Optical wave fields with lateral and longitudinal periodicity

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The propagation of stationary wave fields that exhibit simultaneously lateral and longitudinal periodicity is investigated. As a model, we use a Fabry–Perot resonator with periodically structured mirrors under monochromatic plane wave illumination. The resonator leads to a longitudinal periodicity, the grating mirrors to a lateral periodicity. The angular spectrum of the transmitted wave field is given as the product of two terms, one related to the lateral, the other to the longitudinal properties. Its modal structure can vary significantly depending on the ratio of the lateral and longitudinal periods and the reflectivity of the resonator’s mirrors. For example, it is possible to generate bandgap behavior despite the fact that the periods may be significantly larger than the wavelength. The results of this investigation apply to the design of phase-coupled array resonators and multiplexers. © 2009 Optical Society of America

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1. Introduction: Wave Fields with Lateral and Longitudinal Periodicity

The properties of three-dimensional (3D) wave fields have stimulated interest for a long time. In his paper published in 1914, von Laue [1] described spatial and temporal properties of wave fields and introduced the concept of the time–bandwidth product. Since then, numerous aspects of the properties of wave fields have been discussed and applied. A specific subclass of 3D wave fields are periodic and quasi-periodic wave fields and, closely related, the phenomenon of self-imaging. For an overview of the field in general, and self-imaging wave fields in particular, two review papers might be helpful [2,3].

In this paper, we consider a specific aspect of the self-imaging phenomenon not discussed earlier, namely, the simultaneous occurrence of lateral and longitudinal periodicity, i.e., a periodic structure in the direction of propagation and in the transverse direction. Our considerations originate in part out of a fundamental interest in wave propagation, but there are also practical devices that exhibit this property, for example, laser arrays. There is significant interest in the task of coherently combining several laser beams in order to obtain high output power and good beam quality [4–10]. In order to be able to control the phases of the different beams or modes precisely, one has to apply interferometric techniques. A variety of different approaches have been suggested over the years, which are briefly discussed, for example, in Ref. [10]. One can distinguish between intracavity and intercavity coupling, depending on whether the modes are generated in a single resonator or originate from several individual resonators. The latter case includes, for example, so-called coherently coupled diode laser arrays. An attractive solution for the coupling of the different resonators is, for example, self-imaging, since it leads to rather compact implementations [6–9]. In general, self-imaging phenomena occur in wave fields composed of discrete modes. In a multimodal resonator, the optical feedback mechanism may lead to a coupling between
longitudinal and transverse modes. As we show here, this may result in unwanted effects on the modal structure of the wave field.

Wave fields with lateral or longitudinal periodicity have been widely discussed. There is a direct connection between the symmetries of a wave field and the self-imaging phenomenon [11]. Wave fields with a lateral periodicity exhibit the well-known Talbot effect [12]. Wave fields with a longitudinal periodicity are related to the more general case of Montgomery self-imaging [13]. Here, we consider the combination of both situations. This means that we are interested in the propagation of wave fields that are periodic in both lateral and longitudinal directions as is typically the case in a multimodal resonator. The situation that we investigate is exemplified in Fig. 1(a). It shows a Fabry–Perot resonator with periodically structured mirrors denoted as M₁ and M₂. The aforementioned case of the coherently coupled laser resonator array is a specific example of the general problem. However, other practical situations may occur as, for example, in a multimode waveguide device with added longitudinal periodicity [Fig. 1(b)]. The purpose of this manuscript is to provide an analysis of the propagating wave field \( u(x,y,z) \). Since the problem is strongly related to the self-imaging phenomenon, we begin in Section 2 with a simple review of Talbot and Montgomery self-imaging. In Section 3, we present some general arguments on interferometers in order to inspire an intuitive view of the problem. Section 4 presents a graphical solution to the problem based on the model of the Ewald sphere. This is complemented by an analytical model and simulation results in Section 5. Finally, we discuss and summarize our results in Section 6.

### 2. Mathematical Formulation

Periodic boundary conditions in a multimode wave field are directly related to the self-imaging properties. In its weaker form, self-imaging of paraxial wave fields can be expressed in the following way. If a monochromatic wave field with wavelength \( \lambda \) is laterally periodic with period \( p_z \), then it is also periodic in the longitudinal (\( z \)) direction with a period \( z_T \) or, mathematically,

\[
\begin{align*}
  u_0(x) &= u_0(x+p_z) \Rightarrow u_z(x) = u_{z+z_T}(x),
\end{align*}
\]

with

\[
  z_T = \frac{2p_z^2}{\lambda}. \tag{1a}
\]

For simplicity, we have dropped the second lateral coordinate at this point. The longitudinal period \( z_T \) is called the “Talbot distance.” Equation (1) describes a sufficient condition for self-imaging. Montgomery [13] considered the necessary condition: the requirement that a wave field be periodic in the \( z \) direction with a period \( p_z \); it follows that the lateral components of the \( k \) vector must obey a certain condition. For a 2D wave field with \( x \) and \( z \) components, this is expressed by

\[
  u_z(x) = u_{z+p_z}(x) \Rightarrow u_0(x) = \sum_m A_me^{ik_{x,m}x}, \tag{2a}
\]

with

\[
  k_{x,m}^2 = (2\pi)^2\left[ \left( \frac{1}{\lambda} \right)^2 - \left( \frac{m}{p_z} \right)^2 \right].
\]

In the general case, the “Montgomery condition” of Eq. (2a) does not lead to a simple lateral periodicity. However, it includes the case of “Talbot self-imaging” for the paraxial case. In the following, we will sometimes assume that the paraxial case is valid for the sake of a simplified presentation. In that case, the Montgomery wave field becomes periodic. This can be understood since for large values of \( m \), the difference \( \Delta k_z = k_{x,m} - k_{x,m+1} \approx \text{const} \), which leads to a lateral period \( p_z,M \) by \( p_z,M = 2\pi/\Delta k_z \). It should be noted that, in general, the lateral period of a self-imaging wave field is (much) smaller than the longitudinal, i.e., \( p_x \ll p_z \).
As said earlier, although Talbot self-imaging is a special case of the more general view expressed by Montgomery’s theory, it is useful for our discussion to treat them as separate cases here. The reason is that we will consider the situation where a wave field obeys both lateral and longitudinal periodicity and where the experimental parameters for both cases can be varied independently (see Fig. 1). The wave field under consideration will be modulated with a lateral period $p_x$ from which follows a longitudinal period $z_T$ according to Eq. (1). At the same time, we impose on it a longitudinal period $p_z \neq z_T$ from which follows a different condition for the $k_x$ components. The fact that Talbot and Montgomery conditions do not have to match makes the situation interesting. First, we would like to understand the situation from a fundamental point of view. Second, the situation does, in fact, occur in practical devices. We already mentioned the case of coherently coupled laser resonators [6–9]. In the cited literature, one finds reference to the Talbot effect; however, the aspect that the resonator also adds a longitudinal periodicity to the wave field has been neglected so far. Furthermore, the results may be useful to other situations that include self-imaging, for example, in interferometry [14], and in optical analog and quantum information processing [15,16]. These examples deal with stationary wave fields. Recently, we suggested the use of Talbot and Montgomery self-imaging for the filtering of ultrashort optical pulses, which are nonstationary wave fields [17–19]. However, here we will restrict ourselves to the case of stationary and paraxial wave fields.

3. Simple Consideration about Interferometers

Any interferometer can be understood as a device that generates two or more virtual light sources with the purpose of filtering the wave field, spatially or temporally. We may classify interferometers as shown in Table 1 according to two categories: first, the shift of the virtual light sources (i.e., lateral or longitudinal) and second, the splitting factor $N$. For a two-beam interferometer $N = 2$, and in multiple beam interferometers $N > 2$. Examples for the different categories are given in the table. The case of the grating interferometer (GI) and the Fabry–Perot interferometer (FPI) are shown in Fig. 2.

In this paper, we combine both situations to form an FPI with periodically structured mirrors (Fig. 3). The FPI mirrors lead to a longitudinal periodicity of a stationary wave field, as observed in the context of Montgomery wave fields by Indebetouw [20], and the gratings to a lateral. In order to analyze the transmitted wave field, we will first write up the equations for the GI and the FPI separately.

4. Graphical Explanation

In this section, we describe a wave field transmitted through a structured Fabry–Perot resonator by using the Ewald sphere. The $k$ vectors representing a 3D wave field have a magnitude given by

$$k_x^2 + k_y^2 + k_z^2 = (2\pi/\lambda)^2,$$

which defines the Ewald sphere. We first consider two separate situations: first, the case of a wave field with lateral periodicity, and second, the case of a
longitudinal interferometer. Then, we will combine both results to describe the case of the structured FPI. For simplicity, we use a 2D description first, which facilitates the graphical explanation.

First, let us consider a wave field \( u(x,z) \) generated from a plane monochromatic wave diffracted by a grating as shown in Fig. 4(a). The transmitted wave field is laterally periodic, i.e., \( u(x,z) = u(x + p_x, z) \) for any \( z > 0 \). This results in the well known fact that the angular spectrum contains only discrete frequencies given by \( k_x = m 2\pi/p_x \) (with \( m = 0, \pm 1, \pm 2, ..., m_{\text{max}} \), where \( m_{\text{max}} \) is the largest integer for which \( m/p_x \leq 1/\lambda \)). This expression defines a set of planes in 3D \( k \) space or lines in a 2D representation, respectively; see Fig. 4(b). The \( k \) vectors of \( u(x,z) \) are obtained graphically by intersecting these lines with the Ewald sphere. For \( k_y = 0 \), as in the figure, the \( z \) components of the \( k \) vectors representing the various diffraction orders are given as

\[
k_{T,z}^m = \pm 2\pi \left( \frac{1}{\lambda} \right)^2 - \left( \frac{m}{p_x} \right)^2, \quad m = 0, \pm 1, \pm 2, ....
\]  

(4)

Next we turn to the case of the longitudinally periodic wave field as transmitted by a conventional FPI. We denote the separation of mirrors \( M_1 \) and \( M_2 \) as \( L \) (Fig. 5(a)). The transmitted wave field has a periodicity imposed on it with a period in the \( z \) direction \( p_z = 2L \). In frequency space this leads to a discrete modal spectrum with \( k_z = n 2\pi/p_z \) [Fig. 5(b)], which we refer to as the Montgomery condition. In this case, the \( k_x \) components of the discrete angular spectrum are given as

Here, the subscript “T” stands for Talbot. For the paraxial case \( (m/p_x < 1/\lambda) \), this can be reduced to the expression \( k_{T,z}^m = 2\pi(1/\lambda - m^2/z_T) \).

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\[ k_{M,x}^n = \pm 2\pi \sqrt{\left(\frac{1}{\lambda}\right)^2 - \left(\frac{n}{p_z}\right)^2}, \quad n = \pm 1, \pm 2, \ldots \quad (5) \]

Now we combine a grating and a FPI to form a resonator with structured mirrors [shown schematically in Fig. 6(a)]. Lateral period \( p_x \) and longitudinal period \( p_z \) may be varied independently. The transmitted wave field exhibits both lateral and longitudinal periodicity, and both conditions for the modes as expressed by Eqs. (4) and (5) apply. In our graphical picture this means that only those modes can propagate that satisfy both Talbot and Montgomery conditions: \( k_x = m2\pi/p_x \) and \( k_z = n2\pi/p_z \). In addition, of course, Eq. (3) has to hold. This situation is depicted in Fig. 6(b), where the dots on the Ewald sphere combine the three conditions. Mathematically, we can express this for the shown 2D case \( (k_y = 0) \) by

\[ (k_x^M)^2 + (k_z^T)^2 = \left(\frac{2\pi}{\lambda}\right)^2, \quad (6) \]

or, equivalently,

\[ m^2\left(\frac{\lambda}{2L_x}\right)^2 + n^2\left(\frac{\lambda}{2L_z}\right)^2 = 1. \quad (7) \]

In the 3D case, where \( k_y \neq 0 \), the Talbot and the Montgomery condition each define a set of planes that intersect with the Ewald sphere. The combination of both conditions is expressed mathematically by

\[ (k_x^M)^2 + k_y^2 + (k_z^T)^2 = \left(\frac{2\pi}{\lambda}\right)^2. \quad (8) \]

Graphically, this case is depicted in Fig. 7. The equations \( k_x^T = m2\pi/p_x \) (green/horizontal lines) and \( k_z^M = n2\pi/p_z \) describe equidistant planes orthogonal to the \( k_x \) and the \( k_z \) axis, respectively. Each set of planes forms a set of circles on the Ewald sphere. The propagating modes in the 3D wave field are given by the intersection points of these circles.

Fig. 6. (Color online) Fabry–Perot resonator with periodically structured mirrors. (a) Setup with structured mirrors. (b) Construction of the angular spectrum. The dots indicate positions on the Ewald sphere, where Talbot and Montgomery conditions are both (approximately) satisfied. Note that since in general, \( p_x \ll p_z \), it is usually \( \Delta k_x \gg \Delta k_z \).

Fig. 7. (Color online) 3D representation of the Ewald sphere and the equations \( k_x^T = m2\pi/p_x \) (green/horizontal lines) and \( k_z^T = n2\pi/p_z \) (red/vertical lines). For simplicity, \( \Delta k_x = \Delta k_z \) in this figure.
5. Analytical Model and Simulations

In this section, we present a model for the setup based on a modal description. For simplicity, we again use a 2D presentation. As in the previous section, we first treat each interferometer individually and then combine the results. The basic consideration is that we describe each interferometer by its transfer function for the angular spectrum. The overall transfer function, taking into account both periodicities, is then given by the product of the two individual transfer functions.

Under normal plane wave illumination, the transmitted field of the grating interferometer is according to the Kirchhoff approximation given by the complex transmission function of the grating, \( g(x) \). We denote the transfer function of the grating interferometer by \( T_{\text{GI}}(k_x) \). It is related to the complex transmission function by a Fourier transformation:

\[
T_{\text{GI}}(k_x) = \frac{1}{2\pi} \int g(x) \exp(-ik_x x) dx. \tag{9}
\]

Since \( g(x) = g(x + p_x) \), the transfer function is discrete with maxima at \( k_x^{\text{GI}} = 2\pi/p_x \). We first assume the situation of a conventional Fabry–Perot resonator without structured mirrors. Furthermore, we assume that it is symmetric; this means both mirrors have the same amplitude reflectivity \( r \) and reflectance \( R = r^2 \). The dependency of the transmitted field as a function of the components of the \( k \) vector of the input beam can be described as

\[
\begin{align*}
\mathbf{u}_{\text{FPI}}(x, z) &= (1 - R) \cdot \exp[i(k_x^n x + k_z^n z)] \\
&\quad \cdot \sum_{j=0}^{\infty} R^j \exp(ik_z j p_z).
\end{align*}
\]

Note that energy conservation is satisfied due to \( R < 1 \). The \( x \) components of the \( k \) vectors, \( k_x^n \), are given by Eq. (5). Furthermore, it is \( (k_x^n)^2 + (k_z^n)^2 = (2\pi/\lambda)^2 \). For the paraxial case, it is convenient to replace the index \( n \) as shown in Fig. 5(b) by an index \( \bar{n} \) as shown in Fig. 8 with

\[
k_x^{\bar{n}} = \pm \frac{2\pi}{p_z} \left( \frac{2n p_z - \bar{n}^2}{\lambda^2} \right)^{1/2}, \tag{11}
\]

with \( \bar{n} = N - n \) and \( N \approx p_z/\lambda \). This means for \( \bar{n} = 0 \) it is \( k_x = 0 \) and for \( \bar{n} = N \) it is \( k_x = 2\pi/\lambda \).

Figures 9(a) and 9(b) show examples of the normalized transfer functions for GI and FPI, respectively, as a function of \( k_x \). With respect to the modal spectra shown in the figures, it should be noted that their purpose is to represent the directions under which light propagates behind a grating and a FPI, respectively. The weighting of the modes is left out of

\[
\begin{align*}
\Delta k_z &= 2\pi/p_z, \\
\end{align*}
\]

![Fig. 8](Color online) Ewald sphere representation of the propagating \((k_z > 0)\) and reflected \((k_z < 0)\) modes in the paraxial domain of the FPI. Note: First, the angular range for the paraxial regime is exaggerated here. Second, the largest value of \( k_z \) does not necessarily occur exactly at \( k_z = 0 \). However, for sufficiently large resonator length \( L \), the modal separation \( \Delta k_z \) will be very small compared to the radius of the Ewald sphere.

![Fig. 9](Color online) Exemplary transfer functions for (a) grating and (b) Fabry–Perot interferometer. For simplicity, the transfer function of the GI is shown without a variation in the height of the maxima, since the main purpose is to consider the \( k_x \) positions.
consideration. This means that we do not imply specifically the use of a specific grating that would, for example, generate a uniform array of diffraction orders.

To describe the output field that emerges from the FPI with structured mirrors, we make use of the concept of linear filter theory. In other words, we describe the total transfer function of a FPI with periodically structured mirrors as the product of the individual transfer functions:

$$T(k_x) = T_{GI}(k_x) T_{FPI}(k_x).$$ (12)

We would like to emphasize that this model is limited to the paraxial case. In the paraxial case, diffraction at the gratings leads to orders with $k_x$ values that are multiples of $2\pi/p_x$. The paraxial condition is warranted by the limited numerical aperture of the setup; i.e., higher order modes, which do not satisfy the paraxial condition, are diffracted out of the system. We would like to emphasize that in a 2D resonator, of course, light can couple into modes with $k_y \neq 0$. Here, however, we will continue with our 1D consideration.

In the following, we use Eq. (12) for a few simulations. As mentioned earlier, the important parameter is the reflectance $R$ of the mirrors. The following simulation results were obtained for four different values: $R = 0.01, 0.1, 0.5, \text{ and } 0.9$. Obviously, the increase of $R$ has a significant influence on the angular spectrum. In this particular case, one can notice that some of the lower orders of the GI spectrum are strongly suppressed, even for the moderate value of $R = 0.5$. This results in a “bandgap structure” of the angular spectrum as shown in our examples. Of course, the choice of parameters is arbitrary, since we can vary $p_x$ and $p_z$ independently. For other values of these two parameters, the effects may be less significant. Nonetheless, this example clearly shows that the simultaneous occurrence of symmetries in transverse and longitudinal directions can have a significant influence on the modal spectrum of the wave field.

6. Conclusion

Self-imaging is a fascinating aspect of the propagation of wave fields. Here, we considered the case of a wave field with imposed periodicities in transverse
and longitudinal directions, so that Talbot and Montgomery conditions for self-imaging occur independently. As a result, some of the modes in the angular spectrum may be suppressed, depending on the specific situation. This suppression of modes and the occurrence of “bandgaps” in the angular spectrum is somewhat reminiscent of the occurrence of the bandgap structure in photonic crystals, although the structural dimensions that we consider here are much larger than the wavelength. Nevertheless, by using the same concept of filter theory, it should also be possible to explain the bandgap structure of photonic crystals. In this paper, we have discussed the spatial properties of stationary, monochromatic wave fields. Numerical studies of light propagation in periodically modulated multimode waveguides have simultaneously been carried out [21]. The extension of these results to the nonstationary case, including short pulses in particular, is another interesting case to consider.

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