/\pi F\). In particular, just as we expected, the diffraction efficiency \(\eta\) reduces to \(1/F^2\) the paraxial approximation \(F = f\). As another example, we now evaluate the diffraction efficiency of a MFZP that produces a Gaussian focal spot. As we state in Subsection 3,C, for such a TFZP, the aperture function is given by \(W(r) = 2\beta\lambda F\exp[-(r^2 - s_1^2)/(\pi d_1^2)]\). This aperture function can reduce to \(W(r) \approx 2\beta\lambda F\exp(-r^2/\sigma^2)(\pi D_1)\) by use of the approximations \(f_1 \approx f\) and \(\exp(\sigma/r^2) \approx 1\). Substituting this aperture function into relation (22), employing the approximation \(Q \approx F\), integrating, and using the approximation \(\exp(-2A^2/\sigma^2) \approx 1\), we obtain \(\eta \approx 2\beta\lambda^2 F\sigma^2/(\pi A^2 D_1)\). Usually, the diffraction efficiency of such a MFZP is much lower than that of a TFZP. To understand this property better, we now consider the example given in Section 4 of Ref. 22, in which we used the relations \(\beta = 0.96\), \(\alpha = A/2\), \(D_1 = 4\lambda f\), \(A \approx 100 \mu m\), and \(f = 500 \mu m\). Then, further using the approximation \(F \approx [f + (f^2 + A^2)^{1/2}]/2 \approx 1.01f\), one can get \(\eta \approx 1/(34\pi^2)\), which is much lower than the diffraction efficiency of \(1/(1.01\pi^2)\) of the corresponding TFZP. However, the absolute intensity at the focal point of such a MFZP may be higher than that of a TFZP with the same minimum diffraction structure. In fact, in the comparison presented in Section 4 of Ref. 22, in which the narrowest open rings of both the MFZP and the TFZP are approximately 30 nm wide, the absolute intensity at the focal point of the MFZP is 8.57 times higher than that of the TFZP. The reason is that the absolute intensity at the focal point is proportional to \(A^4\) and that the radius of the MFZP is approximately five times larger than that of the TFZP in the above example.\[22\]

E. Suppression of High-Orders and Appearance of “Fractional Orders”

It is well-known that, besides the intensity peak at the principal focus, there also appear significant intensity peaks at the higher-order foci of a TFZP. These higher-order peaks can be effectively suppressed by a photon sieve.\[17\]\[21\] It has been explained that,\[21\] for a photon sieve, the suppression of higher orders results from the use of different ratios \(d/\omega\) for different pinholes, where \(d\) is the diameter of an individual pinhole and \(\omega\) is the width of the corresponding local half-zone of the underlying TFZP. In the first-order focus, all the pinholes have constructive contributions to the focusing. However, the sign of an individual diffracted field reverses three times more often in the third-order focus than in the first-order focus. Because the ratios \(d/\omega\) are different for different pinholes, some pinholes still have constructive contributions to the focusing in the third order but others have destructive contributions to the focusing in the third-order focus. As a consequence, the total field value in the third-order focus tends to zero (compared with that in the first order) and the focusing in the third-order focus is therefore suppressed. The same thing happens in other higher-order foci. It can be proved that this mechanism also works for a MFZP that uses different values of \(kd_\omega/(2f_\omega)\) for different open rings. To save space, we here do not further discuss this mechanism. Instead, we explore another potential mechanism for suppression of higher orders: nonparaxial phase mismatch.

To investigate the suppression of higher orders of a FZP, one needs to know the on-axis field distributions \(U_n(0, z)\) of the individual diffracted fields. This field distribution \(U_n(0, z)\) can be simply obtained by replacing the \(f\) and the \(f_\omega\) in relation (13) by \(z\) and \((z^2 + r_\omega^2)^{1/2}\), respectively, where \(z\) is the distance from the center of the FZP to the on-axis observation point. After these replacements, one gets (we also present a complete derivation in Appendix B for readability)

\[
U_n(0, z) \approx \frac{2z}{\sqrt{z^2 + r_\omega^2}} \exp[jk(\sqrt{z^2 + r_\omega^2} - z)]
\]

\times \sin\left(\frac{kd_\omega}{2\sqrt{z^2 + r_\omega^2}}\right).
\] (23)

Similar to relation (13), a factor \(\exp(jkz)\) has been ignored in relation (23). Relation (23) has three factors. The first factor, \(2z/(\sqrt{z^2 + r_\omega^2})\), is not critical for the suppression of higher orders because it is always positive. The mechanism stated in Ref. 21 can be deduced from the third factor, \(\sin(kd_\omega/[2(\sqrt{z^2 + r_\omega^2})])\), in the approximation \((z^2 + r_\omega^2)^{1/2} \approx z\). We are now focusing on the second factor, \(\exp(jk(z^2 + r_\omega^2)^{1/2} - z)\). It can be proved that (see Appendix C), for a TFZP, this factor still has the same value for all those open rings at the third-order focus in the paraxial approximation if it has the same value for all those open rings in the first-order focus. However, this is no longer true for the nonparaxial case of a high-NA FZP working in the spectral regions of soft-x-ray and EUV radiation. In this case, the nonparaxiality can produce significant phase mismatch at the third-order focus, which in turn leads to the suppression of the third order. For the third-order focus of a TFZP, the mismatched phase \(\delta_n\) of the \(n\)th open ring can be given by (see Appendix C)

\[
\delta_n = \frac{k}{3} \left\{ [f^2 + 18(n + c)\lambda f + 9(n + c)^2\lambda^2 f^2 - f] - 6(n + c)\pi \right\},
\] (24)

where \(c = 0\) for a transparent-center TFZP and \(c = -0.5\) for an opaque-center TFZP. To have an intuitive impression of the mismatched phase, we now consider a
transient center TFZP with $f = 1500 \mu m$, $\lambda = 2.4 \text{ nm}$, $N = 722$, and $A \approx 72.1 \mu m$. The wrapped $\delta_n$ values are drawn in Fig. 4 as a function of $n$. One can see that the mismatched phase values $\delta_n$ are different for different open rings. It is interesting that these different $\delta_n$ values are extensively distributed in the whole range from $-\pi$ to $\pi$. As a consequence, it can be expected that the focusing at the third-order focus will be significantly suppressed.

To check the above prediction, we now further consider three concrete opaque-center TFZPs, for which the mechanism stated in Ref. 21 disappears in the third order. For all of them, the focal length $f$ and the wavelength $\lambda$ are chosen such that $f = 500 \mu m$ and $\lambda = 2.4 \text{ nm}$. The first one has 300 open rings, and the radius is 26.84 $\mu m$; the second one has 600 open rings, and the radius is 37.97 $\mu m$; and the third one has 900 open rings, and the radius is 46.53 $\mu m$. The normalized on-axis intensity distributions of these three TFZPs are drawn in Fig. 5. One can see that the greater the number of open rings is, the more significant the suppression of higher orders is, because the stronger the nonparaxial effect is. In particular, for the TFZP with 900 open rings, the on-axis intensity peaks at the higher-order focus almost completely disappear. These results explicitly confirm the suppression effect of nonparaxial phase mismatch on higher orders.

The existence of a nonparaxial suppression effect on higher orders shows that the composite FZPs\textsuperscript{16} with high NA are not suitable for the focusing and the imaging of soft x rays and EUV radiation because these TFZPs use higher orders. To overcome this drawback, we here suggest a modified composite FZP that uses the phase-match condition $\arg[U_n(0)] = \text{constant}$ at the desired focal point $(R = 0, z = f)$ but allows $d_n = n f_n/2$, $d_n = 3 f_n/2$, $d_n = 5 f_n/2$, etc. It can be expected that such a modified FZP is better than a composite FZP. As a check, we now consider a composite FZP and a corresponding modified composite FZP. Each is composed of two regions: an inner region and an outer region. They have the same inner region, which consists of 238 open rings, for which the parameters are chosen such that $\lambda = 2.4 \text{ nm}$, $f = 500 \mu m$, $1 \leq n \leq 238$, $m_n = n$, $f_n = f + (m_n - 0.5)\lambda$, and $d_n = n f_n/2$. The outer region of the composite FZP consists of 642 open rings, for which the parameters are chosen such that $\lambda = 2.4 \text{ nm}$, $f = 1500 \mu m$, $1 \leq n \leq 642$, $m_n = 79 + n$, $f_n = f + m_n\lambda$, and $d_n = n f_n/2$. The outer region of the modified composite FZP consists of 639 open rings, for which the pa-
Parameters are chosen such that $\lambda = 2.4 \, \text{nm}$, $f = 500 \, \mu\text{m}$, $1 \leq n \leq 639$, $m_n = 237 + 3n$, $f_n = f + m_n \lambda$, and $d_n = 3f_n/2$. By the way, for each of the two FZPs, the width of the narrowest open ring is approximately 25 nm and the total radius $A$ is approximately 72.1 $\mu$m. We accurately calculate the on-axis intensity distributions of the two FZPs, which are shown in Figs. 6(a) and 6(b), respectively. We found that the absolute intensity at the desired focal point of the modified composite FZP is more than ten times higher than that of the composite FZP. This result explicitly shows that a modified composite FZP is superior to a composite FZP in the nonparaxial case, although they are the same in the paraxial case.

We emphasize that the differences between the two FZPs in the above example are quite large. To understand this statement better, we draw the difference $\Delta_n = r_{n,mc} - r_{n,ce}$ as a function of $n$ in Fig. 7(a), where $r_{n,mc}$ is the characteristic coordinate of the $n$th open ring in the outer region of the modified composite FZP and $r_{n,ce}$ is that corresponding to the composite FZP. We find that the difference $\Delta_n$ starts from $\Delta_n = 6.14 \, \text{nm}$ for $n = 1$ and becomes as large as $\Delta_n = 164.96 \, \text{nm}$ for $n = 639$. These $\Delta_n$ values are far larger than the positioning accuracy of approximately a few nanometers of current fabrication technology. Therefore there is no problem in distinguishing the modified structure from the unmodified version in fabrication. To further have an intuitive impression of the difference of the two FZPs, in Figs. 7(b) and 7(c) we draw their transmittance functions at the boundaries between the inner regions and the outer regions and at the outermost parts of the total elements. From these plots, one can clearly see the difference of the two FZPs.

From Fig. 6(b), one can see that, for the modified composite FZP, the intensity peak at the desired focal point $z = 500 \, \mu\text{m}$, there also appears a small intensity peak at the axial position of approximately $z = 1500 \, \mu\text{m}$. This small intensity peak results from the outer region, whose slowly varying period is approximately three times that of the inner region in the $s = r^2$ coordinate. For this reason, we call this intensity peak the “1/3 order” intensity peak. Out of curiosity, we revisit the example presented in Section 4 of Ref. 22 because the MFZP considered there also contains multiregions. In particular, we pay much attention to the on-axis intensity distribution in the range $z > f$, because it was not paid attention to in Ref. 22. The whole on-axis intensity distribution is drawn in Fig. 6(c). To our surprise, there appear seven additional small intensity peaks in the range $z > f$. Their central positions are located at $z = 2530 \, \mu\text{m}$, $1504 \, \mu\text{m}$, $1265 \, \mu\text{m}$, $1001 \, \mu\text{m}$, $840 \, \mu\text{m}$, $751 \, \mu\text{m}$, and $628 \, \mu\text{m}$, respectively. If one uses the concept of “fractional orders,” one can find that the above-mentioned positions are well consistent with those of the following fractional orders: $z = 2500 \, \mu\text{m}$ (1/5 order), $1500 \, \mu\text{m}$ (1/3 order), $1250 \, \mu\text{m}$ (2/5 order), $1000 \, \mu\text{m}$ (1/2 order), $833 \, \mu\text{m}$ (3/5 order), $750 \, \mu\text{m}$ (2/3 order), and $622 \, \mu\text{m}$ (4/5 order), respectively. We even find that the two small intensity peaks in the range $z < f$, whose central positions are located at $z = 414 \, \mu\text{m}$ and $374 \, \mu\text{m}$, respectively, are also well consistent with the 6/5 order ($z = 417 \, \mu\text{m}$) and the 4/3 order ($z = 375 \, \mu\text{m}$), respectively. These consistencies show that the concept of fractional order is useful. From the viewpoint of physics, all these small intensity peaks result from the three different regions of the MFZP considered in Ref. 22. Concretely, the orders 1/5, 2/5, 3/5, 4/5, and 6/5 result from the outer region, whose slowly varying period is approximately five times that of the underlying TFZP; the orders 1/3, 2/3, and 4/3 result from the middle region, whose slowly varying period is approximately three times that of the underlying TFZP; and the order 1/2 results from the inner region, whose slowly varying period is approximately two times that of the underlying TFZP. Therefore, to say it simply, the quasiperiodicity of the three different regions of the MFZP results in those small intensity peaks of fractional orders. If one intends to suppress these small intensity peaks, one needs to destroy the above-mentioned quasi-periodicity. This could be an interesting topic.

4. CONCLUSION AND DISCUSSION

We have presented a series-form expression for the diffracted field of an individual annular ring. It can be used for the accurate and fast simulation of any diffractive focusing element composed of concentric transparent rings. Working from the leading term of the series-form expression, the linear superposition principle, and some appro-
private physical considerations, we have introduced the equivalent aperture function \( W(r) \) for a FZP with many open rings. The total diffracted field distribution at the main focal plane, except for the background noise, is simply the Fourier–Bessel transform of the equivalent aperture function \( W(r) \). This relation clearly explains the appearance of the Fraunhofer diffraction patterns of a circular aperture at the main focal plane of a TFZP. In terms of the equivalent aperture function \( W(r) \), we have analytically investigated the construction of desired focal spot patterns and the diffraction efficiency of the first order. We have also discussed the suppression of higher orders and the appearance of “fractional orders.” In particular, we have revealed the suppression mechanism of nonparaxial phase mismatch on the higher orders. Most of the results presented in this paper are also applicable, maybe with some small changes, to the multilevel DLs,24–27 because these diffractive elements have the same working principle as that for FZPs.

All the analyses of this paper are based on the Rayleigh–Sommerfeld diffraction integral,28–30 which is the integral-form solution of the scalar Helmholtz wave equation. This formula does not take into account the vectorial property of the electromagnetic field. Strictly speaking,26–30 the purely linear polarization field with only one electric (or magnetic) field component cannot exist in the real world because it can never satisfy the condition \( \mathbf{E} \cdot \mathbf{E} = 0 \). There must exist an associated longitudinal field component along the propagation direction. However, the longitudinal field component can be ignored provided that26–30 the focal spot size is larger than the order of the wavelength. Usually, this condition is well satisfied for the FZPs working in the spectral regions of soft x-rays and EUV radiation. Therefore, to state it simply, the scalar diffraction theory is highly valid for the analysis of the considered FZPs.

It should be pointed out that, in derivation of relation (19), we have used the condition \( A^2 \ll f^2 \), which implies that the absolute quantity of the NA is not large. About this issue, we would like to state that, for the FZPs working in the spectral regions of soft x-rays and EUV radiation, a NA value of approximately 0.05 can be regarded as high15 and a NA value of approximately 0.2 can be regarded as very high.22 For these high-NA FZPs, the paraxial Fresnel diffraction integral is invalid19,22 because of the huge wave number \( k \). In this case, just as we do in this paper, one needs to use the nonparaxial scalar diffraction theory, although a vectorial treatment is not necessary.

**APPENDIX A**

By use of the relation \( ds = 2rdr \), one can obtain

\[
\frac{d^m}{ds^m} = \frac{d}{ds}^m = \left( \frac{d}{2rdr} \right)^m = \frac{1}{2^m} \left( \frac{d}{rdr} \right)^m.
\]  

(A1)

If one lets \( \xi = es^{1/2} = er \), one can get \( \left[ d/(rdr) \right]^m J_m(\xi r) = e^{im\xi}[d/\xi] J_m(\xi) \). By taking this relation and Eq. (A1) into account, we obtain

\[
\frac{d^m}{ds^m} J_m(\xi es^{1/2}) = \frac{e^{2m}}{2^m} \frac{d}{\xi ds}^m J_m(\xi).
\]  

(A2)

On the other hand, from the properties of the Bessel function of the first kind, we know that

\[
\left( \frac{d}{\xi ds} \right)^m J_m(\xi) = \left( \frac{-1}{\xi^n} \right)^m J_m(\xi),
\]  

(A3)

where \( J_m(\xi) \) is the \( m \)-th order Bessel function of the first kind. Substituting Eqs. (A2) and (A3) into Eq. (6), we obtain

\[
J_m(\xi es^{1/2}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!2^m} J_m(\xi s - s_j)^m,
\]  

(A4)

where \( \alpha = kr_n/f_n \). If one further substitutes Eq. (A4) into Eq. (5) and integrates, one obtains

\[
G(R) = \sum_{m=0}^{\infty} \frac{1}{m!2^m} \sum_{j=1}^{m+1} \int_{-j}^{j} t^m \exp(t) dt,
\]  

(A5)

where \( \gamma = kd_n/(2f_n) \) and \( t = jk(s - s_j)/(2f_n) \). Using integration by parts, one can find that Eq. (A5) is just Eq. (9). This relation shows that we have already reached the results of Eqs. (7)–(9).

It may be inconvenient to directly employ Eq. (9) for \( m > 1 \) in numerical simulation because \( m + 1 \) terms are needed to calculate the parameter \( h_m \). To more conveniently calculate the parameter \( h_m \), we now present a simple recurrence formula. Using integration by parts, we derive

\[
h_m = \text{Im}[(j\gamma)^m \exp(j\gamma)] - m h_{m-1},
\]  

(A7)

where \( \text{Im}(\cdot) \) expresses the imaginary part. By use of this recurrence formula, one needs to calculate only one new term to get \( h_m \) if the coefficient \( h_{m-1} \) has been previously calculated. This is always the case because the lower-order terms in Eq. (7) are more important than the higher-order terms.

**APPENDIX B**

From the Rayleigh–Sommerfeld diffraction integral,28–30 we know that the individual diffracted field \( U_n(0, z) \) on the propagation axis is given by

\[
U_n(0, z) = \frac{1}{\lambda} \int \int_{A_n} \frac{z}{\rho^2} \exp(jk\rho) r dr d\theta,
\]  

(B1)

where \( z \) is the longitudinal coordinate, \( \rho = (z^2 + r^2)^{1/2} = (z^2 + r_n^2 + (r^2 - r_n^2))^{1/2} \), and a constant factor \( -j \) has been ignored. We then approximately express \( \rho \) as the Taylor expansion

\[
\rho \approx \sqrt{z^2 + r_n^2 + \frac{r^2 - r_n^2}{2z^2 + r_n^2}}.
\]  

(B2)
because the quantity $|r^2 - r_n^2|$ is far smaller than the quantity $z^2 + r_n^2$ for the $n$th open ring. Substituting relation (B2) and the approximation $\rho \approx (z^2 + r_n^2)^{1/2}$ into the exponential and the denominator in Eq. (B1), respectively, and using the relation $\int_0^{2\pi} d\theta = 2\pi$, one gets

$$U_n(0, z) = \frac{kz}{z^2 + r_n^2} \exp(\frac{jk}{\sqrt{z^2 + r_n^2}}) \frac{b_n}{r_n} \times \exp(\frac{jk}{\sqrt{r^2 - r_n^2}}) r dr. \quad (B3)$$

Using the relations $r dr = ds/2$, $r^2 - r_n^2 = s - s_n$, $b_n^2 = s_n + d_n$, and $d_n^2 = s_n - d_n$, one can prove that

$$\int_0^{b_n} \exp(\frac{jk}{\sqrt{r^2 - r_n^2}}) r dr = \frac{2}{k} \sqrt{z^2 + r_n^2} \sin\left(\frac{kd_n}{2(z^2 + r_n^2)}\right). \quad (B4)$$

Substituting Eq. (B4) into Eq. (B3), one can immediately obtain relation (23).

APPENDIX C

In the paraxial approximation, the phase $\phi_n(z) = k[(z^2 + r_n^2)^{1/2} - z]$ is simply given by $\psi_n(z) = kr_n^2/(2z)$. For a TFZP, the phase $\phi_n(f)$ at the main focus $z = f$ satisfies the relation

$$\phi_n(f) = kr_n^2/(2f) = 2n \pi + 2c \pi, \quad (C1)$$

where $c = 0$ for a transparent-center TFZP and $c = -0.5$ for an opaque-center TFZP. By taking this property into account and using the relation $z = f/\beta$, one can prove that the phases $\phi_n(f/\beta)$ at the third-order focus are $6n \pi + 6c \pi$. Because of the relation $\exp(\beta n \pi) = 1$, the factor $\exp(\frac{j \phi_n(f/\beta)}{2})$ is equal to $\exp(\beta c \pi)$, which is independent of $n$ at the third-order focus $z = f/\beta$.

For a nonparaxial TFZP, the phase $\phi_n(f)$ at the main focus $z = f$ satisfies the relation

$$\phi_n(f) = k\sqrt{f^2 + r_n^2} - f = 2(n + c) \pi. \quad (C2)$$

From Eq. (C2), one can get

$$r_n^2 = 2(n + c) \lambda f + (n + c)^2 \lambda^2. \quad (C3)$$

Using Eq. (C3) and the relation $z = f/\beta$, one gets

$$\psi_n(f/\beta) = \frac{k}{3} \left[ f^2 + 18(n + c) \lambda f + 9(n + c)^2 \lambda^2 \right]^{1/2} - f. \quad (C4)$$

On the other hand, we know that the paraxial approximation is always valid for those open rings located at the central part, because their characteristic coordinates $r_n$ are still far smaller than the focal length $f$. For these open rings, their phases at the third-order focus are still equal to $6c \pi$. Therefore the mismatched phase $\delta_n$ at the third-order ring at the third-order focus can be given by $\delta_n = \phi_n(f/\beta) - 6c \pi$. This expression can also be written as $\delta_n = \psi_n(f/\beta) - 6(n + c) \pi$ because of the relation $\exp(-j6n \pi) = 1$. We use $6(n + c) \pi$ because the former is the corresponding unwrapped phase in the paraxial approximation. Substituting Eq. (C4) into the latter expression for $\delta_n$, one can immediately obtain Eq. (24).

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REFERENCES AND NOTES

The opinion that the suppression of higher orders results from the use of different ratios $d/w$ for different pinholes is presented in this reference, where $d$ is the diameter of an individual pinhole and $w$ is the width of the corresponding local half-zone of the underlying TFZP.


