Area law for the entanglement entropy of the free Fermi gas at nonzero temperature

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The leading asymptotic large-scale behavior of the spatially bipartite entanglement entropy (EE) of the free Fermi gas infinitely extended in multidimensional Euclidean space at zero absolute temperature, $T=0$, is by now well understood. Here, we announce and discuss the first rigorous results for the corresponding EE of thermal equilibrium states at $T>0$. The leading large-scale term of this EE turns out to be twice the leading finite-size correction to the infinite-volume thermal entropy (density). Not surprisingly, this correction is just the thermal entropy on the boundary surface of the bipartition. However, it is given by a rather complicated analytical expression derived from semiclassical functional calculus and differs, at least at high temperature, from simpler expressions previously obtained by arguments based on conformal field theory. In the zero-temperature limit $T\downarrow 0$, the leading large-scale term of the thermal EE considerably simplifies and displays a $\ln(1/T)$-singularity which one may identify with the known logarithmic correction at $T=0$ to the so-called area-law scaling. Our results extend to the whole one-parameter family of (quantum) Rényi entropies.

PACS numbers: 03.65.Ud, 03.67.Mn, 05.30.Fk

Introduction and results. In recent years, entanglement entropy has turned out to be a useful and much studied quantifier of nonclassical correlations between subsystems in composite quantum systems [1]. In particular, given the (pure) ground state of a spatially large many-particle system and reducing (or restricting) it to a spatial subregion $\Omega$, we denote the von Neumann entropy of the resulting mixed state by $S(0,\Omega)$ and call it the local ground-state entropy. The spatially bipartite entanglement entropy (EE), defined for a bounded $\Omega$ as the (quantum) mutual information relative to the complement of the bipartition, we are confronted with the fact that $S(0,\Omega)$ is the spatial dimension of $\Omega$. If the particles within $\Omega$ are correlated with all those outside $\Omega$, it is therefore, for example, well suited to detect the appearance of long-range correlations near the critical point of a quantum phase transition by enlarging $\Omega$, [2,3]. For a many-particle system without long-range interactions the ground-state EE

$$H(0,L\Omega) = 2S(0,L\Omega) , \quad L \geq 1$$

is widely believed [4] to grow to leading order proportional to the area $\partial \Omega |\partial \Omega|^{d-1}$ of the boundary surface $\partial L\Omega$ of the scaled region $L\Omega$ as the (dimensionless) scaling parameter $L$ tends to infinity, $L \to \infty$. Here, $d=1,2,3,\ldots$ is the spatial dimension of $\Omega$. If the particles are fermions and if there is no spectral gap above their ground-state energy in the infinite-volume limit, the effective long-range correlations lurking in the Fermi–Dirac statistics are expected to slightly enhance the large-scale behavior of $H(0,L\Omega)$ by a logarithmic factor $\ln(L)$. In fact, for the free Fermi gas infinitely extended in the $d$-dimensional Euclidean space $\mathbb{R}^d$ such a large-scale behavior of the ground-state EE with a precise and rather explicit prefactor has been proved recently with full mathematical rigor [5], thereby confirming a stimulating conjecture by Gioev and Klich [6].

In this paper, we announce and discuss the first rigorous results on the EE of the free Fermi gas in $\mathbb{R}^d$ in the state of thermal equilibrium at arbitrary nonzero absolute temperature $T>0$ and with chemical potential $\mu \in \mathbb{R} := [0,\infty)$. The latter we will mostly suppress for notational simplicity, but also because we will often consider thermal properties for fixed (mean) particle density $\rho > 0$. In contrast with the ground-state or $T=0$ case, the EE at $T>0$, denoted by $H(T,\Omega)$, must not be expected to be just twice the local thermal entropy $S(T,\Omega) < \infty$, because the thermal state is a mixed one. Moreover, in trying to define $H(T,\Omega)$ as the mutual information

$$H(T,\Omega) = S(T,\Omega) + \frac{1}{2} (S(T,\mathbb{R}^d \setminus \Omega) - S(T,\mathbb{R}^d))$$

of the bipartition, we are confronted with the fact that $S(T,\mathbb{R}^d \setminus \Omega) = S(T,\mathbb{R}^d) = \infty$ by the extensivity (that is, macroscopic additivity) of thermal entropy at $T>0$. We will overcome this problem by subtracting (or regularizing) these two infinities in a physically natural way, see [18] and [19] below. By construction, $H(T,L\Omega)$ will then turn out to be not infinite and will not have an “extensive” term of the order $L^d$, but will exhibit a leading term proportional to $L^{d-1}$ as $L \to \infty$. Given that, our general line of arguments is similar to that of Ref. [7] devoted to noninteracting fermions in the $d$-dimensional cubic lattice $\mathbb{Z}^d$.

Our main results may be summarized as follows. For the (spinless) free Fermi gas in $\mathbb{R}^d$ we find at any temperature $T>0$ the following two asymptotic large-scale
expansions: the local thermal entropy satisfies
\[ S(T,L\Omega) = s(T)|\Omega|L^d + \eta(T,\partial\Omega)L^{d-1} + \ldots \] (3)
and the thermal EE satisfies
\[ H(T,L\Omega) = 2\eta(T,\partial\Omega)L^{d-1} + \ldots \] (4)
up to terms growing slower than \( L^{d-1} \) as \( L \to \infty \). Here, the bounded subregion \( \Omega \subset \mathbb{R}^d \) may be rather general except that it should consist of only finitely many connected components and its boundary surface \( \partial\Omega \) (if \( d \geq 2 \)) should be piecewise sufficiently smooth. For further assumptions see our theorem below. There, we will also make the definitions of the entropies \( S(T,\Omega) \) and \( H(T,\Omega) \) as well as of the coefficient \( \eta(T,\partial\Omega) \) more precise. For the time being, we just define \( s(T) \) and offer some explanations and comments.

Not surprisingly, the leading asymptotic coefficient \( s(T)|\Omega| \) in (3) is nothing but the thermal entropy contained in \( \Omega \) and \( s(T) \geq 0 \) is the (infinite-volume) thermal entropy density. It can be written as
\[ s(T) = \frac{\partial}{\partial T} p(T), \quad p(T) := \int_{\mathbb{R}} dE \mathcal{N}(E) f_T(E - \mu), \] (5)
where the integral is the (grand canonical) pressure of the free Fermi gas as a function of \( T \) (and \( \mu \)). The quantity
\[ \mathcal{N}(E) := (2\pi\hbar)^{-d} \int_{\mathbb{R}^d} dp \Theta(E - \varepsilon(p)) \] (6)
defines the integrated density of states \( \mathcal{N} : \mathbb{R} \to [0,\infty[ \) of the energy-momentum (dispersion) relation \( \varepsilon : \mathbb{R}^d \to [0,\infty[ \) which characterizes the translation-invariant one-particle Hamiltonian of the free Fermi gas. For convenience, we have assumed \( \varepsilon \geq 0 \). This is no loss of generality as long as \( \varepsilon \) is bounded from below. As usual, \( 2\pi\hbar \) is Planck’s constant and \( \Theta \) is Heaviside’s unit-step function. The second factor of the integrand in (6) involves the Fermi function, \( f_T : \mathbb{R} \to [0,1] \), given by \( f_T(E) := (1 + \exp(E/T))^{-1} \). Here and in the following, we have put Boltzmann’s constant \( k_B \) equal to one. From Sommerfeld’s asymptotic low-temperature expansion (written in distributional form),
\[ f_T(E) = \Theta(-E) - (\pi^2/6) T^2 \Theta''(E) + \ldots, \] (7)
we recall the well-known formula
\[ s(T) = (\pi^2/3) \mathcal{N}''(\varepsilon_F) T + \ldots \] (8)
up to terms vanishing faster than \( T \) as \( T \downarrow 0 \). Eq. (8) holds, as it stands, for fixed chemical potential \( \mu \in \mathbb{R} \). As usual, if the particle density \( \rho > 0 \) is kept fixed instead of \( \mu \), one has to invert the relation \( \rho = \partial\varepsilon/\partial\mu \) between \( \rho \) and \( \mu \). By [7], one thus finds at low temperatures that
\[ \mu(T,\rho) = \varepsilon_F - (\pi^2/6) [\mathcal{N}''(\varepsilon_F)/\mathcal{N}'(\varepsilon_F)] T^2 + \ldots, \] (9)
where \( \varepsilon_f := \lim_{T\to0} \mu(T,\rho) > 0 \) is the Fermi energy which satisfies \( \rho = \mathcal{N}(\varepsilon_f) \). As a consequence, Eq. (8) implies \( s(T) = (\pi^2/3) \mathcal{N}'(\varepsilon_F) T + \ldots \) for fixed \( \rho > 0 \). For the ideal (free) Fermi gas, corresponding to the prime example \( \varepsilon(p) = p^2/(2m) \) with \( m \) the mass of each particle, we recall the well-known power-law form
\[ \mathcal{N}(E) = \Theta(E) [mE/(2\pi\hbar^2)]^{1/2}/(d/2)! \] (10)
of its integrated density of states. Quantum effects dominate at low temperatures and become increasingly weaker at intermediate and high temperatures. Accordingly, the thermal properties of the ideal Fermi gas approaches those of the ideal Maxwell–Boltzmann gas in the high-temperature limit, \( T \to \infty \). For example, in this limit the thermal entropy density of the ideal Fermi gas grows, to leading order, proportional to \( T^{d/2} \) for fixed \( \mu \) and proportional to \( \ln(T) \) for fixed \( \rho \), in symbols,
\[ s(T) \sim T^{d/2} (\mu \text{ fixed}), \quad s(T) \sim \ln(T) (\rho \text{ fixed}) \] (11)
Now we turn to the other asymptotic coefficient, called \( \eta(T,\partial\Omega) \). It is non-negative and shows up in (3) and (4). On the one hand, it is the thermal entropy on the boundary surface \( \partial\Omega \) and determines the leading finite-size correction to the infinite-volume entropy (density). On the other hand, \( \eta(T,\partial\Omega)L^{d-1} \) is one half of the EE to leading order in \( L \). As a consequence, Eqs. (3) and (4) together show that the EE of the free Fermi gas at temperature \( T > 0 \) displays a large-scale behavior in agreement with a strict “area law”: although the two subregions \( L\Omega \) and its complement \( \mathbb{R}^d \setminus L\Omega \) carry (contrary to the case \( T = 0 \)) extremely different local entropies (namely \( S(T,L\Omega) < \infty \) and \( S(T,\mathbb{R}^d \setminus L\Omega) = \infty \)), the entropy on their common boundary \( \partial L\Omega \) is the same and proportional to \( L^{d-1} \) as \( L \to \infty \). Roughly phrased, the logarithmic correction \( \ln(L) \) present in the ground-state EE disappears when the temperature is raised from \( T = 0 \) to \( T > 0 \), because the Fermi surface “grows soft”. We stress that the coefficient \( \eta(T,\partial\Omega) \geq 0 \) does not depend on the choice of a condition imposed on the domain of the Hamiltonian at the boundary \( \partial\Omega \). This is because we work from the outset in the infinitely extended position space \( \mathbb{R}^d \) and view all operators to act self-adjointly on the associated one-particle Hilbert space \( L^2(\mathbb{R}^d) \) of square-integrable functions \( \psi : \mathbb{R}^d \to \mathbb{C}, q \mapsto \psi(q) \).

The coefficient \( \eta(T,\partial\Omega) \) turns out to be given by a rather complicated expression, see [20–23] below (with \( \alpha = 1 \)). It can be derived from Refs. [8] and [9] with additional work [14]. To the best of our knowledge, this coefficient has not been written down before in the present context. Interestingly, under mild assumptions additional to those mentioned in the theorem below (see [14]) the coefficient \( \eta(T,\partial\Omega) \) displays a logarithmic singularity in the zero-temperature limit and takes, for fixed \( \mu > 0 \), the simple form
\[ \eta(T,\partial\Omega) = (1/12) J(\partial\Gamma_p, \partial\Omega) \ln(\mu/T) + \ldots \] (12)
up to terms remaining bounded as $T \downarrow 0$. Here, the level set \( \partial \Gamma_{\mu} := \{ p \in \mathbb{R}^d : \varepsilon(p) = \mu \} \) in momentum space is the (effective) Fermi surface corresponding to \( \mu \). The factor $J(\partial \Gamma_{\mu}, \partial \Omega)$ is defined as in Ref. [5] and for $d \geq 2$ given by the double-surface integral

$$J(\partial \Gamma_{\mu}, \partial \Omega) := (2\pi \hbar)^{1-d} \int d\sigma(p) d\tau(q) |m(p) \cdot n(q)|. \quad (13)$$

The vectors $m(p), n(q) \in \mathbb{R}^d$ denote the exterior unit normals at the points $p \in \partial \Gamma_{\mu}$ and $q \in \partial \Omega$, respectively. The canonical $(d-1)$-dimensional area measures on the boundary surfaces $\partial \Gamma_{\mu}$ and $\partial \Omega$ are denoted by $\sigma$ and $\tau$, respectively. [If $\mu < 0$, then $\eta(T, \partial \Omega)$ simply vanishes as $T \downarrow 0$.] If we fix the particle density $\rho > 0$ and use [9] in [12], we arrive at [12] with $\mu$ replaced by $\varepsilon_F$. Accordingly, if one identifies the large ratio $\mu/T$ (resp. $\varepsilon_F/T$) inside the logarithm in [12] with the scaling parameter $L$, Eq. [5] gives

$$S(0, L\Omega) = (1/12) J(\partial \Gamma_{\mu}, \partial \Omega) L^{d-1} \ln(L) + \ldots \quad (14)$$

in agreement with the result for $T = 0$ in Refs. [5, 6] (resp. the corresponding expression with $\mu$ replaced by $\varepsilon_F$). For an isotropic (dispersion) function $\varepsilon$ we know from Ref. [5] that $J(\partial \Gamma_{\mu}, \partial \Omega)$ is proportional to the area $|\partial \Omega|$, note [11]. This is even true for $\eta(T, \partial \Omega)$ itself, at arbitrary temperature $T > 0$. However, the emerging prefactor, the thermal entropy surface density, remains to be given by a multifold integral, see below Eq. [25].

As for the entropy density $s$, the leading high-temperature behavior of the coefficient $\eta$ of the ideal Fermi gas depends on whether $\mu$ or $\rho$ is kept fixed. More precisely, it can be shown that

$$\eta(T, \partial \Omega) \sim T^{(d-1)/2} \quad (\mu \text{ fixed}), \quad (15a)$$

$$\eta(T, \partial \Omega) \sim T^{-1/2} \quad (\rho \text{ fixed}) \quad (15b)$$

as $T \to \infty$. The last asymptotics reflects the fact that for fixed particle density and sufficiently high temperature the particles become uncorrelated.

**Precise definitions and formulation of results for general Rényi entropies.** In order to define the local thermal entropy and the thermal EE we recall that the infinite-volume equilibrium state of the free Fermi gas at temperature $T > 0$ with chemical potential $\mu \in \mathbb{R}$ is completely determined by the one-particle density operator given by the Fermi operator $f_T(\varepsilon(P) - \mu \mathbb{1})$ on $L^2(\mathbb{R}^d)$. Here, $P := -i\hbar \partial / \partial \varepsilon$ is the canonical momentum operator, $\varepsilon(P) \geq 0$ the self-adjoint one-particle Hamiltonian, and $\mathbb{1}$ the identity operator. The local version

$$D(f_T, \Omega) := \chi_\Omega(Q) f_T(\varepsilon(P) - \mu \mathbb{1}) \chi_\Omega(Q) \quad (16)$$

of the Fermi operator then characterizes the equilibrium state after spatial reduction to $\Omega \subset \mathbb{R}^d$. Here, $\chi_\Omega$ denotes the indicator function of $\Omega$ and $Q$ the usual position operator so that the operator $\chi_\Omega(Q) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ acts as a projection according to $(\chi_\Omega(Q) \psi)(q) = \psi(q)$ if $q \in \Omega$ and $0$ otherwise for all $\psi \in L^2(\mathbb{R}^d)$. For $\alpha > 0$ with $\alpha \neq 1$ we define the function $h_\alpha : [0, 1] \to [0, \ln(2)]$ by $h_\alpha(t) := (1 - t)^{-1} \ln \left( \frac{(1 + (1-t)^\alpha)}{\alpha} \right)$. For $\alpha = 1$ we set $h_1(t) := -t \ln(t) - (1-t) \ln(1-t)$ if $t \in [0, 1]$ and $h_1(0) := h_1(1) := 0$. The local thermal Rényi entropies are then given as the traces

$$S_\alpha(T, \Omega) := Tr h_\alpha(D(f_T, \Omega)), \quad \alpha > 0. \quad (17)$$

Whenever we suppress the Rényi index $\alpha > 0$, as in the Introduction, then we mean the von Neumann (or ordinary thermal) limiting case $\alpha = 1$. For the definition of the thermal Rényi EE’s we first introduce for each $\alpha > 0$ the regularized (and local) thermal entropy operator

$$\Delta_\alpha(T, \Omega) := h_\alpha(D(f_T, \Omega)) - D(h_\alpha \circ f_T, \Omega). \quad (18)$$

Here, the operator $D(h_\alpha \circ f_T, \Omega)$ is obtained from [16] by replacing the Fermi function $f_T$ with the composition $h_\alpha \circ f_T$ defined by $(h_\alpha \circ f_T)(E) := h_\alpha(f_T(E))$ for all $E \in \mathbb{R}$. Now we are able to define the thermal $\alpha$-Rényi EE for a bounded $\Omega \subset \mathbb{R}^d$ as the sum of two traces

$$H_\alpha(T, \Omega) := Tr \Delta_\alpha(T, \Omega) + Tr \Delta_\alpha(T, \mathbb{R}^d \setminus \Omega). \quad (19)$$

This is the precise version of Eq. [2]. By the concavity of $h_\alpha$ and the Jensen–Berezin inequality (cf. Eq. (8.18) in Ref. [12]) we see that $Tr \Delta_\alpha(T, \Omega) \geq 0$ and $Tr \Delta_\alpha(T, \mathbb{R}^d \setminus \Omega) \geq 0$ so that $H_\alpha(T, \Omega) \geq 0$. It is nontrivial to show that the second trace on the right-hand side of (19) is not infinite and hence $H_\alpha(T, \Omega) < \infty$, cf. [10]. The symmetry $H_\alpha(T, \Omega) = H_\alpha(T, \mathbb{R}^d \setminus \Omega)$ is then obvious by definition [19]. In order to explain the coefficient $\eta_\alpha(T, \partial \Omega)$ of the leading asymptotic behavior of $H_\alpha(T, \Omega)$ as $L \to \infty$ we have to introduce some more definitions. Firstly, we define the function $U_\alpha : [0, 1] \times [0, 1] \to [0, \infty]$, $(r, t) \mapsto U_\alpha(r, t)$ by

$$U_\alpha(r, t) := \int_0^1 d\lambda \frac{h_\alpha((1 - \lambda)r + \lambda t) - (1 - \lambda)h_\alpha(r) - \lambda h_\alpha(t)}{\lambda(1 - \lambda)}, \quad (20)$$

and, secondly, we consider for a function $g : \mathbb{R} \to [0, 1]$ the integral

$$U_\alpha[g] := \frac{1}{8\pi^2} \int_{\mathbb{R} \times \mathbb{R}} du dv \frac{U_\alpha(g(u), g(v))}{(u - v)^2}. \quad (21)$$

We note that $U_\alpha(r, t) \geq 0$ and hence $U_\alpha[g] \geq 0$ by the concavity of $h_\alpha$.

Now, if $d \geq 2$, we consider for $g$ the $2(d-1)$-parameter family of functions $f_{T,(p,q)} : \mathbb{R} \to [0, 1]$ defined (for given $T > 0$) in terms of the Fermi function by

$$f_{T,(p,q)}(v) := f_T(\varepsilon(p + v n(q)) - \mu), \quad (22)$$
where \( v \in \mathbb{R}, q \in \partial \Omega \) and \( p \in T_q^* (\partial \Omega) \) with \( T_q^* (\partial \Omega) \cong \mathbb{R}^{d-1} \) being the dual space of the tangent space of \( \partial \Omega \) at the point \( q \). Moreover, as in [13] the vector \( n(q) \in \mathbb{R}^d \) is the exterior unit normal at \( q \in \partial \Omega \) and \( \tau \) is the canonical area measure on \( \partial \Omega \). Finally, we are ready to define

\[
\eta_\alpha (T, \partial \Omega) := (2\pi \hbar)^{-1/d} \int_{\partial \Omega} dt(q) \int_{T_q^* (\partial \Omega)} dp \, \mathcal{U}_\alpha (f_T (p, q)) .
\]  

(23)

If the function \( \varepsilon \) is isotropic, then \( f_T (p, q) \) does not depend on \( q \) by the orthogonality of \( p \in T_q^* (\partial \Omega) \) and \( n(q) \). Consequently, the area \( |\partial \Omega| \) can be factored out on the right-hand side of (23). Nevertheless, by the definitions of \( U_\alpha \) and \( \eta_\alpha \) the remaining integral underlying \( \eta_\alpha (T, \partial \Omega) \) is seen to be still a fourfold one for \( d \geq 2 \). For \( d = 1 \), the set \( \Omega \subset \mathbb{R} \) is a finite union of pairwise disjoint bounded intervals. Then \( |\partial \Omega| \) equals the (even) number of all endpoints of these intervals and one has \( \eta_\alpha (T, \partial \Omega) = U_\alpha (f_T \circ (\varepsilon - \mu)) |\partial \Omega| \), which involves a threefold integral.

Now we are prepared to state our main result as the following

**Theorem.** Let the function \( \varepsilon \) be smooth and polynomially bounded in the sense that (i) \( c_1 |p|^{\gamma_1} \leq \varepsilon (p) \leq c_2 |p|^{\gamma_2} \) for all \( p \in \mathbb{R}^d \) outside a ball centered at the origin with some radius \( p_0 > 0 \) and (ii) \( |\nabla^n \varepsilon (p)| \leq c_3 |p|^{\gamma_3} \) for all \( n \in \{0, 1, \ldots, (d+1)/\alpha + 1\} \) with suitable constants \( c_i, \gamma_i \in ]0, \infty[ \), \( \gamma_1 \geq \gamma_2 \) and \( |x| \) denoting the integer part of \( x > 0 \). Furthermore, let \( \Omega \subset \mathbb{R} \) be a (bounded) \( C^1 \)-smooth Lipschitz domain having (at most) finitely many connected components (implying that its boundary surface \( \partial \Omega \) is piecewise \( C^1 \)-smooth). Finally, let \( \Lambda = \Omega \) or \( \Lambda = \mathbb{R}^d \setminus \Omega \). Then, \( \text{Tr} \Delta_\alpha (T, \Lambda) \geq 0 \) exists for any \( T > 0 \) and has the asymptotic large-scale expansion

\[
\text{Tr} \Delta_\alpha (T, \Lambda) = \eta_\alpha (T, \partial \Omega) L^{d-1} + \ldots
\]  

(24)

up to terms growing slower than \( L^{d-1} \) as \( L \to \infty \). [For the notion of a “\( C^1 \)-smooth Lipschitz domain” see Ref. [13].]

This theorem and formula (12) will be proved in forthcoming mathematical publications, Refs. [10], where we will also deal with the non-smoothness of the function \( h_\alpha \) similarly as in Ref. [5]. Combining Eqs. (19) and (24) immediately gives the large-scale behavior of the thermal \( \alpha \)-Rényi EE, namely

\[
H_\alpha (T, L \Omega) = 2 \eta_\alpha (T, \partial \Omega) L^{d-1} + \ldots
\]  

(25)

For \( \alpha = 1 \) it reduces to Eq. (1). From (24) we also infer the two-term large-scale behavior of the local thermal \( \alpha \)-Rényi entropy (17) as follows

\[
S_\alpha (T, L \Omega) = \text{Tr} D(h_\alpha \circ f_T, L \Omega) + \text{Tr} \Delta_\alpha (T, L \Omega)
\]

\[
= s_\alpha (T) L^{d} + \text{Tr} \Delta_\alpha (T, L \Omega)
\]

\[
= s_\alpha (T) L^{d} + \eta_\alpha (T, \partial \Omega) L^{d-1} + \ldots
\]  

(26)

with the thermal \( \alpha \)-Rényi entropy density

\[
s_\alpha (T) := (2\pi \hbar)^{-d} \int_{\mathbb{R}^d} dp \, h_\alpha (f_T (p) - \mu)
\]

\[
= \int_{\mathbb{R}^d} dE \, \mathcal{N}'(E) h_\alpha (f_T (E - \mu)) .
\]  

(27)

For \( \alpha \neq 1 \), an integration by parts gives

\[
s_\alpha (T) = \frac{\alpha}{(\alpha - 1)T} \left[ p(T) - p(T/\alpha) \right]
\]  

(28)

with \( p(T) \) being the pressure, given by the integral in [5]. As \( \alpha \to 1 \), Eq. (28) turns into the derivative in [5], and (26) turns into [3]. The leading large-scale behavior of the local entropy \( S_\alpha (T, \Omega) \) at \( T > 0 \) was first proved (for \( \alpha = 1 \)) in [14]. The sub-leading correction of the order \( L^{d-1} \) in (26) is new, even for \( \alpha = 1 \).

The leading low-temperature behaviors of \( s_\alpha (T) \) and \( \eta_\alpha (T, \partial \Omega) \) are given by the right-hand sides of (8) and (12), respectively, when multiplied by \( (1 + \alpha)/(2\alpha) \). The leading high-temperature behaviors of \( s_\alpha (T) \) and \( \eta_\alpha (T, \partial \Omega) \) are for fixed \( \mu \), up to a different prefactor, the same as for \( \alpha = 1 \) as given by (11) and (15a), respectively. For fixed \( \rho \) their leading high-temperature behaviors turn out to be \( \alpha \)-dependent and given by \( s_\alpha (T) \sim T^{(d/2) \max(0,1-\alpha)} \) (if \( \alpha \neq 1 \)) and \( \eta_\alpha (T, \partial \Omega) \sim T^\delta \) with \( \delta := (d-1)/2 - (d/2) \min(\alpha, 2) \). The corresponding prefactors are explicitly computable, like the ones in (11) and (15).

**Summary and discussion.** For the free Fermi gas in multidimensional continuous space \( \mathbb{R}^d \) we have studied the spatially bipartite entanglement entropy (EE) of the infinite-volume thermal equilibrium state characterized by a nonzero temperature \( T > 0 \) (and a chemical potential \( \mu \) which may have either sign). Given a bipartition \( \mathbb{R}^d = \Omega \cup (\mathbb{R}^d \setminus \Omega) \) of \( \mathbb{R}^d \) into a bounded subregion \( \Omega \subset \mathbb{R}^d \) and its (unbounded) complement, it is more or less straightforward to define [13] the local thermal entropy \( S(T, \Omega) < \infty \) as the von Neumann entropy of the equilibrium state reduced to \( \Omega \). The definition of the thermal EE, which we have denoted by \( H(T, \Omega) \), is more complicated. We have defined it as in (2) by regularizing the difference “...” of two infinities in a natural way. We have found the \( a \ priori \) estimates \( 0 \leq H(T, \Omega) < \infty \) and the symmetry \( H(T, \mathbb{R}^d \setminus \Omega) = H(T, \Omega) \). Moreover, in the zero-temperature limit \( T \downarrow 0 \) we get (back) the ground-state EE, \( H(0, \Omega) = 2S(0, \Omega) \).

In replacing \( \Omega \) by the scaled region \( L \Omega \) we have obtained the large-scale formulas (3) and (4) for the local entropy and the EE at nonzero temperature. These formulas contain the ordinary thermal entropy density \( s(T) \) and a new asymptotic coefficient \( \eta(T, \partial \Omega) \). Although the new coefficient is given by a rather complicated analytical expression, it is exact and a characteristic of the free Fermi gas—a basic and very useful model system of Theoretical Physics for nearly 90 years now.
In the zero-temperature limit \( T \downarrow 0 \) the coefficient \( g(T, \partial \Omega) \) drastically simplifies, displays (at fixed particle density \( \rho \)) a \( \ln(\varepsilon F/T) \) singularity, and takes the explicit form \( (12) \) with \( \mu \) replaced by \( \varepsilon_{F} \). In this form it provides the logarithmic factor of the leading large-scale behavior of the ground-state EE, if one identifies the (large) ratio \( \varepsilon_{F}/T \) with the scaling parameter \( L \). A similar observation (without an explicit prefactor) has recently been made \[7\] for noninteracting fermions in the one-dimensional lattice \( \mathbb{Z}^{1} \).

We have presented rigorous results not only for the von Neumann entropy, but for the whole one-parameter family of (quantum) Rényi entropies.

Finally, we are going to compare our findings for the large-scale behavior of the thermal entropy \( S_{\alpha}(T, L\Omega) \) with predictions of so-called universal crossover formulas obtained \[15\] \[16\] by arguments based (at least for \( d = 1 \)) on conformal field theory \[17\] \[18\]. Such a formula has the simple and appealing form

\[
S_{\alpha}(T, L\Omega) = L^{d-1}A_{\alpha} \ln \left[ \frac{T_{0}}{T} \sinh \left( \frac{L}{L_{0}} \frac{T_{0}}{T} \right) \right] \tag{29}
\]

with suitable constants \( A_{\alpha} \geq 0, L_{0} > 0, \) and \( T_{0} > 0 \) not depending on \( L \) and \( T \). Remarkably, for any \( L \geq 1 \) formula \( (29) \) implies

\[
S_{\alpha}(0, L\Omega) := \lim_{T \downarrow 0} S_{\alpha}(T, L\Omega) = L^{d-1}A_{\alpha} \ln(L/L_{0}), \tag{30}
\]

whereas for any \( T > 0 \) it gives

\[
\lim_{L \rightarrow \infty} \frac{1}{L^{d}} S_{\alpha}(T, L\Omega) = \frac{A_{\alpha}}{T} \frac{T_{0}}{L_{0}} \tag{31}
\]

and

\[
\lim_{L \rightarrow \infty} \frac{1}{L^{d-1}} \left[ S_{\alpha}(T, L\Omega) - L^{d} \frac{A_{\alpha} T}{L_{0} T_{0}} \right] = A_{\alpha} \ln \left( \frac{T_{0}}{2T} \right). \tag{32}
\]

Comparing these three limiting equations with the exact results \[14\] \[26\], \[5\], and \[12\] in combination with those in the last paragraph of the previous section, we note the following: Eq. \( (30) \) is consistent with \( (14) \) if \( A_{\alpha} \) is chosen to be \( (1 + \alpha)/(24\alpha) \) \( J(\partial \Omega_{\mu}, \partial \Omega) \) and if \( L \gg L_{0} \). Eq. \( (31) \) is then consistent with \( (26) \) if the “crossover temperature” \( T_{0} \) is chosen to be \( 1/\sqrt{A_{\alpha}}(\mu/|\Omega|) \), \( L_{0} \) equals \( 3A_{\alpha}/\pi^{2} \), and if \( T \ll T_{0} \), see \( (8) \). Finally, Eq. \( (32) \) is consistent with \( (26) \) if \( T \ll T_{0} \), see \( (12) \). We conclude that formula \( (29) \) correctly reflects properties of the free Fermi gas in \( \mathbb{R}^{d} \) if \( L \) is large and \( T \) is small, but does, for example, not reproduce the high-temperature behaviors \[11\] and \[15\] (of the ideal Fermi gas). At present, we do not know how to derive \( (29) \) or a similar formula rigorously from the microscopic model underlying the free Fermi gas, not even for \( d = 1, \alpha = 1, \) and small \( T \).

Acknowledgement. We thank Ingo Peschel (FU Berlin) for valuable discussions.

11. In particular, for the ideal Fermi gas, corresponding to \( \varepsilon(p) = p^{2}/(2m) \), one simply has \( J(\partial \Omega_{\mu}, \partial \Omega) = 2N_{d-1}(\mu)/|\Omega| \), where \( N_{d-1}(E) \) is given by the right-hand side of \( (10) \) with \( d \) replaced by \( d - 1 \).